Maximal symplectic packings in $\mathbb{P}^2$.

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Abstract

In this paper we describe the intersections between the balls of maximal symplectic packings of $\mathbb{P}^2$. This analysis shows the existence of singular points for maximal packings of $\mathbb{P}^2$ by more than three equal balls. It also yields a construction of a class of very regular examples of maximal packings by five balls.

1 Introduction.

The symplectic packing problem is the question of identifying the conditions on the radii of k balls for being able to pack them symplectically in a given manifold. It was first considered by Gromov as a problem whose answer singularizes symplectic geometry from the volume-preserving one : the restrictions are sometimes stronger than the volume obstruction alone [6]. For instance, the non-squeezing theorem asserts that no ball of radius bigger than one can be packed (or embedded in this single ball situation) in the infinite volume cylinder $B^2(1) \times \mathbb{R}^{2n}$. In the special case of $\mathbb{P}^2$, this problem has been given a complete answer. Apart from the volume obstruction, symplectic packings by less than eight balls are submitted to finitely many purely symplectic obstructions discovered by Gromov [6] and McDuff-Polterovich [9]. Biran also proved that the symplectic obstructions disappear for more than nine balls [3].

This paper is aimed at describing what the packings of $\mathbb{P}^2$ by balls of maximal radii look like. We are particularly interested in understanding their intersection properties. Let us first define our setting. Throughout this paper, the symplectic structure on $\mathbb{P}^2$ is given by the standard Fubini-Study form normalized so that $\int_{\mathbb{CP}^1} \omega = \pi$. With this normalization, $\mathbb{CP}^2 \backslash \mathbb{CP}^1$ is symplectomorphic to the standard unit *open* ball $B^4(1)$ in $\mathbb{C}^2$.

Definition 1.1. *A maximal symplectic packing of M by k balls is a symplectic embedding $\varphi : (B(r_1) \coprod \cdots \coprod B(r_k), \omega_{st}) \hookrightarrow (M, \omega)$ where the radii are such that there exists no symplectic packing of M by balls of radii $(r_1, \dots, r_i + \varepsilon, \dots, r_k)$. It will be said smooth if each $\varphi_i := \varphi_{|B(r_i)}$ extends to a smooth embedding of the closed balls in M. It will be said regular if these maps have only a finite number of singular points on the boundary : each φ_i extends to a topological embedding of the closed ball which is a smooth embedding of $\overline{B(r_i)} \backslash \{p_i^1, \dots, p_i^{n_i}\}$.*

The space of regular maximal symplectic packings by a fixed number of balls is naturally endowed with the Hausdorff topology for compact sets (for instance). Throughout this paper, genericity is meant with respect to this topology, and should be understood in a strong sense : a property is generic if it is true for an open dense set. Our first theorem deals with smooth maximal symplectic packings of $\mathbb{P}^2$ (it still holds partially in $\mathbb{P}^n$).

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Theorem 1. *Below are the generic pattern of intersection between the balls of smooth maximal symplectic packings of $\mathbb{P}^2$.*
a. Generically, the closed balls of a smooth maximal symplectic packing of $\mathbb{P}^2$ by two balls intersect precisely along one common Hopf circle of their boundary.
b. Generically, any two closed balls of a smooth maximal symplectic packing of $\mathbb{P}^2$ by three equal balls intersect precisely along one common Hopf circle of their boundary.
c. Generically, the two smallest balls of a smooth maximal symplectic packing of $\mathbb{P}^2$ by three non-equal balls do not intersect, while the intersection of the biggest ball with any of the others is exactly one common Hopf circle of their boundaries.
d. There exist no smooth maximal symplectic packing of $\mathbb{P}^2$ by more than three equal balls.

The Hopf circles of a ball are the characteristic leafs of its boundary (see section 3.1). Theorem 1 is better understood in the light of Karshon's examples [9]: it says that generic smooth maximal symplectic packing look very much like those she produced (see section 2). The idea consists in translating the maximality property to the existence of characteristic circles in the intersections of the boundary spheres. These characteristics give rise to (maybe singular) symplectic spheres, whose intersection properties lead to the desired uniqueness. The approach is based on a strong connection observed by Paternain-Polterovich-Siburg [12] or Laudenbach-Sikorav [7] between symplectic non-removable intersection and closed characteristics.

The importance of the characteristic foliation in the present study is precisely the reason for our definition of smooth or regular maximal symplectic packing to be so restrictive. Before going further, a discussion about the existence of the objects under consideration is needed. A non-contructive argument due to McDuff shows that there always exist symplectic packings by *open* balls of maximal radii, with no guaranteed boundary regularity. In the other hand, several explicit examples are available. We already mentionned Karshon's smooth packings by two or three balls. Generalizing her construction, Traynor [15] and Schlenk [13] produced examples of maximal symplectic packings of $\mathbb{P}^2$ by five and six balls, which unfortunately fail from far to be regular. It is not completely surprising in view of theorem 1.d. As far as we are concerned, the achieved boundary regularity is nevertheless as difficult to handle as McDuff's abstract maximal packings : no convenient notion of characteristic foliation on the boundary of the balls can be defined. The second result of this paper concerns precisely the relevance of our definition of regularity. Allowing only finitely many singularities enables us to produce interesting maximal packings of $\mathbb{P}^2$, at least by five balls.

Theorem 2. *There exist regular maximal symplectic packings of $\mathbb{P}^2$ by five equall balls.*

The construction relies on a decomposition theorem of Kähler manifolds due to Biran [4].

Our third result generalizes theorem 1 to the regular setting. The intersections between the balls of a regular maximal symplectic packing is a union of *Hopf circles* of the boundary spheres which provides a "grid" of a topological 2-fold, symplectic away from its singularities.

Theorem 3. *Given a regular maximal symplectic packing of a symplectic manifold, there exists at least one "supporting surface" : a closed topological surface covered by the balls of the packing and whose intersection with any ball is a union of smooth symplectic discs bounded by Hopf circles. Generically, the balls of the packing intersect exactly along the Hopf circles contained in the supporting surfaces.*

The existence result above can be sharpened when the singularities are simple enough (see definition 5.3 of packings of simple type). For five balls for instance, the intersection pattern must be the same as in the constructed examples (theorem 2).

Theorem 4. *Regular maximal symplectic packings of $\mathbb{P}^2$ by five equal balls which have simple type have exactly one supporting surface, of symplectic area 2π, intersecting each ball through exactly one Hopf disc. Generically these maximal packings thus intersect along exactly one Hopf circle of each of the boundary spheres.*

The paper is organized as follows. In section 2, we discuss previously known examples of maximal symplectic packings and we construct new ones (theorem 2). We hope this section to shed light on the statements of theorem 1, 3 and 4 by providing relevant illustrations. In the third section, we explain the link between non-removable intersections and characteristic foliations in the setting of smooth balls, and prove theorem 1. The purpose of section 4 is to adapt to non-smooth objects the tools we use in the preceding section. We prove theorem 3 in section 5 and conclude by a technical paragraph aimed at smoothening the supporting surfaces in view of proving theorem 4.

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2 Examples of maximal symplectic packings.

The aim of this part is to provide the reader with examples of smooth or regular symplectic packings. We describe them in the light of the results stated in the above introduction.

Karshon's construction. Examples of smooth maximal symplectic packing of $\mathbb{P}^2$ were shown by Karshon [9] or Traynor [15]. They can be described in the following way. The momentum map - or the action-angle coordinates - presents $\mathbb{P}^2$ as a singular bundle over a closed triangle with 2-dimensional tori as generic fibers. The balls forming the packing are fibered by these tori and project by the momentum map to close triangles (see figure 1).

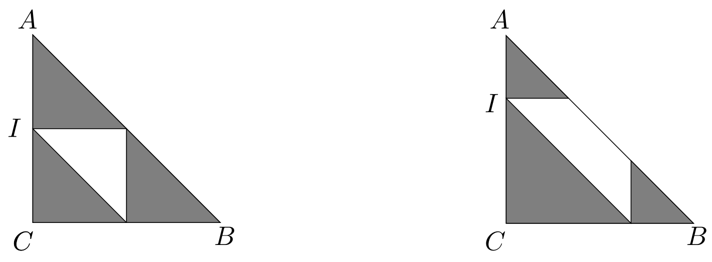

Figure 1: Maximal symplectic packings of $\mathbb{P}^2$ by 3 balls.

Note that the intersection between the closed balls in these examples are precisely one common circle of Hopf fibration of the boundary spheres (point I). Moreover, the fiber over the line AC is a 2-dimensional sphere covered by the union of the two balls B_1, B_2, and whose intersection with each ball is a disc bounded by a Hopf circle. It is thus precisely one of the "supporting surfaces" of the packing. Theorem 1 says that every maximal packings of $\mathbb{P}^2$ by two or three balls always have the same intersection patterns as those examples.

From this example, it is also easy to see that the assumption concerning the equality of the balls in theorem 1.d cannot be dropped. Consider actually a symplectic packing $\{B, B_1\}$ of $\mathbb{P}^2$ by smooth balls of radii $r \ll 1$ and $r_1 = \sqrt{1-r^2}$, centered at the points A and C with the notation of figure 1. Looking at them in $B^4(1)$ rather than $\mathbb{P}^2$, (they do not intersect the $\mathbb{P}^1$ at infinity) we see one big ball B_1' centered at the origin and one thin compact set B' (not a ball any more) very close to the Hopf circle of $\partial B^4(1)$ corresponding to A. There clearly exist approximately $1/r^2$ unitary transformations of $\mathbb{C}^2$ - leaving B_1' invariant - which take B' to disjoint compact sets. After collapsing the Hopf circles of

$\partial B^4(1)$ to get $\mathbb{P}^2$, these compact sets transform to disjoint symplectic balls of the same radius r. Together with B_1, they provide a smooth maximal symplectic packing of $\mathbb{P}^2$ by approximately $1/r^2$ balls.

Biran's decomposition theorem (see [4]). The examples of regular packings to come are based on a decomposition result due to Biran which we describe now briefly. Given an integral Kähler manifold (M, ω, i) (*i.e.* with $\omega \in H^2(M, \mathbb{Z})$), there always exist a complex hypersurface X which is Poincare dual to $k\omega$ for some big enough integer multiplier k. Biran showed that there exists a skeleton $\Delta_X \subset M$ (of empty interior) associated to this hypersurface, whose complement in M is a *standard symplectic bundle over* X. It turns out that some maximal packings appear very clearly in these special coordinates. In order to explain this point, let us discuss briefly the structure of these standard bundles. Their symplectic type is that of the unit disc bundle associated to a Hermitian line bundle of first Chern class $c_1 = [k\tau]$, where $\tau = \omega_{|X}$. The symplectic stucture itself is explicitly given by

$$\omega := \pi^*\tau + d(r^2\alpha)$$

where r is the radial coordinates in the fiber and α is the 1-form whose restriction to the fibers is $\alpha_{|\pi^{-1}(x)} = 1/k\, d\theta$ on $E\backslash\{r = 0\}$ and $d\alpha = -\pi^*\tau$. Notice that although θ is not defined globally on E because of non-vanishing Chern class, the differential form "$d\theta$" is perfectly defined out of the zero section.

The connection between standard symplectic bundles and balls result from the following simple observation. The restriction of these bundles to a symplectic ellipsoïd of X is symplectomorphically equivalent to an ellipsoïd. In the following lemma, $\mathcal{E}_a = \mathcal{E}_{(a_1,\dots,a_n)}$ denotes the standard ellipsoïd of $\mathbb{C}^n$:

$$\mathcal{E}_a := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid \frac{|z_1|^2}{\pi a_1} + \dots + \frac{|z_n|^2}{\pi a_n} < 1 \right\}.$$

Lemma 2.1. *Consider the trivial disc bundle $\pi : \mathcal{E}_a \times \mathbb{D} \longrightarrow \mathcal{E}_a$ over an ellipsoïd of $\mathbb{C}^n$, equipped with a symplectic structure defined by $\omega := \pi^*\omega_{st} + d(r^2\alpha)$, $\alpha_{|\{x\}\times\mathbb{D}} = 1/kd\theta$ and $d\alpha = -\pi^*\omega_{st}$, where (r, θ) are the polar coordinates on $\mathbb{D}$. Then there exists a smooth function $h : \mathcal{E}_a \longrightarrow \mathbb{R}$ such that the map*

$$\begin{array}{rcccl} \Phi & : & (\mathcal{E}_a \times \mathbb{D}, \omega) & \longrightarrow & (\mathcal{E}_{a,\frac{1}{k}}, \omega_{st}) \\ & & (z, w) & \longmapsto & (\sqrt{(1-|w|^2)}z, \frac{1}{\sqrt{k}}e^{i\,h(z)}w) \end{array}$$

is a symplectomorphism.

Proof : Consider the coordinates $(z, w)=(r_1, \theta_1, \dots, r_n, \theta_n, r, \theta)$ on $\mathcal{E}_a \times \mathbb{D}$. The symplectic form ω is given in these coordinates by

$$\omega = \sum_{i=1}^{n} dr_i^2 \wedge d\theta_i + d(r^2\alpha).$$

Taking into account the identities $\alpha_{|\{x\}\times\mathbb{D}} = 1/k\, d\theta$ and $d\alpha = -\sum dr_i^2 \wedge d\theta_i$, we get :

$$\begin{array}{rcl} \omega & = & (1-r^2)\sum_{i=1}^n dr_i^2 \wedge d\theta_i + dr^2 \wedge \alpha \\ & = & \sum_{i=1}^n d[(1-r^2)r_i^2] \wedge d\theta_i + dr^2 \wedge [\alpha + \sum_{i=1}^n r_i^2 d\theta_i] \\ & = & \displaystyle\sum_{i=1}^{n} d[(1-r^2)r_i^2] \wedge d\theta_i + \frac{1}{k}dr^2 \wedge [d\theta + \beta], \end{array}$$

where $\beta := k[\alpha - 1/k\,d\theta + \sum_{i=1}^n r_i^2 d\theta_i]$. The form β is defined on $\mathcal{E}_a \times (\mathbb{D}\backslash\{0\})$ and is closed. Moreover, its action on the fundamental group of $\mathcal{E}_a \times \mathbb{D}\backslash\{0\}$ is trivial because $\beta_{|\{x\}\times\mathbb{D}} = 0$. Hence there is a smooth function $h : \mathcal{E}_a \times \mathbb{D}\backslash\{0\} \longrightarrow \mathbb{R}$ such that $\beta = dh$. Notice now that h does not depend on w because β vanishes on the vertical discs. This function thus extends to $\mathcal{E}_a \times \mathbb{D}$ and depends only on z. We finally get :

$$\begin{aligned} \omega &= \sum_{i=1}^n d[(1-r^2)r_i^2] \wedge d\theta_i + \frac{1}{k} dr^2 \wedge d(\theta + h) \\ &= \Phi^* \left[\sum_{i=1}^n dr_i^2 \wedge d\theta_i + dr^2 \wedge d\theta \right], \end{aligned}$$

where Φ is the announced map. It clearly sends $\mathcal{E}_a \times \mathbb{D}$ to the ellipsoïd $\mathcal{E}_{a,\frac{1}{k}}$. □

Since a complex hypersurface of a Kähler manifold is Kähler, an obvious iteration leads to the following corollary. It seems to hold true also in general compact symplectic manifolds due to Donaldson's results on the existence of symplectic hypersurfaces [5].

Corollary 2.2. *Every Kähler manifold has full packing by one ellipsoïd.*

Maximal full packing of $\mathbb{P}^n$ by k^n balls. We make a digression at this point to explain how to construct a full symplectic packing of $\mathbb{P}^2$ by four *open* balls of radius $1/\sqrt{2}$. The generalization to k^n balls of maximal radius $1/\sqrt[n]{k}$ in $\mathbb{P}^n$ is straightforward. As far as I know, although their existence is well known from McDuff-Polterovich's or Traynor's work [9, 15] *via* McDuff's argument [8], no example of such packings was available up to now.

Consider the quadric $Q := \{z_0^2 + z_1^2 + z_2^2 = 0\}$ in $\mathbb{P}^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$. Then $\pi : \mathbb{P}^2\backslash\mathbb{RP}^2 \longrightarrow Q$ is a standard disc bundle with fibers of area $\pi/2$. Divide Q in four open discs D_i of area $\pi/2$, $i = 1, \ldots, 4$. Then the sets $B_i := \pi^{-1}(D_i) \subset \mathbb{P}^2$ are obviously disjoint. Moreover, the previous lemma shows that they are symplectic balls of radius $1/\sqrt{2}$. Notice also that the boundary singularities of these balls (inavoidable by theorem 1.d) are easily descriptible in terms of the singularities of the discs D_i thanks to the explicit formula for the symplectomorphism Φ of lemma 2.1.

Regular maximal packing of $\mathbb{P}^2$ by five balls (see figure 2). As above consider the quadric $Q := \{z_0^2 + z_1^2 + z_2^2 = 0\}$, and the projection $\pi : \mathbb{P}^2\backslash\mathbb{RP}^2 \longrightarrow Q$ which gives $\mathbb{P}^2\backslash\mathbb{RP}^2$ the structure of a standard disc bundle with fibers of area $\pi/2$. Cover Q by five closed discs of area $2\pi/5$ with finite number of singularities on their boundaries. We claim that we can find the desired balls of the packing inside the saturated sets upon these discs. To see this, we construct a regular symplectic embedding of a ball of radius R inside $\pi^{-1}(D)$, where D is any closed disc in Q of area $\pi R^2 < \pi/2$, with a finite number of singularities $p_1, \ldots, p_k \in \partial D$. First identify $\pi^{-1}(D)$ with $D \times \mathbb{D}$. Then $(\pi^{-1}(D), \omega_{FS})$ is symplectomorphic to a standard ellipsoïd $\mathcal{E} := \mathcal{E}_{R,1/\sqrt{2}}$ *via* a map Ψ which is the composition of a fibered map sending $D \times \mathbb{D}$ to the standard bidisc $\mathbb{D}_R \times \mathbb{D}$ with the map Φ of lemma 2.1. The boundary regularity of the symplectomorphism $\Psi : \pi^{-1}(D) \longrightarrow \mathcal{E}$ can be easily described. First it extends to a homeomorphism between $\overline{D} \times \mathbb{D}$ (*i.e.* $\overline{\pi^{-1}(D)}$ minus the section at infinity) and $\overline{\mathcal{E}}\backslash C_\infty$ where $C_\infty := \{|z_2|^2 = 1/\sqrt{2},\ z_1 = 0\}$. Moreover, this extension is a local diffeomorphism except at the singular points $\cup\{p_i\} \times \mathbb{D}$ of $\pi^{-1}(D)$. Now the ellipsoïd $\overline{\mathcal{E}}$ contains an euclidean closed ball B of radius R whose boundary intersects $\partial\mathcal{E}$ only along the "zero section" $\{z_2 = 0,\ |z_1| = R\}$. The map $\Psi^{-1} : B \longrightarrow \pi^{-1}(D)$ is therefore a regular symplectic embedding of a ball of radius R in $\pi^{-1}(D)$. □

It may be worth noticing that this construction confirms the intuition that the space of maximal symplectic packings of $\mathbb{P}^2$ by four or five balls is not connected (in contrast with the non-maximal situation, see [2, 8]). It is less clear for two or three balls.

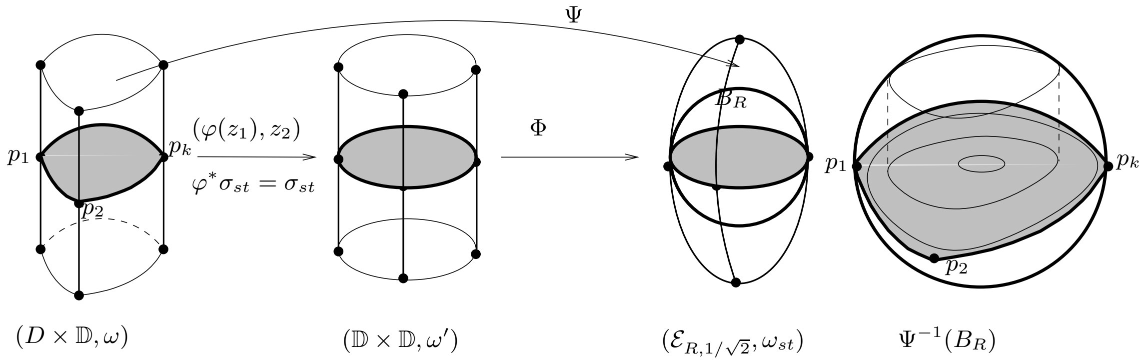

Figure 2: Regular packing of a standard disc bundle over a disc by a ball.

3 Non-removable intersections in smooth maximal packings.

The aim of this section is to introduce the main tool of the paper - namely the link between non-removable intersections in symplectic geometry and characteristic foliations - following [12, 7]. We also prove theorem 1. Dealing only with smooth objects first permits to avoid the technical difficulties arising in the context of regular packings. The ideas should be more transparent to the reader.

Let us explain first how the genericity is achieved in theorem 1. The Hamiltonian transformations of $\mathbb{P}^2$ act trivally on the space of maximal symplectic packings by moving all balls together. More generally, one can also move on this space by considering independant Hamiltonian transformation of each ball of the packing, as long as they preserve the packing property (the interior of the balls do not intersect). Precisely, we define a *Hamiltonian perturbation of a maximal packing* $\{\varphi_i\}_{i=1,\dots,k}$ as a family $\{\varphi_i^t\} := \{\Phi^t_{X_{H_i}} \circ \varphi_i\}$, where the time-dependent Hamiltonian functions H_i on M are such that the open balls $\varphi_i^t(B(r_i))$ remain disjoint sets for all $t \in [0, \varepsilon]$. Note that, unlike global Hamiltonian transformations, these Hamiltonian perturbations of packings enable to break intersections. The properties stated in theorem 1 are generic precisely because they are always true after a possible Hamiltonian perturbation of the packing.

3.1 Digging in a ball by a Hamiltonian : Sullivan's lemma.

As explained in [12] and [7], the characteristic foliation plays a central role in the phenomena of "symplectic non-removable intersection". Loosely speaking, the reason for a compact set inside the smooth boundary of an open set U not to be displacable inside $\overline{U}$ by a Hamiltonian vector field is that it contains a closed invariant set of the characteristic foliation of ∂U.

Definition 3.1. *The characteristic distribution $\mathcal{L}$ of a hypersurface S in a symplectic manifold (M, ω) is the kernel of the restriction of ω to S (i.e. $\forall x \in S$, $\mathcal{L}_x := \ker \omega_{x|T_xS}$). The characteristic foliation of S is the integral foliation associated to this one-dimensional characteristic distribution.*

The characteristic foliation is obviously preserved by any map $\varphi : S \subset (M, \omega) \longrightarrow S' \subset (M', \omega')$ with $\varphi^*\omega'_{|TS'} = \omega_{|TS}$.

Example : On the euclidean sphere $S(r) \subset \mathbb{C}^n$, the characteristic distribution is given by $\mathcal{L}_x = \mathrm{Span}_{\mathbb{R}}(i\vec{N}(x))$ where $\vec{N}(x) := x/\|x\|$. The characteristic foliation is thus the classical foliation of $S(r)$ by Hopf circles. We call *smooth (or regular) symplectic closed ball* of (M,ω) any symplectic smooth (or regular) embedding of a euclidean closed ball of $\mathbb{C}^n$ in M. If B is a symplectic closed ball in M corresponding to a regular embedding $\varphi : \overline{B(r)} \longrightarrow M$, the characteristic leaves of $\partial B = \varphi(S(r))$ are the images by φ of the Hopf circles of $S(r)$ and will be called the Hopf circles of B. Finally, the Hopf discs of a symplectic ball B are the images of the intersections of $B(r)$ with the complex lines of $\mathbb{C}^n$.

The vector field $i\vec{N}(x)$ defines and orients the characteristic foliation of the euclidean sphere S in $\mathbb{C}^n$. The Hamiltonian vector field $\vec{X}_H$ associated to a smooth function $H : \mathbb{C}^n \longrightarrow \mathbb{R}$ points inside B at a point x of S if and only if $i\vec{N} \cdot H(x) < 0$. Actually,

$$\vec{X}_H \cdot \vec{N}(x) < 0 \Longleftrightarrow \omega(\vec{X}_H(x), i\vec{N}(x)) < 0 \Longleftrightarrow dH_x(i\vec{N}(x)) < 0.$$

In other terms, the Hamiltonian vector field $\vec{X}_H$ "digs" in B at a point x of S if H *is decreasing along the characteristic foliation at this point.* The following lemma explains along which compact sets of S one can dig in B in a Hamiltonian way. It is a particular case of a more general result due to Sullivan [14] (see also [7]) :

Lemma 3.2. *Let K be a compact set of $S^{2n-1} \subset \mathbb{C}^n$ and $\mathcal{C}$ the set of Hopf circles belonging to K. There exists a smooth function $H : S^{2n-1} \longrightarrow \mathbb{R}$ such that $i\vec{N} \cdot H < 0$ on $K \backslash \mathcal{C}$ and $dH = 0$ on $\mathcal{C}$.*

<u>Proof :</u> We first reduce Lemma 3.2 to finding convenient functions on solid tori by an argument of partition of unity. Denote $\pi : S^{2n-1} \longrightarrow \mathbb{P}^{n-1}$ the Hopf projection. Given an open cover by balls $\mathcal{U} := \{U_\alpha\}$ of $\pi(K\backslash\mathcal{C})$, choose an open set U which completes $\mathcal{U}$ to a cover of $\mathbb{P}^{n-1}$. Consider a smooth partition of unity $\{\Phi_\alpha, \Phi\}$ associated to (U_α, U). Also choose a smooth non-negative function θ on $\mathbb{P}^{n-1}$ which vanishes on $\pi(\mathcal{C})$ at order two and is positive out of $\pi(\mathcal{C})$. We claim that if we can find functions $h_\alpha : \pi^{-1}(U_\alpha) \longrightarrow \mathbb{R}$ such that $i\vec{N} \cdot h_\alpha < 0$ on $K\backslash\mathcal{C} \cap \pi^{-1}(U_\alpha)$ then the function $H = \theta \circ \pi \sum \Phi_\alpha \circ \pi \cdot h_\alpha$ is of the required type. Actually,

$$\begin{aligned} i\vec{N} \cdot H &= \theta \circ \pi \sum \Phi_\alpha \circ \pi \, i\vec{N} \cdot h_\alpha < 0 \text{ on } K\backslash\mathcal{C} && (1) \\ \text{and } dH &= \left(\sum \Phi_\alpha \circ \pi . h_\alpha\right) d(\theta \circ \pi) + \theta \circ \pi d \left(\sum \Phi_\alpha \circ \pi . h_\alpha\right) = 0 \text{ on } \mathcal{C}. && (2) \end{aligned}$$

The first equality holds because $\theta \circ \pi$ and $\Phi_\alpha \circ \pi$ are constant along $i\vec{N}$. The second because both $\theta \circ \pi$ and $d\theta \circ \pi$ vanish on $\mathcal{C}$.

Thus it only remains to build the cover of $\pi(K\backslash\mathcal{C}) \subset \mathbb{P}^{n-1}$ by balls and the suitable functions on the solid tori which projects to these balls by π. In the neighbourhood of $x \in \pi(K\backslash\mathcal{C})$, the compacity of K allows to find a local section $s : B_{\delta(x)} \longrightarrow S^{2n-1}$ of π and constants $\varepsilon(x) > 0$ such that the intervals

$$I(\tilde{x}) := \left\{\Phi^t_{i\vec{N}}(s(\tilde{x})) \ , \ t \in]-\varepsilon(x), \varepsilon(x)[\right\}, \qquad \tilde{x} \in B_{\delta(x)}$$

contain no points of K. These balls $B_{\delta(x)}$ provide a cover of $\pi(K\backslash\mathcal{C})$, for which we construct now the functions h_x. For this, first identify $\pi^{-1}(B_{\delta(x)})$ with $B_{\delta(x)} \times S^1$ by the map $\varphi : B_{\delta(x)} \times S^1 \longrightarrow \pi^{-1}(B_{\delta(x)})$ defined by $\varphi(\tilde{x}, t) := \Phi^{t+\varepsilon(x)}_{i\vec{N}}(s(\tilde{x}))$. By φ, $I_{\tilde{x}}$ is taken to $]-2\varepsilon(x), 0[$ and $i\vec{N}$ to $\partial/\partial t$. In these coordinates, we can define the function h_x on $B_{\delta(x)} \times S^1$ by :

$$\begin{cases} h_x(\tilde{x}, t) = -t \quad \text{for } t \in [0, 1-2\varepsilon(x)], \\ h_x \text{ is smooth on } B_{\delta(x)} \times S^1. \end{cases}$$

Such a function can obviously be defined, and it fits with the requirement $i\vec{N} \cdot h_\alpha < 0$ on $K\backslash\mathcal{C} \cap \pi^{-1}(U_\alpha)$. □

Corollary 3.3 (Perturbation procedure for smooth balls). *Let M be a symplectic manifold, $B \subset M$ a smooth symplectic closed ball and U an open set of M. Assume that $U \cap B = \emptyset$, denote $K := \partial U \cap B$ and $\mathcal{C}$ the Hopf circles of ∂B contained in K. Then there is a Hamiltonian function H on M such that $\Phi^{\varepsilon}_{X_H}(B) \cap \overline{U} = \Phi^{\varepsilon}_{X_H}(B) \cap \partial U = \mathcal{C}$ for any small positive real number ε.*

Proof : It is well-known that there exists a symplectomorphism Φ between a neighbourhood $N(\partial B)$ of ∂B in M and a neighbourhood $N(S)$ of $S(r)$ in $\mathbb{C}^n$. This map sends K to $K' = \Phi(K)$, $\mathcal{C}$ to $\mathcal{C}' = \Phi(\mathcal{C})$ where $\mathcal{C}'$ is exactly the set of Hopf circles of $S(r)$ contained in K'. Consider a function h on $S(r)$ associated to $(K', \mathcal{C}')$ as in lemma 3.2. Extend it to a smooth function of $\mathbb{C}^n$ with compact support in $N(S)$. The function $H := h \circ \Phi$, *a priori* defined on $N(\partial B)$, can be extended to M by setting $H = 0$ outside $N(\partial B)$. As explained above, the vector field X_H points inside B on $K \backslash \mathcal{C}$ and vanishes on $\mathcal{C}$. $\square$

This corollary is unfortunately only useful when dealing with smooth balls. In our context, B will usually be one of the (only regular) balls B_i and U the union of the other balls $\cup_{j \neq i} B_j$. We will prove the counterpart of this perturbation procedure for regular symplectic closed balls in next section (see proposition 4.3).

Remark : a $\mathcal{C}^0$-definition of the closed characteristics. Since a function cannot grow along an entire closed curve, no Hamiltonian isotopy can take a whole Hopf circle of the boundary of a ball inside this ball for all positive time. On the other hand, given two points x, y in the same Hopf circle of $\partial B(1)$, it is not hard to construct a Hamiltonian isotopy which does not move $B(1)$ except on ε-neighbourhoods of x and y, takes x inside and y outside $B(1)$. These observations generalize obviously to any closed characteristic in the boundary of any smooth domain. They therefore suggest an alternative definition of closed characteristic which does not refer to the characteristic distribution. Given an open set U with smooth boundary and a point $x \in \partial U$, define

$$C_x := \cap_{\varepsilon > 0} C_{x,\varepsilon}, \text{ where}$$
$$C_{x,\varepsilon} := \left\{ y \in \partial U \,|\, \exists H \in \mathcal{C}^{\infty}(M, \mathbb{R}),\ \forall t > 0,\ \left\{ \begin{array}{l} \Phi^t_{X_H}(x) \in U, \\ \text{and } \Phi^t_{X_H}(\overline{U} \backslash B_{\varepsilon}(y)) \subset \overline{U} \end{array} \right. \right\}.$$

It seems that C_x is the closure of the characteristic leaf passing through x in ∂U. For instance it is a full Lagrangian torus when x is a generic point of the boundary of a generic complex ellipsoid in $\mathbb{C}^n$. Anyway, the above discussion guarantees that it coincides with the characteristic passing through x when this characteristic is closed. It would be interesting to know wether this definition gives a symplectic invariant when ∂U is not smooth any more. For instance, do the C_x and the Hopf circles coincide for a sufficiently regular but non-smooth symplectic ball? Equivalently, one can ask if this definition gives non-trivial subsets of ∂U when continuous Hamiltonians rather than smooth ones are considered (see [11] for a definition of continuous Hamiltonian isotopies).

3.2 Symplectic spheres in maximal packings.

We show here that any Hopf circle in the intersection of two symplectic balls along their boundaries gives rise to its own supporting sphere.

Lemma 3.4. *Let B_1, B_2 be regular symplectic closed balls of a symplectic manifold (M, ω) with disjoint interior, of radii r_1, r_2 and centers O_1, O_2. Through any common Hopf circle C of ∂B_1 and ∂B_2 there is a topological 2-sphere S_C of M passing through O_1, O_2, smooth except along C and with symplectic area $\pi(r_1^2 + r_2^2)$. In dimension four, any two such spheres intersect precisely in O_1 and O_2 with intersection number 1 at each point.*

Proof : Let $\varphi_i : \overline{B(r_i)} \longrightarrow B_i$ the corresponding symplectic embeddings and $C_i := \varphi_i^{-1}(C)$. Then C_i is a Hopf circle of $\partial B(r_i)$ and bounds a holomorphic disc $\mathbb{D}_{C_i} \subset B(r_i)$. Its image by φ_i is a symplectic disc D_i in B_i of area πr_i^2 passing through O_i and bounded by C (a Hopf disc). When gluing D_1 with D_2 along their common boundary, we get a topological 2-sphere S_C which is smooth except on C and has area $\mathcal{A}(S_C) = \pi r_1^2 + \pi r_2^2$. By construction, two such spheres S_C, $S_{C'}$ intersect only at O_1, O_2 and in dimension 4, their intersection numbers at O_i is 1 because they are the same as the intersection numbers of $\mathbb{D}_{C_i}$, $\mathbb{D}_{C'_i}$ in $B^4(r_i)$. □

3.3 Smooth maximal packings : proof of theorem 1.

Proof of theorem 1.a. Fix a smooth maximal symplectic packing of $\mathbb{P}^2$ by two balls $\{B_1, B_2\}$ of radii r_1, r_2 checking the maximality condition $r_1^2 + r_2^2 = 1$. Denote by $K := \partial B_1 \cap \partial B_2$ and $\mathcal{C}$ the set of Hopf circles of ∂B_1 contained in K. Note that such a circle is also a Hopf circle of ∂B_2 because ∂B_1 and ∂B_2 are tangent along K. The perturbation procedure 3.3 shows that after a possible Hamiltonian perturbation of the packing, we can assume that $K = \mathcal{C}$.

Gromov's work shows that the balls of a smooth maximal packing of $\mathbb{P}^2$ by two balls cannot be disjoint (we also prove it for proposition 4.1 in the more general setting of regular mapping). It implies of course that no Hamiltonian perturbation of our packing can lead to disjoint balls. In view of the perturbation procedure 3.3, K must contain at least one Hopf circle of ∂B_1, so $\mathcal{C}$ is not empty. To prove that $\mathcal{C}$ is exactly one Hopf circle, we argue by contradiction and assume that $\mathcal{C}$ contains two circles C and C'. The topological 2-spheres S_C and $S_{C'}$ given by Lemma 3.4 intersect precisely at O_1, O_2 with intersection numbers one at each point. On the other hand, they both have symplectic area $\pi(r_1^2 + r_2^2) = \pi$ so they are in the homology class of a line L in $\mathbb{P}^2$. We thus have $2 = S_{C_1} \cdot S_{C_2} = L \cdot L = 1$ which is a contradiction. □

Proof of theorem 1.b,c. Let B_1, B_2, B_3 be the closed balls of our smooth maximal packing. We can assume that their radii check $r_1 \geq r_2 \geq r_3$. Then the maximality condition is :

$$r_1^2 + r_2^2 = r_1^2 + r_3^2 = 1.$$

Define as above $K_{ij} := B_i \cap B_j = \partial B_i \cap \partial B_j$ and the sets $\mathcal{C}_{ij}$ consisting of the Hopf circles of K_{ij}. The first point is to prove that we can get rid of $K_{ij} \backslash \mathcal{C}_{ij}$ by a Hamiltonian perturbation. Unlike the two-balls situation, it is not obvious at first glance because a Hopf circle of ∂B_1 could *a priori* be covered by $\partial B_2 \cup \partial B_3$ without being in any K_{ij}. But notice that the K_{ij} are pairwise disjoint compact sets (and remain so after perturbation) because the intersection of two of the K_{ij} is precisely $B_1 \cap B_2 \cap B_3$. This intersection must be empty because any point of it would be a singular point of at least one of the balls. By connexity, we deduce that $K_{12} \cup K_{13}$ contains no Hopf circles away from those in $\mathcal{C}_{12}$ and $\mathcal{C}_{13}$. After a Hamiltonian perturbation of B_1 according to corollary 3.3, we can assume that $K_{i1} = \mathcal{C}_{i1}$. After a second Hamiltonian perturbation, this time of B_2, we can also assume that $K_{23} = \mathcal{C}_{23}$.

Proof of theorem 1.b ($r_1 = r_2 = r_3 = 1/\sqrt{2}$). In this case, any two balls of B_1, B_2, B_3 form a maximal symplectic packing of $\mathbb{P}^2$ by two balls. As such, and in view of theorem 1.a, $K_{ij} = \mathcal{C}_{ij}$ contains exactly one circle.

Proof of theorem 1.c ($r_1 > r_2 = r_3$). In this case, (B_1, B_2) and (B_1, B_3) form maximal symplectic packings of $\mathbb{P}^2$ by two balls. As behind we conclude that $\mathcal{C}_{12}$ and $\mathcal{C}_{13}$ contain exactly one circle. Finally, if $\mathcal{C}_{23}$ were to contain a circle, Lemma 3.4 would associate to it a toplogical 2-sphere of symplectic area $\pi(r_2^2+r_3^2)\in]0,\pi[$. It is clearly impossible, so $\mathcal{C}_{23}$ is empty. $\square$

Remark 3.5. *This last argument shows that any two closed symplectic balls of radii r_1, r_2 in $\mathbb{P}^2$ with disjoint interiors can intersect along a full Hopf circle of the boundary of one of them (therefore of both) only if $r_1^2+r_2^2\in\mathbb{N}$.*

Proof of theorem 1.d Suppose by contradiction that n closed balls $B_1,\dots,B_n$ of the same radius r constitute a maximal packing of $\mathbb{P}^2$ ($n\geq 4$).

We know from [9] and [3] that for $n=4$ and $n\geq 9$, such a packing should fill the space. Then any point $p\in\overline{\partial B_1\cap\partial B_i}\backslash\mathrm{Int}_{\partial B_1}\partial B_1\cap\partial B_i$ of the boundary of $\partial B_1\cap\partial B_i$ in ∂B_1 belongs to a third ball B_j. So our packing by $B_1,\cdots,B_n$ is not smooth as already noticed in last paragraph. Moreover, the radius of the balls is less than $\sqrt{2/5}<\sqrt{1/2}$ for $n=5,6,7,8$. As before, we are in a non-removable intersection situation : the ball B_1 cannot be disjointed from the union of the other ones $\cup_{j\neq 1}B_j$ (see proposition 4.1). In view of corollary 3.3, it means that there is a Hopf circle C of ∂B_1 which is also covered by $\cup_{j\neq 1}\partial B_j$. Assume to fix the notation that $C\cap\partial B_2\neq\emptyset$. Since $r^2<1/2$, remark 3.5 shows that C cannot be entirely contained in ∂B_2. So there is a point p in the boundary of ∂B_2 in C. Such a point also belongs to another ball, so it is an intersection point of $\partial B_1,\partial B_2$ and ∂B_i for $i\neq 1,2$. Our packing cannot be smooth. $\square$

3.4 A digression on Gromov's two-packings theorem.

We end this section with a remark. The previous analysis on non-removable intersection provides an interesting interpretation of Gromov's theorem on two-packings. Consider a packing of $\mathbb{P}^2$ by two smooth balls $\{B_1,B_2\}$ of radii r_1,r_2. Assume we know the following assertion to hold true :

$$\text{Every two-packings of } \mathbb{P}^2 \text{ extend to smooth maximal packings.} \qquad (*)$$

Then from the previous analysis, we conclude that the maximal extensions $\{B_1',B_2'\}$ of $\{B_1,B_2\}$ intersect along a common Hopf circle of their boundaries, so their radii check ${r_1'}^2+{r_2'}^2\in\mathbb{N}$ by remark 3.5. Taking into consideration the volume restriction ${r_1'}^4+{r_2'}^4\leq 1$, we immediately get that ${r_1'}^2+{r_2'}^2=1$, so $r_1^2+r_2^2\leq 1$. In this approach, the whole difficulty of the two-packing theorem thus lies on the assertion $(*)$. This assertion is *a posteriori* true, when knowing precisely the two-packings theorem, Karshon's examples and McDuff's extension argument.

4 Regularizations and extensions of regular embeddings.

This purely technical section is aimed at extending the perturbation procedure described in corollary 3.3 to non-smooth balls. We first explain how to extend slightly a symplectic open ball in an open manifold M. It should be noticed that the following proposition applies to regular embeddings of a ball.

Proposition 4.1. *Let $\varphi : B(r)\hookrightarrow M$ be a symplectic embedding of an euclidean ball. Assume φ extends smoothly to an embedding of $\overline{B(r)}\backslash K$ into M where K is a closed set of $\partial B(r)$. If K contains no Hopf circle of $\partial B(r)$ then for any open neighbourhood V of $\overline{\mathrm{Im}\,\varphi}$, there is a symplectic embedding $\widetilde{\varphi}_\varepsilon : B(r+\varepsilon)\hookrightarrow V$ for ε small enough.*

If $K=\emptyset$ then $\widetilde{\varphi}_\varepsilon$ can be choosen to be an extension of φ i.e. $\widetilde{\varphi}_{\varepsilon|B(r)}=\varphi$.

The non-squeezing theorem shows that this proposition is sharp in the sense that K actually has to be assumed not to contain any Hopf circle.

Proof : When the singular locus is empty ($K = \emptyset$), this proposition is very classical. We show however its brief proof so that it serves as a basis for the non-smooth case. In this situation, the image $\varphi(\partial B(r))$ is a smooth hypersurface of M. Denoting by $S := \partial B(r)$, the regular neighbourhood theorem gives a symplectomorphism $\psi : N_\varepsilon(S) \longrightarrow N(\varphi(S))$ from an ε-neighbourhood of the standard sphere of radius r in $\mathbb{C}^n$ to a neighbourhood of $\varphi(S)$ in M (see [16] or [10], p.101). Morevover, this map can be chosen in such a way that $\Psi_{|S} = \varphi_{|S}$ and $\psi'(x)\vec{N}(x) = \varphi'(x)\vec{N}(x)$ for any point $x \in S$ ($\vec{N}(x)$ denotes again $x/\|x\|$). The map $\varphi_\varepsilon : B(r+\varepsilon) \longrightarrow M$ defined by

$$\varphi_\varepsilon(x) := \begin{cases} \varphi(x) \text{ for } x \in B \\ \Psi(x) \text{ for } x \in N_\varepsilon(S)\backslash B \end{cases}$$

is $\mathcal{C}^1$-smooth, and hence a symplectomorphism from the ball of radius $r+\varepsilon$ inside M.

For singular hypersurfaces, the standard neighbourhood theorem is not valid anymore, hence the above proof has no straightforward generalization. However it can be easily adapted by using a regularization trick explained in Lemma 4.2 below. It states that under the extension condition on φ of proposition 4.1, there exists a "regularization" $\widetilde{\varphi} : \overline{B(r)} \hookrightarrow M$ which is a symplectic embedding of the *closed* ball inside the prescribed neighbourhood V of $\varphi(\overline{B(r)}\backslash K)$. Now the extension $\widetilde{\varphi}_\varepsilon$ of $\widetilde{\varphi}$ constructed in the case $K = \emptyset$ gives the desired map. □

The result we have used in this proof is a particular case of the following lemma for $\mathcal{C} = \emptyset$. This extended version will nevertheless be usefull in order to generalize corollary 3.3.

Lemma 4.2. *Let $\varphi : \overline{B(r)}\backslash K \hookrightarrow M$ be a symplectic embedding of a closed ball minus a singular compact set $K \subset \partial B(r)$ inside an open manifold M. Denote by $\mathcal{C}$ the set of Hopf circles of $\partial B(r)$ contained in K. There exists a symplectic embedding $\widetilde{\varphi}$ of $\overline{B(r)}\backslash \mathcal{C}$ inside any prescribed neighbourhood V of $Im\,\varphi = \varphi(\overline{B(r)}\backslash K)$.*

Proof : Consider an open set $U \Subset \partial B(r)\backslash K$ which contains at least one point of each Hopf circle of $\partial B(r)\backslash \mathcal{C}$. As in the previous proof, φ can be extended as a symplectic embedding to a shell $U_\varepsilon := \{[x, (1+\varepsilon(x))x[,\ x \in U\}$, where ε is a small positive function on U. Provided ε is sufficiently small, φ actually sends $B'_\varepsilon := B(r)\backslash K \cup U_\varepsilon$ into V. Lemma 3.2 shows that there exist a smooth function $h : \partial B(r) \longrightarrow \mathbb{R}$ such that $i\vec{N} \cdot h < 0$ on $\partial B(r)\backslash(U \cup \mathcal{C})$ with $dh = 0$ on $\mathcal{C}$. When extending this function to $\mathbb{C}^n$, we get a Hamiltonian function whose flow has the property that for t small enough, the set $\Phi^t_{\vec{X}_h}(\overline{B(r)}\backslash \mathcal{C})$ is contained inside B'_ε. The map

$$\widetilde{\varphi} := \varphi \circ \Phi^t_{\vec{X}_h} : \overline{B(r)}\backslash \mathcal{C} \hookrightarrow V$$

gives the desired symplectic embedding of $\overline{B(r)}\backslash \mathcal{C}$ inside V. □

Before stating the anounced generalization of the perturbation procedure 3.3 to regular balls, we need to broaden slightly the notion of Hamiltonian perturbation. We say that a path $\{\varphi_t, B_t\}$ of regular symplectic balls in M is a *regular Hamiltonian deformation* if there exist smooth functions H_t defined in the interior of B_t such that $\varphi_t = \Phi^t_{X_{H_t}} \circ \varphi$. The point is that we do not impose to these functions to be defined in all of M, so they may have singularities on the boundary. We however demand the whole path to be made of regular symplectic balls. Now, a regular Hamiltonian perturbation of a packing is a regular Hamiltonian perturbation of each of its balls such that the configuration of the balls at any time t remains a packing. To illustrate this definition, consider the special case of lemma

4.2 when M is an open manifold with boundary ∂M and $\varphi : (B(r), K) \hookrightarrow (M, \partial M)$ is a regular symplectic ball with a part K of its boundary sent in ∂M. Then the map $\widetilde{\varphi}$ is in fact a regular Hamiltonian deformation of φ associated to the function $h \circ \varphi^{-1}$. Moreover, $\widetilde{\varphi}_{|\mathcal{C}} = \varphi_{|\mathcal{C}}$. This obvious remark is exactly the content of next proposition.

Proposition 4.3 (Perturbation procedure for regular balls). *Let U be an open set of a symplectic manifold M and B a regular symplectic ball in $M \backslash U$. Denote $K := \partial B \backslash \partial U$ and $\mathcal{C}$ the set of Hopf circles of ∂B contained in K. Then there is a regular Hamiltonian perturbation $(B_t)_{t<\varepsilon}$ of B with $B_t \cap \partial U = \mathcal{C}$ for any positive t.*

As a corollary of proposition 4.1, we get a non-removable intersection property for regular maximal symplectic packings. No ball of such a packing is completely disjoint from the other balls.

Corollary 4.4 (Non-removable intersection). *Let $(B_i)_{i=1\ldots k}$ be a regular maximal symplectic packing of M. Then*

$$\forall i = 1 \ldots k, \qquad B_i \cap (\cup_{j \neq i} B_j) \neq \emptyset.$$

This intersection even contains at least one Hopf circle of ∂B_i.

Proof : Call φ_i the symplectic embeddings of $B(r_i)$ in M corresponding to B_i. Suppose by contradiction that $K := B_i \cap (\cup_{j \neq i} B_j)$ contains no Hopf circle of ∂B_i. The map φ_i is a regular symplectic embedding of $\overline{B(r_i)} \backslash K$ inside the open symplectic manifold $M' = M \backslash \cup_{j \neq i} B_j$. By proposition 4.1, φ_i can be "extended" to a symplectic embedding $\widetilde{\varphi}_i : B(r_i + \varepsilon) \hookrightarrow M'$. The maps $\varphi_1, \ldots, \widetilde{\varphi}_i, \ldots, \varphi_k$ thus provide a symplectic packing of M by k balls of radii $(r_1, \ldots, r_i + \varepsilon, \ldots, r_k)$. It is clearly in contradiction with the definition of a maximal symplectic packing. □

5 Non-smooth symplectic packings.

We turn to the problem of identifying non-smooth (but regular) maximal symplectic packings. We first prove the existence of supporting surfaces of such packings (theorem 3). We then remark that passing from theorem 3 to the precise statement of theorem 4 is a matter of being able to perturb the topological supporting surface to a smooth symplectic surface (lemma 5.1). We finally show that the smoothening is possible under the hypothesis of theorem 4, thus proving it.

5.1 Existence of a supporting surface : proof of theorem 3.

The idea of the proof is very simple. The non-removable intersection property of the packing implies the existence of a Hopf circle C_1 of ∂B_1 contained in the union of the other balls. Let us denote $\mathcal{C}_{1i}$ the set of circles of B_i intersecting C_1 along an open set ($\mathcal{C}_{1i}$ may be empty or contain several circles). Denote S_0 the Hopf disc in B_1 bounded by C_1 and S_1 the surface obtained by gluing the Hopf discs corresponding to $\mathcal{C}_{1i}$ to S_0. Obviously no point of C_1 is a boundary point of S_1. Now there are two possibilities. Either the $\mathcal{C}_{1i}$ are covered by the balls B_j for $j \neq i$ or not. In the first case, we can glue to S_1 the Hopf discs corresponding to the circles of $\mathcal{C}_{1ij}$ ($\mathcal{C}_{1ij}$ is the set of circles of ∂B_j intersecting a circle of $\mathcal{C}_{1i}$ in an open set). We obtain a surface S_2 with boundary points neither in C_1 nor in $\mathcal{C}_{1i}$, which we can use in order to iterate the construction. In the latter, we can get rid of one of the circles of $\mathcal{C}_{1i}$ from the intersections between the balls by a small Hamiltonian perturbation. The effect of this transformation is that C_1 is not covered by the other balls any more. It actually means that C_1 was not relevant for our purpose, but there clearly exists another circle of ∂B_1 to which we can apply the previous procedure. The reason for which this iteration process stops and produces a closed surface is the finiteness condition on the singularities. In order to make rigorous and clear the preceding iterative process, we encode the situation into a graph.

Let $B_1, \ldots, B_k$ be the balls of a regular maximal symplectic packing of M. We already know how to associate a supporting topological sphere to any common Hopf circle of two balls of the packing (see lemma 3.4). We thus consider in the following only those Hopf circles not concerned by this basic construction. Define

$$\begin{array}{l} \mathcal{S}_i := \{x \in \partial B_i \mid \exists j \neq i\,,\ C_x \subset \partial B_j\}, \\ \mathcal{C}_i := \{x \in \partial B_i \mid C_x \subset \bigcup_{j \neq i} \partial B_j\ ,\ C_x \not\subset \partial B_j\ \ \forall j \neq i\}, \end{array}$$

where C_x denotes the possibly singular Hopf circle of ∂B_i passing through x. Clearly, if one of the $\mathcal{C}_i$ is empty, then corollaries 4.4 and 3.4 show that there must be a supporting 2-sphere of the packing passing through B_i. As before, our aim is to explain to what $\mathcal{C}_i$ can be reduced after Hamiltonian perturbation of the packing. Notice that by definition, each circle of $\mathcal{C}_i$ contains at least one triple intersection point between the balls of the packing, and there are only finitely many such points because of the regularity condition on the packing. Since any two Hopf circles of a given ball are disjoint, each $\mathcal{C}_i$ thus contains only a finite number of Hopf circle of ∂B_i. Consider henceforth the finite graph G whose vertices are the Hopf circles contained in one of the $\mathcal{C}_i$, and the edges are the pairs of such circles which share an open arc. Also colour the vertices black when they represent a circle $C \in \mathcal{C}_i$ which is also contained in $\cup_{j \neq i} \mathcal{C}_j$ and red otherwise.

A red vertex is a Hopf circle C of ∂B_i which is covered by the other balls but not by the union of the $\mathcal{C}_j$ for $j \neq i$. Since each $\mathcal{C}_j$ is compact, there must be an open arc $I \subset C$ which is a piece of a Hopf circle of ∂B_j, not covered itself by the other balls of the packing. The perturbation procedure then allows to produce a new packing $(\widetilde{B}_1, \ldots, \widetilde{B}_k)$ very close to the original one, with intersection between the balls unchanged except that $\widetilde{B}_i \cap \widetilde{B}_j = B_i \cap B_j \backslash I$. The graph $\widetilde{G}$ associated to the perturbed maximal packing is thus a subgraph of G obtained by erasing the vertex C together with all its adjacent edges, and turning all its neighbours in G to red. In particular, $\widetilde{G}$ has one vertex less than G. This process can be iterated as long as there is a red vertex in the graph. Because the initial graph is finite, there must be a stabilization after a finite number of steps. We conclude that some Hamiltonian perturbation of the packing leads to a graph which is only black. We will now suppose that G itself has only black vertices. Applying the perturbation procedure once more, we can arrange so that

$$B_i \cap \cup_{j \neq i} B_j = \mathcal{C}_i \cup \mathcal{S}_i.$$

We claim that each connected component $\hat{G}$ of G corresponds to a supporting surface. Actually, let S be the union of the Hopf discs corresponding to the Hopf circles of $\hat{G}$. It is obviously a connected space, covered by the closed balls of the packing. We need however a brief discussion of the regularity of this space before we can assure it is actually a topological surface. Inside the balls first, S is an immersed symplectic surface, whose only self-intersection points are positive and located at the origin of the balls. Consider then a point x of $S \cap \partial B_i$ which is not a singular point of any ball. In particular it is not a triple intersection point of the packing, so S cannot be made of more than two discs in a neighbourhood. Moreover x belongs to a Hopf circle of $\mathcal{C}_i$. The fact that this Hopf circle is black coloured implies precisely that x is in the boundary of at least two Hopf discs. It follows that S is locally made of exactly two smooth Hopf discs glued along a interval around x. It is easily checked that S is even locally diffeomorphic to a cylinder $\{y=|x|\}\times\mathbb{R}$ at these points. We thus conclude that S is as announced a closed topological surface with finitely many possible singular points located at the singular points of the packing. $\square$

5.2 Refinement of theorem 3 when the supporting surfaces are smooth.

In the concrete case of five balls in $\mathbb{P}^2$, we explain now how to sharpen theorem 3 and get theorem 4. We consider for the remaining of this paragraph a regular maximal symplectic packing of $\mathbb{P}^2$ by five equal balls $\varphi_1,\dots,\varphi_5$ (or $B_1,\dots,B_5$) - of radii $\sqrt{2/5}$ - and a supporting surface S of the packing. Then S is made of several Hopf discs, each of which is of area $2/5\pi$ because its boundary is a Hopf circle of the boundary of a ball. Since the Fubini-Study form is integral and S is a closed surface, the number of discs is a multiple $5k$ of five, the symplectic area of S is $2k\pi$, and its homology class is $2k[L]$ (L is a line in $\mathbb{P}^2$).

Lemma 5.1. *Assume there is a smooth symplectic immersion $\mathcal{C}^0$-close to S, whose only self-intersections are at the origins of the balls. Then S is of area 2π, made of one Hopf disc in each ball of the packing, and it is the unique supporting surface of the packing.*

Proof : Denote by k_i the number of Hopf discs of $S\cap B_i$, so that $k_1+k_2+\cdots+k_5=5k$. The self-intersection of S near the origin O_i of B_i (namely the number of double-points of generic perturbations of S near O_i) is then given by the formula :

$$\delta_i=\frac{k_i(k_i-1)}{2}.$$

The assumption on the smoothening of S means that one can find a symplectically immersed surface $\widetilde{S}$ homologous to S (hence in the homology class $2k[L]$) with positive self-intersections. They are located in small neighbourhoods of the O_i and are the same as those of $S\cap B_i$. The total self-intersection number of $\widetilde{S}$ is thus

$$\delta:=\sum_{i=1}^{5}\delta_i=\sum_{i=1}^{5}\frac{k_i(k_i-1)}{2}.$$

Taking into account that $\sum k_i=5k$, we easily get that

$$\delta\geq\frac{5k(k-1)}{2}. \tag{3}$$

The positivity of the self-intersections together with the fact that $\widetilde{S}$ is symplectic imply that $\widetilde{S}$ is actually a J-holomorphic curve for an almost complex structure on $\mathbb{P}^2$ (see [1]). It must therefore verify the adjunction inequality, which gives in our present situation :

$$\delta\leq\frac{(2k-1)(2k-2)}{2}=(2k-1)(k-1). \tag{4}$$

It follows from (3) and (4) that $k=1$, and $\delta=0$. We thus conclude that S was made of five Hopf discs, one in each ball. Finally, we argue by contradiction to prove that S is the only supporting surface of the packing. Assume that there is a supporting surface S' distinct of S, made of k_i' Hopf discs in each balls, and of total symplectic area $2k'\pi$ (so that $\sum k_i = 5k'$). Since the only intersection points between S and S' are the center of the balls, we get :

$$4k' = 2L \cdot 2k'L = S \cdot S' = \sum_{i=1}^{5} S \cdot_{O_i} S' = \sum_{i=1}^{5} k_i \cdot 1 = 5k'.$$

This is the desired contradiction. $\square$

Note that the previous computation can be made for seven (*resp.* eight) balls. When they are symplectically smoothable in the previous sense, the supporting surfaces are at most seven (*resp.* eight), all of area 3π (*resp.* 6π). Each one intersects six of the balls through one Hopf disc and the last one through two Hopf dics (*resp.* intersects seven of the balls through two Hopf discs and the last one through three).

5.3 Smoothening of the supporting surfaces for packings of simple type.

In this paragraph, we show that the smoothening required by lemma 5.1 can be achieved when precise conditions on the singularities of the packings are given. Recall that the singularities of S are the union of a finite set of singularities of the packing, segments of the characteristic foliations joining precisely these points, and more exceptionnaly full Hopf circles. The first step is to deal with the generic singular points of S.

Lemma 5.2. *Let S be a symplectic surface singular along a segment or a circle Γ. Assume that a neighbourhood of Γ in S is*
- *either globally diffeomorphic to*

$$\{y = \alpha(z)|x|\} \subset \mathbb{R}^2(x, y) \times]0, 1[(z), \tag{5}$$

where α is a continuous function which vanishes at O and 1 if Γ is a segment;
- *or locally diffeomorphic to $\{y = \alpha(z)|x|\}$ where α is continuous if Γ is a circle.*

Then S can be smoothened to a symplectic surface by a $\mathcal{C}^0$-perturbation.

Proof : Consider first the case of a segment Γ of singularities. Using Moser's argument, a neighbourhood of Γ can be presented symplectically as the cylinder $V_\varepsilon := \{|x_1| < 1 + \varepsilon\ ,\ |y_1| < \varepsilon\ ,\ |z_2| < \varepsilon\} \subset \mathbb{C}^2(z_1 = x_1 + iy_1, z_2)$ in such a way that Γ corresponds to $[-1, 1] \times \{0\}$ and S corresponds to a union of two symplectic surfaces S_1, S_2 with common boundary Γ. Since S_1 and S_2 are symplectic, the identification can be done in order to achieve also $T_qS_1 = \text{Span}_{\mathbb{R}}(\partial/\partial x_1, \partial/\partial y_1)$ and $T_qS_2 = \text{Span}_{\mathbb{R}}(\partial/\partial x_1, -\partial/\partial y_1 - u(x_1)\partial/\partial z_2)$ ($q \in \Gamma$) where $u(x_1)$ is a continuous complex valued function along Γ which vanishes for $x_1 \notin] - 1, 1[$. Now if the neighbourhood of Γ is small enough, S_1 can be straightened to the strip $A = \{|x_1| < 1 + \varepsilon\,,\, 0 \leq y_1 < \varepsilon\,,\, z_2 = 0\}$ by a map h which is $\mathcal{C}^1$-close to the identity (and even tangent to the identity along Γ). This map may distort ω but by no more than an ε-factor. We produce our symplectic smoothening of S by cutting S_1 and replacing it by a "very symplectic" surface $\Sigma_u \subset V_\varepsilon$ which interpolates smoothly between $\big(\Gamma, \text{Span}_{\mathbb{R}}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} + u(x_1)\frac{\partial}{\partial z_2})\big)$ and A.

To this purpose, we consider a smooth "profile" of maps $\varphi_v : [0, \varepsilon[\longrightarrow \mathbb{C}$ parameterized by the field of complex numbers, such that $\varphi_v(0) = 0$, $\dot{\varphi}_v(0) = v$, $\varphi_v \equiv 0$ on $[\varepsilon/2, \varepsilon[$ and $\varphi_0 \equiv 0$. Up to shrinking these maps, we can also arrange so that $\partial\varphi_v/\partial v$ and $\dot{\varphi}_v$ are very small on big compact sets (for v). Then the surface

$$\Sigma_u := \{\big(x_1, y_1, \varphi_{u(x_1)}(y_1)\big),\ |x_1| < 1 + \varepsilon\ ,\ 0 \leq y_1 < \varepsilon\} \subset \mathbb{C}^2$$

obviously interpolates ($\mathcal{C}^1$-)smoothly between $(\Gamma, \frac{\partial}{\partial y_1}+u(x_1)\frac{\partial}{\partial z_2})$ and A. Observe moreover that the tangent vectors to Σ_u

$$v_1 := \frac{\partial}{\partial x_1} + \left(\frac{\partial \varphi_v}{\partial v}\cdot\frac{\partial u}{\partial x_1}\right)\frac{\partial}{\partial z_2} \qquad \text{and} \qquad v_2 := \frac{\partial}{\partial y_1} + \left(\frac{\partial \varphi_v}{\partial y_1}\right)\frac{\partial}{\partial z_2}$$

are small perturbations of $\partial/\partial x_1$ and $\partial/\partial y_1$ respectively provided that φ is sufficiently small in $\mathcal{C}^1$-norm. The tangent planes to Σ_u are thus far from being Lagrangian.

If Γ is now a circle of singularities, consider a subsegment $\Gamma_1 \subset \Gamma$. As above, a neighbourhood of Γ_1 in S is the union of two smooth strips S_1 and S_2 glued along $\Gamma_1 = [-1,1]\times\{0\} \subset \mathbb{C}^2$. Their tangent planes along Γ are $T_qS_1 = \mathrm{Span}_{\mathbb{R}}(\partial/\partial x_1, \partial/\partial y_1)$ and $T_qS_2 = \mathrm{Span}_{\mathbb{R}}(\partial/\partial x_1, -\partial/\partial y_1 - u(x_1)\partial/\partial z_2)$, where $u(x_1)$ is a continuous complex valued function. The only difference with the previous situation is that u may not vanish at the boundary of Γ_1. Consider anyway any smooth function $v : [-1,1] \longrightarrow \mathbb{C}$ vanishing at -1 and 1, which coincides with u on $\Gamma_2 = [-1/2, 1/2]$. Modifying S exactly as above by Σ_v (instead of Σ_u), we obtain a symplectic $\mathcal{C}^0$-perturbation $\widetilde{S}$ of S which is no longer singular on the circle Γ but on the segment $\Gamma\backslash\Gamma_2$. By construction, the singularity of $\widetilde{S}$ has the form (5) and a $\mathcal{C}^0$-perturbation of $\widetilde{S}$ is a smooth symplectic surface. $\square$

Hence all the difficulty of our desingularization problem concentrates at the (unavoidable) intersection of S with singular points of the packing. To understand the situation in the greatest generality, we need local models for the singularities that may arise in regular packings. We do not pretend to find them in this paper. Instead, and rather as an illustration, we focus on the very special type of singularities which appear in the examples we constructed in section 2.

Definition 5.3. *We say that a boundary singularity $p = \varphi(q)$ of a symplectic ball $\{\varphi, B\}$ is simple if φ is continuously differentiable at q, with non-vanishing derivative in any directions but the charachteristic one. A regular packing is said to be of simple type if all the singularities are simple, and if there exist no intersection point between any four balls (we only allow triple intersection points).*

Before proving theorem 4, let us discuss the meaning of this definition in our context. Consider a regular symplectic ball $\{\varphi, B\}$ of a symplectic manifold and a simple singularity p of B. Denote by C_p and D_p the Hopf circle and disc passing through p. For $q = \varphi(\eta) \in C_p$, note also $T^c(q) := \varphi'(\eta)\cdot T^{\mathbb{C}}_\eta \partial B$ and $\pi(q) := T^c(q)^{\perp\omega}$. Then D_p is tangent to $\pi(q)$ along C_p and from the definition of a simple singularity, both symplectic plane distributions $T^c(q)$ and $\pi(q)$ have a well-defined limit $T^c(p)$ and $\pi(p)$ when q goes to p. In particular, D_p has a tangent plane $\pi(p)$ at p. Although T_qC_p may not have limit at p, C_p is tangent to $\pi(p)$ at p. Observing that the tangent plane to ∂B at $q \in C_p$ is $T^c(q) \oplus T_qC_p$ we see that $\partial B \cap U$ is $\mathcal{C}^0$-close to the cylinder $\pi(C_p) \times T^c(p)$ (and $B \cap U$ is close to $D_p \times T^c(p)$). This remark immediately yields the following lemma :

Lemma 5.4. *Let B be a symplectic ball with a simple singularity at $p \in \partial B$. With the notation above, there is a neigbourhood U of p such that the linear projection $\pi : \overline{D_p} \cap U \longrightarrow \pi_p$ along $T^c(p)$ is an injective map.*

This lemma prevents D_p from spiraling too much above its tangent plane, creating problematic cone singularities. The proof of theorem 4 is achieved in two steps. First, we smoothen the supporting surface at each (simple type) singularity of the packing. Being cautious enough in the first step allows to use lemma 5.2 to get rid of the remaining circles or segments of singularities.

Proof of theorem 4 : Consider a maximal symplectic packing $\{B_1, \ldots, B_k\} \subset \mathbb{P}^2$ of simple type, and one of its supporting surfaces S. Denote $\{p_1, \ldots, p_n\}$ the singularities of the packing which belongs to S. Then the singularities of S consist of the points p_i themselves, together with open segments of the characteristic foliation linking the p_i. The differentiable model of the singularities along these segments is locally $\{y = |x|\} \times \mathbb{R} \subset \mathbb{R}^3$. As announced, we are going to smoothen S around the p_i in such a way that the remaining segments of singularities have the global form (5). Consider henceforth one of the singular point $p := p_i$. Recall that p can only be an intersection between two or three balls. Assume first that p lies in the intersection between two balls only, say B_1 and B_2. Let us distinguish between two cases.

Case a : $\pi_1(p) = \pi_2(p)$, $T_1^c(p) = T_2^c(p)$. All indices refer to the balls of the packing (for instance $\pi_1(p)$ is the plane defined above for the ball B_1). Consider local symplectic coordinates taking p to the origin in $\mathbb{C}^2$ and $\pi(p)$ to $\{z_2 = 0\}$. Inside a small bidisc $Q_\varepsilon := \{|z_1| < \varepsilon\} \times \{|z_2| < \varepsilon\}$, the surface S together with its tangent planes are very close to $\{z_2 = 0\}$. The projection $\pi : S \cap Q_\varepsilon \longrightarrow \{z_2 = 0\}$ is therefore a covering map, which must be injective from lemma 5.4. The intersection of S with ∂Q_ε is therefore the graph over $\partial \mathbb{D}_\varepsilon$ of a piecewise $\mathcal{C}^1$-smooth complex valued function with small $\mathcal{C}^1$-norm. This function can obviously be extended to $\mathbb{D}_\varepsilon$ in such a way that its graph $\Sigma \subset Q_\varepsilon$ is tangent to S on ∂Q_ε (except at the singularities of $S \cap \partial Q_\varepsilon$), that its singularities are located on segments inside $\{\varepsilon/2 < |z_1| < \varepsilon\}$ and have the form of lemma 5.2. Moreover, this extension can be choosen $\mathcal{C}^1$-small, so that Σ remains a symplectic surface. Cutting $S \cap Q_\varepsilon$ and replacing it by Σ thus gives the desired smoothening of S at p.

Case b : $\pi_1(p) \neq \pi_2(p)$. Consider a small neighbourhood U of p such that the common Hopf circle $C_p := C_{p1} = C_{p2}$ of ∂B_1, ∂B_2 passing through p is the union of two smooth (open) arcs ℓ_l and ℓ_r meeting at p. Assume also that $\pi_1(q) \neq \pi_2(q)$ for all $q \in \ell_l \cup \ell_r$. Inside U we have $T_q C_p = \pi_1(q) \cap \pi_2(q)$ so that ℓ_l and ℓ_r have common tangency $\pi_1(p) \cap \pi_2(p)$ at p. Since $D_i(p)$ has limit tangent plane $\pi_i(p)$ at p, if $\ell_l \cup \ell_r$ is $\mathcal{C}^1$-smooth then the model of the singularity of S at p is the same as at any generic point of C_p, and we can forget it. Else $\ell_l \cup \ell_r$ is a $\mathcal{C}^1$-cusp, meaning that its projection to $\pi_1(p) \cap \pi_2(p)$ is a half-line (see figure 3). Then $C_p \times T_1^c(p)$ ($= C_p \times T_2^c(p)$) separates U in two cylinders, a big one (with aperture 2π at p) and a small one, in the whereabouts of which each ball is contained. To fix the ideas, suppose that B_1 is the "big ball" in U, and consider symplectic coordinates in U such that $p = 0$, $\pi_1(p) = \{z_2 = x_2 + iy_2 = 0\}$ and $\pi_1(p) \cap \pi_2(p) = \mathrm{Span}_{\mathbb{R}}(\partial/\partial y_1)$. Note that $\pi_2(p)$ is transverse to $\{z_1 = 0\}$ because it is symplectic and contains $\partial/\partial y_2$. So the projection of S on $\pi_1(p)$ along $T_1^c(p)$ inside a small bidisc Q_ε is a covering map, injective by lemma 5.4. The intersection $\partial Q_\varepsilon \cap S$ is henceforth the graph of a piecewise $\mathcal{C}^1$-smooth complex valued function over $\partial \mathbb{D}_\varepsilon$. It is easy to see that this map has bounded derivatives and small $\mathcal{C}^0$-norm. The same procedure as in case *a.* above produces the smoothening.

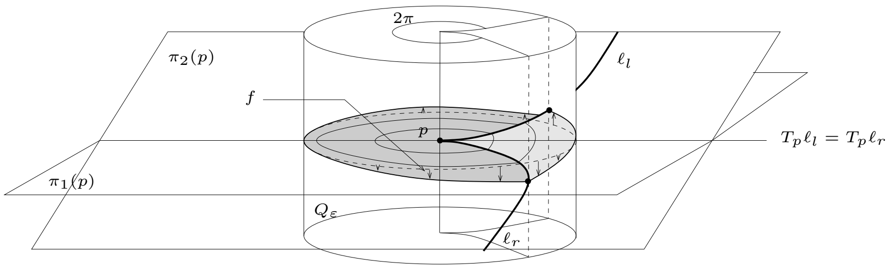

Figure 3: Singularity of type $\pi_1(p) = \pi_2(p)$ and C_p is not smooth.

Assume now that p lies in the boundary of three balls B_1, B_2, B_3. If looking inside a sufficiently small neighbourhood U of p, each intersection $B_i \cap B_j \cap U$ is a smooth open arc ℓ_{ij} ending at p. Reasoning as in cases $a.$ and $b.$ it is easy to smoothen S at p when $\pi_1(p) = \pi_2(p) = \pi_3(p)$ or $\pi_1(p) = \pi_2(p) \neq \pi_3(p)$. Thus it only remains to investigate the situation of three different tangent planes for $D_i(p)$ at p (see figure 4). The curves ℓ_{ij} have then well-defined tangencies at p :

$$T_p\ell_{ij} = \pi_i(p) \cap \pi_j(p).$$

Consider a parameterization of the ℓ_{ij} by smooth maps $\gamma_{ij} : [0, \varepsilon[\longrightarrow U \subset \mathbb{C}^2$ with $\gamma_{ij}(0) = p$. Notice that $\omega(\dot\gamma_{ij}(0), \dot\gamma_{jk}(0)) > 0$ because $\pi_j(p) = \mathrm{Span}_{\mathbb{R}}(\dot\gamma_{ij}(0), \dot\gamma_{jk}(0))$ is a symplectic plane. Applying a symplectic linear change of coordinates and a rescaling of the parameterizations, we can suppose that

$$\dot\gamma_{12}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \dot\gamma_{23}(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \dot\gamma_{31}(0) = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \subset \mathbb{C}^2 \approx \mathbb{R}^4.$$

Looking sufficiently close to p, there is a diffeomorphism Φ $\mathcal{C}^1$-close to the identity taking S to the cone over the triangle :

$$\Sigma := \left\{ \begin{pmatrix} z\varphi_1(\theta) \\ z \\ 0 \end{pmatrix}, \; 0 < h < 1 \right\}$$

where $\varphi_1(\theta)$ is the counterclockwise parameterization of the triangle T_1 spanned by the points $(1,0)$, $(0,1)$, $(-1,-1)$ of $\mathbb{R}^2$. Consider then the polar parameterizations φ_ρ of a family of convex curves T_ρ with $T_\rho = \{x^2 + y^2 = \rho^4\}$ for $\rho \ll 1$, $T_\rho = \rho \cdot T_1$ for $\rho \simeq 1$ and the angles of the singularities of T_ρ vary smoothly with ρ (see figure 4). Then the surface

$$\widetilde{\sigma} := \{(\varphi_h(\theta), h, 0)\}$$

is smooth near the origin, has tangent planes far from being Lagrangian and coincide with Σ in a neighbourhood of $h = \{y_2 = 1\}$. Cutting Σ from S and replacing it by $\Phi^{-1}(\widetilde{\Sigma})$, we get a symplectic smoothening of S at p. We were also sufficiently cautious in the extrapolation from T_1 to T_ρ, $\rho \simeq 0$ to ensure that the remaining singularities have the form (5).

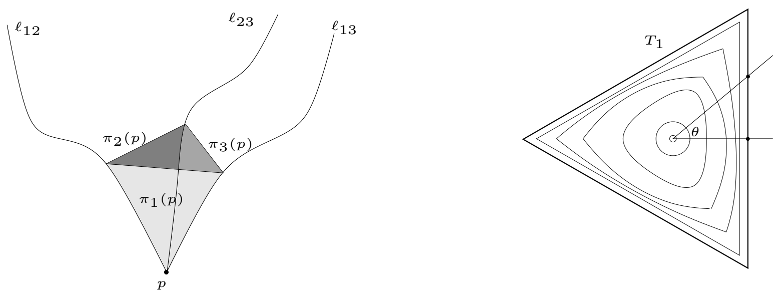

Figure 4: Singularity of S and extrapolation between T_1 and 0.

After a small perturbation of S, we have thus been able to obtain a symplectic surface with singularities along circles or segments, of the type required to apply lemma 5.2. The surface S can therefore be perturbed to a smooth symplectic surface, and applying lemma 5.1 proves theorem 4. $\square$

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