OBSERVABILITY OF COUPLED SYSTEMS

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Abstract. By applying the theory of semigroups, we generalize an earlier result of Komornik and Loreti [5] on the observability of compactly perturbed systems. As an application, we answer a question of the same authors concerning the observability of weakly coupled linear distributed systems.

1. Introduction

Consider the evolutionary problem

$$x' = (A+B)x, \qquad x(0) = x_0$$

where A and B are linear operators in a complex separable Hilbert space H. B is supposed to be compact, it is a so-called compact perturbation. We study the observability of the system, that is, given a finite number of seminorms $p_1, \ldots, p_m$ in H (the observations) and a finite number of intervals $I_1, \ldots, I_m$ in $\mathbb{R}$, (here every interval is finite and not reduced to a point) we are wondering whether these observations are sufficient to distinguish solutions corresponding to different initial data. More precisely, we ask whether we have

$$c\|x_0\|^2 \le \sum_{j=1}^{m} \int_{I_j} p_j(x(t))^2 \, dt \tag{1.1}$$

with some positive constant c independent of the particular choice of x_0, which may be different at different places. We also study the estimates

$$\sum_{j=1}^{m} \int_{I_j} p_j(x(t))^2 \, dt \le c\|x_0\|^2.$$

Here we suppose that the unperturbed system (i.e. with $B = 0$) is observable, at least if the initial data belong to a certain finite codimensional subspace, and thus one can ask whether the perturbed system is also observable. In many concrete cases, A is a skew-adjoint operator having a compact resolvent and thus A is diagonalisable with an orthonormal basis which is an excellent framework to study the

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estimates. However, orthonormal bases don't often resist to compact perturbations. In fact, looking only for norm equivalences, we can extend the framework to bases which are the images of orthonormal ones by a Banach isomorphism (i. e without keeping necessarily the orthogonality) : the Riesz bases. In fact, if there exists a Riesz basis formed by ordinary and generalized eigenvectors of $A+B$, we can, under natural additional assumptions conclude to the observability. Nevertheless, it is not always easy to prove that the perturbed operator admits a Riesz basis of eigenvectors and sometimes it is not even the case. In order to understand this phenomenon, let us consider a class of operators which are stable under a Riesz sum of finite dimensional spaces. To be more precise, fix a doubly indexed Riesz basis $\{e_{k,l} \ : \ k \geq 1, 1 \leq l \leq m_k\}$ with a bounded sequence (m_k) of positive integers, and introduce the finite dimensional spaces

$$Z_k = \{\text{Vect } e_{k,l} \ : \ 1 \leq l \leq m_k\}.$$

Then we build an operator C, stable under the Z_k, by the giving of endomorphisms $A_k : Z_k \to Z_k$:

$$D(C) := \left\{x = \sum x_{k,l}e_{k,l} : \sum A_k x_{k,l} e_{k,l} \in H\right\},$$
$$Cx := \sum A_k x_{k,l} e_{k,l}.$$

We can show that C is closed and that if an unbounded linear operator is closable and stable under the Z_k then it coincides with C on its domain. Furthermore, the initial value problem

$$x'(t) = Cx(t), \qquad t \in \mathbb{R},$$
$$x(0) = x_0 \in D(C)$$

has a unique continuously differentiable solution such that

$$\|x(t)\| \leq c\|x_0\|$$

with a constant c, (which may depend on the time t, but remains independent of the initial data x_0), if and only if $\exp(tA_k)$ is bounded (for a certain norm: we can choose an arbitrary norm on each $\mathbb{C}^{m_k}$ since (m_k) is bounded, the same norm in $\mathbb{C}^{m_k}$ and $\mathbb{C}^{m_\ell}$, if $k \neq \ell$, but $m_k = m_\ell$), for each t $\in \mathbb{R}$. We say then that the problem is well posed for C, and that C generates a strongly continuous group (see [7] for a general definition).

For instance, the problem is well posed for a closed operator A if the latter has a Riesz basis of (generalized) eigenvectors with bounded real parts of their eigenvalues. However, this property may be lost in case of compact perturbations:

Example. Setting

$$A_k = \begin{pmatrix} \lambda_k & k(-\lambda_k + \mu_k) \\ 0 & \mu_k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & 1/k \end{pmatrix} \begin{pmatrix} \lambda_k & \\ & \mu_k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1/k \end{pmatrix}^{-1}.$$

The problem is well-posed for A if the sequences

$$\Re(\lambda_k), \Re(\mu_k) \text{ and } k(-\lambda_k + \mu_k) \text{ are bounded}$$

(it is a bounded perturbation of a C^0 semi-group), but the eigenvectors

$$e_{k,1}, e_{k,1} + \frac{1}{k} e_{k,2}$$

don't form a Riesz basis. (We see here that bringing together the eigenvalues may lead to the loss of the independence of the eigenvectors at infinity.) In particular, we notice that if

$$k(-\lambda_k + \mu_k) \to 0$$

and

$$\Im(\lambda_k), \Im(\mu_k) \to \infty,$$

then we have a compact perturbation of a skew adjoint operator with a compact resolvant.

In [5], general observability results were established for compactly perturbed operators under the assumption that there exists a Riesz basis of generalized eigenvectors. The purpose of this paper is to extend that result so as to include cases like the above example. We will also give a concrete application where this more general result is essential.

2. Observability results

Let $A : D(A) \subset H \to H$ be an unbounded linear operator in a separable Hilbert space H and $B : H \to H$ a continuous linear operator. We suppose that A generates a strongly continuous group S_A. Since B is continuous, $A + B$ also generates a strongly continuous group S_{A+B}. See for example [7].

Let L be a finite-codimensional subspace of H. Concerning the direct inequality, we assume that:

$$\sum_{j=1}^{m} \int_{I_j} p_j(S_A(t)x_0)^2 dt \le c \|x_0\|^2 \text{ for all } x_0 \in L, \tag{2.1}$$

and we want to deduce from this the estimate

$$\sum_{j=1}^{m} \int_{J_j} p_j(S_{A+B}(t)x_0)^2 dt \le c \|x_0\|^2 \quad \text{for all } x_0 \in H, \tag{2.2}$$

for every choice of intervals J_j.

Concerning the inverse inequality, we assume that

$$c \|x_0\|^2 \le \sum_{j=1}^{m} \int_{I_j} p_j(S_A(t)x_0)^2 dt \text{ for all } x_0 \in L. \tag{2.3}$$

We then want to deduce

$$c\,\|x_0\|^2 \leq \sum_{j=1}^{m} \int_{J_j} p_j(S_{A+B}(t)x_0)^2 dt \text{ for all } x_0 \in \widetilde{L}, \tag{2.4}$$

where J_j are intervals such that they contain the closure of I_j in their interior, and $\widetilde{L}$ is a finite codimensional subspace as big as possible, that is,

$$H = \widetilde{L} \oplus \overline{M}$$

where M (respectively $\overline{M}$) is the (respectively closed) linear hull of all vectors $x \in H$ which satisfy for some complex number λ and for some nonnegative integer k the equalities

$$p_j((A + B - \lambda \mathrm{Id}\,)^{\ell} x) = 0, \tag{2.5}$$

for all $\ell = 0, \ldots, k$, $j = 1, \ldots, m$, and

$$(A + B - \lambda \mathrm{Id}\,)^{k} x = 0. \tag{2.6}$$

Indeed, we have:

Lemma 2.1. *If $x_0 \in M$, then*

$$p_j(S_{A+B}(t)x_0) = 0$$

and therefore (2.4) *doesn't hold if $x_0 \in M \setminus \{0\}$.*

Concerning the direct equality, we have the following result:

Proposition 2.2. *We suppose* (2.1)*, then we have* (2.2).

Concerning the inverse equality, we have two results. Let us first introduce the following definition.

Definition. $(f_k)_{k\geq 1}$ is a *pseudo-basis* if Vect $\{f_k\}$ is dense in H and if, for every bounded sequence (x_k) such that

$$x_k \in \text{Vect}\ \{f_j \ : \ j \geq k\},$$

we have

$$x_k \rightharpoonup 0.$$

Lemma 2.3. *$\{f_{k,\ell} \ : \ k \geq 1,\ 1 \leq \ell \leq m_k\}$ is a pseudo-basis, if there exists a Riesz basis $\{e_{k,\ell} \ : \ k \geq 1, 1 \leq \ell \leq m_k\}$ such that*

$$\text{Vect}\ \{e_{k,\ell} \ : \ 1 \leq \ell \leq m_k\} = \text{Vect}\ \{f_{k,\ell} \ : \ 1 \leq \ell \leq m_k\} \tag{2.7}$$

for each k.

Then we have the following result:

Proposition 2.4. *We suppose* (2.1), (2.3)*, that B is compact. Then there exists a finite codimensional subspace $L' \subset L$ such that*

$$c\|x_0\|^2 \leq \sum_{j=1}^{m} \int_{J_j} p_j(S_{A+B}(t)x_0)^2\ dt \quad \textit{for all} \quad x_0 \in L'. \tag{2.8}$$

Moreover for every pseudo-basis $(f_k)_{k\geq 1}$ *such that* $L = \mathrm{Vect}\ \{f_j\ :\ j \geq k'\}$ *for some* k'*, we can take* $L' = \mathrm{Vect}\ \{f_j\ :\ j \geq k''\}$ *with a sufficiently large integer* $k'' \geq k'$*.*

If $A+B$ satisfies some spectral properties, then we will obtain a better result. For this, let us recall, e.g., from [2] that a vector $x \in H$ is called a *generalized eigenvector* with eigenvalue $\lambda \in \mathbb{C}$ of a linear operator C in H if

$$(C - \lambda \mathrm{Id})^m x = 0$$

for some positive integer m. Furthermore, an eigenvalue $\lambda \in \mathbb{C}$ is called *of finite type* if the corresponding generalized eigenvectors form a finite dimensional subspace M, and if

$$H = M \oplus S$$

with M and S stable by C.

Let us now formulate our main result:

Theorem 2.5. *Assume that*

- A *is a skew-adjoint operator having a compact resolvant,*
- B *is compact,*
- $A+B$ *has a pseudo-basis of generalized eigenvectors, whose eigenvalues are of finite type,*
- (2.1) *and* (2.3) *are satisfied with a finite codimensional subspace* L *generated by some generalized eigenvectors of* $A+B$*.*

Then (2.4) *holds true and* M *is finite dimensional.*

Remark 2.6. *In particular, this theorem asserts that the cases of non observability coming from such compact perturbations are those for which* $M \neq \{0\}$*. In fact, we can easily see that* $M \neq \{0\}$ *is equivalent to the existence of a non zero vector* $x \in H$ *which satisfies the inequalities*

$$p_j(x) = 0$$

for all $j = 1, \ldots, m$*, and*

$$(A+B)x = \lambda x,$$

for some complex number λ*.*

We prove the above formulated results in the next section. Then, in the last section of the paper we apply these results in order to answer a question left open in [5].

3. Proof of the results.

3.1. **Proof of Lemma 2.1.** Let $x_0 \in M$, then we compute:

$$S_{A+B}(t)x_0 = \sum_{\lambda \in \mathbb{C}} \sum_{j=0}^{k_\lambda - 1} \frac{t^j e^{\lambda t}}{j!} (A + B - \lambda \mathrm{Id}\ )^j x_0,$$

with a finite number of integers $k_\lambda \geq 1$. Since p_k are semi-norms, we have the result.

3.2. **Proof of Proposition 2.2.** We will first prove that

$$\sum_{j=1}^{m} \int_{J_j} p_j(S_A(t)x_0)^2 \ dt \leq c\|x_0\|^2 \text{ for all } x_0 \in H. \tag{3.1}$$

We fix an orthonormal basis $(e_l)_{l\geq 1}$ such that $L = \text{Vect}_{\ l\geq k}(e_j)$ for a certain integer k. We denote by π_1 (resp. π_2) the orthogonal projection onto $L^\perp$ (resp. onto L). From (2.1), we have, for each $t \in \mathbb{R}$,

$$\int_{I_j} p_j(S_A(s)\pi_2 S_A(t)x_0)^2 \ ds \leq c\|\pi_2 S_A(t)x_0\|^2.$$

Since S_A is a strongly continuous group, there exist numbers ω and M such that

$$\|S_A(t)\| \leq Me^{\omega|t|} \text{ for all } t \in \mathbb{R} \tag{3.2}$$

and therefore

$$\int_{I_j} p_j(S_A(s)\pi_2 S_A(t)x_0)^2 \ ds \leq cM^2 e^{2\omega|t|}\|x_0\|^2.$$

Given an interval I, which we will fix later, we integrate this inequality over I:

$$\int_I \int_{I_j} p_j(S_A(s)\pi_2 S_A(t)x_0)^2 \ ds \ dt \leq cM^2 \int_I e^{2\omega|t|} \ dt\|x_0\|^2.$$

Then, applying the Fubini-Tonelli theorem, we have

$$\int_{I_j} \left(\int_I p_j(S_A(s)\pi_2 S_A(t)x_0)^2 dt \right) \ ds \leq cM^2 \int_I e^{2\omega|t|} \ dt\|x_0\|^2.$$

Hence, there exists $s_0 \in I_j$ (which may depend on I) such that

$$\int_I p_j(S_A(s_0)\pi_2 S_A(t)x_0)^2 dt \leq \frac{2cM^2}{|I_j|} \int_I e^{2\omega|t|} \ dt\|x_0\|^2 \tag{3.3}$$

On the other hand,

$$\pi_1 S_A(t)x_0 = \sum_{l=1}^{k} (S_A(t)x_0|e_l)\, e_l$$

and then, using the inequalities between the arithmetic and quadratic means, we obtain

$$p_j(S_A(s_0)\pi_1 S_A(t)x_0)^2 \leq k \sum_{l=1}^{k} |(S_A(t)x_0|e_l)|^2\, p_j(S_A(s_0)e_l)^2.$$

Hence, thanks to (3.2), we have
(3.4)
$$\int_I p_j(S_A(s_0)\pi_1 S_A(t)x_0)^2 dt \le kM^2 \int_I e^{2\omega|t|}\ dt \sum_{l=1}^k p_j(S_A(s_0)e_l)^2\|x_0\|^2.$$

Combining (3.3) and (3.4) we obtain that

$$\int_I p_j(S_A(s_0)S_A(t)x_0)^2 dt$$
$$\le 2\max\left(\frac{2cM^2}{|I_j|}\int_I e^{2\omega t}dt,\ kM^2\int_I e^{2\omega t}dt\sum_{l=1}^k p_j(S_A(s_0)e_l)^2\right)\|x_0\|^2.$$

Since $S_A(s_0)S_A(t) = S_A(s_0+t)$, then we have

$$\int_{I+s_0} p_j(S_A(t)x_0)^2 dt$$
$$\le 2\max\left(\frac{2cM^2}{|I_j|}\int_I e^{2\omega t}dt,\ kM^2\int_I e^{2\omega t}dt\sum_{l=1}^k p_j(S_A(s_0)e_l)^2\right)\|x_0\|^2.$$

Now, let J_j be an interval; we can choose I such that $J_j \subset I + s_0$. For example, if $J_j = (a,b)$ and $I_j = (c,d)$, since $s_0 \in I_j$, we may take $I = (a-d, b-c)$.

So we obtain
$$\int_{J_j} p_j(S_A(t)x_0)^2\ dt \le c\|x_0\|^2$$
and (3.1) follows.

Next we prove (2.2). Let $x_0 \in H$. Thanks to (3.1), we only have to show that

$$\int_{J_j} p_j(S_{A+B}(t)x_0 - S_A(t)x_0)^2 dt \le c\|x_0\|^2, \tag{3.5}$$

because

$$\int_{J_j} p_j(S_{A+B}(t)x_0)^2\ dt$$
$$\le 2\left(\int_{J_j} p_j(S_{A+B}(t)x_0 - S_A(t)x_0)^2\ dt + \int_{J_j} p_j(S_A(t)x_0)^2 dt\right).$$

Suppose at first that $J_j \subset \mathbb{R}^+$. We begin with

$$S_{A+B}(t)x_0 - S_A(t)x_0 = \int_0^t S_A(t-s)BS_{A+B}(s)x_0\ ds.$$

Hence, putting $J_j = (a, b)$ we have

$$\begin{aligned}\int_{J_j} p_j(S_{A+B}(t)x_0 - S_A(t)x_0)^2 \ dt \\ &= \int_a^b p_j \left(\int_0^t S_A(t-s) B S_{A+B}(s) x_0 ds \right)^2 \ dt \\ &\leq \int_a^b \left(\int_0^t p_j \left(S_A(t-s) B S_{A+B}(s) x_0 \right) \ ds \right)^2 \ dt \\ &\leq \int_a^b t \int_0^t p_j \left(S_A(t-s) B S_{A+B}(s) x_0 \right)^2 \ ds \ dt\end{aligned}$$

by using successively the Minkowski inequality for p_j and the Cauchy-Schwarz inequality.

Next, using the Fubini-Tonelli theorem we have

$$\int_{t=a}^{t=b} \int_{s=0}^{s=t} = \int_{t=a}^{t=b} \int_{s=0}^{s=a} + \int_{t=a}^{t=b} \int_{s=a}^{s=t} = \int_{s=0}^{s=a} \int_{t=a}^{t=b} + \int_{s=a}^{s=b} \int_{t=s}^{t=b},$$

and thus

$$\begin{aligned}\int_{J_j} p_j(S_{A+B}(t)x_0 - S_A(t)x_0)^2 \ dt \\ &\leq \int_0^a \left(\int_{a-s}^{b-s} t p_j \left(S_A(t) B S_{A+B}(s) x_0 \right)^2 dt \right) \ ds \\ &\quad + \int_a^b \left(\int_0^{b-s} t p_j \left(S_A(t) B S_{A+B}(s) x_0 \right)^2 dt \right) \ ds \\ &\leq c \int_0^a \| B S_{A+B}(s) x_0 \|^2 \ ds + c \int_a^b \| B S_{A+B}(s) x_0 \|^2 \ ds,\end{aligned}$$

thanks to (3.1)

$$\begin{aligned}\int_{J_j} p_j(S_{A+B}(t)x_0 - S_A(t)x_0)^2 \ dt \\ &\leq \int_0^a \left(\int_{a-s}^{b-s} t p_j \left(S_A(t) B S_{A+B}(s) x_0 \right)^2 dt \right) \ ds \\ &\quad + \int_a^b \left(\int_0^{b-s} t p_j \left(S_A(t) B S_{A+B}(s) x_0 \right)^2 dt \right) \ ds \\ &\leq c \int_0^a \| B S_{A+B}(s) x_0 \|^2 \ ds + c \int_a^b \| B S_{A+B}(s) x_0 \|^2 \ ds \\ &\leq c \| x_0 \|^2,\end{aligned}$$

Since B is continuous, we obtain (3.5). We recall that we have supposed $J_j = (a, b) \subset \mathbb{R}^+$. Now, if $J_j \subset \mathbb{R}^-$, we proceed alike, by changing t, a, b into $-t, -b, -a$. At last, we conclude to (3.5) in the general case, by cutting the interval into two parts, one included in $\mathbb{R}^+$ and the other included in $\mathbb{R}^-$.

3.3. **Proof of Lemma 2.3.** Set a bounded sequence $(x_{k,\ell})$ such that

$$x_{k,\ell} \in \mathrm{Vect}\left\{\mathrm{f_{j,i}} : (j,i) \geq (k,\ell)\right\}.$$

(Here we use the lexicographic order). Thanks to (2.7), we have

$$x_{k,\ell} \in \mathrm{Vect}\ \left\{\mathrm{e_{j,i}} : (\mathrm{j},\mathrm{i}) \geq (\mathrm{k},1)\right\}$$

Since $\{e_{k,l} : k \geq 1, 1 \leq l \leq m_k\}$ is a Riesz basis, there exists a Banach space automorphism Φ and an orthonormal basis

$$\{u_{k,l} : k \geq 1, 1 \leq l \leq m_k\}$$

such that $\Phi(e_{k,l}) = u_{k,l}$. Thus, we have

$$\Phi^{-1}x_{k,\ell} \in \mathrm{Vect}\ \left\{\mathrm{u_{j,i}} : (\mathrm{j},\mathrm{i}) \geq (\mathrm{k},1)\right\},$$

that is, we can find numbers $(y_{j,i}^{(k,\ell)})$ such that

$$\Phi^{-1}x_{k,l} = \sum_{(j,i)\geq(k,1)} y_{j,i}^{(k,\ell)} u_{j,i}$$

Now, let $x \in H$ and compute:

$$\begin{aligned} (x_{k,l}|x) &= \left(\Phi^{-1}x_{k,l}|\Phi^* x\right) = \sum_{(j,i)\geq(k,1)} y_{j,i}^{(k,l)} \left(u_{k,l}|\Phi^* x\right) \\ &\leq \left\|\Phi^{-1}x_{k,l}\right\| \left(\sum_{(j,i)\geq(k,1)} \left|(u_{j,i}|\Phi^* x)\right|^2\right)^{1/2} \end{aligned}$$

thanks to the Cauchy-Schwarz inequality. Now, $\Phi^{-1}x_{k,\ell}$ remains bounded and $(u_{j,i}|\Phi^* x)$ is square summable by the Parseval identity.

We obtain therefore that

$$(x_{k,l}|x) \to 0$$

as k tends to infinity. Thus, we have the result, since, thanks to (2.7), Vect $\mathrm{f_{k,\ell}}$ is also dense in H.

3.4. **Proof of Proposition 2.4.** We fix a pseudo-basis $(f_k)_{k\geq 1}$ such that $L = \mathrm{Vect}\ _{j\geq k'}(f_j)$ for some integer k'. We fix an integer $k \geq k'$, which we will choose later and a vector $x_0 \in \mathrm{Vect}\left\{\mathrm{f_j} : \mathrm{j} \geq \mathrm{k}\right\}$.

Then we have:

$$\begin{aligned} &\int_{J_j} p_j(S_A(t)x_0)^2\ dt \\ &\leq 2\left(\int_{J_j} p_j(S_A(t)x_0 - S_{A+B}(t)x_0)^2\ dt + \int_{J_j} p_j(S_{A+B}(t)x_0)^2\ dt\right). \end{aligned}$$

Since

$$S_{A+B}(t)x_0 - S_A(t)x_0 = \int_0^t S_A(t-s)BS_{A+B}(s)x_0\ ds,$$

we obtain

$$(3.6)\quad \int_{J_j} p_j(S_{A+B}(t)x_0)^2\ dt$$
$$\geq \frac{1}{2}\int_{J_j} p_j(S_A(t)x_0)^2\ dt - \int_{J_j} p_j(\int_0^t S_A(t-s)BS_{A+B}(s)x_0\ ds)^2\ dt.$$

We write $J_j = (a,b)$, and we consider only the case where $J_j \subset \mathbb{R}^+$ (the general case follows with the same argument as in the preceding proof). Thanks to (2.1), we have like in the Proposition 2.2:

$$(3.7)\quad \int_{J_j} p_j(\int_0^t S_A(t-s)BS_{A+B}(s)x_0\ ds)^2\ dt$$
$$\leq c\int_0^a \|BS_{A+B}(s)x_0\|^2\ ds + c\int_a^b \|BS_{A+B}(s)x_0\|^2\ ds$$
$$\leq c\int_0^b \left(\sup_{\substack{x\in \mathrm{Vect}\ \{\mathrm{f_j:j\geq k}\}\\ \|x\|\leq 1}} \|BS_{A+B}(s)x\|\right)^2 ds\,\|x_0\|^2.$$

Now, for each fixed $s \in \mathbb{R}$, let (x_k) be an approximation of the supremum

$$\sup_{\substack{x\in \mathrm{V}ect\ \{f_j:j\geq k\}\\ \|x\|\leq 1}} \|BS_{A+B}(s)x\|.$$

Since $(f_k)_{k\geq 1}$ is a pseudo-basis, (x_k) converges weakly to zero. Since B is compact, so is $BS_{A+B}(s)$ and therefore, $BS_{A+B}(s)x_k$ converges strongly to zero. So, we can easily conclude that the approximation and thus the supremum (3.4) converges to zero. We also notice that (3.4) is dominated by $\|BS_{A+B}(s)\|$, which is integrable as B is continuous. So, by applying Lebesgue's dominated convergence theorem, we obtain that

$$(3.8)\quad \varepsilon_k := \int_0^b \left(\sup_{\substack{x\in \mathrm{V}ect\ \{f_j:j\geq k\}\\ \|x\|\leq 1}} \|BS_{A+B}(s)x\|\right)^2 ds \to 0 \text{ as } k\to\infty.$$

Keeping in mind from (3.6) and (3.7) that:

$$\int_{J_j} p_j(S_{A+B}(t)x_0)^2 dt \geq \frac{1}{2}\int_{J_j} p_j(S_A(t)x_0)^2 dt - c\varepsilon_k \|x_0\|^2.$$

thanks to (2.3) and (3.8), we can now choose k independent from x_0 such that (2.8) holds true with $k'' = k$.

3.5. **Proof of Theorem 2.5.** Since the eigenvalues of $A+B$ are of finite type, we know that H decomposes into a direct sum:

$$H = \oplus_{i\geq 1}\text{Ker } (A+B-\lambda_i \text{Id})^{m_i}$$

with distinct λ_i. For further use, we denote by π_λ the projection onto

$$E_\lambda := \oplus_{\substack{i\geq 1\\ \lambda_i\neq\lambda}}\text{Ker } (A+B-\lambda_i \text{Id})^{m_i}.$$

Now, since L is a finite codimensional space generated by generalized eigenvectors of $A+B$, we may assume, by "diminishing" L if necessary, that L is of the form

$$L = \oplus_{i\geq r}\text{Ker } (A+B-\lambda_i \text{Id})^{m_i}$$

with some integer r (this only weakens our assumption concerning the estimates (2.1) and (2.3)).

Thanks to Proposition 2.4, since $A+B$ has a pseudo-basis of generalized eigenvectors, there exists $r' \geq r$, such that (2.8) holds true with

$$L' = \oplus_{i\geq r'}\text{Ker } (A+B-\lambda_i \text{Id})^{m_i}$$

In order to prove the theorem, we will use a transformation due to Haraux [3]: given $\delta > 0$, $\lambda \in \mathbb{C}$ and $x_0 \in H$, set

$$I_{\delta,\lambda}(x_0) := x_0 - \frac{1}{\delta}\int_0^\delta e^{-\lambda s} S_{A+B}(s) x_0 ds.$$

We first recall some properties of this transformation.

Lemma 3.1.

(a) $I_{\delta,\lambda} S_{A+B}(t)x_0 = S_{A+B}(t) I_{\delta,\lambda} x_0$.

(b) For any seminorm p in H, and for any interval (a,b) we have the estimates

(3.9)
$$\int_a^b p(I_{\delta,\lambda}S_{A+B}(t)x_0)^2\ dt \leq c\int_a^{b+\delta} p(S_{A+B}(t)x_0)^2\ dt, \quad \textit{for all } x_0\in H.$$

(c) For any $m \in \mathbb{N}^*$, we have the inclusion:

(3.10)
$$I_{\delta,\lambda}\left(\text{Ker}(A+B-\lambda\text{Id})^m\right) \subset \text{Ker}(A+B-\lambda\text{Id})^{m-1}.$$

Proof.

(a) By uniqueness of the Cauchy problem.

(b) For every fixed $t \in \mathbb{R}$, by setting $x(t) = S_{A+B}(t)x_0$, we have

$$\begin{aligned} p(I_{\delta,\lambda}x(t))^2 &\leq 2p(x(t))^2 + 2p\Big(\frac{1}{\delta}\int_0^\delta e^{-\lambda s}x(t+s)\ ds\Big)^2 \\ &\leq 2p(x(t))^2 + \frac{1}{\delta^2}\Big(\int_0^\delta e^{-\lambda s}p(x(t+s))\ ds\Big)^2 \\ &\leq 2p(x(t))^2 + \frac{1}{\delta^2}\int_0^\delta |e^{-\lambda s}|^2\ ds \int_0^\delta p(x(t+s))^2\ ds \\ &\leq 2p(x(t))^2 + \delta^{-1}e^{2|\Re\lambda|\delta}\int_t^{t+\delta} p(x(s))^2\ ds. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_a^b p(I_{\delta,\lambda}x(t))^2\ dt \\ &\quad\leq 2\int_a^b p(x(t))^2\ dt + \delta^{-1}e^{2|\Re\lambda|\delta}\int_a^b\int_t^{t+\delta} p(x(s))^2\ ds\ dt \\ &\quad= 2\int_a^b p(x(t))^2\ dt + \delta^{-1}e^{2|\Re\lambda|\delta}\int_{a-\delta}^{b+\delta}\int_{\max\{a,s-\delta\}}^{\min\{b,s\}} p(x(s))^2\ dt\ ds \\ &\quad\leq 2\int_a^b p(x(t))^2\ dt + e^{2|\Re\lambda|\delta}\int_{a-\delta}^{b+\delta} p(x(s))^2\ dt, \end{aligned}$$

and (3.16) follows with

$$c = 2 + e^{2|\Re\lambda|\delta}.$$

(c) Let $x_0 \in \mathrm{Ker}(A+B-\lambda\mathrm{Id})^m$. Then we have

$$S_{A+B}(t)x_0 = \sum_{j=0}^{m-1}\frac{t^je^{\lambda t}}{j!}(A+B-\lambda Id)^jx_0,$$

and thus

$$I_{\delta,\lambda}x_0 = \frac{-1}{\delta}\sum_{j=1}^{m-1}\int_0^\delta t^jdt(A+B-\lambda Id)^jx_0,$$

so that

$$(A+B-\lambda Id)^{m-1}I_{\delta,\lambda}x_0 = 0. \qquad \square$$

We now prove a deeper property of the Haraux transformation.

Lemma 3.2. *For all but countably many $\delta > 0$, we have*

$$\|\pi_\lambda x_0\|^2 \leq c\,\|\pi_\lambda I_{\delta,\lambda}(x_0)\|^2, \quad \textit{for all } x_0 \textit{ in } H \tag{3.11}$$

Proof. We fix an integer r'' which will be chosen later and we suppose at first that $x_0 \in L'' := \oplus_{i\geq r''}\mathrm{Ker}\ (A+B-\lambda_i\mathrm{Id})^{m_i}$. We know that A is a skew-adjoint operator having a compact resolvant, thus, we can fix an orthonormal basis $(e_k)_{k\geq 1}$ of eigenvectors for A, with purely imaginary eigenvalues μ_k which tend to infinity. We construct a sequence (ε_k)

which tends to zero and such that all numbers $\mu_k+\varepsilon_k$ are distinct from λ,and we define a closed operator B_0 by $B_0 e_k = \varepsilon_k e_k$. Now,we have $x_0 = \sum x_k e_k$ and we introduce the Haraux transformation for $A+B_0$:

$$J_{\delta,\lambda}(x_0) := x_0 - \frac{1}{\delta}\int_0^\delta e^{-\lambda s} S_{A+B_0}(s)x_0 ds = \sum x_k a(k,\delta)e_k,$$

with,

$$a(k,\delta) := 1 - \frac{1}{\delta}\int_0^\delta e^{(\mu_k+\varepsilon_k-\lambda)s}ds.$$

The quantity $a(k,\delta)$ tends to 1 as k tends to infinity, and the set of the δ such that there exist $k\in\mathbb{N}$ cancelling $|a(k,\delta)|$ is countable, since $a(k,\delta)$ is analytic in $\delta>0$. Thus for all but countably many $\delta>0$, $\inf_{k\in\mathbb{N}}|a(k,\delta)|$ is strictly positive and thus

$$\|x_0\|^2 \le c\,\|J_{\delta,\lambda}(x_0)\|^2 . \tag{3.12}$$

Now we have:

$$S_{A+B_0}(t)x_0 - S_{A+B}(t)x_0 = \int_0^t S_{A+B_0}(t-s)(B_0-B)S_{A+B}(s)x_0 ds.$$

Hence, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\|J_{\delta,\lambda}(x_0) - I_{\delta,\lambda}(x_0)\|^2\\ &\le \frac{1}{\delta^2}\int_0^\delta e^{-2\Re(\lambda)t}dt\int_0^\delta \left\|\int_0^t S_{A+B_0}(t-s)(B_0-B)S_{A+B}(s)x_0 ds\right\|^2 dt.\end{aligned}$$

and thus,

$$\begin{aligned}&\|J_{\delta,\lambda}(x_0) - I_{\delta,\lambda}(x_0)\|^2\\ &\qquad\le c\int_0^\delta\left(\sup_{x\in L'',|x|\le 1}\|(B-B_0)S_{A+B}(s)x\|\right)^2 ds\,\|x_0\|^2\end{aligned} \tag{3.13}$$

Now, collecting (3.12) and (3.13), we obtain:

$$\|x_0\|^2 \le c\,\|J_{\delta,\lambda}(x_0)\|^2 \le 2c\,\|I_{\delta,\lambda}(x_0)\|^2 + 2c\,\|J_{\delta,\lambda}(x_0) - I_{\delta,\lambda}(x_0)\|^2 ,$$

Now, since $A+B$ has a pseudo-basis of generalized eigenvectors, by proceeding like in the preceding proof, we can choose r'', such that

$$\|x_0\|^2 \le c\,\|I_{\delta,\lambda}(x_0)\|^2 .$$

By increasing r'', if necessary, since λ_i tend to infinity, because B is compact and A has a compact resolvant, we can suppose that $\lambda\ne\lambda_i$ for $i\ge r''$.Thus, for all $x_0\in L''$, $x_0\in E_\lambda$, $I_{\delta,\lambda}x_0\in E_\lambda$ and the preceding inequality reduces to (3.11). Now, let $z_0 = x_0 + y_0 \in H$ with

$$x_0\in L'' \quad\text{and}\quad y_0\in\oplus_{j<r''}\text{Ker }(A+B-\lambda_j\text{Id})^{m_j}$$

Suppose once that

$$\|\pi_\lambda y_0\|^2 \le c\,\|\pi_\lambda I_{\delta,\lambda}(y_0)\|^2 . \tag{3.14}$$

We then obtain the inequality

$$\|\pi_\lambda z_0\|^2 \leq c\,\|\pi_\lambda I_{\delta,\lambda}(x_0)\|^2 + c\,\|\pi_\lambda I_{\delta,\lambda}(y_0)\|^2 .$$

By the tool of a Riesz basis such that some of its members generate $\oplus_{j<r''}\text{Ker }(A+B-\lambda_j \text{Id})^{m_j}$ and the others L'', we obtain:

$$\|\pi_\lambda z_0\|^2 \leq c\,\|\pi_\lambda I_{\delta,\lambda}(z_0)\|^2$$

Now, it remains to prove (3.14). Since $\oplus_{j<r''}\text{Ker }(A+B-\lambda_j \text{Id})^{m_j}$ is a finite dimensional space, it suffices to verify that

$$\pi_\lambda I_{\delta,\lambda}(z_0) = 0 \Rightarrow \pi_\lambda z_0 = 0.$$

for all but countably many $\delta > 0$. □

We now can prove a weaker form of the estimate (1.1).

Lemma 3.3. *Set*

$$\pi := \prod_{i=1}^{r} \pi_{\lambda_i}$$

Then

$$c\,\|\pi x_0\|^2 \leq \sum_{j=1}^{m} \int_{J_j} p_j(S_{A+B}(t)x_0)^2 \; dt \textit{ for all } x_0 \in H$$

Proof. Set

$$M = \sum_{k<r'} m_k$$

and fix a sufficiently small $\delta > 0$ so that writing $I_j = (a_j, b_j)$ we have

$$(a_j - M\delta, b_j + M\delta) \subset J_j \quad \text{for} \quad j = 1, \ldots, m.$$

We can choose δ such that the estimate (3.11) of the lemma 3.2 is satisfied for every λ_k with $k < k'$. Let us introduce the linear operator

$$I = \prod_{k<r'} I_{\delta,\lambda_k}^{m_k}$$

(composition of M linear operators). It follows from the definition of $I_{\delta,\lambda}$ that the factors I_{δ,λ_k} and π_{λ_k} commute. Hence, by a repeated application of the lemma 3.2 we obtain that

$$\|\pi x_0\|^2 \leq c\,\|\pi I(x_0)\|^2 \tag{3.15}$$

and on the other hand, by a repeated application of (3.9), we obtain:

$$\sum_{j=1}^{m} \int_{I_j} p_j(S_{A+B}(t)Ix_0)^2 dt \leq c \sum_{j=1}^{m} \int_{J_j} p_j(S_{A+B}(t)x_0)^2 dt, \quad \forall x_0 \in H \tag{3.16}$$

It turns out by a repeated application of (3.10) that $I(x_0) \in L'$. It follows that $\pi I(x_0) = I(x_0)$ and that (2.8) holds true. Thus, we have:

$$c\,\|\pi I(x_0)\|^2 = c\,\|I(x_0)\|^2 \leq \sum_{j=1}^{m} \int_{I_j} p_j(S_{A+B}(t)Ix_0)^2 dt$$

By collecting this, (3.15) and (3.16), we obtain the result. □

Now we are ready to prove our main theorem.

proof of theorem 2.5. We first show that M is finite dimensional. Let $x_0 \in H$ satisfying (2.5) and (2.6). Thanks to (2.6), there exists an integer i such that $x_0 \in \mathrm{Ker}\ (\mathrm{A}+\mathrm{B}-\lambda_{\mathrm{i}}\mathrm{Id}\ )^{\mathrm{m}}_{\mathrm{i}}$. Since (2.5) holds, according to lemma 2.1, (1.1) doesn't hold. Therefore, from (2.8), we must have $i < r'$. We then see that M is included in $M' := \oplus_{i<r'}\mathrm{Ker}(A+B-\lambda_i\mathrm{Id})^{m_i}$ and is therefore finite dimensional.

We now fix a supplementar S of M in M' and take $\widetilde{L} = S \oplus L'$ Let $x_0 = y_0 + z_0 \in \widetilde{L}$, with $x_0 \in S$ and $z_0 \in L'$.

Assume for a moment that

$$\|y_0\|^2 \leq c \sum_{j=1}^{m} \int_{I_j} p_j(S_{A+B}(t)y_0)^2\ dt. \tag{3.17}$$

Then

$$\begin{aligned}\|x_0\|^2 &\leq 2\|y_0\|^2 + 2\|z_0\|^2\\ &\leq c\sum_{j=1}^{m}\int_{I_j} p_j(S_{A+B}(t)y_0)^2\ dt + 2\|z_0\|^2\\ &\leq c\sum_{j=1}^{m}\int_{I_j} 2p_j(S_{A+B}(t)x_0)^2 + 2p_j(S_{A+B}(t)z_0)^2\ dt + 2\|z_0\|^2.\end{aligned}$$

(We used in the first step the triangle inequality.) Applying (2.8), for z_0, it follows that

$$\|x_0\|^2 \leq c\sum_{j=1}^{m}\int_{I_j} p_j(S_{A+B}(t)x_0)^2\ dt + c\|z_0\|^2.$$

Applying the preceding lemma, since $\pi x_0 = z_0$, we conclude that

$$\|x_0\|^2 \leq c\sum_{j=1}^{m}\int_{J_j} p_j(S_{A+B}(t)x_0)^2\ dt.$$

It remains to prove (3.17). Since $\oplus_{i<r'}\mathrm{Ker}(A+B-\lambda_i\mathrm{Id})^{m_i}$ is finite dimensional, it suffices to prove that

$$p_j(S_{A+B}(t)y_0) = 0 \quad \text{in} \quad I_j \Rightarrow y_0 = 0. \tag{3.18}$$

So, we suppose that

$$p_j(S_{A+B}(t)y_0) = 0 \quad \text{in} \quad I_j : \text{for } j = 1, \ldots, m.$$

By a translation argument, we obtain

$$p_j(S_{A+B}(t)y_0) = 0 \quad \text{in} \quad \mathbb{R}^+ \quad \text{for} \quad j = 1, \dots, m. \tag{3.19}$$

Thus

$$p_j(I_{\delta,\lambda} y_0) = 0.$$

The solution has the form

$$S_{A+B}(t)y_0 = \sum_{i<r'} \sum_{j=0}^{m_i-1} \frac{t^j e^{\lambda_i t}}{j!}(A + B - \lambda_i Id)^j y_{0,i}$$

with $y_{0,i} \in \text{Ker}(A + B - \lambda_i Id)^{m_i}$.

Let $I_{(i)} := \prod_{\substack{k<r' \\ k\neq i}} I_{\delta,\lambda_k}^{m_k}$. We then have:

$$p_j(I_{(i)} y_0) = 0$$

and

$$I_{(i)} y_0 = \sum_{j=0}^{m_i-1} \alpha_{i,j}(A + B - \lambda_i Id)^j y_{0,i}$$

with some numbers $\alpha_{i,j}$.

We have more generally:

$$p_j(S_{A+B}(t)I_{(i)} y_0) = 0 \quad \text{in} \quad \mathbb{R}^+ \quad \text{for} \quad j = 1, \dots, m.$$

Now let L be defined by $L_i y(t) := y'(t) - \lambda_i y(t)$. Then we have:

$$p_j(L_i S_{A+B}(t)I_{(i)} y_0) = 0.$$

Suppose now that $y_{0,i} \neq 0$ and let j_0 be the first indice such that $\alpha_{i,j_0} \neq 0$. Thus

$$p_j(L_i^{m_i-1-j_0} S_{A+B}(t)I_{(i)} y_0) = 0$$

and

$$L_i^{m_i-1-j_0} S_{A+B}(t)I_{(i)} y_0 = \alpha_{i,j_0}(A + B - \lambda_i Id)^{m_i-1} y_{0,i}.$$

So

$$p_j((A + B - \lambda_i Id)^{m_i-1} y_{0,i}) = 0$$

We go on:

$$p_j(L_i^{m_i-2-j_0} S_{A+B}(t)I_{(i)} y_0) = 0$$

and

$$\begin{aligned} &L_i^{m_i-2-j_0} S_{A+B}(t)I_{(i)} y_0 \\ &\quad = \alpha_{i,j_0} t(A + B - \lambda_i Id)^{m_i-2} y_{0,i} + \alpha_{i,j_0+1}(A + B - \lambda_i Id)^{m_i-1} y_{0,i}; \end{aligned}$$

thus

$$p_j((A + B - \lambda_i Id)^{m_i-2} y_{0,i}) = 0.$$

By recurrence, we then obtain

$$p_j((A + B - \lambda_i Id)^k y_{0,i}) = 0, k = 0, 1, \dots$$

So we conclude that $y_{0,i} \in M$. Thus, y_0 belongs to M. On the other hand, y_0 belongs to S, so $y_0 = 0$ and we have (3.18). □

4. Application

As an application of our result, we improve a theorem given in [5].

Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset of boundary Γ. We fix two integers m and n, numbers $a_1, \dots, a_{m+n} > 0$ and complex numbers $\alpha_{i,j}$ $(1 \le i, j \le m+n)$. We consider the following system:

$$
\begin{cases}
u_i'' = a_i^2 \Delta u_i - \sum_{j=1}^{m+n} \alpha_{i,j} u_j \ \text{ in } \mathbb{R} \times \Omega, 1 \le i \le m, \\
u_i'' = -a_i^2 \Delta^2 u_i - \sum_{j=1}^{m+n} \alpha_{i,j} u_j \text{ in } \mathbb{R} \times \Omega, m < i \le m+n, \\
u_i = 0 \text{ on } \mathbb{R} \times \Gamma, 1 \le i \le m, \\
u_i = \Delta u_i = 0 \text{ on } \mathbb{R} \times \Gamma, m < i \le m+n, \\
u_i(0) = u_{i0}, u_i'(0) = u_{i1}, \ \text{ in } \Omega, 1 \le i \le m+n.
\end{cases}
\tag{4.1}
$$

We can verify by standard methods that, if $(u_{i0}, u_{i1}) \in H_0^1(\Omega) \times L^2(\Omega)$, for $1 \le i \le m$, and $(u_{i0}, u_{i1}) \in H_0^1(\Omega) \times H^{-1}(\Omega)$, for $m < i \le m+n$, then (4.1) has a unique weak solution $u = (u_1, ..., u_m, ..., u_{m+n})$ which satisfies:

$$
\begin{aligned}
&u_i \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega)), \quad 1 \le i \le m. \\
&u_i \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, H^{-1}(\Omega)), \quad m < i \le m+n.
\end{aligned}
$$

Let E_0 be the *initial energy* of the solution defined by

$$
E_0 := \frac{1}{2}\Big(\sum_{i=1}^{m} \|u_{i0}\|^2_{H_0^1(\Omega)} + \|u_{i1}\|^2_{L^2(\Omega)} + \sum_{i=m+1}^{m+n} \|u_{i0}\|^2_{H_0^1(\Omega)} + \|u_{i1}\|^2_{H^{-1}(\Omega)}\Big).
$$

$L^2(\Omega)$ and $H_0^1(\Omega)$ are endowed with the norm:

$$
\|v\|^2_{L^2(\Omega)} = \int_\Omega |v|^2 \, dx, \qquad \|v\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla v|^2 \, dx
$$

and $H^{-1}(\Omega)$ is endowed with the dual norm of $H_0^1(\Omega)$.

We denote by H the underlying Hilbert space:

$$
H := H_0^1(\Omega)^m \times L^2(\Omega)^m \times H_0^1(\Omega)^n \times H^{-1}(\Omega)^n.
$$

Let ν be the normal exterior unit vector to Γ, and $\Gamma_1, \dots, \Gamma_{m+n}$ be open subsets of Γ, $\omega_1, \dots, \omega_{m+n}$ be open subsets of Ω, $I_1, \dots, I_{m+n}$ intervals of $\mathbb{R}$.

We look for the internal observability estimates:

$$
c_1 E_0 \le \sum_{i=1}^{m+n} \int_{I_i} \int_{\omega_i} |u_i'|^2 \, dx \, dt \le c_2 E_0, \tag{4.2}
$$

and the boundary observability estimates:

$$
c_1 E_0 \le \sum_{i=1}^{m+n} \int_{I_i} \int_{\Gamma_i} |\partial_\nu u_i|^2 \, d\Gamma dt \le c_2 E_0. \tag{4.3}
$$

Theorem 4.1. *We suppose that* (4.2), *respectively* (4.3), *holds for every solution* u *satisfying* (4.1) *with* $\alpha_{i,j} = 0$. *Then, given any other choice of* $\alpha_{i,j}$, *there exists a decomposition of the underlying Hilbert space* H *such that*

$$H = M \oplus L$$

with a finite dimensional space M *satisfying the following conditions:*

(i) for all initial data belonging to L, (4.2), *respectively* (4.3), *holds for a solution* u *satisfying* (4.1) *with this particular choice of* $\alpha_{i,j}$, *this initial data, and intervals* J_j *instead of* I_j, J_j *containing the closure of* I_j *in its interior;*

(ii) for all initial data belonging to $M\backslash\{0\}$, (4.2), *respectively* (4.3), *doesn't hold for any solution* u *satisfying* (4.1) *with the same choice of* $\alpha_{i,j}$, *and this other initial data.*

Proof. We rewrite the problem (4.1) in the form

$$y' = (A+B)y,$$
$$y(0) = y_0$$

with

$$y = (u_1, ..., u_m, u_1', ..., u_m', u_{m+1}, ..., u_{m+n}, u_{m+1}', ..., u_{m+n}')$$

and A corresponding to the case $\alpha_{i,j} = 0$.

B then is a compact perturbation of A and A is a skew adjoint operator having a compact resolvent and it generates a group.

Set z_k be an orthonormal basis in $L^2(\Omega)$, satisfying

$$-\Delta z_k = \gamma_k^2 z_k \quad \text{in } \Omega,$$
$$z_k = 0 \quad \text{on } \Gamma.$$

Since $Z_k := \{\beta \cdot z_k, \beta \in \mathbb{C}^{2m+2n}\}$ is stable by $A+B$, we obtain a Riesz basis of subspaces generated by generalized eigenvectors for $A+B$ and we thus can apply the abstract theorem with

$$p_j(x) := \left\| x_j' \right\|_{L^2(\omega_i)},$$

in the case of internal observability, and

$$p_j(x) := \left\| \partial_\nu x_j \right\|_{L^2(\gamma_i)},$$

in the case of boundary observability, for all $j = 1, \dots, m+n$. □

Example. Let us give a concrete example when the compactly perturbed operator $A+B$ does not have a Riesz basis of eigenvectors. Choosing

$$m = 3, \quad n = 0, \quad a_1 = 2 < a_2 = a_3 = 4.$$

$$(\alpha_{i,j}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

the eigenvectors of $A+B$ are given up to a multiplicative factor by the following formulae

$$\begin{aligned}
e^+_{k,1} &= (1, 2\gamma_k^2, -1, 2i\gamma_k, 4i\gamma_k^3, -2i\gamma_k)z_k,\\
e^+_{k,2} &= (\delta_k, -1, 0, \lambda_k\delta_k, -\lambda_k, 0)z_k,\\
e^+_{k,3} &= (\delta_k^{-1}, 1, 0, \mu_k\delta_k^{-1}, \mu_k, 0)z_k,\\
e^-_{k,1} &= (1, 2\gamma_k^2, -1, -2i\gamma_k, -4i\gamma_k^3, 2i\gamma_k)z_k,\\
e^-_{k,2} &= (\delta_k, -1, 0, -\lambda_k\delta_k, \lambda_k, 0)z_k,\\
e^-_{k,3} &= (\delta_k^{-1}, 1, 0, -\mu_k\delta_k^{-1}, -\mu_k, 0)z_k,
\end{aligned}$$

where we put:

$$\lambda_k := \sqrt{-3\gamma_k^2 + \sqrt{\gamma_k^4+1}}, \quad \mu_k := i\sqrt{3\gamma_k^2 - \sqrt{\gamma_k^4+1}},$$

and

$$\delta_k := \gamma_k^2 + \sqrt{\gamma_k^4+1}$$

for brevity.

Since for example

$$\frac{(e^+_{k,1}|e^+_{k,3})}{\|e^+_{k,1}\|\,\|e^+_{k,3}\|} \to 1,$$

they cannot be normalized so as to form a Riesz basis.

One interesting question, now, is to determine the dimension of the parameters $\alpha_{i,j}$ for which we do not have observability, i.e., for which $M \neq \{0\}$.

Concerning internal observability, we have the following proposition:

Proposition 4.2. *The parameters for which $M \neq \{0\}$ form a countable union of hypersurfaces; hence their set has zero Lebesgue measure.*

Remark 4.3. *These special parameters correspond exactly to those which ensure the existence of constant solutions different from zero; in order not to have such parameters, we must observe*

$$\sum_{i=1}^{m+n} \int_{I_i}\int_{\omega_i} |u_i'|^2\,dx\;dt + \sum_{i=1}^{m+n} \int_{I_i}\int_{\omega_i} |u_i|^2\,dx\;dt$$

instead of

$$\sum_{i=1}^{m+n} \int_{I_i}\int_{\omega_i} |u_i'|^2\,dx\;dt.$$

Proof. We distinguish two cases. If 0 is not an eigenvalue of $A+B$, then it follows from the structure of $A+B$ that every eigenvector of $A+B$ with eigenvalue λ has the form

$$e = \beta^1 z_1 + \cdots + \beta^k z_k, \tag{4.4}$$

with a minimal k , where

$$z_l \in H_0^1(\Omega),\ \ z_l \neq 0,$$
$$-\Delta z_l = \gamma_l^2 z_l \quad \text{in} \quad \Omega,$$
$$\beta^l \in \mathbb{C}^{2m+2n} \quad \text{with}$$
$$\beta^l = \left(\beta_1^l, \dots, \beta_m^l, \beta_1'^l, \dots, \beta_m'^l, \beta_{m+1}^l, \dots, \beta_{m+n}^l, \beta_{m+1}'^l, \dots, \beta_{m+n}'^l\right)$$
$$\beta_j'^l = \lambda \beta_j^l, \quad j = 1, \dots, m+n.$$

We may assume that $z_1,\dots,$ z_k are linearly independent. We may also assume that the β^ℓ associated with the same γ_j are linearly independent. Otherwise, we can diminish k. Indeed, if, for example $\gamma_1 = \gamma_2 = \gamma_3$, and $\beta^3 = \beta^1 + \beta^2$, we have

$$\beta^1 z_1 + \beta^2 z_2 + \beta^3 z_3 = \beta^1(z_1 + z_3) + \beta^2(z_2 + z_3)$$

and, since $z_1 + z_3$ and $z_2 + z_3$ remain independent and satisfy (4)-(4), we can use the vectors $z_1 + z_3$ and $z_2 + z_3$ in (4.4) instead of z_1, z_2, z_3: that is, we diminish k. So, since 0 is not an eigenvalue, we have the equivalence:

$$\beta_1^l = \cdots = \beta_{m+n}^l = 0 \Longleftrightarrow \beta_1'^l = \cdots = \beta_{m+n}'^l = 0. \tag{4.5}$$

If $p_1(e) = \cdots = p_{m+n}(e) = 0$, then

$$\beta_j^1 z_1 + \cdots + \beta_j^k z_k = 0 \quad \text{in} \quad \omega_j, \quad 1 \le j \le m+n.$$

Applying $-\Delta$ repeatedly to these equations, we obtain for each $1 \le j \le m+n$ the linear system

$$(\gamma_1^2)^i \beta_j^1 z_1 + \cdots + (\gamma_k^2)^i \beta_j^k z_k = 0 \quad \text{in} \quad \omega_j, \quad i = 0, \dots, k-1$$

for the variables $\beta_j^1 z_1,\dots,$ $\beta_j^k z_k$.

If the numbers γ_l are pairwise distinct, the determinant of this system is different from zero, and therefore

$$\beta_j^1 z_1 = \cdots = \beta_j^k z_k = 0 \quad \text{in} \quad \omega_j, \quad 1 \le j \le m.$$

In the general case, we only obtain for every $\gamma > 0$ the equality

$$\sum_{\gamma_\ell = \gamma} \beta_j^\ell z_\ell = 0 \quad \text{in} \quad \omega_j, \quad 1 \le j \le m+n.$$

Now, for each $j = 1, \dots, m$, putting $u_j(t) = e^{i\gamma_\ell t} \sum_{\gamma_\ell = \gamma} \beta_j^\ell z_\ell$ and $u_p(t) = 0$ for all other $1 \le p \le m+n$, we obtain a solution of (4.1) in the uncoupled case $\alpha_{i,j} = 0$. Hence, applying the hypothesis we conclude that

$$\sum_{\gamma_\ell = \gamma} \beta_j^\ell z_\ell = 0 \text{ in } \Omega.$$

We obtain the same conclusion for $j = m+1, \dots, m+n$ by changing γ_ℓ to γ_ℓ^2 in the definition of $u_j(t)$ above. Since $z_1, \dots, z_k$ are linearly independent, it follows that

$$\beta_j^1 = \dots = \beta_j^k = 0.$$

Using (4.5) hence we conclude that $e = 0$, which implies, by (2.6) that $M = \{0\}$. Now, suppose that 0 is an eigenvalue of $A + B$ and let y_0

be a corresponding nonzero eigenvector. Then the constant function $y(t) := y_0$ solves (4.1) and $p_j(y(t)) \equiv 0$ for all $j = 1, \dots, m+n$. Thus $M \neq \{0\}$.

It remains to prove that the parameters $\alpha_{i,j}$, for which 0 is an eigenvalue of $A + B$ form a countable union E of surfaces of codimension 1. In fact E consists of all matrices $(\alpha_{i,j})$ such that 0 is an eigenvalue of $A + B|_{Z_k}$ for some k, because the subspaces Z_k, $(k = 1, 2, \dots)$ are stable by $A + B$ and that determine, for some k a hypersurface in $\mathbb{C}^{(m+n)^2}$.

□

Now, consider the case of boundary observability.

Proposition 4.4. *The parameters for which $M \neq \{0\}$ are contained in countably many surfaces of codimension $n + m$ of $\mathbb{C}^{(m+n)^2}$.*

Remark 4.5. *If we suppose that the parameters $\alpha_{i,j}$ belong to $\mathbb{R}$ instead of $\mathbb{C}$, we cannot prove the analogous proposition, the real case generating some extra difficulties.*

Proof. We suppose that $M \neq \{0\}$. We fix an orthonormal basis of the Laplacien-Dirichlet operator. So, keeping in mind the preceding proof, we can find an integer k, and k elements $z_1, \dots, z_k$ of the fixed orthonormal basis and k nonzero elements $\beta^1, \dots, \beta^k \in \mathbb{C}^{m+n}$ such that

$$\beta_j^1 \partial_\nu z_1 + \dots \beta_j^k \partial_\nu z_k = 0, \quad \text{on } \Gamma_j, \text{for } j = 1, \dots, m+n. \tag{4.6}$$

The vectors $\beta^1, \dots, \beta^k$ also have to satisfy the relations:

$$\Big((\alpha_{i,j}) - \lambda^2 I_{m+n} \Big) \beta^\ell = a G_\ell \beta^\ell \text{for } \ell = 1, \dots, k, \tag{4.7}$$

with

$$a = \begin{pmatrix} a_1^2 & & \\ & \dots & \\ & & a_{m+n}^2 \end{pmatrix}, \quad G_\ell = \begin{pmatrix} \gamma_\ell^2 I_m & \\ & \gamma_\ell^4 I_n \end{pmatrix}.$$

We keep here the notations of the preceding proof for the definition of γ_ℓ. Suppose once that, for this given sequence $z_1, \dots, z_k$, the parameters $c_{i,j}$ defined by

$$C := (c_{i,j}) := a^{-1} \left(\alpha_{i,j} - \lambda^2 I_{m+n} \right)$$

are described by at most $(m+n)^2-(m+n+1)$ parameters (*). Then, we sum over all the countable sequences $z_1,\dots,z_k$ and we add the the parameter λ to describe all the parameters $\alpha_{i,j}$. So, if we prove (*), we conclude that the exceptional parameters are contained in countable many surfaces of dimension less than or equal $(m+n)^2-(m+n+1)+1$, that is, of codimension superior or equal to $m+n$. It remains now to prove (*); we distinguish two cases.
Suppose that the vectors $\beta^1,\dots,\beta^k$ form a free family. For each $j=1,\dots,m+n$, there exists a point $x\in\Gamma_j$ where $\partial_\nu z_1(x)\neq 0$ by our hypothesis of observability in the uncoupled case. This allows us to express β_j^1 by the variables $\beta_j^2,\dots,\beta_j^k$, via the equation (4.6). On the other hand, we can suppose that $\beta_1^2\in\{0,1\}$ by dividing all the equations (4.6) and (4.7) by β_1^2, if necessary. This doesn't change the definition of the parameters $c_{i,j}$. Hence, the set of parameters (β_j^ℓ) is described by at most $(k-1)(m+n)-1$ parameters. For each such choice of the vectors (β_j^ℓ), the parameters $(c_{i,j})$ are the solutions of the linear system

$$C\beta^\ell = G_\ell\beta^\ell, \ell=1,\dots,k, \tag{4.8}$$

which is the union of $m+n$ uncoupled linear systems

$$c_{i,1}\beta_1^\ell+\dots+c_{i,m+n}\beta_{m+n}^\ell=\begin{cases}\gamma_\ell^2\beta_i^\ell, & i\le m\\ \gamma_\ell^4\beta_i^\ell, & i>m\end{cases}\quad,\quad \ell=1,\dots,k$$

of rank k for each $i=1,\dots,m+n$. It follows that the parameters $(c_{i,j})$ form an affine subspace described by $(m+n)(m+n-k)$ parameters. Summarizing, the parameters $(c_{i,j})$ are given by at most

$$(k-1)(m+n)-1+(m+n)(m+n-k)=(m+n)^2-(m+n+1)$$

parameters.
Suppose now, that the vectors $\beta^1,\dots,\beta^k$ are linked and consider a relation with a minimum of indices, say $1,\dots,r+1$, by rearrangering the indices if necessary (r is less than or equal to the rank of the system of vectors). We recall that the β^ℓ associated with the same γ_j are linearly independent. Thus, by rearrangering again the indices, we may assume that $\gamma_{r+1}\neq\gamma_1$. In order to determine the parameters $c_{i,j}$, we just consider the relations (4.7) for $\ell=1,\dots,r+1$. (In reality, the $c_{i,j}$ should also satisfy the other relations from (4.6) and (4.7), but that will diminish the numbers of parameters which give the $c_{i,j}$ still further). Now we can suppose that

$$\beta^{r+1}=\beta^1+\dots+\beta^r, \tag{4.9}$$

by multiplying each relation (4.7) for $\ell=1,\dots,r+1$ by a suitable multiplicative factor.

From this, we also can suppose that $\beta_1^2\in\{0,1\}$. Indeed, we only have to divide all the relations we need (i.e: (4.7) for $\ell=1,\dots,r+1$ and (4.9)) by β_1^2, if necessary. Again, this doesn't change the definition of the $c_{i,j}$. So, we first choose the $(r-1)(n+m)-1$ parameters for $\beta^2, \dots, \beta^r$. Then, since $G_{r+1} \neq G_1$ holds, β^1 is determined by the compatibility condition:

$$(G_{r+1} - G_1)\beta^1 + \dots + (G_{r+1} - G_r)\beta^r = 0,$$

from (4.7). Here, we have implicitely supposed that $r \geq 2$. In fact r cannot be equal to 1, according to the preceding equality. Hence, the set of parameters (β_j^ℓ) is described by at most $(r-1)(m+n)-1$ parameters. In each such (β_j^ℓ), the parameters $(c_{i,j})$ are the solutions of the linear system (4.8) with $k = r$. Repeating the above arguments, we obtain $(r-1)(m+n)-1-(m+n)(m+n-r) = (m+n)^2-(m+n+1)$ again.

□

Now, if we do not couple the Petrovsky and wave systems, and if we observe in a common region for all the equations, there are not exceptional parameters:

Proposition 4.6. *If $n = 0$ or $m = 0$ and if $\bigcap_1^{m+n} \Gamma_i$ has nonempty interior, then there are no parameters for which $M \neq \{0\}$.*

Proof. The condition of the intersection ensures that β^i are linked. On the other hand, we may suppose, following the proof of the last proposition, that the β^ℓ corresponding to the same γ_i are independent. In fact, even the vectors β^ℓ corresponding to different γ_i are independent. Indeed, G_ℓ is a multiple of the identity matrix and therefore the β^ℓ are now eigenvectors corresponding to different eigenvalues and have no other choice than being independent. So the β^i cannot be linked, that is: there is no exceptional parameters. □

Remark 4.7. *If $\bigcap_1^{m+n} \Gamma_i$ has empty interior, then there may exist special parameters. For example, consider the case: $n = 2$, $m = 0$, $N = 1$, $\Omega =]0, \pi[$, $\Gamma_1 = \{0\}$, $\Gamma_2 = \{\pi\}$, $a_1 = a_2 = 1$. We then have $u_1 = 2sinx + sin2x$, $u_2 = 2sinx - sin2x$ satisfy the system (4.1) with $\alpha_{1,1} = \alpha_{2,2} = \frac{5}{2}$ and $\alpha_{2,1} = \alpha_{1,2} = -\frac{3}{2}$, and $\partial_\nu u_1(0,t) = \partial_\nu u_2(\pi, t) = 0$*

Now we look at the special case where Ω is a ball.

Proposition 4.8. *We suppose that Ω is a ball. Then, if $n \geq 1$ and $m \geq 1$, the parameters for which $M \neq \{0\}$ contain countable many surfaces of codimension $m+n$.*

Proof. If Ω is a ball, we recall that each eigenfunction of the Laplacian-Dirichlet operator is given by the product of a radius function with an hyperspherical harmonic, and for each such hyperspherical harmonic, there exist countable many independent eigenfunctions of the Laplacian-Dirichlet operator. Thus, we can choose $n+m+1$ eigenfunctions z_k corresponding to different γ_k such that the $\partial_\nu z_k$ are colinear on $\partial\Omega$. So, the set of the exceptional values contains the set $\mathcal{E}$ of the parameters $\alpha_{i,j}$ such that there exists $\mu \in \mathbb{C}$

Indeed, if these equations are satisfied, we can choose $m+n+1$ nonzero vectors $\beta^1, \dots, \beta^{m+n+1}$ which agree with (4.7). Now, these $m+n+1$ vectors of $\mathbb{C}^{m+n}$ are automatically linked, and thanks to the colinearity of the $\partial_\nu z_k$ on $\partial\Omega$,, the other condition (4.6) is also satisfied.

Now, it remains to prove that the set $\mathcal{E}$ contain a variety of codimension $m+n$ (*). Suppose at first that the set E of the parameters $\alpha_{i,j}$ such that

$$\det\left((\alpha_{\mathrm{i,j}}) - \mathrm{aG}_\ell\right) = 0 \quad \text{for } \ell = 1, \dots, m+n+1$$

contains non isolated points(**). Then, if these $n+m+1$ equations are independent, that is, if the differentials of the functions defining these equations evaluated at some point of E are independent linear forms, then E is a variety of dimension $(n+m)^2 - (n+m+1)$. In the general case, we can consider a non isolated point x_0 of E where the rank of these linear forms is maximal (we take the maximum along all the non isolated points of E). Then the rank r remains constant in a neighborhood of x_0, because x_0 is not isolated, and E will contain a variety of codimension r, thanks to the constant rank theorem; thus, in any case, E contains a variety of codimension $m+n+1$. Now, each element of $\mathcal{E}$ is the sum of an element of E and an arbitrary multiple of the identity, say μI_{m+n}. So, in order to prove (*), we must prove in a way that the parameter μ is independent of $(n+m)^2 - (n+m+1)$ parameters which defines the variety of codimension $n+m+1$ included in E. So, if we can choose a non isolated point x in E such that I_{m+n} (which represents a tangent vector corresponding to the parameter μ) does not belong to the tangent space of E at the point x, then the tangent space of $\mathcal{E}$ at the point x will be of enough dimension to have (*) and (**) at the same time. So the proposition will be proved if we find an example of such x. Following the case $n = 1$ and $m = 1$ in [4], we can find $\alpha_{1,1}, \alpha_{1,n+1}, \alpha_{n+1,1}, \alpha_{n+1,n+1}$ such that

$$\det \begin{pmatrix} \alpha_{1,1} - a_1^2\gamma_\ell^2 & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} - a_{n+1}^2\gamma_\ell^4 \end{pmatrix} = 0 \quad \text{for } \ell = 1, 2, 3.$$

and Now, we take for the other parameters: $\alpha_{i,j} = 0$ if $i \neq j$,$\alpha_{2,2} = \gamma_4^2, \dots = \alpha_{n,n} = \gamma_{n+2}^2$ and $\alpha_{n+2,n+2} = \gamma_{n+3}^4, \dots, \alpha_{n+m} = \gamma_{n+m+1}^4$. We can easily verify that with this choice $x = \alpha_{i,j}$, (4) is satisfied and x is also not isolated, since the parameters $\alpha_{1,1}, \alpha_{1,n+1}, \alpha_{n+1,1}, \alpha_{n+1,n+1}$ form a surface of dimension 2. On the other hand, I_{m+n} doesn't belong to tangent space of E. In fact, if it would be the case, we would have:

$$\mathrm{tr}\big(\mathrm{Com}(\mathrm{A} - \mathrm{G}_\ell)\big) = 0 \quad \text{for } \ell = 1, \dots, n+m+1.$$

In particular, for $\ell = 1, 2, 3$, we would obtain

$$\alpha_{n+1,n+1} - a_{n+1}^2\gamma_\ell^4 + \alpha_{1,1} - a_1^2\gamma_\ell^2,$$

as the γ_ℓ are all distinct and that is impossible.

□

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