About the Ingham type proof for the boundary observability of a square and its generalization in N-d

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Wave equation on a product of intervals

Let $N \in \mathbb{N}^*$, reals $L_i > 0,\ i = 1, \dots, N$, $\Omega = \prod_{i=1}^{N}]0, L_i[$ and $u(t, x)$ solution of (1).

$$\begin{cases} u'' = \Delta u, \quad 0 < t < T, \quad x \in \Omega \\ u = 0, \quad 0 < t < T, \quad x \in \partial\Omega \\ u(0, x) = u_0(x), \quad u'(0, x) = u_1(x), \quad x \in \Omega. \end{cases} \tag{1}$$

The system is well defined for initial conditions satisfying $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

We have the classical proposition

Proposition

It exists $T_0 > 0$ such that for $T > T_0$, system (1) is observable: there exists a constant $c > 0$ such that we have (2)

$$\|u_0\|^2_{H^1_0(\Omega)} + \|u_1\|^2_{L^2(\Omega)} \leq c \int_0^T \int_\Gamma |\partial_\nu u(t,x)|^2 \, d\Gamma dt, \tag{2}$$

for all $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$.
$\Gamma = \cup_{j=1}^N \left(\prod_{i=1}^{j-1} [0, L_i] \times \{0\} \times \prod_{i=j+1}^N [0, L_i] \right)$

The aim here is to prove this Proposition with Ingham techniques
It is a result of 2008
It follows the extension of the case of a square, presented in Rome, in 2006

Ingham's theorem (1936)

Let $\gamma > 0$ and $(\lambda_n)_{n\in\mathbb{N}}$ satisfying the gap condition

$$\lambda_{n+1} - \lambda_n > \gamma,$$

then for all $T > \frac{2\pi}{\gamma}$, we have

$$c\sum_k |a_k|^2 \leq \int_0^T |\sum_k a_k e^{i\lambda_k t}|^2 \mathrm{d}t \leq C\sum_k |a_k|^2,$$

where constants $c, C > 0$ are independent of the sequence (a_k).

- The gap is needed :

$$2c \leq \int_0^T |1 \cdot e^{i\lambda_n t} - 1 \cdot e^{i\lambda_{n+1} t}|^2 \mathrm{d}t \leq |\lambda_n - \lambda_{n+1}|^2 \int_0^T t^2 \mathrm{d}t$$

Expression of the solution

Notations

$$\mathbf{k} = (k_1, \ldots, k_N)\,,\ \mathbf{x} = (x_1, \ldots, x_N)\,,\ \omega_{\mathbf{k}} = \sqrt{\sum_{j=1}^{N} \left(\frac{k_j \pi}{L_j}\right)^2}.$$

Expression of the solution of (1)

$$u(t,x) = \sum_{\mathbf{k} \in (\mathbb{N}^*)^N} \left(\alpha_{\mathbf{k}} e^{i\omega_{\mathbf{k}} t} + \alpha_{-\mathbf{k}} e^{-i\omega_{\mathbf{k}} t}\right) \prod_{j=1}^{N} \sin\left(\frac{\pi k_j x_j}{L_j}\right), \qquad (3)$$

with adequate coefficients α_k.

Outline

- Give a Ingham type theorem
- Apply it to prove the Proposition
- Conclusion

A Ingham type theorem (1/2)

Hypothesis

Let $d \in \mathbb{N}^*$, reals $(\lambda_{\mathbf{k}})_{\mathbf{k}\in(\mathbb{N}^*)^d}$, and complex numbers $(p_\ell)_{\ell\in\mathbb{N}^*}$.
We suppose that we have the following spectral gap : for $j = 1, \dots, d$, it exists $\gamma_j > 0$ such that

$$\left|\lambda_{k_1,\dots,k_{j-1},k_j,k_{j+1},\dots,k_d} - \lambda_{k_1,\dots,k_{j-1},k'_j,k_{j+1},\dots,k_d}\right| \geq \gamma_j \left|k_j - k'_j\right|, \tag{4}$$

$$\left|\lambda_{k_1,\dots,k_{j-1},k_j,k_{j+1},\dots,k_d} + \lambda_{k_1,\dots,k_{j-1},k'_j,k_{j+1},\dots,k_d}\right| \geq \gamma_j \left|k_j + k'_j\right|, \tag{5}$$

for all indices $k = (k_1, \dots, k_d) \in (\mathbb{N}^*)^d$ and $k'_j \in \mathbb{N}^*$ such that the weights $(p_\ell)_{\ell\in\mathbb{N}^*}$ satisfy

$$\max_{i=1,\dots,d,\ i\neq j} \left|p_{k_i}\right| \leq \max\left(\left|p_{k_j}\right|, \left|p_{k'_j}\right|\right). \tag{6}$$

A Ingham type theorem (2/2)

Conclusion

Then, for $T > 2\pi\sqrt{\sum_{i=1}^{d}\frac{1}{\gamma_i^2}}$, it exists a constant $c_1 > 0$ such that we have,

$$\sum_{j=1}^{d}\sum_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_d\in\mathbb{N}^*}\int_0^T\left|\sum_{k_j\in\mathbb{N}^*}p_{k_j}\left(\beta_{\mathbf{k}}e^{i\lambda_{\mathbf{k}}t}+\beta_{-\mathbf{k}}e^{-i\lambda_{\mathbf{k}}t}\right)\right|^2\mathrm{d}t$$
$$\geq c_1\sum_{\mathbf{k}\in(\mathbb{N}^*)^d}\left(|\beta_{\mathbf{k}}|^2+|\beta_{-\mathbf{k}}|^2\right)\left(\sum_{j=1}^{d}\left|p_{k_j}\right|^2\right), \quad (7)$$

for all complex numbers$(\beta_{\mathbf{k}})_{\mathbf{k}\in(\mathbb{N}^*)^d}$ and $(\beta_{-\mathbf{k}})_{\mathbf{k}\in(\mathbb{N}^*)^d}$, such that the sums in consideration are finite.

Classical Ingham material (1/3)

First Ingham function

$$f(t) = \begin{cases} \cos\frac{\pi t}{T} & \text{if } |t| \le T/2 \\ 0 & \text{if } |t| > T/2 \end{cases},$$

Fourier transform

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{ixt} \mathrm{d}t = -\frac{2T\pi\cos(xT/2)}{x^2T^2 - \pi^2}.$$

Classical Ingham material (2/3)

Diagonal dominant type estimates

$$\hat{f}(0) = \frac{2T}{\pi}$$

For $\gamma > \frac{2\pi}{T}$, $\ell \in \mathbb{N}^*$ and $|x| \geq \ell\gamma$,

$$|\hat{f}(x)| \leq \frac{2T}{\pi} \frac{1}{|\frac{x^2 T^2}{\pi^2} - 1|} = \frac{2T}{\pi} \left(\frac{2\pi}{\gamma T}\right)^2 \frac{1}{|4\left(\frac{x}{\gamma}\right)^2 - \left(\frac{2\pi}{\gamma T}\right)^2|} \leq \frac{2T}{\pi} \left(\frac{2\pi}{\gamma T}\right)^2 \frac{1}{4\ell^2 - 1}. \tag{8}$$

$$\sum_{\ell=1}^{\infty} \frac{1}{4\ell^2 - 1} = \frac{1}{2} \sum_{\ell=1}^{\infty} \left(\frac{1}{2\ell - 1} - \frac{1}{2\ell + 1}\right) = \frac{1}{2}. \tag{9}$$

Classical Ingham material (3/3)

- As $f(t) \leq 1_{[-T/2,T/2]}$, we begin with

$$A := \int_{-T/2}^{T/2} \left| \sum_{k_j \in \mathbb{N}^*} p_{k_j} b_{\mathbf{k}}(t) \right|^2 \mathrm{d}t \geq \int_{\mathbb{R}} f(t) \left| \sum_{k_j \in \mathbb{N}^*} p_{k_j} b_{\mathbf{k}}(t) \right|^2 \mathrm{d}t,$$

with

$$b_{\mathbf{k}}(t) := \beta_{\mathbf{k}} e^{i\lambda_{\mathbf{k}} t} + \beta_{-\mathbf{k}} e^{-i\lambda_{\mathbf{k}} t}.$$

First step: decomposition

- We decompose

$$\sum_{k_j\in\mathbb{N}^*} = \sum_{k_j\in\mathbb{N}^*,\ \delta_j\le\left|p_{k_j}\right|} + \sum_{k_j\in\mathbb{N}^*,\ \delta_j>\left|p_{k_j}\right|},$$

with

$$\delta_j := \max_{i=1,\dots,d,\ i\neq j} \left|p_{k_i}\right|.$$

Second step: decompose the products

$$\Lambda := \left\{(k_j, k_j') \in \mathbb{N}^*,\ \delta_j \leq \left|p_{k_j}\right| \text{ or } \delta_j \leq \left|p_{k_j'}\right|\right\},$$

$$\Lambda := \left\{(k_j, k_j') \in \mathbb{N}^*,\ \delta_j > \left|p_{k_j}\right|,\ \delta_j > \left|p_{k_j'}\right|\right\},$$

$$\sum_{(k_j,k_j')\in\mathbb{N}^*} = \sum_{(k_j,k_j')\in\Lambda} + \sum_{(k_j,k_j')\in\Lambda}.$$

No gap estimates on red terms

Third step: neglect red terms

Positivity of f

$$A \geq \sum_{(k_j,k_j')\in\Lambda} B_{\mathbf{k},k_j'},\ B_{\mathbf{k},k_j'} := \int_{\mathbb{R}} f(t) p_{k_j} b_{\mathbf{k}}(t) \overline{p_{k_j'} b_{\mathbf{k}'}(t)} \mathrm{d}t,$$

with

$$\mathbf{k}' = (k_1, \ldots, k_{j-1}, k_j', k_{j+1}, \ldots, k_d).$$

Original idea of Loreti-Valente, for partial observability
Also used for two-grid semi-discrete observability, in Loreti-M

Fourth step: separate two cases

Uniform gap on Λ

$$|\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}'}| \geq \gamma_j \left|k_j - k_j'\right|, \ |\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}'}| \geq \gamma_j \left|k_j + k_j'\right|, \ (k_j, k_j') \in \Lambda.$$

Development of $B_{\mathbf{k},k_j'}$: as $\hat{f}(x) = \hat{f}(-x)$, we have

$$B_{\mathbf{k},k_j'} = p_{k_j}\overline{p_{k_j'}}\Big(\big(\beta_{\mathbf{k}}\overline{\beta_{\mathbf{k}'}} + \beta_{-\mathbf{k}}\overline{\beta_{-\mathbf{k}'}}\big)\,\hat{f}(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}'}) + \big(\beta_{\mathbf{k}}\overline{\beta_{-\mathbf{k}'}} + \beta_{-\mathbf{k}}\overline{\beta_{\mathbf{k}'}}\big)\,\hat{f}(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}'})\Big).$$

- treat first $(k_j, k_j) \in \Lambda$
- treat then $(k_j, k_j') \in \Lambda, \ k_j \neq k_j'$

Step 5: Ingham estimates for first case

- For $(k_j, k_j) \in \Lambda$

$$B_{\mathbf{k},k_j} \geq \frac{2T}{\pi}\left|p_{k_j}\right|^2 \left(|\beta_{\mathbf{k}}|^2 + |\beta_{-\mathbf{k}}|^2 - 2\,|\beta_{\mathbf{k}}\beta_{-\mathbf{k}}|\,(\frac{2\pi}{\gamma_j T})^2 \frac{1}{4(k_j + k_j)^2 - 1}\right)$$
$$\geq \frac{2T}{\pi}\left|p_{k_j}\right|^2 \left(|\beta_{\mathbf{k}}|^2 + |\beta_{-\mathbf{k}}|^2\right)\left(1 - (\frac{2\pi}{\gamma_j T})^2 \frac{1}{4(k_j + k_j)^2 - 1}\right)$$

Step 5: Ingham estimates for second case

- For $(k_j, k_j') \in \Lambda,\ k_j \neq k_j'$,

$$B_{\mathbf{k},k_j'} \geq \frac{-2T}{\pi}\left(\frac{2\pi}{\gamma_j T}\right)^2 \left|p_{k_j} p_{k_j'}\right| \left(\left(|\beta_{\mathbf{k}}\beta_{\mathbf{k}'}| + |\beta_{-\mathbf{k}}\beta_{-\mathbf{k}'}|\right)\frac{1}{4(k_j - k_j')^2 - 1} + \left(|\beta_{\mathbf{k}}\beta_{-\mathbf{k}'}| + |\beta_{\mathbf{k}'}\beta_{-\mathbf{k}}|\right)\frac{1}{4(k_j + k_j')^2 - 1}\right)$$

$$B_{\mathbf{k},k_j'} \geq \frac{-2T}{\pi}\left(\frac{2\pi}{\gamma_j T}\right)^2 \frac{1}{2}\left(\left|\beta_{\mathbf{k}} p_{k_j}\right|^2 + \left|\beta_{\mathbf{k}'} p_{k_j'}\right|^2 + \left|\beta_{-\mathbf{k}} p_{k_j}\right|^2 + \left|\beta_{-\mathbf{k}'} p_{k_j'}\right|^2\right) \left(\frac{1}{4(k_j - k_j')^2 - 1} + \frac{1}{4(k_j + k_j')^2 - 1}\right)$$

Step 6: make the sum (1/3)

Summation of the $B_{\mathbf{k},k'_j}$

$$\frac{\pi}{2T}\sum_{(k_j,k'_j)\in\Lambda} B_{\mathbf{k},k'_j} \geq \sum_{(k_j,k_j)\in\Lambda}\left|p_{k_j}\right|^2\left(|\beta_{\mathbf{k}}|^2+|\beta_{-\mathbf{k}}|^2\right)$$

$$-\sum_{(k_j,k'_j)\in\Lambda,\ k_j\neq k'_j}\left(\frac{2\pi}{\gamma_j T}\right)^2\left(\left|p_{k_j}\beta_{\mathbf{k}}\right|^2+\left|p_{k_j}\beta_{-\mathbf{k}}\right|^2\right)$$

$$\frac{1}{4(k_j-k'_j)^2-1}$$

$$-\sum_{(k_j,k'_j)\in\Lambda}\left(\frac{2\pi}{\gamma_j T}\right)^2\left(\left|\beta_{\mathbf{k}}p_{k_j}\right|^2+\left|\beta_{-\mathbf{k}}p_{k_j}\right|^2\right)$$

$$\frac{1}{4(k_j+k'_j)^2-1}$$

Step 6: make the sum (2/3)

$$\frac{\pi}{2T}\sum_{(k_j,k_j')\in\Lambda} B_{\mathbf{k},k_j'} \geq \sum_{(k_j,k_j)\in\Lambda} \left|p_{k_j}\right|^2 \left(|\beta_{\mathbf{k}}|^2 + |\beta_{-\mathbf{k}}|^2\right)$$

$$-\left(\frac{2\pi}{\gamma_j T}\right)^2 \sum_{k_j\in\mathbb{N}^*} \left(\left|p_{k_j}\beta_{\mathbf{k}}\right|^2 + \left|p_{k_j}\beta_{-\mathbf{k}}\right|^2\right)$$

$$\left(\sum_{k_j'\neq k_j\in\mathbb{N}^*} \frac{1}{4(k_j-k_j')^2-1} + \sum_{k_j'\in\mathbb{N}^*} \frac{1}{4(k_j+k_j')^2-1}\right)$$

Step 6: make the sum (3/3)

$$\frac{\pi}{2T}\sum_{(k_j,k_j')\in\Lambda} B_{\mathbf{k},k_j'} \geq \sum_{(k_j,k_j)\in\Lambda} \left|p_{k_j}\right|^2 \left(|\beta_{\mathbf{k}}|^2 + |\beta_{-\mathbf{k}}|^2\right)$$

$$- \left(\frac{2\pi}{\gamma_j T}\right)^2 \sum_{k_j\in\mathbb{N}^*} \left(\left|p_{k_j}\beta_{\mathbf{k}}\right|^2 + \left|p_{k_j}\beta_{-\mathbf{k}}\right|^2\right)$$

$$\geq \sum_{k_j\in\mathbb{N}^*} \left|p_{k_j}\right|^2 \left(|\beta_{\mathbf{k}}|^2 + |\beta_{-\mathbf{k}}|^2\right) \left(1_{\delta_j \leq \left|p_{k_j}\right|} - \left(\frac{2\pi}{T\gamma_j}\right)^2\right).$$

Step 7: make the sum over j

$$B := \sum_{j=1}^{d} \sum_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_d\in\mathbb{N}^*} \int_{-T/2}^{T/2} \left| \sum_{k_j\in\mathbb{N}^*} p_{k_j} \left(\beta_{\mathbf{k}} e^{i\lambda_{\mathbf{k}} t} + \beta_{-\mathbf{k}} e^{-i\lambda_{\mathbf{k}} t}\right)\right|^2 \mathrm{d}t$$

$$\geq \frac{2T}{\pi} \sum_{\mathbf{k}\in(\mathbb{N}^*)^d} \left(|\beta_{\mathbf{k}}|^2 + |\beta_{-\mathbf{k}}|^2\right) \sum_{j=1}^{d} \left(1_{\delta_j \leq |p_{k_j}|} - \left(\frac{2\pi}{T\gamma_j}\right)^2\right) \left|p_{k_j}\right|^2 .$$

As we have
$\sum_{j=1}^d 1_{\delta_j\leq |p_{k_j}|} \left|p_{k_j}\right|^2 \geq \max_{j=1,\dots,d} \left|p_{k_j}\right|^2 \geq \sum_{j=1}^d \left|\frac{p_{k_j}}{\gamma_j}\right|^2 / \sum_{j=1}^d \frac{1}{\gamma_j^2}$, we get

$$B \geq \frac{2T}{\pi} \left(1/\sum_{j=1}^{d} \frac{1}{\gamma_j^2} - \left(\frac{2\pi}{T}\right)^2 \right) \sum_{k\in(\mathbb{N}^*)^d} \left(|\beta_k|^2 + |\beta_{-k}|^2\right) \left(\sum_{j=1}^{d} \left|\frac{p_{k_j}}{\gamma_j}\right|^2 \right) .$$

Application

$$\int_0^T \int_\Gamma |\partial_\nu u(t,x)|^2 \,\mathrm{d}\Gamma \mathrm{d}t = \sum_{j=1}^{N} \int_0^T \int_0^{L_1} \dots \int_0^{L_{j-1}} \int_0^{L_{j+1}} \dots \int_0^{L_N}$$

$$\left| \sum_{k \in (\mathbb{N}^*)^N} \frac{\pi k_j}{L_j} \left(\alpha_k e^{i\omega_k t} + \alpha_{-k} e^{-i\omega_k t} \right) \prod_{\ell=1, \ell \neq j}^{N} \sin\left(\frac{\pi k_\ell x_\ell}{L_\ell} \right) \right|^2$$

$$\mathrm{d}x_1 \dots \mathrm{d}x_{j-1} \mathrm{d}x_{j+1} \mathrm{d}x_N \mathrm{d}t.$$

Orthogonality of $\left(\sin\left(\frac{\pi k_\ell x_\ell}{L_\ell}\right)\right)_{k_\ell \geq 1}$ **in** $L^2(0, L_\ell)$

$$\int_0^T \int_\Gamma |\partial_\nu u(t,x)|^2 \,\mathrm{d}\Gamma \mathrm{d}t \asymp \sum_{j=1}^N \sum_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_N \in \mathbb{N}^*} \int_0^T \left| \sum_{k_j \in \mathbb{N}^*} k_j \left(\alpha_{\mathbf{k}} e^{i\omega_{\mathbf{k}} t} + \alpha_{-\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} \right) \right|^2 \mathrm{d}t. \quad (10)$$

Energie

$$\|u_0\|^2_{H^1_0(\Omega)} + \|u_1\|^2_{L^2(\Omega)} \asymp \sum_{\mathbf{k} \in (\mathbb{N}^*)^N} \left(\sum_{j=1}^N |k_j|^2 \right) \left(|\alpha_{\mathbf{k}}|^2 + |\alpha_{-\mathbf{k}}|^2 \right). \quad (11)$$

$d = N$, $\lambda_k = \omega_k = \sqrt{\sum_{j=1}^{N} \left(\frac{k_j \pi}{L_j}\right)^2}$, for $k \in (\mathbb{N}^*)^N$ and $p_\ell = \ell$, for $\ell \in \mathbb{N}^*$.
Verification of (4)-(5)-(6), for $j \in \{1, \ldots, N\}$. We consider $k = (k_1, \ldots, k_N) \in (\mathbb{N}^*)^N$ and $k'_j \in \mathbb{N}^*$, such that

$$0 < k_i \le \max(k_j, k'_j), \; i = 1, \ldots, N, \; i \ne j, \tag{12}$$

and we have to get (4)-(5), for a certain γ_j.

We have $|\omega_k + \omega_{k'}| \geq \left|k_j + k_j'\right| \frac{\pi}{L_j}$.

$$|\omega_k - \omega_{k'}| = \left|k_j - k_j'\right| \frac{\pi^2}{L_j^2} \frac{\left|k_j + k_j'\right|}{\sqrt{D + \left(\frac{k_j \pi}{L_j}\right)^2} + \sqrt{D + \left(\frac{k_j' \pi}{L_j}\right)^2}},$$

with $D = \sum_{i=1,\ i \neq j}^{N} \left(\frac{k_i \pi}{L_i}\right)^2$.

We can suppose that $k_j < k'_j$ and we have with (12)

$$\frac{|\omega_k - \omega_{k'}|}{\left|k_j - k'_j\right|} \geq \frac{\pi^2}{L_j^2} \frac{\left|1 + (k_j/k'_j)\right|}{\sqrt{\sum_{i=1,\ i\neq j}^d \left(\frac{\pi}{L_i}\right)^2 + (k_j/k'_j)^2 \frac{\pi^2}{L_j^2}} + \sqrt{\sum_{i=1}^d \left(\frac{\pi}{L_i}\right)^2}}$$

$$\geq \frac{\pi^2}{(L_j)^2} \frac{1}{2\sqrt{\sum_{i=1}^d \left(\frac{\pi}{L_i}\right)^2}}.$$

We then get (4)-(5) taking $\gamma_j = \frac{\pi^2}{L_j^2} \frac{1}{2\sqrt{\sum_{i=1}^d \left(\frac{\pi}{L_i}\right)^2}}. < \frac{\pi}{L_j}$. From (10) and (11), we find (2) with $T > 2\pi\sqrt{\sum_{j=1}^N \gamma_j^2} = 4\sqrt{\sum_{j=1}^N (L_j)^4}\sqrt{\sum_{j=1}^N L_j^{-2}}$.

Conclusion/Remarks/perspectives

- Ingham proof of observability for product of intervals
- This technique can be used for other applications: works of Komornik-Loreti, Komornik-Miara, Mauffrey-Münch.
- Improvements have also been done
- Can we have the optimal time with such technique?
- Treatment of other geometries.
- Numerical approach