Vectorial Ingham–Beurling type estimates

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MOTIVATION: AN OBSERVABILITY PROBLEM

We consider the coupled string–beam system

$$\begin{cases} u_{tt} - u_{xx} + au + bw = 0, \\ w_{tt} + w_{xxxx} + cu + dw = 0 \end{cases}$$

with usual initial conditions and with Dirichlet–hinged boundary conditions on a bounded interval $(0, \ell)$, where a, b, c, d are given coupling constants.
Given $T > 0$, we investigate the validity of the estimates

$$c_1 E(0) \leq \int_0^T |u_x(t,0)|^2 + |w_x(t,0)|^2 \, dt \leq c_2 E(0)$$

with suitable positive constants c_1, c_2 where $E(0)$ denotes the usual initial energy ($\mathcal{H} = H_0^1 \times L^2 \times H_0^1 \times H^{-1}$).

ABSTRACT FORM

Following Komornik–Loreti, we may write these estimates in the abstract form

$$c_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \leq \int_0^T |x(t)|^2 \, dt \leq c_2 \sum_{k\in\mathbb{Z}} |x_k|^2$$

where

$$x(t) = \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}$$

with square-summable complex coefficients x_k. Here (U_k) is a given sequence of unit vectors in $\mathbb{C}^4$ and (ω_k) is a given sequence of real numbers, depending on the parameters of the problem (eigenvector traces and eigenvalues).

ASSUMPTIONS AND NOTATIONS

- Let $\Omega := (\omega_k)_{k\in\mathbb{Z}}$ be a family of real numbers satisfying the *gap condition*

$$\gamma := \inf_{k\neq n} |\omega_k - \omega_n| > 0.$$

- Let $(U_k)_{k\in\mathbb{Z}}$ be a corresponding family of unit vectors in some finite-dimensional complex Hilbert space H and consider the sums

$$x(t) = \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}$$

with square summable complex coefficients x_k.

- By the gap condition Ω has a finite upper density defined by

$$D^+ = D^+(\Omega) := \lim_{r\to\infty} n^+(r)/r$$

where $n^+(r)$ denotes the maximum number of terms ω_k contained in an interval of length r. We have $D^+ \leq 1/\gamma$.

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MAIN RESULT

We recall that

$$x(t) = \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}, \quad x_k \in \mathbb{C}.$$

Theorem

(Barhoumi, Komornik, M.)

(a) If $T > 2\pi D^+$, then the estimates

$$c_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \leq \int_0^T \|x(t)\|_H^2 \, dt \leq c_2 \sum_{k\in\mathbb{Z}} |x_k|^2$$

hold with suitable $c_1, c_2 > 0$.

(b) Conversely, if the above estimates hold true and $\dim H = d$, then $T \geq 2\pi D^+/d$.

ONE-DIMENSIONAL EXAMPLES

In the scalar case $d = 1$ the critical length is $T = 2\pi D^+$ by Beurling's original theorem.

- For $\omega_k = k$ we have $D^+ = 1$ and the critical length is 2π in correspondence with Parseval's equality:

$$\int_0^{2\pi} \left| \sum_{k\in\mathbb{Z}} x_k e^{ikt} \right|^2 dt = 2\pi \sum_{k\in\mathbb{Z}} |x_k|^2.$$

- For $\omega_k = k^3$ we have $D^+ = 0$, so that

$$c_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \le \int_0^T |x(t)|^2 \, dt \le c_2 \sum_{k\in\mathbb{Z}} |x_k|^2$$

for any $T > 0$ (the constants $c_1, c_2 > 0$ depend on T).

- Ingham's earlier sufficient condition ensured the preceding estimates for $T > 2\pi/\gamma = 2\pi$. (We recall that $D^+ \le 1/\gamma$.)

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HIGHER-DIMENSIONAL EXAMPLE

- If $d > 1$, (U_k) is d-periodical and $U_1, \ldots, U_d$ is an orthonormal basis of H, then the critical length is $T = 2\pi D^+/d$. Indeed,

$$\int_0^T \Big|\sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}\Big|_H^2 \, dt = \sum_{j=1}^{d} \int_0^T \Big|\sum_{k\in\mathbb{Z}} x_{kd+j} e^{i\omega_{kd+j} t}\Big|^2 \, dt$$

and we may apply the scalar case to each sum on the right side.

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VECTORIAL CASE FROM SCALAR CASE

- Scalar case: Beurling's original theorem
- Vectorial case:
  We fix an orthonormal basis $(E_n)_{n\in N}$ of H and we develop each U_k into Fourier series:

$$U_k = \sum_{n\in N} u_{kn} E_n.$$

- If $T > 2\pi D^+$, then using the scalar case we have

$$\int_0^T \Big\| \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t} \Big\|_H^2 dt = \sum_{n\in N} \int_0^T \Big| \sum_{k\in\mathbb{Z}} x_k u_{kn} e^{i\omega_k t} \Big|^2 dt$$
$$\asymp \sum_{n\in N} \sum_{k\in\mathbb{Z}} |x_k u_{kn}|^2$$
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NECESSITY OF $T \geq 2\pi D^+/d$. NOTATIONS

- Assume by scaling that
$$\int_0^{2\pi} \left| \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t} \right|^2 dt \asymp \sum_{k\in\mathbb{Z}} |x_k|^2.$$
  We need to show that $D^+ \leq d$.
- We adapt a method of Gröchenig and Razafinjatovo.

  Fix $R > 0$, $y \in \mathbb{R}$, $r > 0$ and set
$$V = V_{y,r} := \mathrm{Vect}\{U_k e^{i\omega_k t} \ : \ |\omega_k - y| < r\},$$
$$W = W_{R,y,r} := \mathrm{Vect}\{U e^{ikt} \ : \ U \in H, |k - y| < r + R\}.$$
  Note that
$$\sup_y \dim V = n^+(2r),$$
$$\dim W \leq (2r + 2R)d.$$

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PROOF OF $D^+ \leq d$

- We use a comparison theorem (shown next slide)

$$\dim V \leq (1 + o_R(1)) \dim W \quad \text{as} \quad R \to \infty.$$

- This will imply that

$$n^+(2r) = \sup_y \dim V \leq (2r + 2R)d(1 + o_R(1))$$

- and hence that

$$D^+ = \lim_{r\to\infty} \frac{n^+(2r)}{2r} \leq d(1 + o_R(1))$$

for all $R > 0$. Letting $R \to \infty$ this yields $D^+ \leq d$.

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PROOF OF $\dim V \leq (1 + o_R(1)) \dim W$

Let P, Q be the orthogonal projections of $L^2(0, 2\pi; H)$ onto V and W. Then

$$S := P \circ Q|_V \in L(V, V)$$

has norm ≤ 1 and rank $\leq \dim W$, so that

$$\operatorname{Tr} S \leq \dim W.$$

It remains to show that

$$\operatorname{Tr} S \geq (1 - o_R(1)) \dim V.$$

ESTIMATE OF $\|(Q-I)e_k\|_{L^2(0,2\pi;H)}$

We have the estimate

$$\|(Q-I)e_k\|^2_{L^2(0,2\pi;H)} \leq \frac{2d}{\pi} \sum_{|n-y|\geq r+R} \frac{1}{|\omega_k - n|^2}$$

For k such that $|\omega_k - y| < r$, we have

$$|n-y| \geq r+R \Rightarrow |n-\omega_k| > R$$

It follows that

$$\|(Q-I)e_k\|^2_{L^2(0,2\pi;H)} \leq \frac{4d}{\pi} \sum_{n=0}^{\infty} \frac{1}{(R+n)^2} \approx \frac{4d}{\pi} \int_R^{\infty} \frac{1}{x^2}\, dx = \frac{4d}{\pi R}.$$

PROOF OF $\operatorname{Tr} S \geq (1 - o_R(1)) \dim V$

Let (f_k) be a bounded biorthogonal sequence to $e_k := U_k e^{i\omega_k t}$ in $L^2(0, 2\pi; H)$. Since

$$\operatorname{Tr} S = \sum_{|\omega_k - y| < r} (Se_k, f_k)_{L^2(0,2\pi;H)} = \sum_{|\omega_k - y| < r} (Qe_k, Pf_k)_{L^2(0,2\pi;H)},$$

we have

$$\begin{aligned} \operatorname{Tr} S - \dim V &= \sum_{|\omega_k - y| < r} ((Q - I)e_k, Pf_k)_{L^2(0,2\pi;H)} \\ &\geq -(\sup \|f_k\|)(\dim V) \sup_{|\omega_k - y| < r} \|(Q - I)e_k\|_{L^2(0,2\pi;H)} \\ &= -o_R(1) \dim V \end{aligned}$$

by a direct computation.

PARTITIONS AND UPPER DENSITY

In order to show that the critical value of T may be anything between $2\pi D^+/d$ and $2\pi D^+$, we use the following result:

Theorem

Let Ω be a set of real numbers with a finite upper density D^+ and let $\alpha_1, \alpha_2, \dots$ be a finite or infinite sequence of numbers in $[0,1]$ satisfying

$$\alpha_1 + \alpha_2 + \cdots \geq 1.$$

Then there exists a partition

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots$$

such that the upper density of Ω_j is equal to $\alpha_j D^+$ for every j.

OPTIMALITY OF THE MAIN THEOREM

Given $1/d \le \alpha \le 1$ arbitrarily we choose $\alpha_1, \dots, \alpha_d \ge 0$ such that

$$\alpha_1 + \cdots + \alpha_d = 1 \quad \text{and} \quad \max\{\alpha_1, \dots, \alpha_d\} = \alpha.$$

Applying the above theorem we obtain a partition $\Omega = \Omega_1 \cup \cdots \cup \Omega_d$ such that $D^+(\Omega_j) = \alpha_j D^+$ for all j. Fix an orthonormal basis $E_1, \dots, E_d$ of H and set $U_k = E_j$ if $\omega_k \in \Omega_j$. Then using the identity

$$\int_0^T \Big\| \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \Big\|_H^2 \, dt = \sum_{j=1}^d \int_0^T \Big| \sum_{\omega_k \in \Omega_j} x_k e^{i\omega_k t} \Big|^2 \, dt$$

and applying the scalar case of the theorem we conclude that the required estimates hold if $T > 2\pi\alpha D^+$, and they fail if $T < 2\pi\alpha D^+$.

ℓ INDEPENDENT VECTORS

Proposition

We suppose that $\{U_k\} = \{V_1, V_2, \dots, V_\ell\}$, where $V_1, V_2, \dots V_\ell$ are independent in $\mathbb{C}^d$ (which implies that $\ell \leq d$). Then we partition $\Omega = \cup_{i=1}^{\ell} \Omega_i$, where

$$\Omega_i = (\omega_k)_{k \in I_i}, \; I_i = \{k \in \mathbb{Z}, \; U_k = V_i\}.$$

We suppose that for each $i = 1, \dots, d$, Ω_i satisfies a gap condition. Then, if $T > 2\pi \max_{1 \leq i \leq d} D^+(\Omega_i)$, we have (1) and reciprocally, if (1) holds, then $T \geq 2\pi \max_{1 \leq i \leq d} D^+(\Omega_i)$.

PROOF

Proof.

It follows from the estimates

$$\int_0^T \|x(t)\|^2 dt = \int_0^T \|\sum_{j=1}^{d}\left(\sum_{k\in I_j} x_k e^{i\omega_k t}\right) V_j\|^2 dt$$

$$\asymp \sum_{j=1}^{d}\int_0^T \left|\sum_{k\in I_j} x_k e^{i\omega_k t}\right|^2 dt.$$

□

A INGHAM TYPE SUFFISANT CONDITION

Theorem

We suppose that

$$\sup_{k\in\mathbb{Z}} \sum_{\ell\neq k} G\left((\omega_k - \omega_\ell)\frac{T}{2\pi}\right) |(U_k, U_\ell)| < 1,$$

where G is defined by $G(y) = \left|\frac{\cos(\pi y)}{4y^2-1}\right|$. *Then we have*

$$\int_0^T \|x(t)\|^2 dt \asymp \sum_k |x_k|^2, \; x(t) = \sum_k x_k U_k \exp(i\omega_k t). \qquad (1)$$

PROOF 1/2

We follow Ingham's first method. We consider the function

$$f(t) = \begin{cases} \cos \frac{\pi t}{T} & \text{if } |t| \leq T/2 \\ 0 & \text{if } |t| > T/2. \end{cases}$$

It's Fourier transform $\hat{f}$ satisfies: $\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{ixt} \mathrm{d}t = -\frac{2T\pi \cos(xT/2)}{x^2T^2 - \pi^2}$.
We have $\hat{f}(0) = \frac{2T}{\pi}$ and $|\hat{f}(x)| \leq \frac{2T}{\pi} G(x\frac{T}{2\pi})$.

PROOF 2/2

We then get

$$\int_{-T/2}^{T/2} \|x(t)\|^2 dt \geq \int_{\mathbb{R}} f(t)\|x(t)\|^2 dt = \sum_{k,\ell} x_k \overline{x_\ell} (U_k, U_\ell) \hat{f}(\omega_k - \omega_\ell)$$

$$\geq \frac{2T}{\pi} \left(\sum_k |x_k|^2 - \sum_{k \neq \ell} |x_k||x_\ell| G\left((\omega_k - \omega_\ell) \frac{T}{2\pi} \right) |(U_k, U_\ell)| \right).$$

By using the fact that $|x_k||x_\ell| \leq \frac{|x_k|^2 + |x_\ell|^2}{2}$, we get

$$\int_{-T/2}^{T/2} \|x(t)\|^2 dt \geq \frac{2T}{\pi} \sum_k |x_k|^2 \left(1 - \sum_{\ell \neq k} G\left((\omega_k - \omega_\ell) \frac{T}{2\pi} \right) |(U_k, U_\ell)| \right),$$

and the result follows.

A SPECIFIC CLASS OF EXAMPLES

- Proposition:
  Let $a_i,\ i = 1, \ldots, d$, be real numbers satisfying $0 = a_1 \leq a_2 \leq \cdots \leq a_d < 2\pi$, and we define

$$T_0 = \max(a_2 - a_1, a_3 - a_2, \ldots, a_d - a_{d-1}, 2\pi - a_d).$$

  We suppose that $\omega_k = k\pi$ and $U_k = \left(e^{ika_1}, \ldots, e^{ika_d}\right)/\sqrt{d} \in \mathbb{C}^d$. Then, if $T \geq T_0$, (1) holds, and reciprocally, if (1) holds then $T < T_0$.

- Proof:
  It follows from the identity

$$\int_0^T \|x(t)\|^2 dt = \frac{1}{d} \int_{\cup_{j=1}^d [a_j, a_j+T]} \left| \sum_k x_k e^{ik\pi t} \right|^2 dt.$$

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INGHAM FOR THE LAPLACIAN IN A SQUARE

Hypothesis

Let $d \in \mathbb{N}^*$, real numbers $(\lambda_k)_{k\in(\mathbb{N}^*)^d}$, et complex numbers $(p_\ell)_{\ell\in\mathbb{N}^*}$. We suppose to have the spectral gap : for $j = 1, \dots, d$, there exists $\gamma_j > 0$ such that

$$\left|\lambda_{k_1,\dots,k_{j-1},\mathbf{k_j},k_{j+1},\dots,k_d} - \lambda_{k_1,\dots,k_{j-1},\mathbf{k'_j},k_{j+1},\dots,k_d}\right| \geq \gamma_j \left|\mathbf{k_j} - \mathbf{k'_j}\right|, \tag{2}$$

$$\left|\lambda_{k_1,\dots,k_{j-1},\mathbf{k_j},k_{j+1},\dots,k_d} + \lambda_{k_1,\dots,k_{j-1},\mathbf{k'_j},k_{j+1},\dots,k_d}\right| \geq \gamma_j \left|\mathbf{k_j} + \mathbf{k'_j}\right|, \tag{3}$$

for all the index $k = (k_1, \dots, k_d) \in (\mathbb{N}^*)^d$ and $k'_j \in \mathbb{N}^*$ such that the weights $(p_\ell)_{\ell\in\mathbb{N}^*}$ satisfy

$$\max_{i=1,\dots,d,\ i\neq j} \left|p_{k_i}\right| \leq \max\left(\left|p_{k_j}\right|, \left|p_{k'_j}\right|\right). \tag{4}$$

INGHAM FOR THE LAPLACIAN IN A SQUARE

Conclusion

Then, for $T > 2\pi\sqrt{\sum_{i=1}^{d}\frac{1}{\gamma_i^2}}$, there exists a constant $c_1 > 0$ such that we have,

$$\sum_{j=1}^{d}\sum_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_d\in\mathbb{N}^*}\int_0^T\left|\sum_{k_j\in\mathbb{N}^*}p_{k_j}\left(\beta_k e^{i\lambda_k t}+\beta_{-k}e^{-i\lambda_k t}\right)\right|^2 \mathrm{d}t$$

$$\geq c_1\sum_{k\in(\mathbb{N}^*)^d}\left(|\beta_k|^2+|\beta_{-k}|^2\right)\left(\sum_{j=1}^{d}\left|p_{k_j}\right|^2\right), \quad (5)$$

for all the complex numbers $(\beta_k)_{k\in(\mathbb{N}^*)^d}$ and $(\beta_{-k})_{k\in(\mathbb{N}^*)^d}$, such that the sums involved are finite.

A NON TRANSLATION INVARIANT INEQUALITY

We consider

$$C(I)\sum_{k\geq 1}|a_k|^2 \leq \int_I \left|\sum_{k\geq 1} a_k e^{ikat}\right|^2 \mathrm{d}t + \int_I \left|\sum_{k\geq 1} a_k e^{-ikbt}\right|^2 \mathrm{d}t, \qquad (6)$$

where a, b are positive numbers.

A NON TRANSLATION INVARIANT INEQUALITY

Proposition

(cf Loreti, M.) Let $a, b > 0$, $\alpha \in \mathbb{R}$ *and* $I = [\alpha T, \alpha T + T]$. *We define* T_α *by*

- *If* $\alpha = \ell + s$, *with* $\ell \in \mathbb{N}$ *and* $0 < s < 1$, *then* $T_\alpha = \min(\frac{2\pi}{\max(a,b)}, \frac{2\pi(\ell+2)}{(a+b)(\ell+1+s)})$.
- *If* $\alpha = -\ell + s$, *with* $\ell \in \mathbb{N}$, $\ell \geq 2$ *and* $0 < s < 1$, *then* $T_\alpha = \min(\frac{2\pi}{\max(a,b)}, \frac{2\pi\ell}{(a+b)(\ell-s)})$.
- *If* $\alpha = -1 + s$, *with* $0 < s < 1$, *then* $T_\alpha = \min(\frac{2\pi}{\max(a,b)}, \frac{2\pi}{(a+b)\max(1-s,s)})$.
- *If* $\alpha \in \mathbb{Z}$ *then* $T_\alpha = \min(\frac{2\pi}{\max(a,b)}, \frac{2\pi}{a+b}) = \frac{2\pi}{a+b}$.

Then, for all $T \geq T_\alpha$, *(6) is satisfied, with a constant* $C(I)$ *independent of the coefficients* (a_k). *If* $T < T_\alpha$, *then (6) cannot hold with a constant* $C(I)$ *independent of the coefficients* (a_k).

A NON TRANSLATION INVARIANT INEQUALITY

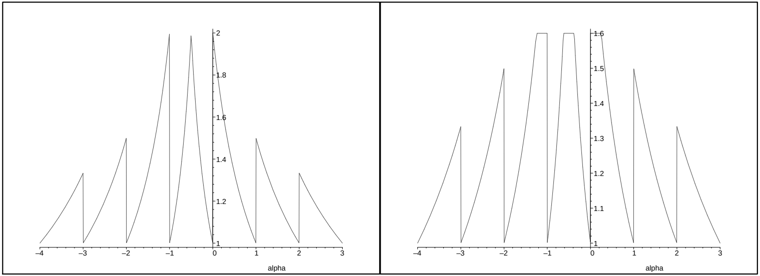

Figure: $T_\alpha \frac{(a+b)}{2\pi}$ versus α, for $a = b = 1$ (left) and for $a = 1$ and $b = 0.6$ (right).