Second order anti-diffusive Lagrange-remap scheme

SIMCAPIAD project

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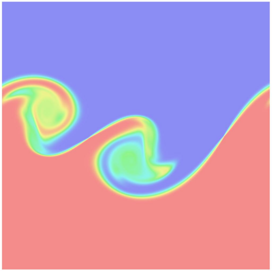
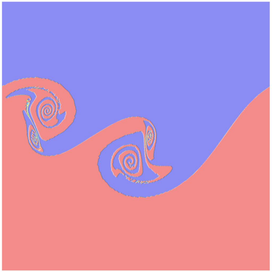
- Numerical simulation of interfaces between two fluids



upwind



anti-diffusive

- **Lagrange-projection** (or Lagrange-remap) schemes

  + **anti-diffusive** tool for the interface

  → 2$^{\text{nd}}$ order schemes

Outline

Outline

The 5 - equations model (1)

Sharp interface model between two fluids in one model

$\rho = \rho_1$	$\rho = \rho_0$	density
$u = u_1$	$u = u_0$	velocity
$e = e_1$	$e = e_0$	energy

- compressible Euler system $(u = u_1 = u_0)$

$$
\begin{array}{llll}
\partial_t \rho & + & \partial_x(\rho u) & = & 0 \\
\partial_t(\rho u) & + & \partial_x(\rho u^2 + p) & = & 0 \\
\partial_t(\rho e) & + & \partial_x[u(\rho e + p)] & = & 0 \\
\\
\partial_t(\rho y) & + & \partial_x(\rho y u) & = & 0 \\
\partial_t z & + & u\partial_x z & = & 0
\end{array}
$$

- $z =$ "color" function **→"indicator function"**
- $y =$ mass fraction of the phase 1 **→"conservative variable"**

The 5 - equations model (2)

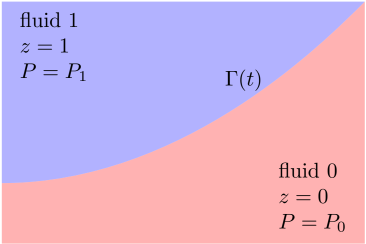
- $z =$ "color" function **→"indicator function"**

$$\begin{array}{llllllll} \rho & = & z & \rho_1 & + & (1-z) & \rho_0 & \text{(total density)} \\ \rho u & = & z & (\rho u)_1 & + & (1-z) & (\rho u)_0 & \text{(total momentum)} \\ \rho e & = & z & (\rho e)_1 & + & (1-z) & (\rho e)_0 & \text{(total energy)} \end{array}$$

- pressure

$$\text{in } [z=1], \quad p = p_1(\rho_1, \varepsilon_1), \quad \varepsilon_1 = e_1 - \frac{u^2}{2}$$

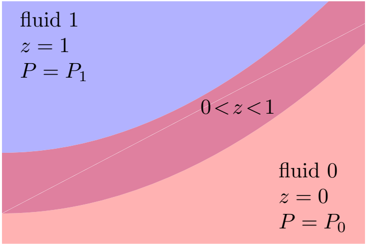
$$\text{in } [z=0], \quad p = p_0(\rho_0, \varepsilon_0), \quad \varepsilon_0 = e_0 - \frac{u^2}{2}$$



- **theoretically $\mathbf{y = z \in \{0, 1\}}$**

The 5 - equations model (3)

- **numerical diffusion of the interface $\mathbf{y \neq z \in [0,1]}$**
- ρ, ρu, ρe, p in the region $[0 < z < 1]$



$$\begin{array}{lcllcll} \rho & = & z & \rho_1 & + & (1-z) & \rho_0 \\ \rho u & = & z & (\rho u)_1 & + & (1-z) & (\rho u)_0 \\ \rho e & = & z & (\rho e)_1 & + & (1-z) & (\rho e)_0 \end{array}$$

if $0 < z < 1, \quad p = p_1(\rho_1, \varepsilon_1) = p_0(\rho_0, \varepsilon_0)$

$$\text{where } \varepsilon_1, \varepsilon_0 \text{ solve } \begin{cases} p_1(\rho_1, \varepsilon_1) = p_0(\rho_0, \varepsilon_0) \\ z\rho_1\varepsilon_1 + (1-z)\rho_0\varepsilon_0 = \rho\varepsilon \end{cases} \quad \text{(isobaric closure)}$$

Outline

The numerical scheme

- Lagrange-projection: **separate the acoustic waves / the "material" waves**

$$
\begin{aligned}
\partial_t \rho &+ \partial_x(\rho u) &= 0\\
\partial_t(\rho u) &+ \partial_x(\rho u^2 + p) &= 0\\
\partial_t(\rho e) &+ \partial_x[u(\rho e + p)] &= 0\\
\\
\partial_t(\rho y) &+ \partial_x(\rho y u) &= 0\\
\partial_t z &+ u\,\partial_x z &= 0
\end{aligned}
$$

The numerical scheme

- Lagrange-projection: **separate the acoustic waves / the "material" waves**

$$
\begin{array}{llllll}
\partial_t \rho & + & \rho\, \partial_x u & + & u\, \partial_x \rho & = 0 \\
\partial_t (\rho u) & + & \rho u\, \partial_x u + \partial_x p & + & u\, \partial_x \rho u & = 0 \\
\partial_t (\rho e) & + & (\rho e)\, \partial_x u + \partial_x (p u) & + & u\, \partial_x (\rho e) & = 0 \\
\\
\partial_t (\rho y) & + & (\rho y)\, \partial_x u & + & u\, \partial_x (\rho y) & = 0 \\
\partial_t z & + & & & u\, \partial_x z & = 0 \\
 & & \underbrace{\hphantom{(\rho e)\, \partial_x u + \partial_x (p u)}}_{\text{acoustic}} & & \underbrace{\hphantom{u\, \partial_x (\rho e)}}_{\text{convection}} &
\end{array}
$$

The numerical scheme

Notation: $[\![X]\!]_i = (X_{i+1/2} - X_{i-1/2})$

Lagrange step $(w^n \rightarrow \tilde{w})$

acoustic scheme

$$L_i \widetilde{\rho}_i = \rho_i^n \qquad \text{with } L_i = 1 + \frac{\Delta t}{\Delta x}[\![u^n]\!]_i$$

$$L_i (\widetilde{\rho u})_i = (\rho u)_i^n - \frac{\Delta t}{\Delta x}[\![p^n]\!]_i$$

$$L_i (\widetilde{\rho e})_i = (\rho e)_i^n - \frac{\Delta t}{\Delta x}[\![p^n u^n]\!]_i$$

$$\widetilde{y}_i = y_i^n, \quad \widetilde{z}_i = z_i^n$$

Projection step $(\tilde{w} \rightarrow w^{n+1})$

convection

$$\rho_i^{n+1} = \widetilde{\rho}_i + \frac{\Delta t}{\Delta x}[\![\widetilde{\rho} u^n]\!]_i - \frac{\Delta t}{\Delta x}\widetilde{\rho}_i [\![u^n]\!]_i$$

$$(\rho u)_i^{n+1} = \widetilde{\rho u}_i + \frac{\Delta t}{\Delta x}[\![\widetilde{\rho u} u^n]\!]_i - \frac{\Delta t}{\Delta x}\widetilde{\rho u}_i [\![u^n]\!]_i$$

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$$z_i^{n+1} = \widetilde{z}_i + \frac{\Delta t}{\Delta x}[\![\widetilde{z} u^n]\!]_i - \frac{\Delta t}{\Delta x}\widetilde{z}_i [\![u^n]\!]_i$$

The fluxes

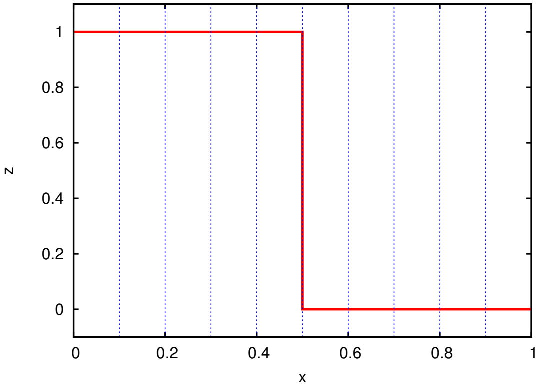
Notation: $[\![X]\!]_i = (X_{i+1/2} - X_{i-1/2})$

Lagrange step ($w^n \longrightarrow \tilde{w}$) **acoustic scheme**	• Compute $[\![u^n]\!]$, $[\![p^n]\!]$? ➞ Roe type fluxes for $u_{i+1/2}$ and $p_{i+1/2}$
Projection step ($\tilde{w} \longrightarrow w^{n+1}$) **convection**	• Compute $[\![\widetilde{\rho} u^n]\!]$? $(\widetilde{\rho} u^n)_{i+1/2} = \widetilde{\rho}_{i+1/2} u^n_{i+1/2}$ $\widetilde{\rho}_{i+1/2} = z_{i+1/2}(\widetilde{\rho}_1)_{i+1/2} + (1 - z_{i+1/2})(\widetilde{\rho}_0)_{i+1/2}$ ➞ upwind flux for $\widetilde{\rho}_1$ and $\widetilde{\rho}_0$ computed from y • Compute $z_{i+1/2}$? ➞ anti-diffusive flux for z

Anti-diffusive flux

Anti-diffusive flux = as downwind as possible
= if $u_{i+1/2} > 0$, $z_{i+1/2}$ is the nearest value to z_{i+1} ensuring stability and consistency

Example: $\partial_t z + u\ \partial_x z = 0$, with $u = 1$



flux at $(x = 0.6)$
$= 0$

$\mathbf{t = 0\Delta t}$

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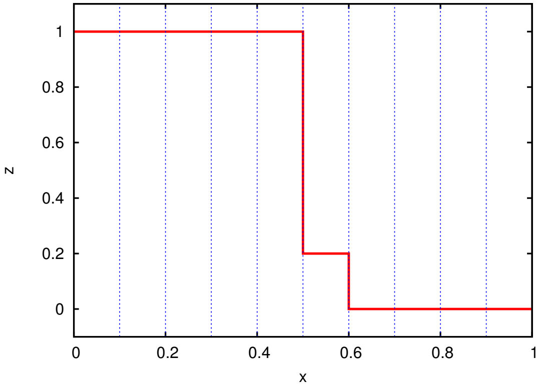

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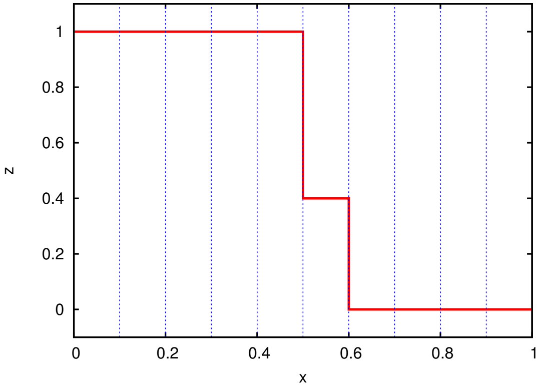

flux at $(x = 0.6)$
$= 0$

$\mathbf{t = 2\Delta t}$

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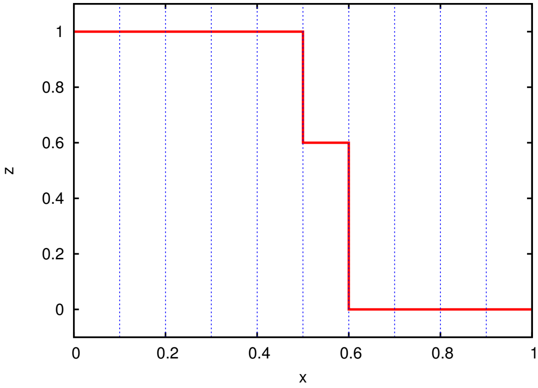

flux at $(x = 0.6)$
$= 0$

$\mathbf{t = 3\Delta t}$

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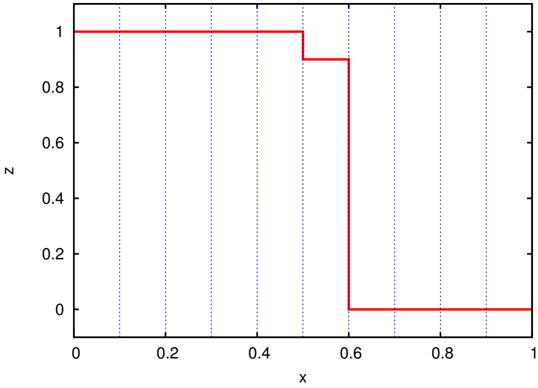

flux at $(x = 0.6)$
$= 1$

$\mathbf{t = 4\Delta t}$

Anti-diffusive flux

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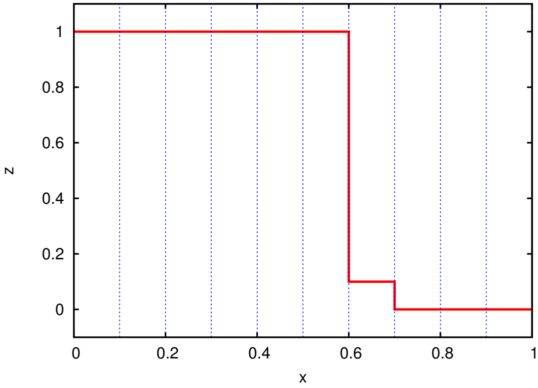
Example: $\partial_t z + u\ \partial_x z = 0$, with $u = 1$



flux at $(x = 0.6)$
$= 1$

$\mathbf{t = 5\Delta t}$

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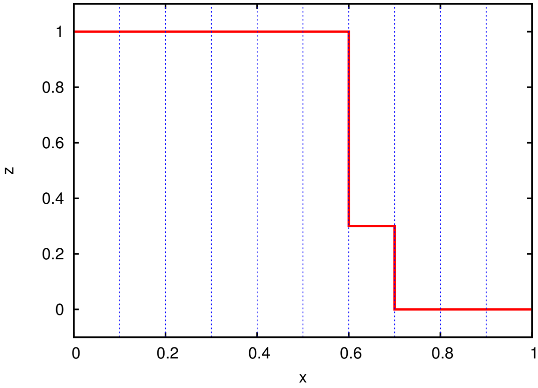


flux at $(x = 0.6)$
$= 1$

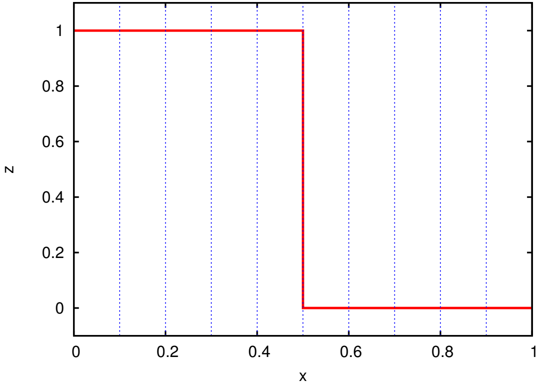
$\mathbf{t = 6\Delta t}$

Upwind flux

Diffusion of the interface with the upwind flux:

$$z_{i+1/2} = \begin{cases} z_i & \text{if } u_{i+1/2} > 0 \\ z_{i+1} & \text{if } u_{i+1/2} < 0 \end{cases}$$

Example: $\partial_t z + u\ \partial_x z = 0$, with $u = 1$



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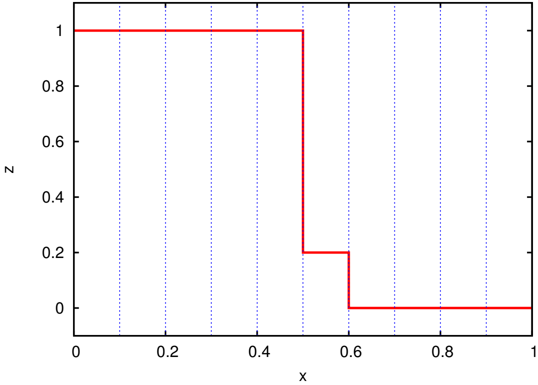

$\mathbf{t = 1\Delta t}$

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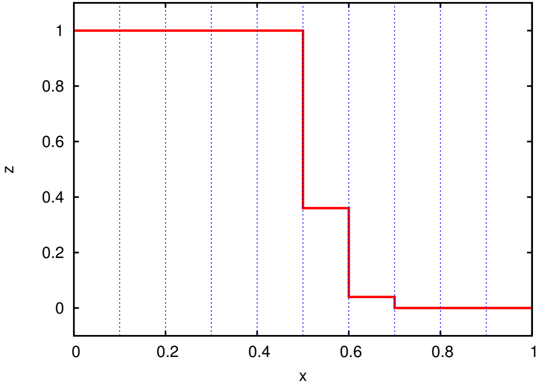

$t = 2\Delta t$

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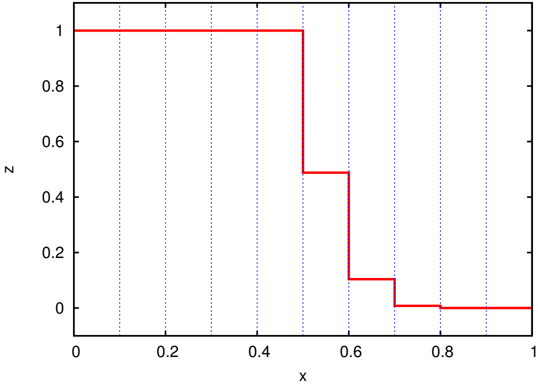
Example: $\partial_t z + u \, \partial_x z = 0$, with $u = 1$



$\mathbf{t = 3\Delta t}$

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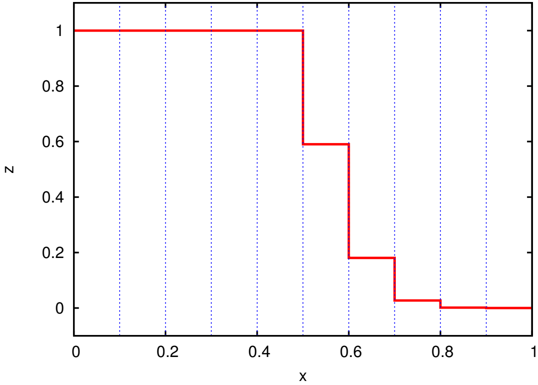

$\mathbf{t = 4\Delta t}$

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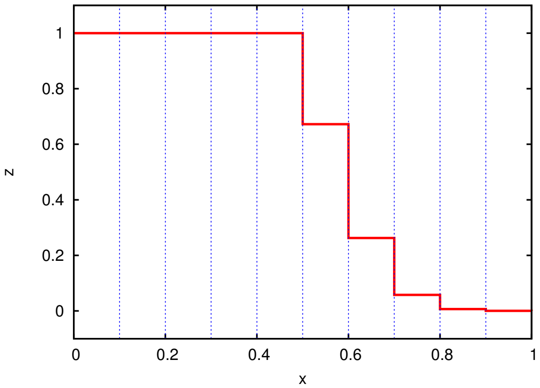

$\mathbf{t = 5\Delta t}$

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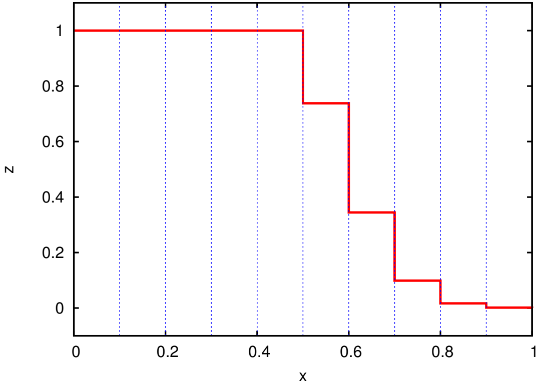
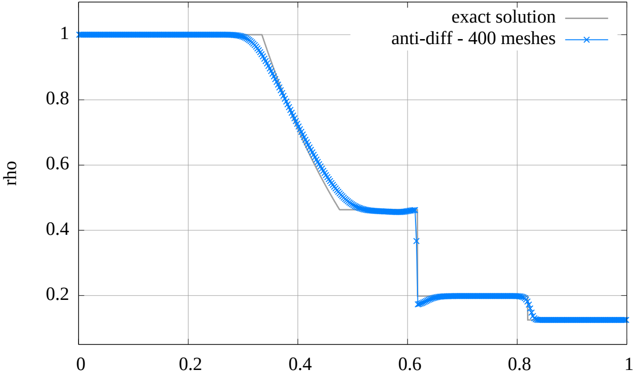


$\mathbf{t = 6\Delta t}$

Numerical exemple

Sod test-case :

$\rho_\ell = 1.0$	$\rho_r = 0.125$
$u_\ell = 0.0$	$u_r = 0.0$
$p_\ell = 1.0$	$p_r = 0.1$
$z = y = 1$	$z = y = 0$



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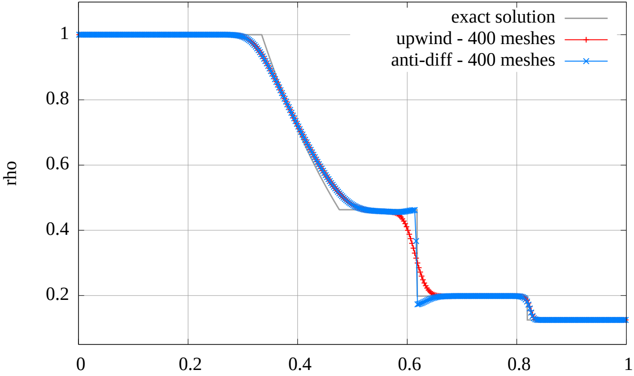
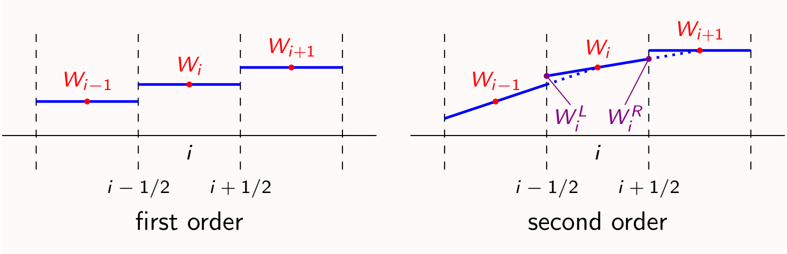


Outline

MUSCL strategy

- linear reconstruction for the fluxes

$$\widetilde{w}_i(x) = w_i + \sigma_i(x - x_i)$$
$$\sigma_i = \text{minmod}\left(\frac{w_i - w_{i-1}}{\Delta x}, \frac{w_{i+1} - w_i}{\Delta x}\right)$$

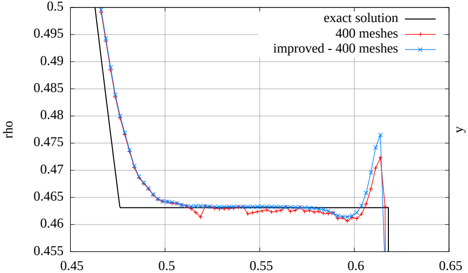


- second order fluxes in the Lagrange step for the Roe fluxes
  in the projection step for the upwind fluxes

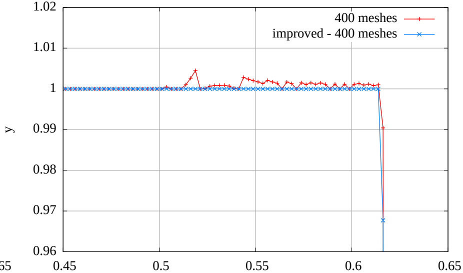
→ **but keep the anti-diffusive flux for z: we can not do better !**

Improvement

- maximum principle on y is not satisfied (Sod test case)



ρ

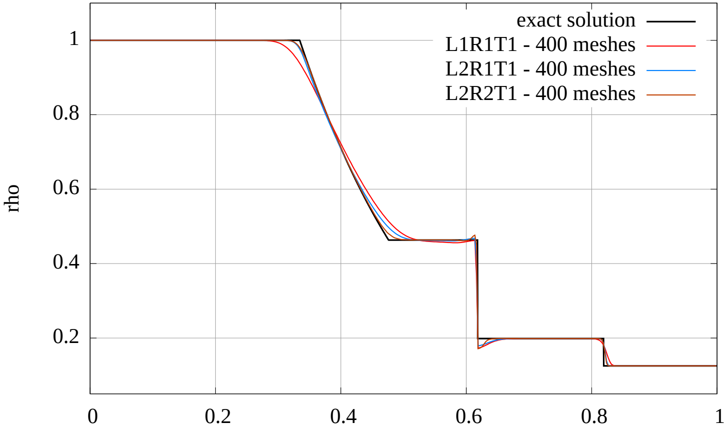


y

- in the projection step, keep the order 1 upwind flux (i.e. for $(\widetilde{\rho}_0)_{i+1/2}$, $(\widetilde{\rho}_1)_{i+1/2}$) in the projection step at the interface $[0 < z < 1]$

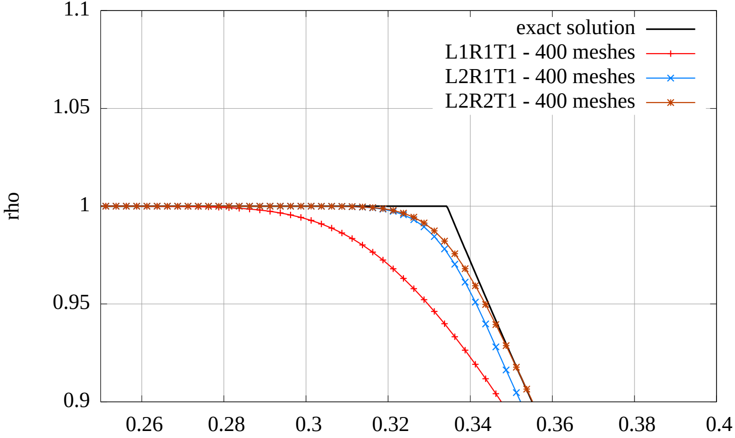
Comparison with order 1

Sod test case



Comparison with order 1

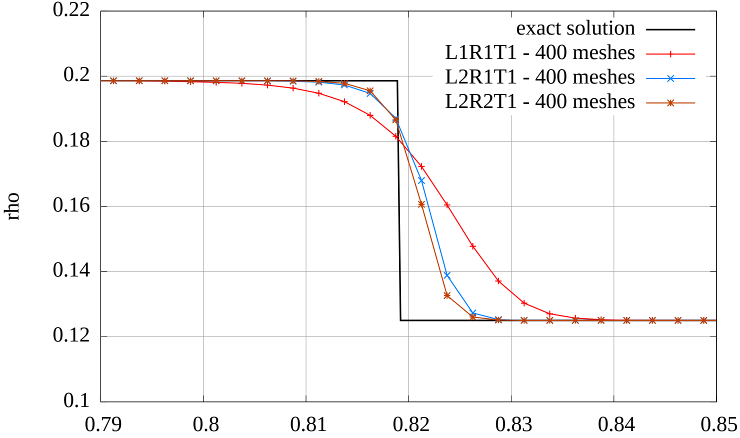
Sod test case: zoom on the rarefaction



the rarefaction is improved

Comparison with order 1

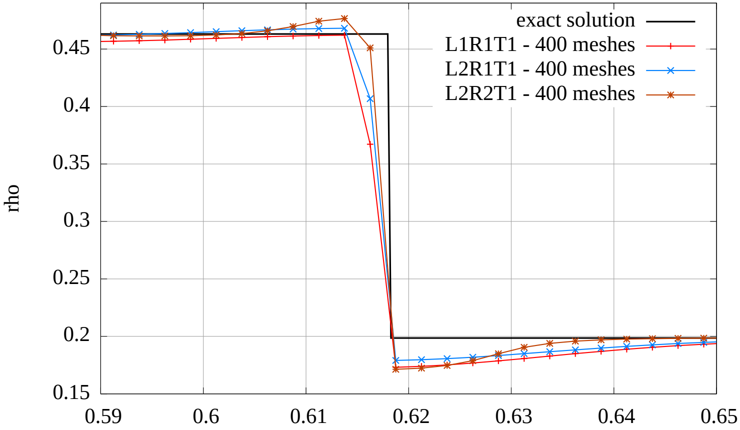
Sod test case: zoom on the shock



the shock is improved

Comparison with order 1

Sod test case: zoom on the contact



the anti-diffusive contact is preserved

Numerical order

Orders for the Sod test-case:

	ρ	u	p
Order 1 upwind	0.63	0.82	0.77
Order 1 anti-diff	0.75	0.82	0.78
Lagrange order 2 anti-diff	0.86	0.88	0.85
Lagrange-projection order 2 upwind	0.83	1.03	1.08
Lagrange-projection order 2 anti-diff	1.08	1.04	1.10

Comments:

1. upwind $\rightarrow$ anti-diffusive : improve the order for ρ
2. with the order 2 method, orders are improved

   but numerical order $\lesssim 1$ (discontinuous solutions)
3. Lagrange order 2 + projection of order 2

   better than Lagrange order 2

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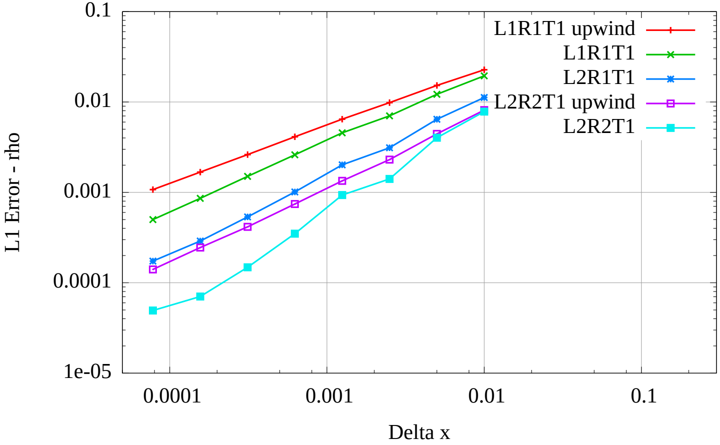
Numerical order



Comments:

1. increasing order + decreasing absolute error
2. no possible order 2 (discontinuous solution)

Conclusion

What has been done

- one-dimensional codes (C and Scilab)
- rewriting the Lagrange-projection scheme to integrate Muscl strategy
- desactivation of the Muscl procedure at the interface for preserving the anti-diffusive property

Results

- space order is improved
- no better order in a test case with continuous solution
- wall-heating disappears with mesh refinement

Work in progress

- order 2 in time: no major improvement
- 2D simulations by directionnal splitting
- reducing the wall-heating phenomenon