Optimal decay rates for the stabilization of a string network

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Controllability and networks
Conference in honour of the 60th birthday of Vilmos Komornik
Rome, May 26-28, 2014

Joint work with M. Jellouli, Faculté des Sciences de Monastir, Tunisie

Outline

- I. Introduction
- II. Theoretical results
- III. Numerical results
- IV. Conclusion

The case of one string

- Vibrating elastic string fixed at end point

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0, \text{ on } (0,\ell), \; u(t,\ell) = 0$$

- Dissipation condition $\partial_x u(t,0) = \alpha \partial_t u(t,0)$ at origin
- Then the energy of the system satisfies

$$E(t) \leq C \exp(-\gamma_\alpha t), \; \gamma = \frac{1}{\ell} \log \left| \frac{1+\alpha}{1-\alpha} \right|$$

- If $\alpha = 1$, then $E(t) = 0$, for $t \geq 2\ell$.

The case of one string

- See V. Komornik, *The Multiplier Method*
- Explicit solution via d'Alembert formula
- Exponential stabilization for $\alpha \neq 1$
- Controllability for $\alpha = 1$ (stabilization in finite time)

Question: what happens for a network of strings?

Some related results

- Star-shaped with stabilization at the common node or tree-shaped with stabilization at the root (Ammari, Jellouli, Khenissi, 2004/5)

  no exponential stability in the non degenerate case
- Pointwise stabilisation of a string viewed as 2 star shaped strings
  - (Ammari, Henrot, Tucsnak, 2001) Dirichlet/Neumann
    - exponential stabilization is obtained if and only if the lengths satisfy $\frac{\ell_1}{\ell_1+\ell_2} = \frac{p}{q}$, with p and q odd numbers
    - best decay rate, fixing total length $\ell_1 + \ell_2$ obtained for equal lengths $\ell_1 = \ell_2 = \ell$
    - best decay rate is $\gamma = \frac{\ln(3)}{\ell}$
  - (Nicaise, Valein, 2010) Dirichlet/Dirichlet, degenerate case
    - Energy limit E_∞ is identified
    - exponential decrease of energy to E_∞

A tree-shaped configuration

- $N \geq 3$ vibrating elastic strings

$$\partial_t^2 u_j(t,x) - \partial_x^2 u_j(t,x) = 0,\ t > 0,\ x \in (0,\ell_j),\ j = 1,\dots,N$$

- Continuity: $u_1(t,\ell_1) = u_j(t,0),\ j = 2,\dots,N$
- Dissipation at the root of the tree

$$\partial_x u_1(t,0) = \alpha \partial_t u_1(t,0),\ t \geq 0\ (\alpha > 0)$$

- Dirichlet at other exterior nodes $u_j(t,\ell_j) = 0,\ t \geq 0,\ j = 2,\dots,N$
- Transfert condition

$$\partial_x u_1(t,\ell_1) = \sum_{j=2}^{N} \partial_x u_j(t,0),\ t \geq 0,$$

- Initial condition

$$u_j(0,x) = a_j(x),\ \ \partial_t u_j(0,x) = b_j(x),\ x \in [0,\ell_j],\ j = 1,\dots,N.$$

Questions

- What is the energy limit E_∞?
- Do we have exponential decrease to E_∞?
- What is the best decay rate?
- Can we have stabilization in finite time?
- How do numerical schemes behave?

In the sequel, we suppose that all the lengths are equal

$$\ell_j = \ell,\ j = 1, \ldots, N.$$

The case $N = 2$ can be recasted to $N = 1$ with length $\ell_1 + \ell_2$.

Energy and energy limit

The energy is defined as

$$E(t) = \frac{1}{2} \sum_{j=1}^{N} \|\partial_x u_j(t)\|^2 + \frac{1}{2} \sum_{j=1}^{N} \|\partial_t u_j(t)\|^2 ,$$

with $\| \ \|$ for the norm of $L^2(0, \ell)$

Theorem

The energy limit is given by

$$E_\infty = \frac{1}{2(N-1)} \sum_{j=2}^{N} \sum_{k=j+1}^{N} \left(\|a'_k - a'_j\|^2 + \|b_k - b_j\|^2 \right) .$$

$E_\infty = 0$, for $a'_j = a'_2, \ b_j = b_2, \ j = 2, \dots, N$.

Exponential decrease to E_∞ and best decay rate

Theorem

If $\alpha \neq \alpha_0$,

$$0 \leq E(t) - E_\infty \leq C_{\alpha,N}\left(\|a_1'\|^2 + \|b_1\|^2 + \left\| \sum_{j=2}^{N} a_j' \right\|^2 + \left\| \sum_{j=2}^{N} b_j \right\|^2 \right) e^{-\gamma t}$$

If $\alpha = \alpha_0$,

$$0 \leq E(t) - E_\infty \leq C_{\alpha,N}\left(\|a_1'\|^2 + \|b_1\|^2 + \left\| \sum_{j=2}^{N} a_j' \right\|^2 + \left\| \sum_{j=2}^{N} b_j \right\|^2 \right) t^2 e^{-\gamma_0 t},$$

where $\gamma_0 = \frac{1}{\ell} \log(1 + 2\frac{\sqrt{N-1}+1}{N-2})$ *and* $\gamma < \gamma_0$ *for* $\alpha \neq \alpha_0 = \frac{2\sqrt{N-1}}{N}$.

Remarks

- No stabilization in finite time, except for special initial data
- Proof based on operators of type τ (Ammari, Jellouli, 2007/10)
- Explicit computations based on d'Alembert formula
- Difficult to make the computations when lengths are different
- Star-shaped configuration should be feasible with same approach
- Explicit values permits to check the code (as in plasma physics...)
- Fixing total length L, we have $\ell = L/N$ and
  - $L\gamma_0$ is the worst for $N = 4$ (value is $4\log(2+\sqrt{3}) \simeq 5.2678$)
  - $L\gamma_{\alpha=1}$ is the worst for $N = \infty$ (value is 2)

Semi-discretization

- Classical finite difference space semi-discretization
- Discrete energy is decreasing; what is the limit?

Proposition

For $\phi \in \mathbb{R}^{2M}$, and numerical solution $V(t) = \exp(tA)\phi$, we can write $\phi = \phi_1 + \phi_2$ where ϕ_1 belongs to the space Λ_1 of eigenvectors relative to the eigenvalues λ of A, with $\Re(\lambda) = 0$ and ϕ_2 belongs to the space Λ_2 of generalized eigenvectors relative to the eigenvalues λ of A, with $\Re(\lambda) < 0$. We then have

$$E_{h,\phi,\infty} := \lim_{t\to\infty} E_{h,\phi}(t) = E_{h,\phi_1}(0),$$

and this value does not depend on $\alpha > 0$.

Numerical results

- We consider $d = 3$ strings of length $\ell = 1$
- N is now the number of points in each string
- initial condition non zero on string 2
  - $u_2^0(x) = \sin^2(\pi x)$
  - $u_2^0(x) = \sin^2(8\pi x)$

exact/semi-discrete time evolution of $E(t) - E_\infty$

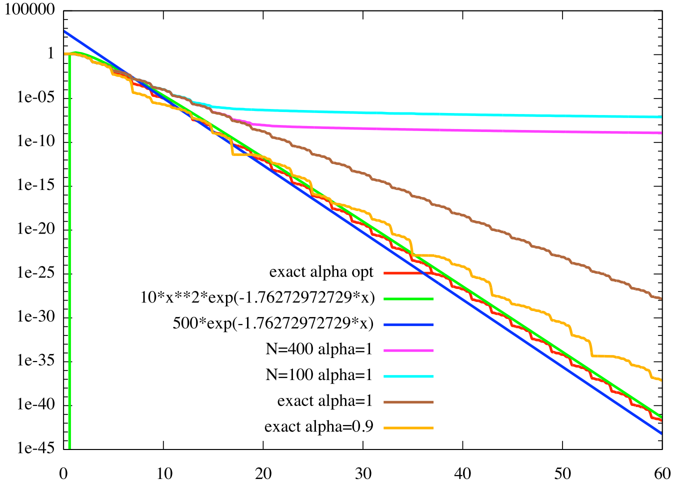

Figure: $\alpha \in \{0.9, 1, \alpha_{\text{opt}} \simeq 0.9428\}$

exact/semi-discrete time evolution of $E(t) - E_\infty$

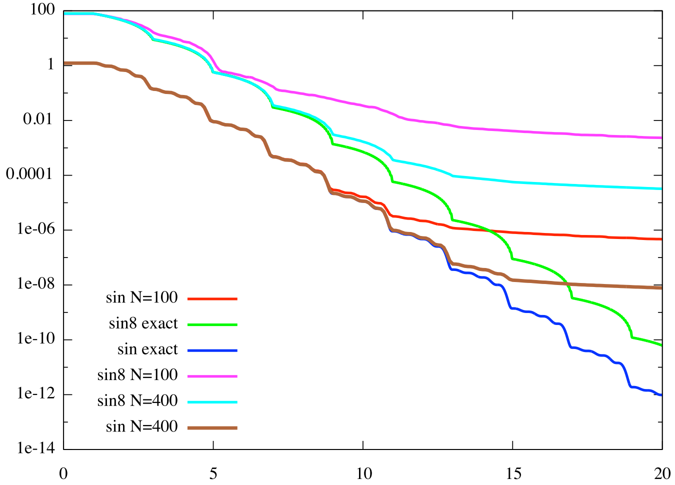

Figure: different initial conditions, $\alpha = \alpha_{\text{opt}}\}$

Conclusion/Perspectives

- Conclusion
  - Study of the stabilization for a tree-shaped network with equal lengths
  - Numerical results are coherent with theoretical ones
- Perspectives
  - Find other configurations where explicit computations are possible
  - Optimal configurations?
  - Design of performant numerical scheme for general configuration