Semi-Lagrangian simulations on polar grids: from diocotron instability to ITG turbulence

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Abstract

While developing a new semi-Lagrangian solver, the gap between a linear Landau run in $1D \times 1D$ and a $5D$ gyrokinetic simulation in toroidal geometry is quite huge. **Intermediate test cases** *are welcome for checking the code. We consider here as building block, a $2D$ guiding center type equation on an annulus:*

$$\partial_t f - \frac{\partial_\theta \Phi}{r}\partial_r f + \frac{\partial_r \Phi}{r}\partial_\theta f = 0,\ t \in [0,T], \qquad (1)$$

with $(r,\theta) \in \Omega = [r_{\min}, r_{\max}] \times [0, 2\pi]$, where the potential Φ solves a Poisson type equation. We first revisit a $2D$ test case previously done with a PIC approach [6] and detail the boundary conditions. We then consider a $4D$ drift kinetic slab simulation [2] for which we give some first results of a new conservative method.

1. Diocotron instability test case [3]

We consider $\rho = f$, solution of (1), with Φ solving Poisson equation:

$$-\partial_r^2\Phi - \frac{1}{r}\partial_r\Phi - \frac{1}{r^2}\partial_\theta^2\Phi = \gamma\rho,$$

where $\gamma \in \{-1, 1\}$, depending on the context (astro/plasma physics). We define the electric energy

$$\mathcal{E}(t) = \int_\Omega r|\partial_r\Phi|^2 + \frac{1}{r}|\partial_\theta\Phi|^2 dr d\theta, \qquad (2)$$

and the mass

$$\mathcal{M}(t) = \int_\Omega r\rho dr d\theta. \qquad (3)$$

We take Dirichlet boundary condition at $r_{\max}$:

$$\Phi(r_{\max}) = 0$$

At $r_{\min}$, we take either Dirichlet boundary condition or the following condition, named **Neumann mode 0** which differ from Neumann boundary condition (the latter being $\partial_r\Phi(r, r_{\min}, \theta) = 0$):

- Neumann boundary condition at $r_{\min}$ for the Fourier mode 0 in θ:

$$\int_0^{2\pi} \partial_r\Phi(t, r_{\min}, \theta)d\theta = 0,$$

- Dirichlet boundary condition at $r_{\min}$ for the other modes, which reads

$$\Phi(t, r_{\min}, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(t, r_{\min}, \theta')d\theta'$$

We then have the following result:

- For such boundary conditions, mass and energy are constant in time
- If we move to Neumann boundary condition at $r_{\min}$, this may not remain true.

We consider the following initial data [Davidson, 1990]

$$\rho(0, r, \theta) = \begin{cases} 0,\ r_{\min} \le r < r^-, \\ 1 + \epsilon\cos(\ell\theta),\ r^- \le r \le r^+, \\ 0,\ r^+ < r \le r_{\max}, \end{cases}$$

where ϵ is a small parameter.
Linearized problem leads then here to explicit solutions

$$\rho(t, r, \theta) \simeq n_0(r) + \varepsilon\widehat{n}_{1,\ell}(r)\exp(i\ell\theta)\exp(-i\omega t/\gamma) + O(\varepsilon^2)$$
$$\Phi(t, r, \theta) \simeq \gamma\Phi_0(r) + \varepsilon\gamma\widehat{\Phi}_{1,\ell}(r)\exp(i\ell\theta)\exp(-i\omega t/\gamma) + O(\varepsilon^2)$$

Analytical growth rates $|\Im(\omega)|$ of the linear phase are obtained for all the discussed boundary conditions and comparison is made with BSL method on 128×128 grid, $\Delta t = 0.05$, $r_{\min} = 1,\ r_{\max} = 10,\ \gamma = 1$.

ℓ	r^-	r^+	$\Im(\omega)$	analytical
2	4	5	$0.2875,\ t \in [26, 55]$	0.288739227554270
3	4	5	$0.3667,\ t \in [26, 72]$	0.367315895142460
4	4	5	$0.3852,\ t \in [26, 51]$	0.384081542249742
7	6	7	$0.3424,\ t \in [26, 61]$	0.337573424025866

Table 1: *Dirichlet boundary condition*

ℓ	r^-	r^+	$\Im(\omega)$	analytical
2	4	5	$0.0719,\ t \in [93, 138]$	0.0717643545314240
3	4	5	$0.2264,\ t \in [33, 87]$	0.226724091815516
4	4	5	$0.2991,\ t \in [34, 69]$	0.298891133128847
7	6	7	$0.3338,\ t \in [35, 53]$	0.330764942142169

Table 2: *Neumann mode 0 boundary condition*

ℓ	r^-	r^+	$\Im(\omega)$	analytical
2	4	5	$0.0672,\ t \in [89, 151]$	0.067338570606553
3	4	5	$0.2263,\ t \in [32, 89]$	0.226665019518819
4	4	5	$0.2991,\ t \in [34, 69]$	0.298891598934172
7	6	7	$0.3338,\ t \in [35, 53]$	0.330764942148054

Table 3: *Neumann boundary condition*

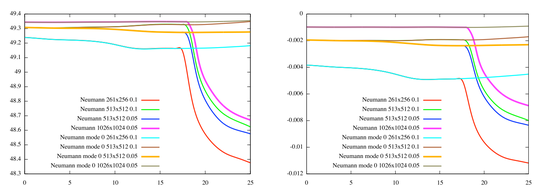


Figure 1: *Evolution of $\mathcal{E}(t)$ (left) and $\mathcal{M}(t)$ (right) for different grid sizes and Δt*

2. 4D slab drift kinetic Vlasov equation

We look for $f = f(t, r, \theta, z, v)$ [2], satisfying

$$\partial_t f - \frac{\partial_\theta\phi}{r}\partial_r f + \frac{\partial_r\phi}{r}\partial_\theta f + v\partial_z f - \partial_z\phi\partial_v f = 0, \qquad (4)$$

for $(r, \theta, z, v) \in \Omega \times [0, L] \times [-v_{\max}, v_{\max}]$.
Potential $\phi = \phi(r, \theta, z)$ solves the quasi neutral equation

$$-(\partial_r^2\phi + (\frac{1}{r} + \frac{\partial_r n_0(r)}{n_0(r)})\partial_r\phi + \frac{1}{r^2}\partial_\theta^2\phi) + \frac{1}{T_e(r)}(\phi - \langle\phi\rangle) = \frac{1}{n_0(r)}\int f dv - 1,\ \langle\phi\rangle = \frac{1}{L}\int_0^L \phi(r, \theta, z)dz.$$

The initial function is

$$f(0, r, \theta, z, v) = f_{\rm eq}(r, v)(1 + \epsilon\exp(-\frac{(r - r_p)^2}{\delta r})\cos(\frac{2\pi n}{L}z + m\theta)),$$

with equilibrium function

$$f_{\rm eq}(r, v) = \frac{n_0(r)\exp(-\frac{v^2}{2T_i(r)})}{(2\pi T_i(r))^{1/2}}.$$

The profiles have here analytical expressions

$$\mathcal{P}(r) = C_{\mathcal{P}}\exp(-\kappa_{\mathcal{P}}\delta r_{\mathcal{P}}\tanh(\frac{r - r_p}{\delta r_{\mathcal{P}}})), \mathcal{P} \in \{T_i, T_e, n_0\}$$

$$C_{T_i} = C_{T_e} = 1,\ C_{n_0} = \frac{r_{\max} - r_{\min}}{\int_{r_{\min}}^{r_{\max}}\exp(-\kappa_{n_0}\delta r_{n_0}\tanh(\frac{r - r_p}{\delta r_{n_0}}))dr}.$$

We consider the parameters of [1] (MEDIUM case)

$$r_{\min} = 0.1,\ r_{\max} = 14.5,\ \kappa_{n_0} = 0.055,\ \kappa_{T_i} = \kappa_{T_e} = 0.27586,$$
$$\delta r_{T_i} = \delta r_{T_e} = \frac{\delta r_{n_0}}{2} = 1.45,\ \varepsilon = 10^{-6},\ n = 1,\ m = 5,$$
$$L = 1506.759067,\ r_p = \frac{r_{\min} + r_{\max}}{2}, \delta r = \frac{4\delta r_{n_0}}{\delta r_{T_i}}.$$

Analytical growth rates are no more available, but we can check the linear phase with recent results [1], p19.

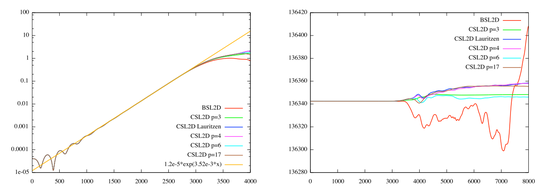


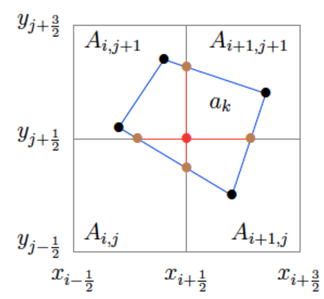
Figure 2: *Evolution of $\sqrt{\int \Phi^2(r_p, \theta, z)d\theta dz}$ (left), mass (right) on $32 \times 32 \times 32 \times 64$ grid, with $\Delta t = 8,\ v_{\max} = 7.32$*

3. Description of the classical method

As in [2], the BSL method is used. The following options are here:

- constant advection in z, v, with cubic splines (periodic)
- $2D$ advection in (r, θ) with cubic splines
  - Verlet algorithm for the characteristics
  - linear interpolation of the fields
- splitting and predictor corrector for time loop
- fields derivatives with cubic splines in r, θ and second order finite differences in z
- Poisson/quasi neutral with FFT in θ and second order finite difference in r.

4. A new $2D$ conservative method, following [5]



- unknowns are cell values $\bar{f}^n_{i,j}$
- backward displacement of cell
- order p compact reconstruction of edge values $\bar{f}^n_{i,j\pm}$, $\bar{f}^n_{i\pm,j}$ and point values $\bar{f}^n_{i\pm,j\pm}$
- polynomial on $A_{i,j}$: $\sum_{\alpha,\beta\le 2} c_{\alpha,\beta}x^\alpha y^\beta$
- contribution of $x^\alpha y^\beta$ on each edge of $a_k \cap A_{i,j}$ with Stokes formula

Adaptations w.r.t. uniform periodic case are needed as in [4] for the FSL method.

- multiply by jacobian, since conservative form is on $g(t, r, \theta) = rf(t, r, \theta)$, satisfying

$$\partial_t g - \partial_r(\frac{\partial_\theta\Phi}{r}g) + \partial_\theta(\frac{\partial_r\Phi}{r}g) = 0$$

- work on $f - f_{\rm eq}$ for non zero boundary conditions (also used for BSL)

5. Conclusion/ future work

- description of $2D$ and $4D$ test cases on polar geometry for semi-Lagrangian solvers
- development of a new conservative method
- ongoing implementation in a dedicated library [7]

References

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