On Mumford's families of abelian varieties

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Abstract

In [Mum69], Mumford constructs families of abelian varieties which are parametrized by Shimura varieties but which are not of PEL type. In this paper we investigate Mumford's families. We notably determine, for each fibre of such a family over a number field, the possible isogeny types and the possible Newton polygons of its reductions. In the process, a classification of the CM points on Mumford's Shimura varieties is obtained.

Introduction

Classically, Shimura varieties are constructed as quotients of bounded symmetric domains by arithmetic groups. It was discovered in an early stage that these varieties have important arithmetic properties. In some cases, a Shimura variety may parametrize a family of abelian varieties and in such a case, this circumstance plays an important role in the study of its properties, notably for the construction of canonical models. Mumford defined a class of Shimura varieties all of which parametrize a family of abelian varieties. He calls such families the 'families of Hodge type'. Somewhat oversimplifying, one can say that these families are characterized by the Hodge classes living on the powers of the abelian varieties. In [Mum69], one can find a characterization of the families of Hodge type. In the same paper, Mumford also gives an interesting and surprisingly simple example of a one dimensional family of Hodge type which is not of PEL type (i. e. not characterized by the existence of algebraic endomorphisms on the abelian varieties). In fact, the generic abelian variety belonging to this family has endomorphism ring equal to $\mathbf{Z}$. It is this example that will be studied in this paper. The precise construction will be recalled in 1.1, but let us give the main idea.

Mumford begins by constructing an algebraic group G' over $\mathbf{Q}$ which is isogenous to a $\mathbf{Q}$-simple form of the real algebraic group $\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SL}_{2,\mathbf{R}}$ and hence to a form of $\mathrm{SL}^3_{2,\mathbf{C}}$. The group G' moreover possesses a faithful 8-dimensional $\mathbf{Q}$-linear representation such that the induced representation of $\mathrm{SL}^3_{2,\overline{\mathbf{Q}}}$ is isomorphic to the tensor product of the standard representations of the factors $\mathrm{SL}_{2,\overline{\mathbf{Q}}}$. The bounded symmetric domain Δ is a quotient of $G'(\mathbf{R})$ by a maximal compact subgroup and the corresponding Shimura curves M are quotients Δ/Γ, for appropriate arithmetic subgroups $\Gamma \subset G'(\mathbf{Q})$. In 1.1 we will use the language of [Del72],

1991 Mathematics Subject Classification 14G35, 11G18, 14K15 but the difference is unimportant for the purposes of this introduction. By means of the above faithful 8-dimensional representation, such a Mumford–Shimura curve parametrizes a family of four dimensional abelian varieties characterized by the presence of certain Hodge classes on their powers.

This property can be rephrased by saying that these abelian varieties are characterized by the fact that their Mumford–Tate groups are 'small'. In this case, this means the following. Let $X/\mathbf{C}$ be a fibre in one of Mumford's families, a 4-dimensional abelian variety. The Mumford–Tate group of X is then contained in a group G which is isogenous to $\mathbf{G}_m \times G'$. Equivalently, the Lie-algebra of the Mumford–Tate group is contained in $\mathrm{Lie}(G)$. The group G satisfies $\mathrm{Lie}(G)_{\overline{\mathbf{Q}}} \cong \mathfrak{c} \oplus (\mathfrak{sl}_2)^3$ (with $\mathfrak{c}$ the 1-dimensional centre) and the representation of $\mathrm{Lie}(G)$ on $\mathrm{H}^1_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$ becomes isomorphic over $\overline{\mathbf{Q}}$ to the tensor product of the standard representations. In what follows we will refer to this property by saying that $(G, \mathrm{H}^1_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q}))$ is a pair of Mumford's type.

The generic fibre of a family of Mumford's kind has Mumford–Tate group G. The above construction therefore shows in particular that there exist abelian varieties over $\mathbf{C}$ with Mumford–Tate group of this type. It follows from [Noo95] that such abelian varieties also exist over number fields. Conversely, any abelian variety for which the representation of its Mumford–Tate group on the first Betti cohomology is of Mumford's type occurs as a fibre in one of Mumford's families. This well known fact is proved in 1.5. It follows from the fact that an abelian variety over $\mathbf{C}$ corresponds to a point of the Shimura variety defined by its Mumford–Tate group and an analysis showing that any of Mumford–Tate group of Mumford's type can be obtained by the construction of [Mum69, §4].

In this paper we study the reductions modulo prime ideals of fibres over number fields in these families. Many results can be deduced from the paper [Noo99], where the reduction properties are investigated of abelian varieties with associated Galois representation of Mumford's type. To begin with, it is noted in remark 1.6 that the results of loc. cit. imply that an abelian variety X over a number field F occurring as a fibre of one of Mumford's families has potentially good reduction at all finite places of the base field.

We next turn our attention to the possible Newton polygons of the reductions. As before, let X be a fibre of one of Mumford's families over a number field F and let v be a finite place of F. Since X has been realized as a fibre of a family over a Mumford–Shimura curve, the base field F is naturally an extension of the reflex field E associated to the Shimura curve in question. A refinement of the results of [Noo99, §3] shows that there are two possibilities, depending only on the restriction to E of v, for the Newton polygon of the reduction of X at a finite place v of F. This result is given in proposition 2.2. Beside the Newton polygons, we also determine the possible isogeny types of the reductions in question. This is accomplished using the results of [Noo99, §4].

In the case where $G_{\mathbf{Q}_p}$ (with p the residue characteristic) is quasi split, it is shown that both possible Newton polygons and all possible isogeny types actually occur. To this end, we analyze the special points of the Mumford–Shimura curves and use proposition 5.1 to pass from a special point to a non-special point where the corresponding abelian variety has the same reduction. The analysis of the special points is carried out using CM theory and is the object of section 3. As a by-product, we are able to give a description of all special points on these Shimura curves, see proposition 3.9. In section 4 we compute the Newton polygons of the reductions of the abelian varieties corresponding to the special points. The existence results are summed up in proposition 5.3 and remark 5.4.

The Shimura varieties and the families considered here have been studied frequently, for see for instance Shimura [Shi70], Morita [Mor81] and, more recently, Reimann [Rei97].

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Contents

1 Mumford's families.

We will recall the necessary facts about Shimura varieties as we go along, basically following the approach of [Del72], to which we refer for a detailed treatment of the subject.

1.1 Define $S = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m)$ and let S^1 be the kernel of the norm map $S \to \mathbf{G}_{m,\mathbf{R}}$. Of course, S^1 is just the circle $\{e^{i\theta} \mid \theta \in \mathbf{R}\} \subset \mathbf{C}^*$, viewed as an algebraic group over $\mathbf{R}$. The character group $X(S)$ has a basis $([\mathrm{id}], [c])$, where c is the complex conjugation. It is thus isomorphic to $\mathbf{Z}^2$, with the complex conjugation acting by exchanging the coordinates.

We recall the construction of Mumford's Shimura curves from [Mum69, §4]. Let K be a totally real number field with $[K : \mathbf{Q}] = 3$ and let D be a quaternion division algebra over K such that $\mathrm{Cor}_{K/\mathbf{Q}}(D) \cong \mathrm{M}_8(\mathbf{Q})$ and $D \otimes \mathbf{R} \cong \mathbf{H} \times \mathbf{H} \times \mathrm{M}_2(\mathbf{R})$. Here $\mathbf{H}$ denotes Hamilton's real quaternion algebra. There exists a natural 'norm' map

$$\mathrm{Nm}\colon D^* \longrightarrow \mathrm{Cor}_{K/\mathbf{Q}}(D)^* \cong \mathrm{GL}_8(\mathbf{Q}),$$

cf. [Mum69, §4]. In what follows we consider D^* as an algebraic group over $\mathbf{Q}$ and Nm as a morphism of algebraic groups. Let $G \subset \mathrm{GL}_{8,\mathbf{Q}}$ be the the image of D^* by this map. By construction G has a natural faithful representation on $V = \mathbf{Q}^8$. The center of G is reduced to $\mathbf{G}_{m,\mathbf{Q}}$, acting on V by scalar multiplication. Let $x \mapsto \bar{x}$ be the standard involution on D and $\widetilde{G}' = \{x \in D \mid x\bar{x} = 1\}$, viewed as algebraic group over $\mathbf{Q}$. By construction, one has $\widetilde{G}'_{\mathbf{R}} \cong \mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SL}_{2,\mathbf{R}}$. Put $\widetilde{G} = \mathbf{G}_{m,\mathbf{Q}} \times \widetilde{G}'$. The product of the morphisms $\mathbf{G}_{m,\mathbf{Q}} \to G$ and $\mathrm{Nm}_{|\widetilde{G}'} \colon \widetilde{G}' \to G$ is a central isogeny $N \colon \widetilde{G} \to G \subset \mathrm{GL}_{8,\mathbf{Q}}$. We define $G' \subset G$ as the image of $\widetilde{G}'$. The space of G'-invariants in $V \wedge V$ is 1-dimensional, so there exists a unique symplectic form $\langle \cdot, \cdot \rangle$ on V which is fixed up to scalars by the action of G.

Let GSU_2 be the real algebraic group generated by $\mathrm{SU}_2 \subset \mathrm{GL}_{2,\mathbf{C}}$ and $\mathbf{G}_{m,\mathbf{R}} \subset \mathrm{GL}_{2,\mathbf{C}}$. We define a map

$$\begin{aligned} \tilde{h}_0 \colon S &\longrightarrow \mathrm{GSU}_2 \times \mathrm{GSU}_2 \times \mathrm{GL}_{2,\mathbf{R}} \cong D^*_{\mathbf{R}} \\ a + bi &\mapsto \left(1, 1, \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right). \end{aligned}$$

The image of the composite $\mathrm{Nm}_{\mathbf{R}} \circ \tilde{h}_0$ lies in $G_{\mathbf{R}}$ so we get a map $h_0 \colon S \to G_{\mathbf{R}}$. By construction, $h'_0 = (h_0)_{|\mathrm{S}^1}$ lifts to a map $\tilde{h}'_0 \colon \mathrm{S}^1 \to \widetilde{G}'_{\mathbf{R}}$. It is rather easy to see (cf. [Mum69]) that these data permit to define one dimensional Shimura varieties

$$M_C(\mathbf{C}) = M_C(G, Y)(\mathbf{C}) = G(\mathbf{Q}) \backslash Y \times (G(\mathbf{A}_f)/C),$$

where $Y \subset \mathrm{Hom}(S, G_{\mathbf{R}})$ is the $G(\mathbf{R})$-conjugacy class of h_0 and $C \subset G(\mathbf{A}_f)$ is any compact open subgroup. Here, as in what follows, $\mathbf{A}_f = \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$ is the ring of finite adeles of $\mathbf{Q}$. The reader is referred to [Del72, 1.5] for the properties that should be verified and to [Mum69] for the verification of these properties. Mumford uses the groups $\widetilde{G}'$ and S^1 rather than G and S, but for our purposes these two approaches are the same. The difference lies in the fact that Mumford's varieties are connected, whereas ours can have many connected components. However, all these components can be obtained by Mumford's construction and any variety arising from Mumford's construction is a connected component of one of the above Shimura varieties. It is not difficult to show that any Shimura variety constructed in this fashion is compact, but we will not make use of this fact.

Suppose that C is sufficiently small. Using the natural representation of G on V, a point $x \in M_C(\mathbf{C})$ determines a 4-dimensional abelian variety $X_x/\mathbf{C}$ endowed with a level structure (depending on C) and some other data. The other data determine a $G(\mathbf{Q})$-class of symplectic isomorphisms $V \cong \mathrm{H}^1_{\mathrm{B}}(X_x, \mathbf{Q})$ such that, for any isomorphism in this class, the morphism $h = h_x \colon S \to G_{\mathbf{R}}$ giving the Hodge structure on $V \otimes \mathbf{R} \cong \mathrm{H}^1_{\mathrm{B}}(X_x, \mathbf{R})$ is conjugate to h_0 by an element of $G(\mathbf{R})$. This property implies that the Mumford–Tate group of X_x is contained in G. We refer to [Del72, §4] for further details.

Let $E(G, Y) \subset \mathbf{C}$ be the reflex field as in [Del72, 3.7] (denoted $E(G, h_0)$ in loc. cit.). The theory of Shimura varieties shows that there exists a canonical model $M_C/E(G, Y)$, which is a quasi-projective (and in this case even projective) $E(G,Y)$-scheme of which $M_C(\mathbf{C})$ is the set of $\mathbf{C}$-valued points. See [Del72, 5.9] for a proof in this case. By choosing C sufficiently small, we can make sure that M_C is smooth and that there exists a polarized abelian scheme $\mathcal{X} \to M_C$ such that, for $x \in M(\mathbf{C})$, the fibre $\mathcal{X}_x$ is isomorphic to the abelian variety X_x we saw above. In what follows, we fix C such that these properties are fulfilled and we write M instead of M_C.

1.2 Lemma. *The reflex field $E(G,Y)$ is the image of K in $\mathbf{R} \subset \mathbf{C}$ under the embedding corresponding to the real place of K where the algebra D is split.*

Proof. By definition, $E(G,Y) \subset \mathbf{C}$ is the field of definition of the conjugacy class of the morphism $\mu_0 = (h_0)_{\mathbf{C}} \circ r \colon \mathbf{G}_{m,\mathbf{C}} \to G_{\mathbf{C}}$, where $r \colon \mathbf{G}_{m,\mathbf{C}} \to S_{\mathbf{C}}$ is the cocharacter dual to $[\mathrm{id}] \in X(S)$. The lemma follows since μ_0 is conjugate to $z \mapsto (1, 1, \mathrm{diag}(1, z))$. $\square$

1.3 Definition Let K be a field of characteristic 0, let G be an algebraic group over K and let V be a faithful K-linear representation of G. We will say that the pair (G,V) is *of Mumford's type* if

- $\mathrm{Lie}(G)$ has one dimensional centre $\mathfrak{c}$,
- $\mathrm{Lie}(G)_{\bar{K}} \cong \mathfrak{c}_{\bar{K}} \oplus \mathfrak{sl}^3_{2,\bar{K}}$ and
- $\mathrm{Lie}(G)_{\bar{K}}$ acts on $V_{\bar{K}}$ by the tensor product of the standard representations.

We do not require G to be connected.

1.4 Lemma. *Let $X/\mathbf{C}$ be a polarized abelian variety, $V = \mathrm{H}^1_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$, let G be the Mumford–Tate group of X and assume (G,V) is of Mumford's type. Then G^{ad} is $\mathbf{Q}$-simple and the morphism $h \colon S \to G_{\mathbf{R}}$ defining the Hodge structure on $\mathrm{H}^1_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$ has the property that the composite*

$$S_{\mathbf{C}} \xrightarrow{h_{\mathbf{C}}} G_{\mathbf{C}} \longrightarrow G^{\mathrm{ad}}_{\mathbf{C}} \xrightarrow{\cong} \mathrm{PSL}^3_{2,\mathbf{C}}$$

projects non-trivially to exactly one of the factors $\mathrm{PSL}_{2,\mathbf{C}}$.

Proof. There is a central isogeny $\widetilde{G} \to G$, with $\widetilde{G} = \mathbf{G}_{m,\mathbf{Q}} \times \widetilde{G}'$ and $\widetilde{G}'_{\overline{\mathbf{Q}}} \cong \mathrm{SL}^3_{2,\overline{\mathbf{Q}}}$. There exists an integer k such that h^k lifts to a map $\tilde{h}^k \colon S \to \widetilde{G}_{\mathbf{R}}$. One has $\tilde{h}^k_{\mathbf{C}} = (\tilde{h}_0, \tilde{h}_1, \tilde{h}_2, \tilde{h}_3)$, where $\tilde{h}_0 \colon S_{\mathbf{C}} \to \mathbf{G}_{m,\mathbf{C}}$ and $h_i \colon S_{\mathbf{C}} \to \mathrm{SL}_{2,\mathbf{C}}$ for $i = 1, 2, 3$. For the action of $S_{\mathbf{C}}$ by $h_{\mathbf{C}}$, the representation $V_{\mathbf{C}}$ is the sum of the eigenspaces for the characters z and $\bar{z}$ respectively. This implies that it is the sum of the z^k and the $\bar{z}^k$-eigenspaces for $h^k_{\mathbf{C}}$ and hence also for the action of $\tilde{h}^k_{\mathbf{C}}$, through the induced representation of $\widetilde{G}$ on V.

Hence one and only one of the h_i (for $i = 1, 2, 3$) is non-trivial, so the composite map from the lemma projects non-trivially to exactly one of the factors of $G^{\mathrm{ad}}_{\mathbf{C}}$. It also follows that if G^{ad} is not simple, then there exists a subgroup $H \subset G$ such that h factors through $H_{\mathbf{R}}$, which contradicts the fact that G is the Mumford–Tate group of X. $\square$

1.5 Proposition. *Let $X/\mathbf{C}$ be a polarized abelian variety, $V = \mathrm{H}^1_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$, let G be the Mumford–Tate group of X and assume (G, V) is of Mumford's type. Then there exist a number field K and a division algebra $D \supset K$ as in 1.1 such that G is isomorphic to the group constructed in loc. cit. using these K and D. Moreover, X is isomorphic to a fibre of the family $\mathcal{X}/M$ over a $\mathbf{C}$-valued point of $M = M_C$ (for any sufficiently small $C \subset G(\mathbf{A}_f)$).*

Proof. Let $\widetilde{G} \to G$ and $\widetilde{G}'$ be as in the proof of lemma 1.4. Since G^{ad} is $\mathbf{Q}$-simple by the lemma, it follows that $\mathcal{G}_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts transitively on the set of factors of the product $(\mathrm{SL}_{2,\overline{\mathbf{Q}}})^3$. Let $\mathcal{H} \subset \mathcal{G}_{\mathbf{Q}}$ be the subgroup fixing the first factor and let $K = (\overline{\mathbf{Q}})^{\mathcal{H}}$. Then K is cubic number field and $\widetilde{G}'$ is the Weil restriction from K to $\mathbf{Q}$ of a K-form of SL_2. It follows that there exists a central K-algebra D of dimension 4 such that $\widetilde{G}' = \{x \in D \mid xx' = 1\}$ (viewed as algebraic group over $\mathbf{Q}$). The fact that $\widetilde{G}'$ acts on an 8 dimensional $\mathbf{Q}$-vector space implies that $\mathrm{Cor}_{K/\mathbf{Q}}(D) \cong \mathrm{M}_8(\mathbf{Q})$.

Since X is polarized, V carries a symplectic form $\langle\cdot,\cdot\rangle$ and if $h\colon S \to G_{\mathbf{R}}$ denotes the morphism defining the Hodge structure on $V \otimes \mathbf{R}$, the symmetric bilinear form $\langle\cdot, h(i)\cdot\rangle$ is positive definite. This implies that if H is the real form of $G_{\mathbf{C}}$ corresponding to the involution $\mathrm{ad}(h(i))$, then H^{der} is compact.

It follows from lemma 1.4 that the projection of h on one and only one of the factors $\mathrm{PSL}_{2,\mathbf{C}}$ of $G^{\mathrm{ad}}_{\mathbf{C}}$ is non trivial and one concludes from this and from the compactness of H^{der} that K is totally real and that at least two of the factors of $\widetilde{G}'_{\mathbf{R}}$ are isomorphic to SU_2. The fact that $\mathrm{Cor}_{K/\mathbf{Q}}(D) \cong \mathrm{M}_8(\mathbf{Q})$ implies that $\widetilde{G}'_{\mathbf{R}} \cong \mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SL}_{2,\mathbf{R}}$. Since H^{der} is compact, it follows that h is conjugate to the map $h_0\colon S \to G_{\mathbf{R}}$ derived from

$$\begin{aligned} \tilde{h}_0\colon S &\longrightarrow \mathrm{GSU}_2 \times \mathrm{GSU}_2 \times \mathrm{GL}_{2,\mathbf{R}} \cong D^*_{\mathbf{R}} \\ a + bi &\mapsto \left(1, 1, \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right). \end{aligned}$$

This terminates the proof of the proposition. ☐

1.6 Remark. Let F be a discretely valued field and let X/F be a 4-dimensional abelian variety with Mumford–Tate group as in 1.5. Then, using [Del82, 2.11 and 2.9], it follows from [Noo99, cor. 2.1] that X has potentially good reduction. This implies in particular that if F is a number field and X/F an abelian variety with such Mumford–Tate group, then X has potentially good reduction at all finite places of F.

2 Possible reductions

For any field F, we write $\mathcal{G}_F = \mathrm{Gal}(\bar{F}/F)$. If p is a prime number, we denote by $\mathbf{Q}_p$ the p-adic completion of $\mathbf{Q}$, by $\mathbf{Q}_p^{\mathrm{nr}}$ the maximal unramified extension of $\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$ and by $\mathbf{C}_p$ the completion of $\overline{\mathbf{Q}}_p$.

2.1 Lemma. *Let $G/\mathbf{Q}_p$ be an algebraic group and V a $\mathbf{Q}_p$-linear representation of G such that (G, V) is of Mumford's type. Suppose that F is a finite extension of $\mathbf{Q}_p^{\mathrm{nr}}$ and that $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ is a continuous, polarizable, crystalline representation with Hodge–Tate weights 0 and 1. Let $\mu_{\mathrm{HT}}\colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{\mathbf{C}_p}$ be the Hodge–Tate cocharacter and let $C_{\mathrm{HT}} \subset G_{\mathbf{C}_p}$ be its geometric conjugacy class. Assume moreover that G is connected and that G^{ad} has exactly two $\mathbf{Q}$-simple factors. Then G^{ad} has a unique simple factor G_1^{ad} such that the image of $\rho(\mathcal{G}_F)$ in $G^{\mathrm{ad}}(\mathbf{Q}_p)$ projects non trivially to $G_1^{\mathrm{ad}}(\mathbf{Q}_p)$ and*

- *$\left(G_1^{\mathrm{ad}}\right)_{\overline{\mathbf{Q}}_p} \cong \mathrm{PSL}_{2,\overline{\mathbf{Q}}_p}$ if and only if the field of definition in $\mathbf{C}_p$ of C_{HT} is $\mathbf{Q}_p$ and*
- *$\left(G_1^{\mathrm{ad}}\right)_{\overline{\mathbf{Q}}_p} \cong \mathrm{PSL}^2_{2,\overline{\mathbf{Q}}_p}$ if and only if the field of definition in $\mathbf{C}_p$ of C_{HT} is of degree 2 over $\mathbf{Q}_p$.*

Proof. Only the last two lines are new compared to [Noo99, 3.5]. Let $G_1^{\mathrm{ad}} \subset G^{\mathrm{ad}}$ be the unique simple factor such that μ_{HT} projects non-trivially on $G^{\mathrm{ad}}_{1,\mathbf{C}_p}$, it is also the $\mathbf{Q}_p$-simple factor such that $\rho(\mathcal{G}_F)$ projects non-trivially to $G_1^{\mathrm{ad}}(\mathbf{Q}_p)$. One has $G^{\mathrm{ad}}_{1,\overline{\mathbf{Q}}_p} \cong \mathrm{PSL}^k_{2,\overline{\mathbf{Q}}_p}$ for $k = 1$ or 2, so there are an extension $K/\mathbf{Q}_p$ with $[K : \mathbf{Q}_p] = k$ and a subgroup $G_2 \subset G_K$ with $G_{2,\mathbf{C}_p} \cong \mathrm{GL}_{2,\mathbf{C}_p}$ such that a conjugate of μ_{HT} factors through

$$\begin{aligned} \mathbf{G}_{m,\mathbf{C}_p} &\longrightarrow G_{2,\mathbf{C}_p} \cong \mathrm{GL}_{2,\mathbf{C}_p} \subset G_{\mathbf{C}_p} \\ z &\mapsto \mathrm{diag}(1, z). \end{aligned}$$

This proves that the degree of the field of definition of C_{HT} is at most k. In the case where $k = 2$, C_{HT} can not be defined over $\mathbf{Q}_p$ because its projection in $\mathrm{PSL}^2_{2,\mathbf{C}_p}$ cannot be. □

2.2 Proposition. *Let K be a totally real cubic number field, let D be a quaternion algebra over K as in 1.1 and let M/K be the Mumford–Shimura constructed in 1.1. Fix a prime number p. Let $F \supset K$ be a number field, $\mathfrak{p}_F$ a prime of F with residue field k of characteristic p and $\mathfrak{p} = \mathfrak{p}_F \cap \mathcal{O}_K$.*

Suppose that $x \in M(F)$ and that X/F is the abelian variety corresponding to x. Then there is a finite extension F' of F such that $X_{F'}$ has good reduction at all places over $\mathfrak{p}_F$ and, depending on $[K_{\mathfrak{p}} : \mathbf{Q}_p]$, there are the following possibilities for the Newton polygon of such a reduction.

- *If $[K_{\mathfrak{p}} : \mathbf{Q}_p] = 1$, then the possible Newton polygons are $4 \times 0, 4 \times 1$ and $8 \times 1/2$.*
- *If $[K_{\mathfrak{p}} : \mathbf{Q}_p] = 2$, then the possible Newton polygons are $2 \times 0, 4 \times 1/2, 2 \times 1$ and $8 \times 1/2$.*
- *If $[K_{\mathfrak{p}} : \mathbf{Q}_p] = 3$, then the possible Newton polygons are $0, 3 \times 1/3, 3 \times 2/3, 1$ and $8 \times 1/2$.*

Proof. The existence of F' follows from remark 1.6. Replace F by F' and $\mathfrak{p}_F$ by a place of F' lying over it. View K as a subfield of $\mathbf{C}$ via the embedding $\varphi\colon K \hookrightarrow \mathbf{R} \subset \mathbf{C}$ corresponding to the real place of K where D is split and extend this embedding to $F \hookrightarrow \mathbf{C}$. Let $\overline{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, so that K and F are subfields of $\overline{\mathbf{Q}}$. We also fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ such that the composite $F \subset \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ induces the place $\mathfrak{p}_F$ on F. Let $\mathcal{I}_F \subset \mathcal{G}_F$ be the inertia subgroup deduced from this embedding and let k be the residue field at $\mathfrak{p}_F$.

Let $G/\mathbf{Q}$ be the algebraic group associated to D as in 1.1. The map $S \to \mathrm{GL}(\mathrm{H}^1_{\mathrm{B}}(X_{\mathbf{C}}, \mathbf{R}))$ defining the Hodge structure on $\mathrm{H}^1_{\mathrm{B}}(X_{\mathbf{C}}, \mathbf{Q})$ factors through $G_{\mathbf{R}}$, so the Mumford–Tate group G_X of $X_{\mathbf{C}}$ is contained in G. Identifying $\mathrm{H}^1_{\mathrm{B}}(X_{\mathbf{C}}, \mathbf{Q}) \otimes \mathbf{Q}_p = \mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$, the p-adic Galois representation associated to X factors through a map $\rho\colon \mathcal{G}_F \to G_X(\mathbf{Q}_p) \subset G(\mathbf{Q}_p)$. This implies that if (p) splits completely in K or if K has only one prime ideal over (p), then the proposition follows from [Noo99, 3.6].

We can therefore assume that K has two ideals $\mathfrak{p}_{K,1}$ and $\mathfrak{p}_{K,2}$ over (p) necessarily with local degrees 1 and 2 respectively. In this case, $G_{\mathbf{Q}_p}$ satisfies the condition of lemma 2.1, so in order to apply [Noo99, 3.6], we have to show that the local degree $[K_{\mathfrak{p}} : \mathbf{Q}_p]$ determines the factor of $G(\mathbf{Q}_p)$ containing the image of ρ. By the lemma and [Noo99, 3.6], the Newton polygon of X_k is either $4 \times 0, 4 \times 1$ or $2 \times 0, 4 \times 1/2, 2 \times 1$ or $8 \times 1/2$. In what follows, we assume that the Newton polygon of X_k is not $8 \times 1/2$.

The Hodge cocharacter

$$\mu_{\mathrm{HdR}}\colon \mathbf{G}_{m,\mathbf{C}} \longrightarrow (G_X)_{\mathbf{C}} \subset G_{\mathbf{C}}$$

is conjugate to the morphism $\mu_0\colon \mathbf{G}_{m,\mathbf{C}} \to G_{\mathbf{C}}$ from the proof of lemma 1.2, so the field of definition in $\mathbf{C}$ of its conjugacy class C_{HdR} in $G_{\mathbf{C}}$ is equal to K.

On the other hand, one has $\rho(\mathcal{I}_F) \subset G_X(\mathbf{Q}_p)$, so the Hodge–Tate decomposition associated to $\rho_{|\mathcal{I}_F}$ is determined by a cocharacter

$$\mu_{\mathrm{HT}}\colon \mathbf{G}_{m,\mathbf{C}_p} \longrightarrow (G_X)_{\mathbf{C}_p} \subset G_{\mathbf{C}_p}.$$

Let C_{HT} be the conjugacy class of μ_{HT} in $G_{\mathbf{C}_p}$. It follows from [Win88], proposition 7 and the fact that conjecture 1 of loc. cit. has been proven by Blasius (see [Ogu90, 4.2]) that $C_{\mathrm{HT}} = C_{\mathrm{HdR}} \otimes_K \mathbf{C}_p$, where the base change is via the inclusion $K \subset \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ fixed above. One concludes that the field of definition of C_{HT} is equal to $\widehat{K} \subset \overline{\mathbf{Q}}_p$, the closure of K in $\overline{\mathbf{Q}}_p$ for the p-adic topology. It is clear that $\widehat{K}$ is isomorphic to $K_{\mathfrak{p}}$.

If the Newton polygon of X_k is $4 \times 0, 4 \times 1$, then it follows from 2.1 that $K_{\mathfrak{p}} = \widehat{K} = \mathbf{Q}_p$ and hence $[K_{\mathfrak{p}} : \mathbf{Q}_p] = 1$. If the Newton polygon is $2 \times 0, 4 \times 1/2, 2 \times 1$, then the same lemma implies that $\widehat{K}$ is of degree 2 over $\mathbf{Q}_p$, so $[K_{\mathfrak{p}} : \mathbf{Q}_p] = 2$. □

2.3 Corollary. *Let notations and hypotheses be as in proposition 2.2. Then there is a finite extension F' of F such that X has good reduction at all places of F' lying over $\mathfrak{p}_F$. Let $\mathfrak{p}'_{F'}$ be such a place, let k be the residue field at $\mathfrak{p}'_{F'}$ and let X_k be the reduction in question. Then either X_k has Newton slopes $8 \times 1/2$ or the Newton polygon of X_k is the one given in table 2.3.□. In the first case (where the Newton polygon is $8 \times 1/2$), $X_{\bar{k}} \sim (X^{(1)})^4$, where $X^{(1)}/\bar{k}$ is an elliptic curve. In the second case (where the Newton polygon is different from $8 \times 1/2$), either $X_{\bar{k}}$ is simple or its simple factors $X^{(1)}$ and $X^{(3)}$ are of dimension 1 and 3 respectively and their Newton polygons are as given in table 2.3.□.*

$[K_{\mathfrak{p}} : \mathbf{Q}_p]$	Newton polygon of X_k	Newton polygon of $X^{(1)}$	Newton polygon of $X^{(3)}$
$[K_{\mathfrak{p}} : \mathbf{Q}_p] = 1$	$4 \times 0, 4 \times 1$	$0, 1$	$3 \times 0, 3 \times 1$
$[K_{\mathfrak{p}} : \mathbf{Q}_p] = 2$	$2 \times 0, 4 \times 1/2, 2 \times 1$	$0, 1$ or $2 \times 1/2$	$0, 4 \times 1/2, 1$ or $2 \times 0, 2 \times 1/2, 2 \times 1$
$[K_{\mathfrak{p}} : \mathbf{Q}_p] = 3$	$0, 3 \times 1/3, 3 \times 2/3, 1$	$0, 1$	$3 \times 1/3, 3 \times 2/3$

Table 2.3.□: Possible Newton polygons and reduction types of X in corollary 2.3

Proof. We can assume that the Newton polygon of X_k is not $8 \times 1/2$. One can choose a prime number ℓ which is inert in K. As the ℓ-adic Galois representation associated to X factors through the Mumford–Tate group of X, it follows from [Noo99, proposition 4.1] that $X_{\bar{k}}$ is either simple or isogenous to a product of an elliptic curve and a simple abelian threefold. This proves the statements about the isogeny types of the reduction and leaves only the statements about the Newton polygons to prove.

If $[K_{\mathfrak{p}} : \mathbf{Q}_p] = 1$, then proposition 2.2 implies that the Newton polygon of X_k has slopes $4 \times 0, 4 \times 1$. If $X_{\bar{k}} \sim X^{(1)} \times X^{(3)}$ then the slopes of $X^{(1)}$ and $X^{(3)}$ are necessarily $0, 1$ and $3 \times 0, 3 \times 1$ respectively.

If $[K_{\mathfrak{p}} : \mathbf{Q}_p] = 2$, then it follows from proposition 2.2 that the Newton polygon of X_k is $2 \times 0, 4 \times 1/2, 2 \times 1$. If $X_{\bar{k}} \sim X^{(1)} \times X^{(3)}$ then this implies that either $X^{(1)}$ has slopes $0, 1$ and $X^{(3)}$ has slopes $0, 4 \times 1/2, 1$ or $X^{(1)}$ has slopes $2 \times 1/2$ and $X^{(3)}$ has slopes $2 \times 0, 2 \times 1/2, 2 \times 1$.

The proof in the case where $[K_{\mathfrak{p}} : \mathbf{Q}_p] = 3$ is completely analogous. If $X^{(1)} \times X^{(3)}$ has Newton polygon $0, 3 \times 1/3, 3 \times 2/3, 1$, then the Newton polygon of $X^{(3)}$ must be $3 \times 1/3, 3 \times 2/3$ and that of $X^{(1)}$ must be $0, 1$. □

2.4 Remark. Let X be a polarized abelian variety over a number field $F \subset \mathbf{C}$, with Mumford–Tate group G such that $(G, \mathrm{H}^1_{\mathrm{B}}(X(\mathbf{C}), \overline{\mathbf{Q}}))$ is of Mumford's type. Assume that the conjugacy class in $G_{\mathbf{C}}$ of the Hodge cocharacter is defined over F and let $K \subset F$ be the field of definition of this conjugacy class. Then a statement analogous to corollary 2.3 holds.

3 The special points

3.1 We return to the notations of 1.1 and consider a Mumford–Shimura curve M constructed using a totally real field K and a K-algebra D. A point $x \in M(\mathbf{C})$ is called a *special point* if the associated map $h\colon S \to G_{\mathbf{R}}$ factors through a ($\mathbf{Q}$-rational) torus $T \subset G$. Equivalently, this means that there exist a torus $T \subset G$, a map $h\colon S \to T_{\mathbf{R}}$ and a sufficiently small compact open subgroup $C_T \subset T(\mathbf{A}_f)$ such x belongs to the Hecke orbit of a point in the image of the map $M_{C_T}(T, \{h\})(\mathbf{C}) \hookrightarrow M(\mathbf{C})$ induced by the inclusion $\{h\} \subset Y$. For every torus $T \subset G$, we fix a sufficiently small subgroup $C_T \subset T(\mathbf{A}_f)$ to have an immersion $M_{C_T}(T, \{h\}) \hookrightarrow M_{E(T,\{h\})}$ and we write $M(T, \{h\})$ instead of $M_{C_T}(T, \{h\})$. If x is a special point, then it lies in $M(\overline{\mathbf{Q}})$ and the corresponding abelian variety is of CM type. As the images of all Hecke conjugates of the points of $M_{C_T}(T, \{h\})(\mathbf{C})$ correspond to isogenous abelian varieties, the CM type of a special point only depends on T and h.

We determine the special points on M, which naturally fall in two classes. In each case, we compute the CM types of the simple factors. We use the following notation. If L is a CM field and $L_0 \subset L$ the totally real subfield with $[L : L_0] = 2$, then the map $N_{L/L_0}\colon z \mapsto z\bar{z}$ defines a map $L^* = \mathrm{Res}_{L/\mathbf{Q}}(\mathbf{G}_m) \to \mathrm{Res}_{L_0/\mathbf{Q}}(\mathbf{G}_m)$. We define T'_L as the kernel of this map and T_L as the inverse image in L^* of $\mathbf{G}_{m,\mathbf{Q}} \subset \mathrm{Res}_{L_0/\mathbf{Q}}(\mathbf{G}_m)$.

3.2 First case. Let L be a maximal subfield of D such that there exists a field E with $L = E \otimes_{\mathbf{Q}} K$. We necessarily have $[E : \mathbf{Q}] = 2$ and since D is not split over $\mathbf{R}$, E is an imaginary quadratic extension of $\mathbf{Q}$. The field L is thus a CM field and the tori T_L and T'_L can be defined as in 3.1. The inclusion $L^* \subset D^*$ induces morphisms of algebraic groups $\tilde{\rho}'\colon T'_L \hookrightarrow \widetilde{G}'$ and $\rho' = N \circ \tilde{\rho}'\colon T'_L \to G'$.

Let $\varphi_1, \varphi_2, \varphi_3\colon L \hookrightarrow \mathbf{C}$ be such that the $(\varphi_i)_{|K}$ (for $i = 1, 2, 3$) are the complex (real) embeddings of K and such that the $(\varphi_i)_{|E}$ (for $i = 1, 2, 3$) are all equal. Put $\varphi = (\varphi_1)_{|E}$. We can assume that $(\varphi_3)_{|K}\colon K \hookrightarrow \mathbf{R}$ is the embedding for which D is split. Write $[\varphi_i]$ and $[\bar{\varphi}_i]$ ($i = 1, 2, 3$) for the induced characters of L^*, of T_L and of T'_L, so $([\varphi_1], [\varphi_2], [\varphi_3])$ is a basis of $X(T'_L)$. We fix an isomorphism

$$L \otimes_{\mathbf{Q}} \mathbf{R} \cong \prod_{K \hookrightarrow \mathbf{R}} L \otimes_K \mathbf{R} \cong \prod_{K \hookrightarrow \mathbf{R}} \mathbf{C} \tag{3.2.*}$$

such that the composite of $E \subset L \hookrightarrow L \otimes_{\mathbf{Q}} \mathbf{R}$ with the projection to any factor $\mathbf{C}$ induces the embedding $\varphi\colon E \hookrightarrow \mathbf{C}$. One deduces an isomorphism of $\mathbf{R}$-algebraic groups $L^*_{\mathbf{R}} \cong S^3$. Let h be the composite

$$S \longrightarrow L^*_{\mathbf{R}} \subset D^*_{\mathbf{R}} \xrightarrow{\mathrm{Nm}} G_{\mathbf{R}},$$

where the map $S \to L^*_{\mathbf{R}}$ is the inclusion on the coordinate corresponding to φ_3. By construction $h' = h_{|\mathrm{S}^1}\colon \mathrm{S}^1 \to G'_{\mathbf{R}}$ lifts to a map $\tilde{h}'\colon \mathrm{S}^1 \to (T'_L)_{\mathbf{R}} \subset \widetilde{G}'_{\mathbf{R}}$ (so $h' = \rho' \circ \tilde{h}'$). If $T \subset G$ is the image of L^*, then T is a ($\mathbf{Q}$-rational) torus of G and h factors through T. Since h is a conjugate of the map h_0 of 1.1 by an element of $G(\mathbf{R})$, this determines a set of special points of $M(\mathbf{C})$. Let X be an abelian variety in the corresponding isogeny class.

3.3 Proposition. *The abelian variety X is isogenous to a product $X^{(1)} \times X^{(3)}$, where $X^{(1)}$ is an elliptic curve and $X^{(3)}$ is a simple abelian threefold. Both $X^{(1)}$ and $X^{(3)}$ are of CM type and one can choose embeddings of E and L into their respective endomorphism algebras such that the CM types are $(E, \{\varphi\})$ and $(L, \{\varphi_1, \varphi_2, \bar{\varphi}_3\})$ respectively.*

The reflex field of $(E, \{\varphi\})$ is $\varphi(E)$ and the reflex norm is $\varphi^{-1}\colon \varphi(E) \to E$. The reflex field of $(L, \{\varphi_1, \varphi_2, \bar{\varphi}_3\})$ is $\varphi_3(L)$ and the reflex norm is the map $N'\colon \varphi_3(L) \to L$ given by

$$\varphi_1(N'(\varphi_3(x))) = \bar{\varphi}_1(x)\varphi_2(x)\varphi_3(x).$$

Proof. Define $\nu\colon T'_L \to T'_L$ by $\nu(x) = N_{L/E}(x)x^{-2}$, where $N_{L/E}\colon T'_L \to T'_E$ is the map induced by the field norm $N_{L/E}$. This gives rise to

$$\pi' = (\nu, N_{L/E})\colon T'_L \longrightarrow T'_L \times T'_E.$$

The natural action of $T'_L \times T'_E$ on $L \oplus E \cong \mathbf{Q}^6 \oplus \mathbf{Q}^2 \cong \mathbf{Q}^8$ makes π' into an 8-dimensional representation of T'_L.

The weights of the representation induced by $N_{L/E}$ are $\pm([\varphi_1]+[\varphi_2]+[\varphi_3])$ and those of the representation induced by ν are $\pm(-[\varphi_1]+[\varphi_2]+[\varphi_3])$, $\pm([\varphi_1]-[\varphi_2]+[\varphi_3])$, $\pm([\varphi_1]+[\varphi_2]-[\varphi_3])$. This implies that the weights of π' are $\pm[\varphi_1]\pm[\varphi_2]\pm[\varphi_3]$. As the weights of the representation $\rho'\colon T'_L \to G' \subset \mathrm{GL}_{8,\mathbf{Q}}$ from 3.2 are also $\pm[\varphi_1]\pm[\varphi_2]\pm[\varphi_3]$, we have shown that the representations ρ' and π' of T'_L are isomorphic. It follows that X is isogenous to a product $X^{(1)} \times X^{(3)}$, with $X^{(i)}$ of dimension i and with E (resp. L) acting on $X^{(1)}$ (resp. $X^{(3)}$).

This also proves the statement about the CM type of $X^{(1)}$. To compute that of $X^{(3)}$, let $h'_L = \nu \circ \tilde{h}'$. Under the isomorphism 3.2.* (with the factors of the product indexed by $\varphi_1, \varphi_2, \varphi_3$), we have $h'_L(\alpha) = (\alpha, \alpha, \alpha^{-1})$, whence the assertion about the CM type of $X^{(3)}$. For the simplicity of $X^{(3)}$, one shows that this CM type is simple, see [Lan83], Chapter I.

The computation of the reflexes is a easy application of [Lan83, I, §5]. $\square$

3.4 Second case. Let L be a maximal subfield of D and assume that L is a totally imaginary extension of K and that there does not exist a field E such that $L = E \otimes K$. Again, L is a CM field and as in 3.2, the inclusion $L^* \subset D^*$ induces morphisms of algebraic groups $\tilde{\rho}'\colon T'_L \hookrightarrow \widetilde{G}'$ and $\rho' = N \circ \tilde{\rho}'\colon T'_L \to G'$.

Let $\varphi_1, \varphi_2, \varphi_3\colon L \hookrightarrow \mathbf{C}$ be such that $\{(\varphi_i)_{|K}\}_{i=1,2,3}$ is the set of complex (real) embeddings of K. As in the first case, we assume that $(\varphi_3)_{|K}\colon K \hookrightarrow \mathbf{R}$ corresponds to the real place where D is split. The complex embeddings of L are $\varphi_i, \bar{\varphi}_i$ for $i = 1,2,3$. As above, we write $[\varphi_i]$ and $[\bar{\varphi}_i]$ $(i = 1,2,3)$ for the induced characters of L^*, of T_L and of T'_L, so that $([\varphi_1],[\varphi_2],[\varphi_3])$ is a basis of $X(T'_L)$. We fix an isomorphism $L \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathbf{C}^3$ as in 3.2.* such that the composite of $L \hookrightarrow L \otimes_{\mathbf{Q}} \mathbf{R}$ with the projection on the ith factor $\mathbf{C}$ induces the embedding $\varphi_i\colon L \hookrightarrow \mathbf{C}$. We deduce an isomorphism $L^*_{\mathbf{R}} \cong S^3$. As in 3.2, one defines h to be the composite

$$S \longrightarrow L^*_{\mathbf{R}} \subset D^*_{\mathbf{R}} \xrightarrow{\mathrm{Nm}} G_{\mathbf{R}},$$

where the map $S \to L^*_{\mathbf{R}}$ is the inclusion on the coordinate corresponding to φ_3. Again, for $h' = h_{|\mathrm{S}^1}\colon \mathrm{S}^1 \to G'_{\mathbf{R}}$, there is a map $\tilde{h}'\colon \mathrm{S}^1 \to (T'_L)_{\mathbf{R}} \subset \widetilde{G}'_{\mathbf{R}}$ such that $h' = \rho' \circ \tilde{h}'$. Again, h is conjugate to the map h_0 of 1.1 and factors through the torus $T \subset G$, image of $L^* \subset D^*$, so h defines a set of special points of M. Let $X/\mathbf{C}$ be an abelian variety in the corresponding isogeny class. To compute its CM type, we need a construction.

Let $\widetilde{K} \subset \widetilde{L} \subset \mathbf{C}$ be normal closures of K and L respectively and let $\mathcal{H} = \mathrm{Gal}(\widetilde{K}/\mathbf{Q})$. According to the case if $\widetilde{K} = K$ or $[\widetilde{K} : K] = 2$, one has $\mathcal{H} \cong \mathrm{A}_3$ or $\mathcal{H} \cong \mathrm{S}_3$. We fix such an isomorphism. In either case, $\mathcal{H}$ operates on the group $\{\pm 1\}^3$ by permutation of the coordinates.

3.5 Lemma. *There is an isomorphism of* $\mathrm{Gal}(\widetilde{L}/\mathbf{Q})$ *with the semi-direct product* $\{\pm 1\}^3 \rtimes \mathcal{H}$ *and thus with a subgroup of* $\{\pm 1\}^3 \rtimes \mathrm{S}_3$*, given as follows.*

The action of $\mathrm{Gal}(\widetilde{L}/\mathbf{Q})$ *on* $\{\varphi_i, \bar{\varphi}_i\}_{i=1,2,3}$ *factors through the 'natural' action of* $\{\pm 1\}^3 \rtimes \mathrm{S}_3$ *on this set, i. e. the action by which* $\{(1,1,1)\} \rtimes \mathrm{S}_3$ *acts by permutation on the sets* $\{\varphi_1, \varphi_2, \varphi_3\}$ *and* $\{\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3\}$ *and* $((\varepsilon_1, \varepsilon_2, \varepsilon_3), \mathrm{id})$ *fixes* φ_i *and* $\bar{\varphi}_i$ *if* $\varepsilon_i = 1$ *and exchanges them if* $\varepsilon_i = -1$*.*

Proof. As $K \subset L$, $\mathrm{Gal}(\widetilde{L}/\mathbf{Q})$ acts on $\{(\varphi_i)_{|K}\}_{i=1,2,3}$ and thus on

$$\big\{\{\varphi_i, \bar{\varphi}_i\}\big\}_{i=1,2,3}\,.$$

This action factors through the action of $\mathcal{H} \subset \mathrm{S}_3$ on $\{(\varphi_i)_{|K}\}_{i=1,2,3}$. For each $i = 1, 2, 3$, we let -1 act on $\{\varphi_i, \bar{\varphi}_i\}$ as above, i. e. by exchanging φ_i and $\bar{\varphi}_i$. This gives an action of $\{\pm 1\}^3$ on $\{\varphi_i, \bar{\varphi}_i\}_{i=1,2,3}$ inducing the action of $\{\pm 1\}^3 \rtimes \mathcal{H}$ on $\{\varphi_i, \bar{\varphi}_i\}_{i=1,2,3}$ from the statement of the lemma. By construction, the action of $\mathrm{Gal}(\widetilde{L}/\mathbf{Q})$ on this set factors through a morphism

$$\mathrm{Gal}(\widetilde{L}/\mathbf{Q}) \longrightarrow \{\pm 1\}^3 \rtimes \mathcal{H} \subset \{\pm 1\}^3 \rtimes \mathrm{S}_3.$$

Since $\widetilde{L}$ is generated by the $\varphi_i(L)$ and the $\bar{\varphi}_i(L)$, it is clear that this morphism is injective.

The group $\mathcal{H} = \mathrm{Gal}(\widetilde{K}/\mathbf{Q})$ contains an element of order 3, so there exists an element $\sigma \in \mathrm{Gal}(\widetilde{L}/\mathbf{Q})$ which cyclicly permutes the $\{\varphi_1, \bar{\varphi}_1\}$, $\{\varphi_2, \bar{\varphi}_2\}$, $\{\varphi_3, \bar{\varphi}_3\}$. After replacing σ by its square if needed, we can assume that σ is of order 3 and, maybe after exchanging φ_i and $\bar{\varphi}_i$ (for $i = 2, 3$) and/or replacing σ by σ^{-1}, we can even assume that $\sigma = (1, 2, 3)$ acting on $\{\varphi_1, \varphi_2, \varphi_3\}$ (and on $\{\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3\}$) by cyclic permutation.

Consider the projection

$$\mathrm{pr}\colon\ \mathrm{Gal}(\widetilde{L}/\mathbf{Q}) \hookrightarrow \{\pm 1\}^3 \rtimes \mathrm{S}_3 \longrightarrow \mathrm{S}_3.$$

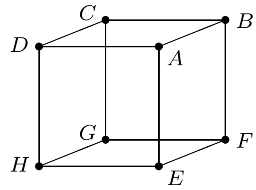
The complex conjugation induces an element of $\mathrm{Gal}(\widetilde{L}/\mathbf{Q})$ corresponding to $(-1, -1, -1)$, so $\{\pm(1, 1, 1)\} \subset \ker(\mathrm{pr})$. If $\ker(\mathrm{pr}) \neq \{\pm(1, 1, 1)\}$ then the lemma is true, because in this case $\mathrm{Gal}(\widetilde{L}/\mathbf{Q})$ contains $\{\pm 1\}^3$. We can thus assume that $\ker(\mathrm{pr}) = \{\pm(1, 1, 1)\}$. Let

$$P = \{(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3 \mid \varepsilon_1\varepsilon_2\varepsilon_3 = 1\}$$

and $\mathcal{H}' = \mathrm{Gal}(\widetilde{L}/\mathbf{Q}) \cap (P \rtimes \mathrm{S}^3)$. Then $\mathrm{Gal}(\widetilde{L}/\mathbf{Q}) \cong \{\pm 1\} \times \mathcal{H}'$ and by our assumption on $\ker(\mathrm{pr})$, the restriction $\mathrm{pr}_{|\mathcal{H}'}\colon \mathcal{H}' \to \mathrm{S}_3$ is injective. The fact that $\mathrm{A}_3 \subset \mathcal{H}'$ implies that one actually has $\mathcal{H}' = \mathcal{H} \subset \mathrm{S}_3$ and therefore $\mathrm{Gal}(\widetilde{L}/\mathbf{Q}) = \{\pm 1\} \times \mathcal{H}$. This is a contradiction because L contains the field $E = \widetilde{L}^{\mathcal{H}}$ and we have $L = EK = E \otimes K$. $\square$

3.6 Let $E = \widetilde{L}^{\mathcal{H}}$ and let $\psi_A \colon E \hookrightarrow \mathbf{C}$ be the complex embedding induced by the inclusions $E \subset \widetilde{L} \subset \mathbf{C}$. One has $[E : \mathbf{Q}] = 8$. We construct a bijection between the set of complex embeddings of E and the set of vertices of the cube, denoted in the following way.

$$\begin{aligned} A &= (1,1,1) & B &= (-1,1,1) \\ C &= (-1,-1,1) & D &= (1,-1,1) \\ E &= (1,1,-1) & F &= (-1,1,-1) \\ G &= (-1,-1,-1) & H &= (1,-1,-1) \end{aligned}$$



These are the vertices the 'standard' cube in $\mathbf{Z}^3$.

The action of $\mathrm{Gal}(\widetilde{L}/\mathbf{Q}) = \{\pm 1\}^3 \rtimes \mathcal{H}$ on $\{(\pm 1, \pm 1, \pm 1)\}$ makes this group act on the cube. The subgroup $\mathcal{H}$ is the stabilizer of A in $\{\pm 1\}^3 \rtimes \mathcal{H}$ and the action of the subgroup $\{\pm 1\}^3$ on the set $\{A, \dots, H\}$ is simply transitive. Define $\psi_A, \dots, \psi_H$ by

$$\psi_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)A} = \sigma_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \circ \psi_A \colon E \longrightarrow \widetilde{L} \subset \mathbf{C},$$

where $\sigma_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \in \mathrm{Gal}(\widetilde{L}/\mathbf{Q})$ corresponds to $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3$ under the isomorphism of lemma 3.5. It is clear that $\mathrm{Hom}(E, \mathbf{C}) = \{\psi_A, \dots \psi_H\}$ and that any complex conjugation acts on $\{\psi_A, \dots \psi_H\}$ as inversion with respect to the centre acts on the vertices of the cube. This implies in particular that E is a CM field.

3.7 Proposition. *Let $h \colon S \to T_{\mathbf{R}} \subset G_{\mathbf{R}}$ be the map constructed in 3.4 and let $X/\mathbf{C}$ be an abelian variety in the corresponding isogeny class. Then X is a simple abelian variety of CM type and one can choose an embedding of E into the endomorphism algebra of X such that the CM type is $(E, \{\psi_A, \psi_B, \psi_C, \psi_D\})$.*

The reflex field of this type is $\varphi_3(L)$ and the reflex norm $N' \colon \varphi_3(L) \to E$ is given by

$$\psi_A(N'(\varphi_3(x))) = \varphi_1(x)\varphi_2(x)\varphi_3(x).$$

Proof. In analogy with 3.2, we write $[\psi_A], \dots [\psi_H]$ for the characters of E^* induced by $\psi_A, \dots \psi_H$. Since E is a CM field, the constructions of 3.1 define tori T_E and T'_E. It is clear that $([\psi_A], [\psi_B], [\psi_C], [\psi_D])$ is a basis of $X(T'_E)$. Define

$$T_{\mathrm{mt}} = \ker([\psi_A] - [\psi_B] + [\psi_C] - [\psi_D]) \subset T_E$$

and $T'_{\mathrm{mt}} = T_{\mathrm{mt}} \cap T'_E$. These are subtori of T_E defined over $\mathbf{Q}$.

Using the above, one verifies that for each $x \in L$, one has $\varphi_1(x)\varphi_2(x)\varphi_3(x) \in E$, so one defines a map $\nu \colon L^* \to E^*$ by $\nu(x) = \varphi_1(x)\varphi_2(x)\varphi_3(x)$. It is easily checked that ν induces a map $\pi' \colon T'_L \to T'_{\mathrm{mt}}$. The natural representation of T'_{mt} on $E \cong \mathbf{Q}^8$ makes π' into an 8-dimensional representation of T'_L, with weights $\pm[\varphi_1] \pm [\varphi_2] \pm [\varphi_3]$. The representation ρ' constructed in 3.4 also has weights $\pm[\varphi_1] \pm [\varphi_2] \pm [\varphi_3]$, so ρ' and π' are isomorphic (as representations of T'_L). This implies in particular that E acts on X.

To finish the proof of the proposition, we have to determine the CM type. Let h'_E be the composite

$$h'_E\colon \mathrm{S}^1 \xrightarrow{\tilde{h}'} (T'_L)_{\mathbf{R}} \xrightarrow{\pi'_{\mathbf{R}}} (T'_{\mathrm{mt}})_{\mathbf{R}} \subset (T'_E)_{\mathbf{R}}.$$

As $([\psi_A],[\psi_B],[\psi_C],[\psi_D])$ is a basis of $X(T'_E)$, we get

$$(T'_E)_{\mathbf{R}} \cong \prod_{i=A}^{D} \mathrm{S}^1 \subset E^*_{\mathbf{R}} \cong \prod_{i=A}^{D} S$$

and under this morphism one has $h'_E(\alpha) = (\alpha,\alpha,\alpha,\alpha)$. Hence the CM type of X is $(E,\{\psi_A,\psi_B,\psi_C,\psi_D\})$. As this CM type is simple, X is simple. Using the lemma 3.5, the computation of the reflex is a straightforward application of [Lan83, I, §5]. □

3.8 Remark. The abelian varieties with this CM type are studied by Pohlman in [Poh68].

3.9 Proposition. *For T and h ranging over the tori and the maps constructed in 3.2 and 3.4, the union of the Hecke orbits of the images of the maps $M(T,\{h\})(\mathbf{C}) \to M(\mathbf{C})$ is equal to the set of special points of $M(\mathbf{C})$.*

Proof. Let $x \in M(\mathbf{C})$ be a special point, let $h\colon S \to G_{\mathbf{R}}$ be the associated morphism and let $T \subset G$ be a torus such that $h\colon S \to T_{\mathbf{R}} \subset G_{\mathbf{R}}$. As h is conjugate to the map h_0 of 1.1 by an element of $G(\mathbf{R})$, the restriction $h'\colon \mathrm{S}^1 \to G'$ lifts to $\tilde{h}'\colon \mathrm{S}^1 \to \widetilde{G}'$. Let $\widetilde{T}' \subset \widetilde{G}'$ be a torus such that $\tilde{h}'$ factors through $\widetilde{T}'$. Considering

$$\widetilde{G}'_{\mathbf{C}} \cong \prod_{K \hookrightarrow \mathbf{C}} \mathrm{SL}_{2,\mathbf{C}} \subset \prod_{K \hookrightarrow \mathbf{C}} \mathrm{M}_{2,\mathbf{C}} \cong D \otimes_{\mathbf{Q}} \mathbf{C}$$

and using the fact that $\tilde{h}'$ is conjugate to the map $\tilde{h}'_0$ of 1.1 by an element of $\widetilde{G}'(\mathbf{R})$, one sees that $\widetilde{T}'_{\mathbf{C}}$ contains a torus of the form $\{1\} \times \{1\} \times T_3 \subset \mathrm{SL}^3_{2,\mathbf{C}}$, where T_3 is a maximal torus of the factor $\mathrm{SL}_{2,\mathbf{C}}$ corresponding to the embedding $K \hookrightarrow \mathbf{R} \subset \mathbf{C}$ where D is split. Since the above isomorphisms and inclusion are $\mathrm{Aut}(\mathbf{C})$ equivariant and since $\widetilde{T}'$ is defined over $\mathbf{Q}$, it follows that it is a maximal torus.

We have $\widetilde{T}'(\overline{\mathbf{Q}}) \subset \widetilde{G}'(\overline{\mathbf{Q}}) \subset D \otimes \overline{\mathbf{Q}}$. Let $L_{\overline{\mathbf{Q}}}$ be the $\overline{\mathbf{Q}}$-subalgebra of $D \otimes \overline{\mathbf{Q}}$ generated by $\widetilde{T}'(\overline{\mathbf{Q}})$ and the centre $K^* \subset D^*$. Since $\widetilde{T}'$ and K^* generate a maximal torus of D^*, it follows that $L_{\overline{\mathbf{Q}}}$ is a maximal commutative subalgebra. As $\widetilde{T}'$ is defined over $\mathbf{Q}$, there exists an algebra $L \subset D$ such that $L_{\overline{\mathbf{Q}}} = L \otimes \overline{\mathbf{Q}}$. By construction, L is a maximal commutative subalgebra and because D is a division algebra, it is a maximal subfield. This implies that it splits D. It is therefore imaginary at the two real places of K where D is non-split. Since

$$\widetilde{T}'(\mathbf{R}) \subset (L \otimes_{\mathbf{Q}} \mathbf{R})^* \cong \prod_{K \hookrightarrow \mathbf{R}} (L \otimes_K \mathbf{R})^*$$

contains $h(\mathrm{S}^1(\mathbf{R}))$ and as this image lies in the factor corresponding to the embedding $K \hookrightarrow \mathbf{R}$ where D is split, it follows that L is imaginary at the third real place as well. □

4 Newton polygons at the special points

4.1 How to compute Newton polygons. Suppose that X is an abelian variety of CM type over a number field $F \subset \mathbf{C}$ with all its $\mathbf{C}$-endomorphisms defined over F and that X is of CM type (E, Φ) for some CM field E of degree $2 \dim X$. Let (E', Φ') be the reflex type. Since all geometric endomorphisms of X are defined over F, one has $E' \subset F$. Assume that $\mathfrak{p}$ is a prime of F, of residue characteristic p, let k be the residue field and suppose that X has good reduction X_k at $\mathfrak{p}$. Then the Newton slopes of X_k can be computed as follows.

Let $N'_\Phi\colon \operatorname{Res}_{E'/\mathbf{Q}}(\mathbf{G}_m) \to T_E$ be the reflex norm. For a sufficiently large integer N, there exists $\alpha \in \mathcal{O}_F$ such that $\mathfrak{p}^N = (\alpha)$. By [Ser68, II, 3.4], see also the proposition in II, 2.3, the eigenvalues of the Nth power of a geometric Frobenius element (acting on the ℓ-adic étale cohomology of X, for any prime number $\ell \neq p$) at $\mathfrak{p}$ are the $\sigma(N'_\Phi \circ N_{F/E'}(\alpha))$, for σ running through the set of complex embeddings of E. It follows that the Newton slopes of X_k are the numbers

$$\frac{v(\lambda)}{v((\mathbf{N}\mathfrak{p})^N)} = \frac{v(\lambda)}{v(N_{F/\mathbf{Q}}(\alpha))},$$

for λ running through the Galois conjugates of $N'_\Phi \circ N_{F/E'}(\alpha)$ and v a fixed p-adic valuation of a normal closure $\widetilde{E}$ of E. One finds the same slopes if one replaces F by E', $\mathfrak{p}$ by $\mathfrak{p} \cap \mathcal{O}_{E'}$ and takes $\alpha \in \mathcal{O}_{E'}$ and $N \in \mathbf{N}$ such that $(\mathfrak{p} \cap \mathcal{O}_{E'})^N = (\alpha)$. This means that the Newton polygon only depends on the intersection of $\mathfrak{p}$ with $\mathcal{O}_{E'}$, so in the sequel we will speak of the Newton slopes and the Newton polygon of X at a prime $\mathfrak{p}$ of E'. Since there exists a finite extension F' of F such that $X_{F'}$ has good reduction at all non-archimedean places of F', we may consider the Newton polygon (slopes) of X at any prime of E'.

4.2 Proposition. *Let E be an imaginary quadratic field and $(E, \{\varphi\})$ a CM type as in 3.3. Let X be an elliptic curve, over a number field containing the reflex field, with this CM type. For any prime $\mathfrak{p}$ of the reflex field $E' = \varphi(E)$, the Newton slopes of X at $\mathfrak{p}$ are determined by E and the residue characteristic p of $\mathfrak{p}$ as follows.*

- *If (p) splits in E then the slopes are* 0, 1.
- *If (p) is inert or ramified in E then the slopes are* $2 \times 1/2$.

Proof. Easy exercise. □

4.3 Proposition. *Let K be a totally real number field of degree 3 as in 1.1, E an imaginary quadratic field, $L = E \otimes K$ and let X be a 3-dimensional abelian variety (over a number field containing the reflex field) with the CM type $(L, \Phi) = (L, \{\varphi_1, \varphi_2, \bar{\varphi}_3\})$ of 3.2. For any prime $\mathfrak{p}$ of the reflex field $L' = \varphi_3(L)$, the Newton slopes of X at $\mathfrak{p}$ are given by the table 4.3.□ . In this table, p is the residue characteristic of $\mathfrak{p}$ and $\mathfrak{p}_K$ is the intersection of $\mathfrak{p}$ with $\varphi_3(\mathcal{O}_K) \subset L'$.*

In the cases marked 'See text', there are two possibilities. If L' has one prime over $\mathfrak{p}_K$ then the slopes are $6 \times 1/2$, if L' has two primes over $\mathfrak{p}_K$ then the slopes are $2 \times 0, 2 \times 1/2, 2 \times 1$.

Nr.	$[K_{\mathfrak{p}_K} : \mathbf{Q}_p]$	(p) splits in E	(p) is inert in E	(p) is ramified in E
1.	$[K_{\mathfrak{p}_K} : \mathbf{Q}_p] = 1$	$3 \times 0, 3 \times 1$	$6 \times 1/2$	$6 \times 1/2$
2.	$[K_{\mathfrak{p}_K} : \mathbf{Q}_p] = 2$	$0, 4 \times 1/2, 1$	See text	See text
3.	$[K_{\mathfrak{p}_K} : \mathbf{Q}_p] = 3$	$3 \times 1/3, 3 \times 2/3$	$6 \times 1/2$	$6 \times 1/2$

Table 4.3.□: The Newton polygon of X in proposition 4.3

Proof. We consider the reflex norm N'_Φ as a map from L' to $\widetilde{L} \subset \mathbf{C}$. It is then given by $N'_\Phi(\varphi_3(\alpha)) = \bar{\varphi}_1(\alpha)\varphi_2(\alpha)\varphi_3(\alpha)$ (for $\alpha \in L$) and its image actually lies in $\varphi_1(L)$.

Let $v_\mathfrak{p}$ be the valuation on L' associated to $\mathfrak{p}$, normalized by $v_\mathfrak{p}(p) = 1$ and let v be a p-adic valuation on $\widetilde{L}$ extending $v_\mathfrak{p}$. Let $N \in \mathbf{N}$ and $\alpha \in \mathcal{O}_{L'}$ be such that $\mathfrak{p}^N = (\alpha)$ and put $q = N_{L'/\mathbf{Q}}(\alpha) = (\mathbf{N}\mathfrak{p})^N$. Then the Newton slopes are the numbers $v(\sigma(N'_\Phi(\alpha)))/v(q)$ for σ running through the set of complex embeddings of L. As the list of $\sigma(N'_\Phi(\alpha))$ is the list of the

$$\varphi_i(\alpha)\varphi_j(\alpha)\bar{\varphi}_k(\alpha) \quad \text{for } \{(i,j,k)\} = \{(1,2,3),(2,3,1),(3,1,2)\}$$

and their complex conjugates, we have to compute the $v(\beta)/v(q)$ for the β in this list.

For example, in case 1, (p) splits completely in K, in the first column, (p) splits in E, one notes that since there are six valuations on L above p, one has $v(\varphi_3(\alpha)) = N$ and for all other complex embeddings φ one has $v(\varphi(\alpha)) = 0$, whence the result in this case.

The verification of the other entries of the table is left to the reader. □

4.4 Proposition. *Let K be a totally real number field of degree 3 as in 1.1, L a totally imaginary quadratic extension of K not containing any quadratic number field and let X be a 4-dimensional abelian variety with the CM type $(E, \Psi) = (E, \{\psi_A, \psi_B, \psi_C, \psi_D\})$ constructed in 3.4 (over a number field containing the reflex field).*

Let $\mathfrak{p}$ be a prime of the reflex field $L' = \varphi_3(L)$ and $\mathfrak{p}_K$ the intersection of $\mathfrak{p}$ with $\varphi_3(\mathcal{O}_K) \subset L'$. If $\mathfrak{p}_K$ is split in L', then the Newton slopes of X at $\mathfrak{p}$ are given by the table 4.4.□. In

Nr.	$[K_{\mathfrak{p}_K} : \mathbf{Q}_p]$	Newton polygon if $\mathfrak{p}_K$ splits in L'
1.	$[K_{\mathfrak{p}_K} : \mathbf{Q}_p] = 1$	$4 \times 0, 4 \times 1$
2.	$[K_{\mathfrak{p}_K} : \mathbf{Q}_p] = 2$	$2 \times 0, 4 \times 1/2, 2 \times 1$
3.	$[K_{\mathfrak{p}_K} : \mathbf{Q}_p] = 3$	$0, 3 \times 1/3, 3 \times 2/3, 1$

Table 4.4.□: The Newton polygon of X in proposition 4.4

all other cases ($\mathfrak{p}_K$ inert in L', $\mathfrak{p}_K$ ramified in L'), the Newton slopes are $8 \times 1/2$.

Proof. The proof closely resembles the proof of proposition 4.3. We use the same notations as above and note that in this case, the list of $\sigma(N'_\Psi(\alpha))$, for σ running through the set of complex embeddings of E, is the list of $\varphi(\alpha)\varphi'(\alpha)\varphi''(\alpha)$ where φ, φ' and φ'' are any three embeddings of L into $\mathbf{C}$ such that $\varphi_{|K}$, $\varphi'_{|K}$ and $\varphi''_{|K}$ are distinct. The reader will have no difficulty to convince herself that the proposition is true. $\square$

5 Existence results

5.1 Proposition. *Let M/K be a Mumford–Shimura curve constructed as in 1.1, with generic Mumford–Tate group G as in loc. cit., $F \supset K$ a number field and $\mathfrak{p}_F$ a prime of F with residue field k. Suppose that $x_F \in M(F)$ is a special point and that X_F/F is the abelian variety corresponding to x_F. Then there exist a finite extension F' of F, a prime $\mathfrak{p}_{F'}$ of F' lying above $\mathfrak{p}_F$, with residue field $k' \supset k$, and a point $y_{F'} \in M(F')$ such that the abelian variety $Y_{F'}/F'$ corresponding to $y_{F'}$ has Mumford–Tate group G and such that $X_{F'}$ and $Y_{F'}$ have good reduction $X_{k'}$ and $Y_{k'}$ at $\mathfrak{p}_{F'}$ satisfying $Y_{k'} \cong X_{k'}$.*

Proof. After replacing F by a finite extension, one may assume that X_F has good reduction at $\mathfrak{p}$. Let $\mathcal{O}_\mathfrak{p}$ be the localization of $\mathcal{O}_F$ at $\mathfrak{p} = \mathfrak{p}_F$ and let p be the characteristic of k. We change the compact open subgroup $C \subset G(\mathbf{A}_f)$ so that it defines a level structure which is prime to p and such that there exists a finite map $M \to (\mathcal{A}_4)_F$, where $\mathcal{A}_4/\mathcal{O}_\mathfrak{p}$ denotes an appropriate fine moduli scheme of 4-dimensional abelian schemes with level structure prime to p. This amounts to replacing M by a quotient of a finite cover. After replacing F by a finite extension, we can assume that the hypotheses of the proposition are still verified and it suffices to give a proof in this case. Since $K \subset F$, we can base change to F and assume that M is an F-scheme. We finally replace F by a finite extension and M by the geometrically irreducible component containing x_F.

Let $\mathcal{M}/\mathcal{O}_\mathfrak{p}$ be the Zariski closure in $\mathcal{A}_4$ of the image of M and let $\mathcal{X}$ be the pull back to $\mathcal{M}$ of the universal abelian scheme on $\mathcal{A}_4$. By construction, the pull back of $\mathcal{X}$ to M is Mumford's family on M. The image in $(\mathcal{A}_4)(F)$ of $x_F \in M(F)$ extends to $x \in \mathcal{M}(\mathcal{O}_\mathfrak{p})$.

Replace $\mathcal{M}$ by an affine open subset containing x and let $\pi\colon \mathcal{M} \to \mathbf{A}^1_\mathcal{O}$ be a map such that π_k is non-constant on any component of $\mathcal{M}_k$ and such that $\pi_F\colon \mathcal{M}_F \to \mathbf{A}^1_F$ is étale. For any prime number ℓ, it follows from [Noo95], proposition 1.3, theorem 1.7 and their proofs that there is a thin subset $\Omega \subset \mathbf{A}^1(F)$ with the following property. For each $y'_F \in \mathbf{A}^1(F)$, $y'_F \notin \Omega$, each finite extension F' of F and each $y_{F'} \in \pi^{-1}(y'_F)(F')$, the image of the ℓ-adic Galois representation associated to $\mathcal{X}_{y_{F'}}$ has a subgroup of finite index which is Zariski dense in $G(\mathbf{Q}_p)$. It follows from [Ser89, 9.6] that there exist infinitely many points $y' \in \mathbf{A}^1(\mathcal{O}_\mathfrak{p})$ with $y'_k = \pi(x)_k$ and such that $y'_F \notin \Omega$. This implies the proposition. $\square$

5.2 Proposition. *Let K and D be as in 1.1. Then there exist maximal subfields L of both the kinds used in 3.2 and 3.4. Moreover, there is the following freedom left.*

In the first case, $L = E \otimes K$ for an imaginary quadratic field E. Fix a finite set P' of prime numbers. For each $p \in P'$, fix a reduced $\mathbf{Q}_p$-algebra $E^{(p)}$ of degree 2 and assume that $K_{\mathfrak{p}} \otimes_{\mathbf{Q}_p} E^{(p)}$ is a field for each prime $\mathfrak{p}$ of K over (p) where D is not split. Then one can choose E such that $E \otimes \mathbf{Q}_p \cong E^{(p)}$ for each $p \in P'$.

In the second case, fix a finite set P' of primes of K and for each $\mathfrak{p} \in P'$, fix a reduced $K_{\mathfrak{p}}$-algebra $L^{(\mathfrak{p})}$ of degree 2. Assume that $L^{(\mathfrak{p})}$ is a field for each $\mathfrak{p} \in P'$ such that D is not split at $\mathfrak{p}$. Then we can choose L such that $L \otimes_K K_{\mathfrak{p}} \cong L^{(\mathfrak{p})}$ for each $\mathfrak{p} \in P'$.

Proof. We first show the existence of maximal subfields of the 'first' kind (3.2). Let P be the set of prime numbers $p \in \mathbf{Q}$ such that D is non-split at at least one place of K lying above p. Let $P_1 \subset P - P'$ be the set of the $p \in P - P'$ that are unramified in K and let P_2 be the complement of P_1 in $P - P'$. Let E be an imaginary quadratic extension of $\mathbf{Q}$ such that E is ramified at each $p \in P_1$, E has residue degree 2 at all $p \in P_2$ and $E \otimes \mathbf{Q}_p \cong E^{(p)}$ for each $p \in P'$. Since $P \cup P'$ is finite, the existence of E follows from the Chinese remainder theorem. By construction, $L = E \otimes K \supset K$ has local degree 2 at each place of K where D is non-split, so $D \otimes_K L$ is split.

For maximal subfields of the 'second' kind (3.4), the proof is similar. Let P be the set of finite places of K where D is not split and let $p \in \mathbf{Q}$ be a prime number that splits completely in K and such that none of the places $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ lying above p are in $P \cup P'$. By the Chinese remainder theorem, there exist a polynomial $Q \in \mathcal{O}_K[X]$ of degree 2 and an ideal $I \subset \mathcal{O}_K$ such that for each $Q' \in \mathcal{O}_K[X]$ which is congruent to Q modulo I, one has

- Q' is irreducible modulo $\mathfrak{p}$ for each $\mathfrak{p} \in P$,
- Q' is irreducible modulo $\mathfrak{p}_1$,
- Q' is reducible modulo $\mathfrak{p}_2$ and
- $K_{\mathfrak{p}}[X]/(Q') \cong L^{(\mathfrak{p})}$ for all primes $\mathfrak{p} \in P'$.

Note that by hypothesis, the first and the last conditions are not in contradiction with each other. For any such Q', the field $L = K[X]/(Q')$ has local degree 2 at all places in P, and L does not contain any subfield E of degree 2 over $\mathbf{Q}$. Moreover, the local extension at each $\mathfrak{p} \in P'$ is what it should be. It suffices to show that we can choose Q' to be irreducible at all real places of K, and for this it suffices to pick a Q' with constant term sufficiently large at all real embeddings of K. This is possible because the image of I under the map $K \hookrightarrow \mathbf{R}^3$ induced by the three real embeddings of K is a lattice. $\square$

5.3 Proposition. *Let K be a totally real cubic number field, D a central K-algebra as in 1.1 and let M/K be the corresponding Shimura curve. Let G as in 1.1 be the generic Mumford–Tate group. Let $\mathfrak{p}$ be a prime of K such that D is split at $\mathfrak{p}$.*

Then, for each possibility listed in 2.3.$\square$, there exist an extension F of K, a prime $\mathfrak{p}_F$ of F lying over $\mathfrak{p}$, with residue field k, and a point $x \in M(F)$ such that the corresponding *abelian variety X has Mumford–Tate group G and such that X has good reduction X_k with the given Newton polygon and of the given isogeny type.*

Proof. It follows from (for example) propositions 3.7, 5.2 and 4.4 that there is a special point in $M(\overline{\mathbf{Q}})$ such that the reduction of the corresponding abelian variety has Newton polygon $8 \times 1/2$. Proposition 5.1 therefore implies the existence of an abelian variety X with Mumford–Tate group G whose reduction has this Newton polygon. The existence of the other reduction types is proven similarly, using 3.7, 5.2 and 4.4 for $X_{\bar{k}}$ simple and 3.3, 5.2, 4.2 and 4.3 if $X_{\bar{k}} \sim X^{(1)} \times X^{(3)}$. $\square$

5.4 Remark. It is left to the reader to restate the above result in terms of Mumford–Tate groups, in analogy with remark 2.4.

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