Symplectic Camel theorems and $\mathcal{C}^0$-rigidity of coisotropic submanifolds

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Abstract

This paper deals with the $\mathcal{C}^0$-rigidity of the reduction of coiostropic submanifolds under the action of symplectic homeomorphism. More precisely, we exhibit several situations where a symplectic homeomorphism that takes a coisotropic submanifold to a smooth submanifold (which are then known to be coisotropic by a result of Humilière-Leclercq-Seyfaddini) abides to the non-squeezing property in the reduction.

1 Introduction

This paper continues the study of the action of the symplectic homeomorphisms on smooth submanifolds as initiated in [Ops09, HLS15, BO16]. Recall that a symplectic homeomorphism between symplectic manifolds M and M' is a homeomorphism that can be approximated by symplectic diffeomorphisms between M and M'. In [HLS15], it was observed that when the image of a coisotropic submanifold by a symplectic homeomorphism is a smooth submanifold, then this image is coisotropic and the symplectic homeomorphism intertwines the charateristic foliations of the source and target submanifolds. As a result, it acts on the reduction, which is locally a symplectic manifold, and one may ask which symplectic properties of the reduction are preserved. In other terms, the set of coisotropic submanifolds is $\mathcal{C}^0$-rigid, as well as the characteristic foliations, and we ask which further symplectic invariants of coisotropic submanifolds are $\mathcal{C}^0$-rigid. As far as we know, three results are already known, prior to this paper:

- When the reduction homeomorphism is smooth, it is symplectic [BO16].
- Let Σ, Σ' be hypersurfaces in symplectic manifolds, h a symplectic homeomorphism that takes Σ to Σ' and $\operatorname{Red} h$ its action at the level of the reduction. Then $\operatorname{Red} h$ preserves the stable displacement energy of subsets of $\operatorname{Red} \Sigma$ [BO16].
- Let h be a symplectic homeomorphism of $\mathbb{T}^{2n}$ that preserves the split coisotropic subtorus $\mathbb{T}^{2k+l} \times \{0\}$. Then $\operatorname{Red} h : \mathbb{T}^{2k} \to \mathbb{T}^{2k}$ preserves the spectral capacity of open subsets of $\mathbb{T}^{2k}$ [HLS16].

In these situations, the symplectic homeomorphism therefore also inherits symplectic properties at the level of the reduction. In this paper, we study the (non)-squeezing properties of the reduction of symplectic homeomorphisms in this setting, namely:

Conjecture 1. *Let h be a symplectic homeomorphism that takes some coisotropic submanifold to a smooth, hence coisotropic, submanifold. Then the reduction of h does not squeeze balls into cylinders of smaller capacities.*

Our first result generalizes the last result cited above.

Theorem 1. *Let N be a closed manifold that admits a metric with non-negative scalar curvature. Assume the existence of a symplectic homeomorphism h defined in a neighbourhood of $B^{2k}(a) \times N \subset \mathbb{C}^k \times T^*N$ such that $h(B^{2k}(a) \times N) \subset Z^{2k}(A) \times N$. Then $A \geq a$.*

In other terms, when the characteristic foliation consists of closed leaves and is trivial, and provided the leaves satisfy some topological property made explicit in remark 2.2, the reduction of a symplectic homeomorphism that preserves this submanifold verifies the non-squeezing property. Theorem 1 relies on the classical non-squeezing theorem *via* a short and easy argument.

By contrast, our study of the general case is less conclusive. In a model situation, we still get the following result:

Theorem 2. *Let h be a symplectic homeomorphism defined in a neighbourhood of $B^{2k}(1) \times [-1,1]^{n-k} \subset \mathbb{C}^n$, with values in $\mathbb{C}^n$, that takes the coisotropic submanifold $B^{2k}(1) \times [-1,1]^{n-k}$ to $\mathbb{C}^k \times \mathbb{R}^{n-k}$. Then there exists $\delta(h) > 0$ such that for all $a < \delta(h)$, if $h\big(B^{2k}(a) \times [-1,1]^{n-k}\big) \subset Z^{2k}(A) \times \mathbb{R}^{n-k}$, $a \leq A$.*

The dependence of the constant $\delta(h)$ with h prevents the previous theorem to provide a satisfactory answer to conjecture 1. The best we would get in the general framework of conjecture 1, within the techniques of the present paper, would be that the balls of small diameters in the reduction of the source are either not-squeezed or highly distorted. When the reduction has dimension 2 however, non-squeezing means area-preservation, and theorem 2 is enough to show that conjecture 1 holds:

Theorem 3. *If C is a coisotropic submanifold of dimension $n+1$ in a symplectic manifold M^{2n}, and if some symplectic homeomorphism takes C to a smooth submanifold C', then C' is coisotropic, h conjugates the characteristic foliations of C, C' and $\operatorname{Red} h : \operatorname{Red} C \to \operatorname{Red} C'$ is area preserving.*

Notice that for a coisotropic submanifold, having a reduction of dimension 2 means being of minimal dimension above the lagrangian case. Thus, this corollary is in some sense orthogonal to the previous local result, obtained in [BO16], that concerned hypersurfaces, since these have maximal dimension below the open case. Theorem 2 relies on a slightly stronger version of the following coisotropic Camel theorem (see also [Bus19] for a very close statement):

Theorem 4. *Let l, n be two integers with $l < n$ and $\varphi : B^{2n}(a) \times \mathbb{R}^l \to \mathbb{R}^{2n} \approx \mathbb{C}^{n-l} \times \mathbb{R}^{2l}_{(x,y)}$ a smooth map such that:*

i) $\forall t \in \mathbb{R}^l$, $\varphi(\cdot, t)$ is a symplectic embedding,

ii) $\varphi(\cdot, t) = \mathrm{Id} + \sum_1^l t_i \frac{\partial}{\partial x_i}$ *for* $|t| \gg 1$,

iii) $\mathrm{Im}\,\varphi \cap \{x_1 = \cdots = x_l = 0\} \subset Z^{2(n-l)}(A) \times [-1,1]^l_y$.

Then $A \geq a$.

For $l = 1$, this is due to Eliashberg [Eli90], and the generalization above is fairly straightforward. The strenghtened version necessary for our application, theorem 3.1, requires more work, but is still not sufficient to handle conjecture 1. In fact, this conjecture would follow if we could localize the symplectic camel theorem in the sense of relaxing assumption iii) above by a knotting assumption.

Definition 5. *We say that a smooth map* $\varphi : B^{2n}(a) \times \mathbb{R}^l \to \mathbb{R}^{2n}$ *is knotted with an* $l-1$*-cycle* Z *if* $\mathrm{Im}\,\varphi \cap Z = \emptyset$ *and* $\varphi(\{0\} \times \mathbb{R}^l)$ *has non vanishing homological intersection with a bounding* l*-chain* B *of* Z *(with* $\partial B = Z$*), where the intersection is computed with* $\mathbb{Z}$ *or* $\mathbb{Z}_2$*-coefficients.*

Conjecture 2. *Theorem 4 holds when iii) is replaced by:*

iii)' ϕ is knotted with $\partial Z^l_{A,R}$*, where*

$$Z^l_{A,R} := \left\{ (z_1, z', x, y) \in \mathbb{C} \times \mathbb{C}^{n-l-1} \times \mathbb{R}^l \times \mathbb{R}^l \;\middle|\; \begin{array}{l} x = 0, \\ \pi|z_1|^2 < A, \\ \|(z', y)\|_\infty < R \end{array} \right\}.$$

In other terms, φ *provides a symplectic isotopy of the ball* $B^{2n}(a)$ *through the window* $Z^l_{A,R}$*, without any further assumption that the isotopy avoids the coisotropic wall* $\mathbb{C}^{2(n-l)} \times \mathbb{R}^l$ *away from this window.*

As far as I am aware of, this conjecture is not confirmed in any dimension: even the first dimension $2n = 4$ and $l = 1$ is unknown.

Organization of the paper: The short section 2 proves theorem 1 and is independent from the rest of the paper. In section 3, we state and prove two versions of the coisotropic versions of the Camel theorem that we have stated above. We provide an extensive argument, although some good references are already available [MT93, Nie14]. The reason is that these references either insist on the dimension 4, or on the pseudoconvexity of the Camel space, which are not relevant in our context. Finally, we prove theorems 2 and 3 in section 4.

Notation: Throughout this paper, we adopt the following notation:

- $D(a)$ denotes the open euclidean 2-dimensional disc of area a centered at 0 in $\mathbb{R}^2$.
- $B^{2n}(a)$ denotes the open euclidean ball centered at the origin of capacity a in $\mathbb{C}^n$. When the dimension is irrelevant or clear from the context, we simply write $B(a)$.

- $\mathbb{D}$ is the open unit disc in $\mathbb{C}$.
- $\mathcal{A}_\omega(S)$ stands for the symplectic area of a surface (S,ω), and if $u : S \to (M,\omega)$, $\mathcal{A}_\omega(u) = \int_S u^*\omega$.
- Given two sets $A \subset B$, $\mathrm{Op}\,(A,B)$ stands for an arbitrary but fixed neighbourhood of A in B.
- A continuous map $\Phi : B^{2n}(a) \times \mathbb{R}^l \to \mathbb{C}^n$ is a parametric symplectic embedding if $\Phi(\cdot,t) : B^{2n}(a) \to \mathbb{C}^n$ is a symplectic embedding for all $t \in \mathbb{R}^l$. We then denote $\Phi_t := \Phi(\cdot,t)$
- We split $\mathbb{C}^n$ as $\mathbb{C}^n = \mathbb{C}^{k+l} = \mathbb{C} \times \mathbb{C}^{k-1} \times \mathbb{R}^l \oplus i\mathbb{R}^l$ and we use coordinates (z_1, z', x, y) on these four factors, respectively.
- We endow $\mathbb{C}^{k-1} \times \mathbb{C}^l$ with the L_∞-norm : $\|(z',x,y)\| := \max\{\|z'\|_\infty, \|x\|_\infty, \|y\|_\infty\}$ (and $\|z'\|_\infty := \max\{\|\mathrm{Re}\, z'\|_\infty, \|\mathrm{Im}\, z'\|_\infty\}$).
- $Z^{k,l}_{A,R} := D(A) \times \{(z',0,y) \,|\, \|(z',0,y)\| < R\} \subset \mathbb{C}^k \times i\mathbb{R}^l$ and $Z^{k,l}_A := Z^{k,l}_{A,\infty}$. When k,l are clear from the context, we omit the superscripts for easier readability.
- $\Gamma^{k,l}_{A,R} := S^1(A) \times \{(z',0,y) \,|\, \|(z',0,y)\| \le R\} \subset \mathbb{C}^k \times i\mathbb{R}^l$ (this is the "horizontal" part of $\partial Z^{k,l}_{A,R}$) and $\Gamma^{k,l}_A := \Gamma^{k,l}_{A,\infty}$.
- $F^{k,l}_{A,R,M} := D(A + \pi M^2) \times \{(z',x,y) \,|\, R - M < \|(z',0,y)\| < R + M$ and $\|x\| < M\} \subset \mathbb{C}^k \times \mathbb{C}^l$. This is an M-neighbourhood of the "vertical" part of $\partial Z^{k,l}_{A,R}$. As a result, notice that any point in $F^{k,l}_{A,R,M}$ is at euclidean distance no less than M' of $\partial F^{k,l}_{A,R,M+M'}$.

The following figure illustrates our notation. It is rather accurate when $k = 1$ and $l = 0$. When $k + l \ge 2$, the figure is hopefully useful, but it is only partially representative. The main difference is that $F_{A,R,M}$ becomes connected. In the figure, $B_\infty(R)$ stands for the R-ball in the L_∞-norm in $\mathbb{C}^{k-1} \times \mathbb{R}^l$.

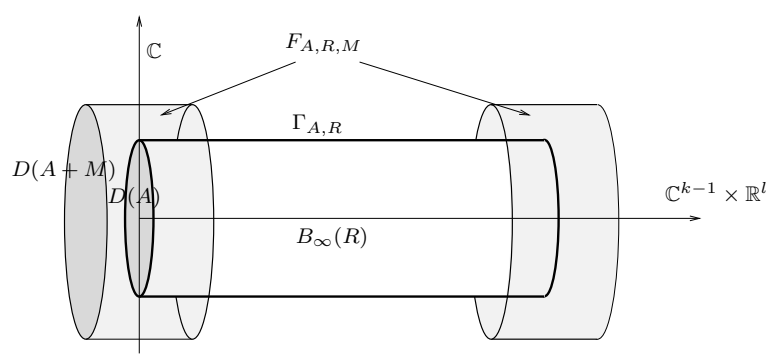

Figure 1: The sets $\Gamma_{A,R}$ and $F_{A,R,M}$ when $k = 1$.

Aknowledgements: Lev Buhovsky was initially engaged in this project and contributed to uncovering a crucial flaw in the initial argument. This paper therefore owes him a lot.

2 Proof of theorem 1

Let g be a metric on N with non-positive sectional curvature, $\varepsilon > 0$, and $h : \mathrm{Op}\,(B^{2k}(a) \times N, \mathbb{C}^k \times T^*N) \to \mathbb{C}^k \times T^*N$ be a symplectic homeomorphism such that $h(B^{2k}(a) \times N) \subset Z^{2k}(A) \times N$. Consider an ε-approximation of h in the $\mathcal{C}^0$-norm by a symplectic diffeomorphism f. Thus, f is defined in a neighbourhood of $B^{2k}(a) \times N$ in $B^{2k}(a) \times T^*N$ and provided this neighbourhood is small enough and the approximation fine enough, we have

$$f(B^{2k}(a) \times T^*_\delta N) \subset Z(A+\varepsilon) \times T^*_{2\varepsilon} N,$$

where δ, ε are small positive numbers. Since g has non-positive curvature, Cartan-Hadamard's theorem guarantees that the universal cover $\tilde{N}$ of N is diffeomorphic to $\mathbb{R}^l$. The covering $\pi : \tilde{N} \to N$ lifts to symplectic coverings $T^*\pi : T^*\tilde{N} \to T^*N$ and $\Pi := \mathrm{Id} \times T^*\pi : \mathbb{C}^k \times T^*\tilde{N} \to \mathbb{C}^k \times T^*N$. The map π obviously being a Riemannian covering with respect to the metrics $(\tilde{g} := \pi^* g, g)$ on the pair $(\tilde{N}, N)$, it is easy to see that $\Pi^{-1}(U \times T^*_\delta N) = U \times T^*_\delta \tilde{N}$, where $T^*_\delta \tilde{N}$ stands for the δ-neighbourhood of the zero section with respect to the lifted metric $\tilde{g}$. Thus, f lifts through Π to a symplectic map

$$\tilde{f} : B(a) \times T^*_\delta \tilde{N} \longrightarrow Z(A+\varepsilon) \times T^*\tilde{N}.$$

Since h is a symplectic homeomorphism, it sends the characteristic leaf $\{0\} \times N$ of $B^{2k}(a) \times N$ to one characteristic leaf $\{*\} \times N$, in a 1-to-1 way. Letting $\mathrm{pr} : \mathbb{C}^k \times N \subset \mathbb{C}^k \times T^*N \to N$ denote the projection to the second factor followed by the projection to the zero section, the projection $\mathrm{pr} \circ h_{|\{0\}\times N} : \{0\} \times N \to N$ therefore has degree 1. This property also holds for f, and as a result, $\tilde{f}$ is injective, hence an embedding

$$\tilde{f} : B^{2k}(a) \times T^*_\delta \tilde{N} \longrightarrow Z^{2k}(A+\varepsilon) \times T^*\tilde{N} \approx Z^{2k}(A+\varepsilon) \times T^*\mathbb{R}^l = Z^{2n}(A+\varepsilon).$$

Claim 2.1. *$T^*_\delta \widetilde{N}$ contains a symplectic ball of capacity a.*

Let thus $\varphi : B^{2l}(a) \hookrightarrow T^*_\delta \mathbb{R}^l$ be a symplectic embedding. The map $\tilde{f} \circ (\mathrm{Id} \times \varphi)$ therefore provides a symplectic embedding of $B^{2k}(a) \times B^{2l}(a)$ into $Z(A+\varepsilon)$. By Gromov's non-squeezing theorem, we get $A + \varepsilon \geq a$ and, letting ε to 0, $A \geq a$. $\square$

Proof of claim 2.1: Let (N, g) be a non-positively curved Riemannian manifold and $(\widetilde{N}, \tilde{g})$ its universal cover. Fix any point $p \in \widetilde{N}$. By Cartan-Hadamard's theorem, the map $f := \exp_p : T_p\widetilde{N} \to \widetilde{N}$ is a diffeomorphism. Moreover, the negative curvature implies that the map

$$D_\lambda(x) := f(\lambda f^{-1}(x))$$

verifies $\|(D'_\lambda)^{-1}\|_{\tilde{g}}^{-1} \geq \sqrt{\lambda}$. Indeed, given $u \in T_x\tilde{N}$ of norm 1, $U(t) := t \mapsto D'_t(x)u$ is the Jacobi vector field along the geodesic issued from p and passing at x at time 1, that vanishes

at 0 and equals u at 1. A classical computation shows that the function $h(t) := g(U(t), U(t))$ is convex in non-positive curvature. Since it verifies $h(0) = 0$ and $h(1) = 1$, $h(t) \geq t$ for $t \geq 1$, so $\|D'_\lambda(x)u\|^2 \geq \lambda$, which is our assertion. Given now any small ball $\mathcal{D} \subset \widetilde{N}$, $T^*\mathcal{D}$ contains a relatively compact symplectic ball $\varphi : B(a) \hookrightarrow T^*\mathcal{D}$. The image of φ lies in $T^*_{\tilde{g},K}\widetilde{N}$ for some large K. Putting $\lambda := \frac{\delta}{K}$, the natural lift of D_λ to $T^*\widetilde{N}$ then takes $\varphi(B(a))$ into $T^*_\delta \tilde{N}$. $\square$

Remark 2.2. *The proof above shows that theorem 1 holds whenever N has the property that any neighbourhood $T^*_{\delta,\tilde{g}}\tilde{N}$ of the universal cover of N equipped with some pull-back metric has infinite Gromov capacity. I do not know whether there is a more conventional characterization of this natural symplectic property. The existence of a non-positive curvature on N is enough, as explained above.*

3 Two Camel theorems

The aim of this section is to state and prove two slightly improved versions of the classical Camel theorem (theorem 4), that we use in our proof of theorem 2. The statements below use the notations introduced in the introduction, p. 3.

Theorem 3.1. *There exists a universal constant $C > 0$ for which the following holds. Let $\Phi : B^{2n}(a) \times \mathbb{R}^l \to \mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^l_{x+iy}$ be a parametric symplectic embedding which verifies:*

(i) *Φ is standard at infinity: $\exists K > 0$ such that for $|t| > K$, $\Phi_t(z) = z + \sum_{i=1}^l t_i \frac{\partial}{\partial x_i}$,*

(ii) *Φ is knotted with $\partial Z^{k,l}_{A,R}$.*

(iii) *$\operatorname{Im}\Phi$ does not intersect $F^{k,l}_{A,R,CA}$.*

Then $A \geqslant a$.

Theorem 3.2. *There exists a universal constant $C > 0$ for which the following holds. Let L be a Lagrangian embedding of $(S^1)^k \times \mathbb{R}^l$ into $\mathbb{C}^n = \mathbb{C}^{k+l}$ which verifies:*

(i) *L is properly embedded, standard at infinity: there is a compact set $K \subset \mathbb{C}^n$ such that*

$$L \cap {}^cK = \left(S^1(a)^k \times \mathbb{R}^l\right) \cap {}^cK = \{\pi|z_i|^2 = a, i = 1 \dots k\} \times \mathbb{R}^l \cap {}^cK \subset \mathbb{C}^k \times \mathbb{C}^l.$$

(ii) *L is knotted with $\partial Z^{k,l}_{A,R}$.*

(iii) *L does not intersect $F^{k,l}_{A,R,CA}$.*

Then $a \leqslant A$.

Obivously, the assumptions on theorem 3.1 are less restrictive than the usual ones, so theorem 3.1 implies theorem 4. Theorem 3.2 is also a slightly localized version of an

existing Lagrangian camel theorem [Bus19]. As already explained, getting a genuinely localized version of both results would consist in removing assumption (iii). We prove both theorems below, because both seem interesting in their own sake, and that none of them seems to follow from the other, but their proofs are very similar. For our application to the $\mathcal{C}^0$-rigidity of the reduction of a symplectic homeomorphism however, only theorem 3.1 is relevant, through the following corollary:

Corollary 3.3. *If there is a compactly supported symplectic diffeomorphism f of $\mathbb{C}^n = \mathbb{C}^{k+l}$ such that*

(i) $f(B^{2k}(a) \times \mathbb{R}^l)$ is knotted with $\partial Z^{k,l}_{A,R}$ for some R,

(ii) $f(B^{2k}(a) \times \mathbb{R}^l) \cap F^{k,l}_{A,R,CA} = \emptyset$,

then $A \geq a$.

Proof: Let $K, \varepsilon > 0$ to be taken very large and very small, respectively. Let $\rho : \mathbb{R}^+ \to [\varepsilon, 1]$ be a smooth function with $\rho(s) \equiv \varepsilon$ for $s < K$ and $\rho(s) \equiv 1$ for $s > K+1$. The map

$$\begin{array}{rcccl} \varphi & : & B^{2k}(a) \times B^{2l}(a) \times \mathbb{R}^l & \longrightarrow & \mathbb{C}^{k+l} \\ & & (z_1, z_2, t) & \longmapsto & (z_1, \frac{1}{\rho(\|t\|)}\mathrm{Re}\, z_2 + t, \rho(\|t\|)\mathrm{Im}\, z_2) \end{array}$$

verifies:

(i) φ is smooth,

(ii) for all $t \in \mathbb{R}^l$, $\varphi(\cdot, t)$ is a symplectic embedding of $B^{2k}(a) \times B^{2l}(a) \supset B^{2n}(a)$,

(iii) for $|t| > K+1$, $\varphi(z_1, z_2, t) = (z_1, z_2) + t$,

(iv) for $|t| < K$, $\varphi(z_1, z_2, t)$ belongs to an ε-neighbourhood of $B^{2k}(a) \times \mathbb{R}^l \subset \mathbb{C}^k \times \mathbb{C}^l$.

Let now f be the symplectic diffeomorphism considered by our statement. For $\varepsilon \ll 1$ and $K \gg 1$, the restriction of the map $\Phi := f \circ \varphi$ to $B^{2n}(a) \times \mathbb{R}^l \subset B^{2k(a)} \times B^{2l}(a) \times \mathbb{R}^l$ verifies all the assumptions of theorem 3.1, so $A \geq a$. □

Before proving theorems 3.1 and 3.2, let us unify their statements. First, we fix k, l, n so we drop all these indices from our notations. Notice also that scaling the symplectic form allows to assume that $A = 1$, which we do henceforth, removing from our notations any reference to A (hence Z_R, $F_{R,M}, \Gamma_R$ stand for $Z^{k,l}_{1,R}, F^{k,l}_{1,R,M}, \Gamma^{k,l}_{1,R}$). Let $f : X \times \mathbb{R}^l \to \mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^l$ be a map that is either a Lagrangian embedding if $X = S^1(a)^k$, or a parametric symplectic embedding of the ball if $X = B^{2n}(a)$. It is also standard at infinity, knotted with ∂Z_R, and its image avoids $F_{R,M}$. We need to find a constant C such that $M \geq C$ implies the inequality $a \leq 1$. Since f is standard at infinity in both cases, its image is contained in $\{(z, x, y) \in \mathbb{C}^k \times \mathbb{R}^l \times \mathbb{R}^l \mid \|(z, y)\| < K\}$ for some large K, that we take larger than M. Moreover, still because f is standard at infinity, we can compactify the triple $(f, \mathrm{Dom}\, f, \mathbb{C}^k \times i\mathbb{R}^l \oplus \mathbb{R}^l)$ by compactifying the $\mathbb{R}^l$-factor to a very large torus, which we choose in any case

to contain $\{|x| \geq \max\{1, M\}\}$. We get a new triple $(\tilde{f}, \mathrm{Dom}\,\tilde{f}, \mathbb{C}^k \times i\mathbb{R}^l \times \mathbb{T}^l)$, where $\mathrm{Dom}\,\tilde{f} = X \times \mathbb{T}^l$, $X = S^1(a)^k$ or $X = B^{2n}(a)$, and $\tilde{f}$ is either a Lagrangian embedding or a parametric symplectic embedding. Moreover, $\mathrm{Im}\,\tilde{f}$ is still knotted with ∂Z_R, and avoids $F_{R,M}$, both sets being unaffected by the compactification because $Z_R \subset \{x = 0\}$ and $K > M$. In order to keep light notation, we drop the tilde from $\tilde{f}$. We call $Y := \mathbb{C}^k \times i\mathbb{R}^l \times \mathbb{T}^l$ our ambient space. It is equipped with the standard symplectic form ω, and with a standard complex structure J_{st}. The almost complex structures that we will consider in this analysis belong to the set of ω-compatible almost complex structures $\mathcal{J}(\omega)$ on Y and are of the following type. First, fix two disjoint neighbourhoods $V_\Lambda \Subset \{\|(z, y)\| < K\}$ of $\Lambda := \mathrm{Im}\,(f)$ and V_Γ of $\Gamma_R \cup F_{R,M}$, and consider some ω-compatible almost complex structure J_Λ on V_Λ. We define

$$\mathcal{J}(R, M, J_\Lambda) := \{J \in \mathcal{J}(\omega) \mid J_{|V_\Lambda} \equiv J_\Lambda \text{ and } J_{|V_\Gamma \cup \{\|(z,y)\| > K\}} \equiv J_{\mathrm{st}}\}.$$

The idea is the same as for the classical Camel theorem: we must fill Γ_R by holomorphic discs of area 1. Since Γ_R is not totally real, the problem is not in a good setting. We therefore introduce for $c \in \mathbb{R}^{k-1}$

$$\begin{cases} \Gamma_{R,c} := \Gamma_R \cap \{\mathrm{Re}\, z' = c\} = S^1 \times \{(z', 0, y) \mid \mathrm{Re}\, z' = c,\ \|(\mathrm{Im}\, z', y)\| \leq R\} \\ \Gamma_c := \Gamma_{\infty,c} = \Gamma \cap \{\mathrm{Re}\, z' = c\} \end{cases}$$

which provide foliation of Γ_R and Γ by Lagrangian leaves, which we will each fill. The main difficulty here is that we do not assume that the map f avoids the coisotropic wall $\mathbb{C}^k \times i\mathbb{R}^l$ away from $Z(1) \times i\mathbb{R}^l$, so the classical filling technique must be adapted. For the reader aware of the classical proof, it might be worth having in mind that the main issue in this setting, compared to the more classical one, is the compactness. Since $J = J_{\mathrm{st}}$ on $F_{R,M}$, the circles $S^1 \times \{(z', 0, y)\}$ are filled by the vertical discs $\mathbb{D} \times \{(z', 0, y)\}$ for $\|(z', 0, y)\| \approx R$ ($\mathbb{D} := \{|z| < 1\} \subset \mathbb{C}$). We need to guarantee that the boundaries of the non-vertical holomorphic discs do not approach $\partial\Gamma_{R,c}$, which will be realized by taking M large enough (see lemma 3.8).

Although the proof of the Camel theorem is folklore, we could not find a complete proof in the litterature, except in dimension 4 [MT93]. We therefore provide a complete proof, and not only of the compactness issue.

3.1 Preliminaries for the compactness

Here is the main statement we will need in order to address the compactness issues.

Proposition 3.4. *There exists $M > 0$ such that for any $R > 0$, $c \in \mathbb{R}^l$, $J_\Lambda \in \mathcal{J}(\omega)$ and $J \in \mathcal{J}(R, 4M, J_\Lambda)$, all the J-holomorphic discs with boundary on $\Gamma_{R,c}$, of area 1, that intersect $F_{R,M}$ are vertical discs (of the form $z \mapsto (z, 0, c) \in \mathbb{C} \times \mathbb{C}^{k-1} \times \mathbb{C}^l$).*

This statement relies on classical area estimation of analytic sets, provided by the next lemmas.

Lemma 3.5. *There exists a relative holomorphic embedding* $\varphi : (B^{2n}(1), \mathbb{R}^n) \hookrightarrow (Y, \Gamma_0)$, *with* $\varphi(0) = (z_1 = 1, z' = x = y = 0) \in \Gamma_0$.

Proof: There exists a holomorphic embedding of $(\mathbb{D}, \mathbb{R}) \to (\mathbb{C}, S^1)$ with $\varphi(0) = 1$ because S^1 is real-analytic. Taking product with the map $(z', x + iy) \mapsto (iz', ix - y)$ provides a holomorphic map on $\mathbb{D} \times \mathbb{C}^{n-1}$ that sends $\mathbb{R}^n$ to $S^1 \times i\mathbb{R}^{k-1} \times i\mathbb{R}^l = \Gamma_0$. We obtain the desired embedding by restricting this map to $B^{2n}(1)$. □

In the next lemma, we say that an analytic subset X has *real boundary* if every local branch of X near a point $x \in \partial X$ can be parametrized by a holomorphic map defined on $\mathbb{D} \cap \{\operatorname{Im} z \geq 0\}$. For us, the main example of an analytic subset with real boundary on Γ_c is the image of a holomorphic disc $u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \Gamma_c)$.

Lemma 3.6. *Let* ω *be a symplectic form on* $B^{2n}(1) \subset \mathbb{C}^n$, *compatible with* J_{st}. *There exists a constant* $\hbar$, *that depends only on* ω, *such that:*

(i) For any proper analytic subset X *of* $B^{2n}(1)$ *of complex dimension* 1, *with* $0 \in X$, *the symplectic area* $\mathcal{A}_\omega(X)$ *verifies* $\mathcal{A}_\omega(X) \geq \hbar$.

(ii) For any proper analytic subset X *of* $B^{2n}(1)$ *of complex dimension* 1, *with real boundary on* $\mathbb{R}^n$ *and* $0 \in \partial X$, $\mathcal{A}_\omega(X) \geq \frac{\hbar}{2}$.

Proof: Both assertions are well-known when $\omega = \omega_{\mathrm{st}}$ (this is the monotonicity property, see for instance [MS12, p.72, section 4.1]). Since ω_{st} and $\omega_0 := \omega(0)$ are J_{st}-compatible, they have symplectic basis of the form $(e_1, J_{\mathrm{st}}e_1, \ldots, e_n, J_{\mathrm{st}}e_n)$, $(f_1, J_{\mathrm{st}}f_1, \ldots, f_n, J_{\mathrm{st}}f_n)$. The J_{st}-linear map A that takes e_i to f_i therefore transports ω_0 to ω_{st}. Then $A(X)$ is an analytic subset passing through 0, proper in $A(B(1)) \supset B(r)$, for some $r > 0$ that depends only on A, hence on ω_0.

Let us prove (i). The monotonicity lemma guarantees that $\mathcal{A}_{\omega_{\mathrm{st}}}(AX \cap B(\varepsilon)) \geq \pi\varepsilon^2$ for all $\varepsilon < r$. Since moreover $A^*\omega_0 = \omega_{\mathrm{st}}$, we have $A^*\omega = \omega_{\mathrm{st}} + R$, where $R \in O(\varepsilon)$ on $B(\varepsilon)$, so

$$\mathcal{A}_{A^*\omega}(AX \cap B(\varepsilon)) = \int_{AX \cap B(\varepsilon)} A^*\omega = \int_{AX \cap B(\varepsilon)} \omega_{\mathrm{st}} + R \geq \mathcal{A}_{\omega_{\mathrm{st}}}(AX \cap B(\varepsilon)) - C\varepsilon\mathcal{A}_{g_{\mathrm{st}}}(AX \cap B(\varepsilon)),$$

where $\mathcal{A}_{g_{\mathrm{st}}}$ stands for the euclidean area, and C depends only on ω. Now since AX is an analytic set, $\mathcal{A}_{g_{\mathrm{st}}}(AX \cap B_\varepsilon) = \mathcal{A}_{\omega_{\mathrm{st}}}(AX \cap B_\varepsilon)$, so we get

$$\mathcal{A}_{A^*\omega}(AX \cap B(\varepsilon)) \geq \mathcal{A}_{\omega_{\mathrm{st}}}\big(AX \cap B(\varepsilon)\big)(1 - C\varepsilon) \geq \pi\varepsilon^2(1 - C\varepsilon).$$

For some small enough $\varepsilon_0 < r$, we therefore see that $\mathcal{A}_{A^*\omega}(AX \cap B(\varepsilon_0)) \geq \frac{\pi}{2}\varepsilon_0^2$. Finally, since $\varepsilon_0 < r$, $B(r) \subset A(B(1))$ and X is analytic,

$$\mathcal{A}_\omega(X \cap B(1)) \geq \mathcal{A}_\omega(X \cap A^{-1}(B(\varepsilon_0))) = \mathcal{A}_{A^*\omega}(AX \cap B(\varepsilon_0)) \geq \frac{\pi}{2}\varepsilon_0^2 = \hbar.$$

In order to prove (ii), we consider a proper analytic set with real boundary on $\mathbb{R}^n$. Recall that this means that $(X, \partial X) \subset (B^{2n}(1), \mathbb{R}^n)$, and that every local branch B of

X near a point $x \in \partial X$ can be parametrized by a holomorphic function f_B defined on a neighbourhood of 0 in $\mathbb{H}$, and we assume here that $f_B(\mathbb{R}) \subset \mathbb{R}^n$. Then $X \cup \sigma(X)$ (where $\sigma(z_1, \dots, z_n) = (\overline{z}_1, \dots, \overline{z}_n)$) is analytic on X, on $\sigma(X)$, and around the points of $\partial X = \partial\sigma(X)$ by the reflection principle, applied to each component of f_B. Thus $X \cup \sigma(X)$ is a proper analytic set of $B^{2n}(1)$, without boundary, which passes through 0, so its area is at least $\hbar$ by the first part of the argument. The first part of the argument also shows that

$$\mathcal{A}_\omega(X) \geq (1 - C\varepsilon_0)\mathcal{A}_{\omega_{\mathrm{st}}}(X \cap B(\varepsilon_0)) = \frac{1}{2}(1 - C\varepsilon_0)\mathcal{A}_{\omega_{\mathrm{st}}}(X \cup \sigma(X) \cap B(\varepsilon_0)) \geq \frac{1}{2}\hbar. \qquad \square$$

Lemma 3.7. *Any non-constant J_{st}-holomorphic disc with boundary on Γ_c is a branched covering of a vertical disc.*

Proof: Let $u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \Gamma_c)$ be a J_{st}-holomorphic disc. We can write $u = (u_1, u', u_2)$, where $u_1 : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}, S^1)$, $u' : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^{k-1}, \{\mathrm{Re}\, z' = c\})$, $u_2 : (\mathbb{D}, \partial\mathbb{D}) \to (i\mathbb{R}^l \times \mathbb{T}^l, \{x = 0\})$, and all of these maps are J_{st}-holomorphic discs. Then u', u_2 are easily seen to be constant discs by Stokes formula for instance (the integrals of λ_{st} over their boundaries vanish). And u_1 takes values in $\mathbb{D}$ by the maximum principle. Finally, $u_1 : \mathbb{D} \to \mathbb{D}$ is a proper map, hence a branched covering of $\mathbb{D}$. $\square$

Proof of proposition 3.4: Let $M, R > 0$, $J_\Lambda \in \mathcal{J}(\omega)$, $J \in \mathcal{J}(R, 4M, J_\Lambda)$ and $u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \Gamma_{R,c})$ be a J-holomorphic disc of area 1 that intersects $F_{R,M}$. Notice that if this disc lies in $F_{R,4M}$, it is J_{st}-holomorphic hence vertical by lemma 3.7. We therefore assume throughout this proof that $u(\mathbb{D})$ intersects both $F_{R,M}$ and $Y \backslash F_{R,4M}$. We are then in one of three possible situations:

- Either $u(\partial\mathbb{D}) \subset F_{R,3M}$. Then since $\mathrm{Im}\, u$ intersects $Y \backslash F_{R,4M}$, there is a euclidean ball B of radius $M/2$ centered on $\mathrm{Im}\, u$ that lies in $F_{R,4M} \backslash F_{R,3M}$. Then $u(\mathbb{D}) \cap B$ is a proper analytic subset of B (because $J = J_{\mathrm{st}}$ on $F_{R,4M}$), so has area at least $\pi M^2/4$ by the monotonicity lemma. Since u has total area 1, this situation does not occur if we choose $M \geq 2$.

- Or $u(\partial\mathbb{D}) \cap F_{R,2M} = \emptyset$. Since u visits $F_{R,M}$, the same argument shows that u has area at least $\pi M^2/4$, so does not occur again if $M \geq 2$.

- Or $u(\partial\mathbb{D})$ intersects both $F_{R,2M}$ and $Y \backslash F_{R,3M}$. Denote by δ_0 the euclidean diameter of the J_{st}-holomorphic ball $\varphi : (B(\varepsilon_0), \mathbb{R}^n) \to (Y, \Gamma_0)$ centered at $p_0 := (z_1 = 1, x = y = z' = 0)$ provided by lemma 3.5. By assumption, $u(\partial\mathbb{D})$ is connected and intersects both $F_{R,2M}$ and $Y \backslash F_{R,3M}$ so its intersection with $F_{R,3M} \backslash F_{R,2M}$ has diameter at least M. Thus, we can center at least $k := \lfloor M/2\delta_0 \rfloor$ disjoint euclidean balls of radii δ_0 on points $p_j \in u(\partial\mathbb{D}) \cap F_{R,3M} \backslash F_{R,2M}$ ($j \in [1, k]$), and these balls lie in $F_{R,4M}$ if $M \geq \delta_0$. Denote by τ_j the composition of a translation in the (z', y)-factor and rotation in the z_1-factor that brings p_0 to p_j. Then, the $\tau_j \circ \varphi(B(\varepsilon_0))$ provide k disjoint J_{st}-balls centered on the p_j. Moreover, since ω is invariant by the τ_j, the pull-backs $(\tau_j \circ \varphi)^*\omega$ are a fixed symplectic form on $B^{2n}(\varepsilon_0)$. As a result, the area of u is at least the sum

of the areas of the intersections of $u(\mathbb{D})$ with these k balls, each of which being at least some constant $\hbar$ by lemma 3.6.(ii) (this constant depends only on φ by the discussion above). Thus the area of u is at least $k\hbar \geq \big({}^{M}/_{2\delta_0} - 1\big)\hbar$. Taking $M > 2\delta_0\big(1 + {}^{1}/_{\hbar}\big)$, the area of u exceeds 1, so this situation does not happen either.

We finally conclude that if $M > \max\{2, \delta_0, 2\delta_0\big(1+\frac{1}{\hbar}\big)\}$, the only discs of area 1 with boundary on $\Gamma_{R,c}$ that intersect $F_{R,M}$ are the vertical ones. $\square$

3.2 Filling by holomorphic discs

Recall our notations: $\mathcal{J}(\omega)$ is the set of ω-compatible almost complex structures on $Y = \mathbb{C}^k \times i\mathbb{R}^l \times \mathbb{T}^l$, $Z_R = \mathbb{D} \times \{(z', 0, y),\ \|(z', y)\| < R\}$, $\Gamma_R = S^1 \times \{(z', 0, y),\ \|(z', y)\| \leq R\}$ is the "horizontal" part of ∂Z_R, while $F_{R,M} = \mathbb{D} \times \{(z', x, y),\ \|x\| \leq M,\ R - M \leq \|(z', y)\| \leq R + M\}$ is the M-neighbourhood of the vertical part of ∂Z_R. The constant M is chosen large enough so that the conclusion of proposition 3.4 holds, and R is then chosen large compared to M, so that $F_{R,M}$ does not intersect Z_R. A map $f : X \times \mathbb{T}^l \to Y$ is given (either a Lagrangian embedding, or parametric embedding of the ball, as dicussed previously), whose image is surrounded by a small neighbourhood V_Λ. We assume that $V_\Lambda \subset Y \backslash (\Gamma_R \cup F_{R,4M})$. We aim at filling Γ_R with holomorphic discs for good complex structures in $\mathcal{J}(\omega)$. In order to apply the Fredholm theory, we rather fill all Lagrangian leaves $\Gamma_{R,c}$, $c \in \mathcal{D} := \overline{B}^{k-1}(R) \subset \mathbb{R}^{k-1}$ in a consistent way. To prove theorem 3.1, we will further need a parametric version of this filling. Throughout this section, T stands for a parameter space, which is a smooth closed manifold, $J_\Lambda : T \times \mathcal{D} \to \mathcal{J}(\omega)$ is any $\mathcal{C}^\ell$-smooth map ($\ell \geq 1$), and $J : T \times \mathcal{D} \to \mathcal{J}(\omega)$ is a smooth map that verifies $J_{(t,c)|V_\Lambda} \equiv J_\Lambda$ and $J \equiv J_{\mathrm{st}}$ on $F_{R,4M}$. Formally, J is a $\mathcal{C}^\ell$-section of the bundle $\mathcal{J}(R, 4M, J_\Lambda) \to T \times \mathcal{D}$, whose fiber at $(t, c) \in T \times \mathcal{D}$ is

$$\mathcal{J}(R, 4M, J_\Lambda(t, c)) := \{J \in \mathcal{J}(\omega),\ J_{|V_\Gamma \cup \{\|(z,y)\| > K\}} \equiv J_{\mathrm{st}},\ J_{|V_\Lambda} \equiv J_\Lambda(t, c)\},$$

where V_Γ is any fixed neighbourhood of $\Gamma \cup F_{R,4M}$. Also, $\pi_2(Y, \Gamma_c)$ is generated by the class of vertical discs (parametrized injectively and holomorphically). The image of this class by the Hurewitz morphism $\pi_2(Y, \Gamma_c) \to H_2(Y, \Gamma_c)$ is denoted by E_c. The spaces of interest for us are

$$\mathcal{M}(J) := \{(t, c, u),\ (t, c) \in T \times \mathcal{D},\ u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \Gamma_{R,c}),\ \overline{\partial}_{J_{(t,c)}} u = 0,\ [u] = E_c\}$$

and what will turn out to be the interior of $\mathcal{M}(J)$, obtained by replacing the closed ball $\mathcal{D}$ by the open ball $\overset{\circ}{\mathcal{D}} := B^{k-1}(R)$ and $\Gamma_{R,c} = S^1 \times \{(\mathrm{Im}\, z', c, y),\ \|(\mathrm{Im}\, z', y)\| \leq R\}$ by $\overset{\circ}{\Gamma}_{R,c} := S^1 \times \{(\mathrm{Im}\, z', c, y),\ \|(\mathrm{Im}\, z', y)\| < R\}$:

$$\mathcal{M}'(J) := \{(t, c, u),\ (t, c) \in T \times \overset{\circ}{\mathcal{D}},\ u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \overset{\circ}{\Gamma}_{R,c}),\ \overline{\partial}_{J_{(t,c)}} u = 0,\ [u] = E_c\}$$

Denoting τ_c the translation of vector $c \in \mathbb{R}^{k-1}$, the spaces $\mathcal{M}(J)$ and $\mathcal{M}'(J)$ are in one-to-one correspondence via the map $(t, c, u) \to (t, c, \tau_{-c} \circ u)$ with the following spaces, more

suited to our analysis:

$$\mathcal{M}_0(J) := \{(t,c,u),\ (t,c) \in T \times \mathcal{D},\ u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \Gamma_{R,0}),\ \overline{\partial}_{\tau_c^* J_{(t,c)}} u = 0,\ [u] = E_0\}.$$
$$\mathcal{M}_0'(J) := \{(t,c,u),\ (t,c) \in T \times \mathring{\mathcal{D}},\ u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \mathring{\Gamma}_{R,0}),\ \overline{\partial}_{\tau_c^* J_{(t,c)}} u = 0,\ [u] = E_0\}.$$

Notice that the almost complex structures $\tau_c^* J_{(t,c)}$ do not belong anymore to $\mathcal{J}(R, 4M, J_\Lambda(t,c))$ but to

$$\mathcal{J}(R, 4M, c, J_\Lambda) := \{J \in \mathcal{J}(\omega) \mid J_{|V_\Lambda} \equiv \tau_c^* J_\Lambda(t,c), J_{|\tau_c^*(V_\Gamma \cup \{\|(y,z)\| > K\})} \equiv J_{\mathrm{st}}\}.$$

Also, $\mathcal{M}(J)$ is endowed with an action of $G := \mathrm{PSL}_2(\mathbb{R}) \simeq \mathrm{Aut}(\mathbb{D}, j)$ by source reparametrization of u.

Lemma 3.8. *If $(t,c,u) \in \mathcal{M}_0(J)$ and $\mathrm{Im}\, u \subset \tau_c^{-1}(F_{R,4M})$, u is a vertical disc, and $\tau_c^* J_{(t,c)}$ is Fredholm-regular with respect to u.*

The Fredholm theory and its usual notations are assumed in the following proof. The reader can consult [MS12], or look at the proof of lemma 3.10 for few details.

Proof: $\tau_c^* J_{(t,c)} = J_{\mathrm{st}}$ on $\tau_c^{-1}(F_{4R,M})$, so the first part of the assertion directly follows from lemma 3.7. Moreover $\tau_c^* J_{(t,c)}$ is linear (constant) on this region, so the derivative

$$D_u \xi := D\overline{\partial}_{\tau_c^* J_{(t,c)}}(u)\xi = \overline{\partial}_{\tau_c^* J_{(t,c)}} \xi = \overline{\partial}\xi.$$

Therefore, if ξ belongs to $\ker D_u$, ξ is easily seen to be a constant in an $n-1$-real parameter space, once modded out the source reparametrization. It follows that $\dim \ker D\overline{\partial}_{\tau_c^* J_{(t,c)}}(u) = n-1$, while the index of $D\overline{\partial}_{\tau_c^* J_{(t,c)}}(u)$ is $n-1$, so $D\overline{\partial}_{\tau_c^* J_{(t,c)}}(u)$ is indeed surjective. $\square$

Lemma 3.9. *If $(t,c,u) \in \mathcal{M}_0(J)$, then u is somewhere injective, and almost all points of $\mathrm{Im}\, u$ have exactly one preimage.*

This point is classical because the class E has least area in $H_2(M, \Gamma_c)$. Here is nonetheless a full proof.

Proof: We first recall that the critical set $\mathcal{C}(u)$ of a non-constant J-holomorphic disc u is a discrete subset of $\mathbb{D}$, hence negligible (see [MS12, Lemma 2.4.1]). Let $(t,c,\tilde{u}) \in \mathcal{M}_0(J)$, so $u := \tau_c \circ \tilde{u}$ is $J_{(t,c)}$-holomorphic. By [Laz00], there exists a holomorphic disc $v : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \Gamma_c)$ with $\mathrm{Im}\, v \subset \mathrm{Im}\, u$, and $v^{-1}(v(z)) = \{z\}$ for almost all points $z \in \mathbb{D}$. Then,

$$0 < \mathcal{A}_\omega(v) \leq \mathcal{A}_\omega(u) = [\omega][E] = 1.$$

Since moreover $H_2(Y, \Gamma_c)$ is generated by the class $[E]$, it follows that $\mathcal{A}_\omega(v) = 1 = \mathcal{A}_\omega(u)$. Let $D' := u^{-1}(v(\mathbb{D}))$. This is an open subset of $\mathbb{D}$ because each local branch of $u^{-1} \circ v$ is holomorphic (its derivatives are $\mathbb{C}$-linear maps) and non-constant. We also claim that it has full measure in $\mathbb{D}$. Indeed,

$$1 = \mathcal{A}_\omega(u) = \int_{D'} u^*\omega + \int_{\mathbb{D}\setminus D'} u^*\omega \geqslant \mathcal{A}_\omega(v) + \int_{\mathbb{D}\setminus D'} u^*\omega \geq \mathcal{A}_\omega(v) = 1.$$

This chain of inequalities consists therefore of equalities only, so

$$0 = \int_{\mathbb{D}\backslash D'} u^*\omega = \int_{\mathbb{D}\backslash D'\cup\mathcal{C}(u)} u^*\omega.$$

On $\mathbb{D}\backslash(D'\cup\mathcal{C}(u))$, u is a local diffeomorphism and $u^*\omega > 0$. Thus

$$0 = \mathcal{L}\mathrm{eb}\big((\mathbb{D}\backslash D')\backslash\mathcal{C}(u)\big) \geq \mathcal{L}\mathrm{eb}(\mathbb{D}\backslash D') - \mathcal{L}\mathrm{eb}(\mathcal{C}(u)) = \mathcal{L}\mathrm{eb}(\mathbb{D}\backslash D'),$$

which shows that D' has full measure. Define now

$$\mathrm{NI} := \{p \in v(\mathbb{D}) \,|\, \#v^{-1}(\{p\}) \neq 1\}, \quad N_v := v^{-1}(\overline{\mathrm{NI}}), \quad N_u := u^{-1}(\overline{\mathrm{NI}}).$$

It is clear that $\overline{\mathrm{NI}} \subset \mathrm{NI} \cup v(\mathcal{C}(v)) = v(v^{-1}(\mathrm{NI}) \cup \mathcal{C}(v))$. Since $\mathcal{C}(v)$ and $v^{-1}(\mathrm{NI})$ are both negligible sets, and v is smooth, it follows that $\overline{\mathrm{NI}}$ has vanishing 2-dimensional Hausdorff measure. Then

$$N_u = u^{-1}(\overline{\mathrm{NI}}) = \big(u^{-1}(\overline{\mathrm{NI}}) \cap \mathcal{C}(u)\big) \cup \big(u^{-1}\big(\overline{\mathrm{NI}}\backslash u(\mathcal{C}(u))\big)\big).$$

Since $\mathcal{C}(u)$ is discrete, the measure of the first set in the right hand side vanishes. On the other hand, u^{-1} is smooth on $\overline{\mathrm{NI}}\backslash u(\mathcal{C}(u))$, so it preserves the vanishing of the 2-dimensional Hausdorff measure, so the latter is also Lebesgue-negligible. We therefore conclude that $\mathcal{L}\mathrm{eb}(N_u) = 0$. Now $\varphi := v^{-1} \circ u : D'\backslash N_u \longrightarrow \mathbb{D}\backslash N_v$ is a holomorphic map, surjective on $\mathbb{D}\backslash N_v$ because $\mathrm{Im}\, v \subset \mathrm{Im}\, u$, so the map $\deg\varphi : \mathbb{D}\backslash N_v \to \mathbb{Z}$ that associates to each point of $\mathbb{D}\backslash N_v$ the algebraic count of its preimages takes values in $\mathbb{N}^*$. Thus,

$$1 = \int_{\mathbb{D}} u^*\omega = \int_{D'\backslash N_u} u^*\omega = \int_{D'\backslash N_u} (v\circ\varphi)^*\omega = \int_{\mathbb{D}\backslash N_v} \deg\varphi\, v^*\omega \geq \int_{\mathbb{D}} v^*\omega = 1$$

(the last inequality holds because $v^*\omega > 0$). Thus $\deg\varphi = 1$ almost everywhere in $\mathbb{D}\backslash N_v$, therefore constantly equals 1 because φ is holomorphic. Thus $\varphi = v^{-1}\circ u$ is injective on $D'\backslash N_u$, so also is $u_{|D'\backslash N_u}$. This proves the lemma because $D'\backslash N_u$ has full measure and u is smooth. □

Lemma 3.10. *For every smooth section $J \in \Gamma^\ell(\mathcal{J}(R, M, J_\Lambda))$, and for every $\varepsilon > 0$, there exists $J' \in \Gamma^\ell(\mathcal{J}(R, M, J_\Lambda))$, with $d_{\mathcal{C}^\ell}(J, J') < \varepsilon$, such that $\mathcal{M}'_0(J')$ is a smooth submanifold.*

Proof : This is standard and lengthy, so we only sketch the proof briefly, referring to [MS12, §3] for details. One defines the universal moduli space

$$\mathcal{M} := \left\{ (u, J, t, c), \;\middle|\; \begin{array}{l} (t, c) \in T \times \overset{\circ}{\mathcal{D}},\ J \in \mathcal{J}(R, M, J_\Lambda), \\ u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \overset{\circ}{\Gamma}_{R,0}),\ \overline{\partial}_{\tau_c^* J_{(t,c)}} u = 0,\ [u] = E_0 \end{array} \right\}.$$

The first point is to see that $\mathcal{M}$ is a Banach manifold. Let $\mathcal{B}^{k,p} := W^{k,p}((\mathbb{D}, \partial\mathbb{D}) \to (Y, \overset{\circ}{\Gamma}_{R,0}))$, $\mathcal{E}^{k-1,p}$ the Banach vector bundle over $\mathcal{B}^{k,p} \times \mathcal{J}(R, M, J_\Lambda) \times T \times \overset{\circ}{\mathcal{D}}$ whose fiber is

$$\mathcal{E}^{k-1,p}(u, J, t, c) := W^{k-1,p}\big((\mathbb{D}, \partial\mathbb{D}) \to (\Lambda^{(0,1)}_{\tau_c^* J}(u^*TY), \Lambda^{(0,1)}_{\tau_c^* J}(u^*T\Gamma_{R,0}))\big).$$

Then $\mathcal{F}(u,J,t,c) := \overline{\partial}_{\tau_c^* J} u$ defines a differentiable section of this Banach bundle, whose zero-section is precisely $\mathcal{M}$. The derivative of $\mathcal{F}$ at the zero-section can be computed explicitely (using any connection on $\mathcal{J}(R,M,J_\Lambda)$, we identify $T_{(t,c,J)}\mathcal{J}(M,J_\Lambda)$ with a subset of $T_tT \times T_c\mathbb{R}^{k-1} \times T_J\mathcal{J}(\omega)$):

$$d\mathcal{F}(\xi,Y,\delta t,\delta c) = D_u\xi + \frac{1}{2}\tau_c^*\left(Y(u) + \frac{\partial J}{\partial c}(u)\delta c + \frac{\partial J}{\partial t}(u)\delta t\right) \circ du \circ j,$$

where $D_u(\xi) := D\overline{\partial}_{\tau_c^* J}(u)\xi$. We claim that this differential is surjective at all $(u,t,c,J) \in \mathcal{M}$. Since D_u is onto for the elements $u \in \mathcal{M}$ which are contained in $\tau_c^{-1}(F_{R,4M})$ by lemma 3.8, $d\mathcal{F}(u,J,t,c)$ is onto for all these elements. Let (u,J,t,c) be another element of $\mathcal{M}$. To show that $d\mathcal{F}(u,J,t,c)$ is surjective, consider an element $\eta \in (T_{\mathcal{F}(u,J,t,c)}\mathcal{E}^{k-1,p})^*$ of the dual that anihiliates the image of $d\mathcal{F}(u,J,t,c)$. Then in particular, for all elements $(\xi,Y) \in T_{(u,J)}\mathcal{B}^{k,p} \times \mathcal{J}(M,J_\Lambda(t,c))$,

$$\langle D_u\xi,\eta\rangle_{L^2} = 0 \text{ and } \langle Y(u)\circ du\circ j,\eta\rangle_{L^2} = 0.$$

The first equation guarantees that if η vanishes on some region, it vanishes identically. Now $\operatorname{Im} u$ intersects the region $Y\backslash(F_{R,4M} \cup V_\Lambda)$, where, by lemma 3.9 there exists a somewhere injective point of u, and where J is unconstrained by definition of $\mathcal{J}(M,J_\Lambda(t,c))$. Using the freedom on Y at such a point, the classical argument shows that $\eta \equiv 0$. It follows that $d\mathcal{F}(u,J,t,c)$ is surjective also at this point $(u,J,t,c) \in \mathcal{M}$. Thus, 0 is a regular value of $\mathcal{F}$ and $\mathcal{M} = \mathcal{F}^{-1}(0)$ is a Banach manifold.

Now the map $\pi : \mathcal{M} \to \mathcal{J}(R,M,J_\Lambda)$ is a Fredholm map, so for any smooth section $J : T \times \overset{\circ}{\mathcal{D}} \to \mathcal{J}(R,M,J_\Lambda)$, there is a $\mathcal{C}^\ell$ ε-small deformation P' of the submanifold $P := \{(J(t,c),t,c)\}$, such that π is transverse to P' [Sma65]. Such a perturbation is obviously the graph over $T \times \mathcal{D}$ of a section J', which verifies $d_{\mathcal{C}^\ell}(J,J') < \varepsilon$. The transversality ensures that $\pi^{-1}(P') = \mathcal{M}_0'(J')$ is a smooth manifold. □

We will call henceforth *generic* the sections $J : T \times \mathcal{D} \to \mathcal{J}(R,M,J_\Lambda)$ such that $\mathcal{M}_0'(J)$, hence $\mathcal{M}'(J)$ is a smooth submanifold. We now address the compactness.

Lemma 3.11. *Let $J \in \mathcal{J}(R,4M,J_\Lambda)$ and $(t_n,c_n,u_n) \in \mathcal{M}(J)$. After extracting a subsequence,*

- *either there exists a sequence $g_n \in G = \mathrm{PSL}_2(\mathbb{R})$ such that $(t_n,c_n,u_n\circ g_n)$ converges in $\mathcal{M}'(J)$,*
- *or the u_n are vertical discs for $n \gg 1$, $\operatorname{Im} u_n = \mathbb{D} \times \{(z_n',y_n)\}$, with $(z_n',y_n) \to \partial B_\infty(R)$.*

Proof: We argue throughout this proof modulo extraction of subsequence. Since T is assumed to be compact, we can assume that (t_n,c_n) converges to $(t,c) \in T \times \mathcal{D}$ so $J_{(t_n,c_n)}$ converges to $J_{(t,c)}$. Then (u_n) is a sequence of $J_{(t_n,c_n)}$-holomorphic discs with boundaries in Γ_{R,c_n}, representing the class E_{c_n}. By proposition 3.4, either the u_n intersect $F_{R,M}$ and then are vertical discs so the conclusion of the lemma is achieved, or their images remain in $Y\backslash F_{R,M}$ (and obviously in a compact set of Y by maximum principle since $J = J_{\mathrm{st}}$ at

infinity). In the latter case, (u_n) is a sequence of $J_{(t_n,c_n)}$-holomorphic discs with boundaries in a subset of Γ_{R,c_n} contained in a neighbourhood of a compact subset of $\Gamma_{R,c}$. Their images are moreover contained in a compact subset of Y, and have bounded symplectic area. Then Gromov's compactness theorem guarantees the existence of an accumulation point of (u_n) (in the sense of Gromov), which is a non-constant $J_{(t,c)}$-bubble tree. Since however $\pi_2(Y) = 0$ and the classes $[u_n] = E_{c_n}$ converge to E_c, which has least area among the non-constant symplectic discs with boundary on Γ_c, the bubble tree can have only one component. This means that there is $g_n \in G$ such that $u_n \circ g_n$ converges to an element of $\mathcal{M}(J)$. $\square$

We fix a generic section $J : T \times \mathcal{D} \to \mathcal{J}(R, 4M, J_\Lambda)$ until the end of this section, and we define

$$\begin{aligned} W &:= \mathcal{M}(J) \times_G \overline{\mathbb{D}} := (\mathcal{M}(J) \times \overline{\mathbb{D}})_{/G}, \\ W' &:= \mathcal{M}'(J) \times_G \mathbb{D}, \end{aligned}$$

where $g \in G$ acts on (u, t, c, w) by $g \cdot (u, t, c, w) := (u \circ g, t, c, g^{-1}(w))$. We also put

$$W^\partial := W \backslash W' = \big(\mathcal{M}(J) \backslash \mathcal{M}'(J)\big) \times_G \overline{\mathbb{D}} \cup \mathcal{M}'(J) \times_G \partial\mathbb{D} =: W^\partial_{\mathrm{V}} \cup W^\partial_{\mathrm{H}}.$$

We now explain that W has the structure of a smooth compact manifold with boundary W^∂ (and corners). Notice first that an elements of $\mathcal{M}(J)$ having area 1, it cannot be a non-trivial covering of a vertical disc, so lemma 3.7 implies:

Lemma 3.12. *The map*

$$\begin{array}{rcccl} i & : & T \times \{\|(z', y)\| = R\} \times \overline{\mathbb{D}} & \longrightarrow & W^\partial_V \\ & & (t, z', y, w) & \longmapsto & [t, \mathrm{Re}\, z', w \mapsto (w, z', y, 0), w] \end{array}$$

is a one-to one correspondence.

Also, lemma 3.10 provides $\mathcal{M}'(J)$ with a structure of smooth manifold, for which the action of G on $\mathcal{M}'(J) \times \overline{\mathbb{D}}$ is smooth, proper and free. As a result, W' is a smooth manifold, W^∂_{H} lies in the closure of W', as a boundary of W.

Lemma 3.13. *W is a compact manifold with W^∂ as boundary and corners along $W^\partial_H \cap W^\partial_V$.*

Proof: Let $[t_n, c_n, u_n, w_n]$ be a sequence in W'. We can extract a subsequence for which $(t_n, c_n) \to (t, c) \in T \times \mathcal{D}$. Lemma 3.11 leaves us with an alternative, up to extracting a subsequence. Either $\exists g_n \in G$ such that $u_n \circ g_n$ converges to $u \in \mathcal{M}'(J)$. We can further extract so that $g_n^{-1}(w_n)$ converges to $w \in \overline{\mathbb{D}}$, and $[t_n, c_n, u_n, w_n] \to [t, c, u, w] \in W' \cup W^\partial_{\mathrm{H}}$. Or the u_n are vertical discs of the form $u_n(z) = (g_n(z), z'_n, y'_n)$, where $(z'_n, y'_n) \to \partial B(R)$ and $g_n : \mathbb{D} \to \mathbb{D}$ are ramified coverings. Since the u_n have area 1 by assumption, these coverings are in fact automorphisms of the discs. Extracting again from $g_n^{-1}(w_n)$ a converging subsequence, we see that $[t_n, c_n, u_n, w_n]$ converge to an element of W^∂_{V}. Notice also that

$$\begin{aligned} [t_n, c_n, u_n, w_n] &= [t_n, c_n, z \mapsto (g_n(z), z'_n, y'_n), w_n] \\ &= [t_n, c_n, z \mapsto (z, z'_n, y'_n), g_n^{-1}(w_n))] \end{aligned}$$

Since t_n, c_n, z'_n, y'_n are arbitrary and $g_n^{-1}(w_n)$ is any point in $\mathbb{D}$ (since g_n is any element in $\mathrm{PSL}_2(\mathbb{R})$ which acts transitively on $\mathbb{D}$), this gives a parametrization of a neighbourhood of $W_{\mathrm{V}}^{\partial}$ by $T \times \mathcal{D} \times \mathbb{D} \times \{R - \varepsilon < \|(z'_n, y'_n)\| \leq R\}$, hence the fact that the points of $W_{\mathrm{V}}^{\partial} \backslash W_{\mathrm{H}}^{\partial}$ are boundary points, and the points of $W_{\mathrm{V}}^{\partial} \cap W_{\mathrm{H}}^{\partial}$ are corners. $\square$

The space $\mathcal{M}(J) \times_G \overline{\mathbb{D}}$ comes with a natural evaluation

$$\begin{array}{rccc} \sigma & : & W & \longrightarrow & T \times Y \\ & & [u,t,c,w] & \longmapsto & (t, u(w)). \end{array}$$

As a quotient of a G-invariant smooth map on $\mathcal{M}(J) \times \overline{\mathbb{D}}$, σ is smooth. It also sends $W_{\mathrm{H}}^{\partial} = \mathcal{M}(J) \times_G \partial\mathbb{D}$ to $T \times \Gamma_R$ by construction. Notice that lemma 3.11 implies that $W_{\mathrm{H}}^{\partial}$ is a smooth manifold with boundary $W_{\mathrm{H}}^{\partial} \cap W_{\mathrm{V}}^{\partial}$. By lemma 3.12, $\sigma_{|\partial W_{\mathrm{H}}^{\partial}} : \partial W_{\mathrm{H}}^{\partial} \to T \times \partial\Gamma_R$ is a degree 1 map, hence so is $\sigma_{|W_{\mathrm{H}}^{\partial}} : W_{\mathrm{H}}^{\partial} \to T \times \Gamma_R$. Lemma 3.12 also states that $\sigma_{|W_{\mathrm{V}}^{\partial}} : W_{\mathrm{V}}^{\partial} \to T \times \mathbb{D} \times \{\|(z', y)\| = R,\ x = 0\}$ has degree 1. A consequence of these two points is that $\mathrm{Im}\,\sigma$ is homologous to $T \times Z_R$, relative to $T \times \partial Z_R$. Indeed, the concatenation $(T \times Z_R) \star \mathrm{Im}\,\sigma$, provides an element of $H_{\dim Z_R + \dim T}(T \times Y)$, well-defined because $\partial\sigma$ has degree 1, and because $T \times Z_R$ and $\mathrm{Im}\,\sigma$ coincide over $T \times \partial Z_R$. But this homology group is

$$H_{2k+l+\dim T}(T \times Y) = H_{2k+l+\dim T}(T \times \mathbb{C}^k \times i\mathbb{R}^l \times \mathbb{T}^l) \simeq H_{2k+l+\dim T}(T \times \mathbb{T}^l) = 0 \text{ (because } k > 0).$$

The image of σ is moreover covered by $J_{(t,c)}$-holomorphic discs, for $(t,c) \in T \times \mathbb{R}^k$. Summarizing the discussion of this paragraph, we have obtained:

Theorem 3.14. *Let $M > 0$ be such that the conclusion of proposition 3.4 holds, and*

- $Y := (\mathbb{C}^k \times i\mathbb{R}^l \times \mathbb{T}^l, \omega = \pi_* \omega_{\mathrm{st}})$, $\quad \mathcal{D}_R := \{\|(z', y)\|_\infty \leq R\} \subset \mathbb{C}^{k-1} \times \mathbb{R}^l$,
- $\Gamma_R := S^1 \times \mathcal{D}_R \times \{x = 0\} \subset Y$, $\quad Z_R = \mathbb{D} \times \overset{\circ}{\mathcal{D}}_R \times \{x = 0\} \subset Y$,
- *$F_{R,M}$ an M-neighbourhood of $\mathbb{D} \times \partial\mathcal{D}_R \times \{x = 0\}$ in the $\|(z_1, z', x, y)\|_\infty$-norm,*
- $\Lambda \Subset Y \backslash (\Gamma_R \cup F_{R,4M})$, $\quad V_\Lambda \in \mathrm{Op}\,(\Lambda, Y \backslash (\Gamma_R \cup F_{R,4M}))$, $\quad J_\Lambda : T \times Y \to \mathcal{J}(\omega)$,
- $\mathcal{J}(R, 4M, J_\Lambda) := \{(t,c,J) \in T \times \mathcal{D} \times \mathcal{J}(\omega) \,|\, J_{|V_\Lambda} \equiv J_\Lambda(t,c),\ J \equiv J_{\mathrm{st}}$ *on* $\{|(z,y)| > K\} \cup F_{R,4M}\}$ *(for some $K \gg 1$).*

For any generic section $J : T \times \mathcal{D}_R \to \mathcal{J}(R, 4M, J_\Lambda)$ - which exists in any $\mathcal{C}^\ell$-neighbourhood of a given smooth section -, there exists a manifold with boundary and corners W and a smooth map $\sigma : (W, \partial W) \to (T \times Y, T \times \partial Z_R)$ such that:

(i) $[\mathrm{Im}\,\sigma] = [T \times Z(R)] \in H(T \times Y, T \times \partial Z_R, \mathbb{Z})$,

(ii) For all $p \in \mathrm{Im}\,\sigma$, there exist $(t,c) \in T \times \mathcal{D}_R$, $u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \Gamma_{R,c})$ such that $p \in \{t\} \times \mathrm{Im}\,u$, $\mathcal{A}_\omega(u) = 1$, $\overline{\partial}_{J(t,c)} u = 0$.

There is nothing to prove here, since all proofs have been done thoughout this section. A remark is however in order. In order to get the point (i), one needs to know the orientability of the moduli spaces, which has not been addressed here. This orientability is

now understood and holds in our specific situation, but if one wishes to forget it so as to consider this paper mostly self-contained, the point (i) has to be replaced by

$$[\operatorname{Im}\sigma] = [T \times \mathbb{Z}_R] \in H(T \times Y, T \times \partial Z_R, \mathbb{Z}_2).$$

This is a weaker statement, which is enough for our applications to the $\mathcal{C}^0$-rigidity of the reduction (see below), and to theorems 3.2 and 3.1, provided the knotting holds in the $\mathbb{Z}_2$-coefficient setting. For theorem 3.1 for instance, it amounts to assuming that $\Phi(\mathbb{R}^l \times \{0\})$ intersects Z_R transversally, an odd number of times.

3.3 Proof of theorem 3.1

Let $\Phi : B^{2n}(a) \times \mathbb{R}^l \to \mathbb{C}^n$ be a parametric symplectic embedding, which satisfies the assumption of theorem 3.1 with $C := 4M$, M being the constant obtained in the previous paragraph, in particular theorem 3.14. Thus, (i) for $|t| \geq K$, $\Phi(\cdot, t) = \mathrm{Id} + t$, (ii) Φ is knotted with $\partial Z_R = S^1 \times \mathcal{D}_R \times \{x = 0\}$ (recall that we rescaled the space so that $A = 1$), and (iii) $\operatorname{Im}\Phi$ does not intersect $F_{R,4M}$. We recall that (ii) means that $\operatorname{Im}\Phi \cap \partial Z_R = \emptyset$ and $\Phi(\{0\} \times \mathbb{R}^l)$ has a non-vanishing homological intersection with Z_R, relative to ∂Z_R (say with coefficients in $\mathbb{Z}$, or in $\mathbb{Z}_2$ if one decides to forget about the orientability of the moduli spaces). Restricting Φ to $B^{2n}(a') \times \mathbb{R}^l$, $a' < a$, we can also assume that $\overline{\operatorname{Im}\Phi} \cap \partial Z_R = \emptyset$.

By (i), we can compactify the domain of Φ to $\overline{B^{2n}(a')} \times \mathbb{T}^l$, getting a parametric symplectic embedding $\Phi : \overline{B^{2n}(a')} \times \mathbb{T}^l \to \mathbb{C}^k \times i\mathbb{R}^l \times \mathbb{T}^l = Y$ which still satisfies (ii), meaning that the homological intersection of $[\Phi(\mathbb{T}^l \times \{0\})]$ with $[Z_R]$ relative to ∂Z_R does not vanish. Since $\operatorname{Dom}\Phi$ is now compact, there exists $K > 0$ such that $\operatorname{Im}\Phi \Subset \{|(z, y)| < K\}$. Taking $T := \mathbb{T}^l$, we define $\Lambda(t) := \Phi(\overline{B^{2n}(a')} \times \{t\})$ and $J_\Lambda(t) := \Phi(\cdot, t)_* J_{\mathrm{st}}$. Since $\mathcal{J}(\omega)$ is a fiber bundle over Y with contractible fiber, there exists a smooth path $J(t) \in \mathcal{J}(R, 4M, \Lambda(t))$. Applying theorem 3.14, we get a filling $\sigma : (W, \partial W) \to (T \times Y, T \times \partial Z_R)$ with holomorphic discs. Since $[\Phi(\{0\} \times T)] \cdot [Z_R] \neq 0$, the graph $\operatorname{Graph}(\Phi) := \{(t, \Phi(0, t)), t \in T\}$ has non-vanishing homological intersection with $T \times Z_R$. Moreover, since by theorem 3.14 (i), our filling $\operatorname{Im}\sigma$ is homologous to $T \times Z_R$ relative to $T \times \partial Z_R$, $[\operatorname{Graph}(\Phi)] \cdot [\operatorname{Im}\sigma] \neq 0$ and there exists $t \in \mathbb{T}^l$ such that $(t, \Phi(0, t)) \in \operatorname{Im}\sigma$. By theorem 3.14 (ii), there exists $t' \in \mathbb{T}^l$ and a disc $u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \Gamma_R)$ such that $(t, \Phi(0, t)) \in \{t'\} \times \operatorname{Im}(u)$ (hence $t = t'$ and $\Phi(0, t) \in \operatorname{Im}(u)$), $\mathcal{A}_\omega(u) = 1$, and $\overline{\partial}_J u = 0$ for some $J \in \Gamma^\ell(\mathcal{J}(M, J_\Lambda))$. This disc is therefore symplectic, has total area 1, and its trace on the symplectic ball $\Phi(B^{2n}(a') \times \{t\})$ is $\Phi(\cdot, t)_* J_{\mathrm{st}}$-holomorphic. The classical monotonicity argument then shows that $1 \geq a'$ (see for instance [MS12, p.72, section 4.1]). Since this holds for all $a' < a$, we get $a \leq 1$. $\square$

3.4 Proof of theorem 3.2

Let $\Phi : \mathbb{T}^k \times \mathbb{R}^l \hookrightarrow \mathbb{C}^{k+l}$ be a Lagrangian embedding that satisfies the assumptions of theorem 3.2: (i) $\Lambda := \operatorname{Im}\Phi$ coincides with $S^1(a)^k \times \mathbb{R}^l$ outside a compact set, and (ii) L is knotted with ∂Z_R, meaning that $[\Phi(\{*\} \times \mathbb{R}^l)] \cdot [Z_R] \neq 0$, where coefficients belong to $\mathbb{Z}$ (or

$\mathbb{Z}_2$) and homology is understood as always in these paragraphs relative to the boundary. Finally, Λ does not intersect $F_{R,4M}$.

By (i), we can compactify the domain of Φ to $\mathbb{T}^k \times \mathbb{T}^l$, getting a Lagrangian embedding $\Phi : \mathbb{T}^n \hookrightarrow \mathbb{C}^k \times i\mathbb{R}^l \times \mathbb{T}^l = Y$, which still verifies (ii), and is compactly contained into $\{\|(y,z)\| < K\}$ for some $K \in \mathbb{R}$. We consider $T = \{*\}$, $\Lambda = \operatorname{Im}\Phi$, $V_\Lambda \in \operatorname{Op}(\Lambda, Y \cap \{\|(y,z)\| < K\} \backslash (\Gamma_R \cup F_{R,4M}))$. Applying theorem 3.14, we get for each compatible almost-complex structure J_Λ defined on V_Λ a complex structure $J \in \mathcal{J}(R, 4M, J_\Lambda)$ and a J-holomorphic disc $u : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \Gamma_c)$ with $[u] = E_c$ and $\operatorname{Im} u \cap \Lambda \neq \emptyset$. Moreover, since J is standard at infinity, this disc lies in the compact domain $\{|(y,z)| \leq K\}$, where K does not depend on J_Λ.

We now proceed to a neck-stretching argument. Recall that Λ is the image by an exact lagrangian embedding Φ of $L := S^1(a)^k \times \mathbb{T}^l$. Consider on L the Euclidean metric $g := g_{\mathrm{st}}$. As usual in SFT, we consider a neighbourhood $\mathcal{U}$ of Λ symplectomorphic to a neighbourhood of the zero section in T^*L, endow it with the metric g induced canonically from g on L, and consider $\Sigma := \{g = \varepsilon_0\} \subset \mathcal{U}$. For small enough ε_0, this is a contact type hypersurface contained in V_Λ which splits Y into two pieces $Y^- \cup Y^+$, where Y^- has concave boundary at X (and contains ∂Z_R), and Y^+ has convex boundary (and contains Λ). Now we consider a neck-stretching along Σ. Since $\Sigma \subset V_\Lambda$ is disjoint from $\Gamma_R \cup F_{R,4M}$ and contained into $\{|(y,z)| < K\}$, we can consider a degeneration J_ε of the complex structure with the following properties.

- $J_\Lambda(\varepsilon)$ is defined in $\operatorname{Op}(\Lambda)$ and stretches the neck of Σ when the parameter ε goes to $+\infty$.
- J_ε is generic in $\mathcal{J}(R, 4M, J_\Lambda(\varepsilon))$,
- J_ε is uniformly bounded in the $\mathcal{C}^\ell$-topology outside the neck-stretching zone.

The previous discussion guarantees the existence of a disc $u_\varepsilon : (\mathbb{D}, \partial\mathbb{D}) \to (Y, \Gamma_{c_\varepsilon})$ for some $c_\varepsilon \leq R$, with $\overline{\partial}_{J_\varepsilon} u_\varepsilon = 0$, $[u_\varepsilon] = E_{c_\varepsilon}$, u_ε intersects Λ, and $\operatorname{Im} u_\varepsilon \subset \{|(y,z)| < K\}$. The compactness theorem in SFT thus implies that there is some ε_n such that $u_n := u_{\varepsilon_n}$ converge to a holomorphic building B [BEH$^+$03], whose main features are summarized below:

- It has a main component, in Y^-, which is a J_∞-holomorphic map $u_\infty^{\mathrm{main}} : S \to Y^-$, where S is a punctured disc (with p punctures, $p \geqslant 1$ because u_n intersects Λ, hence Y^+), and whose boundary is sent to Γ_R and has action 1. The action is computed with respect to the standard Liouville form λ on $Y \simeq \mathbb{C}^k \times T^*\mathbb{T}^l$.
- All other components on Y^-, Y^+ are J_∞-holomorphic maps whose domains are punctured spheres (because the u_n are discs).
- The components of the building in Y^- are asymptotic at each puncture to a negative Reeb orbit of $\partial Y^- = \Sigma$ (the boundary of the image is oriented by the opposite of the Reeb flow). Similarly, the components in Y^+ are asymptotic to positive Reeb orbits of Σ.

- There might be intermediate layers: components of B in $\Sigma \times \mathbb{R}$ (the symplectization of Σ). These components are J-holomorphic for some cylindrical almost complex structure, again punctured spheres, asymptotic to positive Reeb orbits at $\Sigma \times \{+\infty\}$ and negative Reeb orbits at $\Sigma \times \{-\infty\}$.

- The total symplectic area of these components is 1, and they glue together to form a topological disc (see figure 2). We will refer to this last property by saying that the building B forms a disc.

- More generally, a subbuilding S of B is any union of its components. Its underlying topological surface is obtained by blowing-up the punctures of the components of the subbuilding (to boundary components), gluing these different boundary components accordingly with the building structure, and blowing-down back those punctures that have not been glued (see the next point for a more formal description of these subbuildings). We say that a subbuilding is connected if its underlying surface is connected. Since B is a disc, any connected subbuilding is either a punctured sphere, or a punctured disc when it contains the main component u_∞^{main}.

- If the reader misses details in the previous definition of subbuildings, here is a more formal description of B and its subbuildings. The building B is made of several components u_i from a domain C_i to either Y^+, Y^- or the intermediate layers $\Sigma \times \mathbb{R}$. Each C_i is a sphere or a disc with punctures $p_{ij} \in \mathcal{P}_i$, and we denote by $\mathcal{P} := \cup \mathcal{P}_i$ the set of all punctures. Each puncture $p_{ij} \in \mathcal{P}$ is associated to a positive or a negative Reeb orbit γ_{ij} or $-\gamma_{ij}$. Finally, there is a pairing between the indices (i, j) that keeps track of the way u_n degenerate to B: the image of u_n for n large is a small perturbation of the curve obtained by blowing-up all the punctures to boundary components α_{ij} and gluing the α_{ij} with $\alpha_{i'j'}$ when (i, j) is paired with (i', j'). Now the underlying surface $C_{\mathcal{S}}$ of a subbuilding $S = \{u_i\}_{i \in \mathcal{S}}$ is obtained in the following way. Let $\mathcal{P}_{\mathcal{S}}$ be the set of punctures of the components of S that are paired with punctures in components which *are not* in S and $\overline{\mathcal{P}_{\mathcal{S}}}$ the other ones. Blow-up the punctures p_{ij} of each C_i, $i \in \mathcal{S}$ which are in $\overline{\mathcal{P}_{\mathcal{S}}}$ to boundary components α_{ij}. And finally define $C_{\mathcal{S}} := (\sqcup_{i \in \mathcal{S}} C_i)_{/\mathcal{R}}$ where $\mathcal{R}$ is the relation that identifies each α_{ij} with $\alpha_{i'j'}$ when (i, j) is paired with (i', j'). Notice that $C_{\mathcal{S}}$ has $\mathcal{P}_{\mathcal{S}}$ as set of punctures.

Denote by $-\gamma_1, \ldots, -\gamma_p$ the (negative) Reeb orbits to which u_∞^{main} is asymptotic. Since B forms a disc, its other components glue together to form p discs asymptotic to $\gamma_1, \ldots, \gamma_p$. Since the γ_i are Reeb orbits, they project to closed geodesics of L under the natural projection $\pi : Y^+ \simeq T^*_{\varepsilon_0} L \to L$ (recall that $Y^+ \simeq \{g < \varepsilon_0\}$). We write $[\pi\gamma_i] = \sum k_j^{(i)} e_j \in H_1(L)$ for the homology classes of these geodesics, where e_j is the class of the j-th S^1-factor in $L \simeq (S^1)^n$, and $k_j^{(i)} \in \mathbb{Z}$. Observe at this point that since γ_i bounds a topological disc, $k_j^{(i)} = 0$ for $j > k$ (because $Y \simeq \mathbb{C}^k \times T^*\mathbb{T}^l$). We will use the following obvious fact:

Fact 3.15. *If γ is a positive Reeb orbit of Σ, the cylinder*

$$\begin{array}{rcccl} \rho_\gamma & : & [0,1] \times S^1 & \longrightarrow & Y^+ \simeq T^*_{\varepsilon_0} L \\ & & (s,t) & \longmapsto & s\gamma(t) \end{array}$$

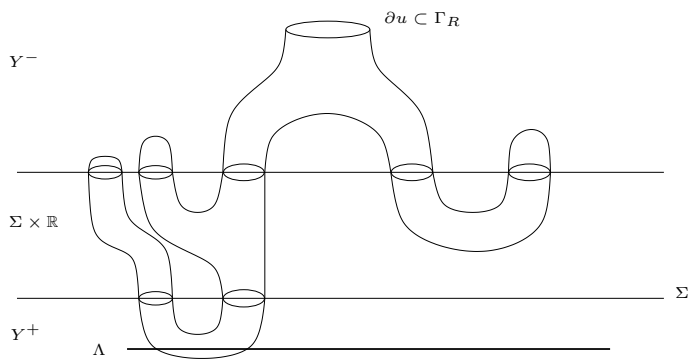

Figure 2: A holomorphic building.

is symplectic, with (oriented) boundaries γ and $-\pi\gamma$.

Lemma 3.16. *For each i, $\sum_{j=1}^{n} k_j^{(i)} > 0$.*

Proof: Let S_i denote the maximal connected subbuilding of B with no component in Y^- and γ_i as a boundary component. In other terms S_i is composed of all the components of B in Y^+ and the intermediate layers $\Sigma \times \mathbb{R}$ that can be connected to γ_i through components of B not in Y^-. This subbuilding cannot have only one puncture because $\pi\gamma_i$ is non-contractible in $Y^+ \subset T^*\mathbb{T}^n$. Therefore, S_i has punctures asymptotic to positive Reeb orbits $\gamma_i, \gamma_i^1, \ldots, \gamma_i^{k_i}$, with $k_i \geqslant 1$. Observe that πS_i provides a chain whose boundary is $[\pi\gamma_i] + \sum[\pi\gamma_i^k]$, so $[\pi\gamma_i] = -\sum[\pi\gamma_i^k]$. But since B forms a disc, there is a disc with positive area - composed of a gluing of different components of B - which is asymptotic to each $-\gamma_i^k$. Gluing to these discs the cylinders $\rho_{\gamma_i^k}$, we get discs with positive area, whose boundaries lie in L and represent the class $-[\pi\gamma_i^k]$ (see figure 3). Therefore,

$$0 < \sum \lambda[-\pi\gamma_i^k] = \lambda[\pi\gamma_i] = \sum_{i=1}^{k} k_j^{(i)}\, a. \qquad \square$$

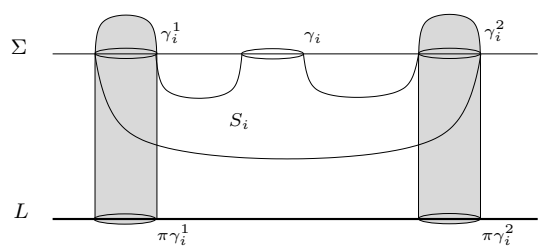

Figure 3: Capping the class $[\pi\gamma_i]$.

The map obtained by gluing to u_∞^{main} the symplectic cylinders ρ_{γ_i} now gives a symplectic surface of area

$$0 < 1 - \sum \lambda[\pi\gamma_i] = 1 - \sum_{i=1}^{p} \left(\sum_j k_j^{(i)} \right) a \leqslant 1 - a,$$

where the last inequality holds because $p \geq 1$ and the previous lemma. We get $1 \geq a$. $\square$

4 Coisotropic $\mathcal{C}^0$-rigidity

4.1 The reduction of a coisotropic submanifold

We now define precisely what we mean by reduction of a coisotropic submanifold. This is just a generalization of the discussion for hypersurfaces, which was held in [BO16].

Definition 4.1. *Let Σ^m be a smooth manifold endowed with a foliation $\mathcal{F}$, where the dimension of each leaf of the foliation equals r.*

1. *We say that an open topological submanifold*[1] *$U^{m-r} \subset \Sigma$ is (topologically) transverse to the foliation $\mathcal{F}$ on Σ if U has a neighbourhood $V \subset \Sigma$ such that U intersects exactly once each leaf of V (where by a leaf of V we mean a connected component of the intersection of V with a leaf of $\mathcal{F}$).*

2. *Let $U, U' \subset \Sigma$ be $(m-r)$-dimensional topological submanifolds, that are transverse to the foliation $\mathcal{F}$. We say that U and U' are equivalent (denoting $U \sim U'$) if there exists a (continuous) homotopy $G : W \times [0,1] \to \Sigma$, $t \in [0,1]$, of a topological manifold W^{m-r}, such that $G_{|W\times\{0\}}$ is a homeomorphism onto U, $G_{|W\times\{1\}}$ is a homeomorphism onto U', and such that for each $x \in W$, the trajectory $t \mapsto G(x,t)$ goes along a leaf of $\mathcal{F}$.*

3. *The reduction of a smooth manifold Σ endowed with a foliation $\mathcal{F}$, denoted by $\mathrm{Red}\,(\Sigma, \mathcal{F})$, is defined as the set of open topological submanifolds $U^{m-r} \subset \Sigma$ which are transverse to the characteristic foliation of Σ, considered modulo the above equivalence relation. If Σ is a coisotropic (or, more generally, a pre-symplectic) submanifold of a symplectic manifold M, and $\mathcal{F}$ is the characteristic foliation on Σ, then we simplify the notation for the reduction to $\mathrm{Red}\,(\Sigma)$.*

Now let us address several points:

Smooth and symplectic structures On a topological submanifold $U \subset \Sigma$ which is transverse to the foliation $\mathcal{F}$, we have a natural structure of a smooth manifold. Moreover, if Σ is a coisotropic (or, more generally, a pre-symplectic) submanifold of a symplectic manifold, and $\mathcal{F}$ is the characteristic foliation on Σ (we will call such situation a "symplectic setting"), then U also inherits a natural symplectic structure. Indeed, let V be a neighbourhood of U in Σ such that U intersects exactly once each characteristics of V, as in definition 4.1. Then any point $z \in U$ has a neighbourhood $U_1 \subset U$ such that U_1

[1] Recall that a topological submanifold of a topological manifold X is a subset $Y \subset X$, such that there exists a topological manifold Z and a map $i : Z \to X$ which is a homeomorphism onto the image $i(Z) = Y$.

lies inside a (smooth) flow-box $\Phi : W_1 \times (0,1)^r \to \Sigma$, where $\operatorname{Im}\Phi \subset V$. Then the map $\varphi := \pi \circ \Phi^{-1} : U_1 \to W_1$ is injective and hence, by the Invariance of Domain theorem, is a homeomorphism onto the open image $\varphi(U_1) \subset W_1$ (here $\pi : W_1 \times (0,1)^r \to W_1$ is the natural projection). The map φ induces a natural smooth structure on U_1, in case of a general foliation, and moreover induces a natural symplectic structure in U_1 if we are in a symplectic setting.

Naturality of the structures If two topological submanifolds $U, U' \subset \Sigma$ are equivalent ($U \sim U'$), then they are diffeomorphic (and moreover symplectomorphic if we are in a symplectic setting) via a homotopy $G : W \times [0,1] \to \Sigma$, as in definition 4.1. Let us describe explicitly the diffeomorphism (resp. symplectomorphism) between U and U'. By continuity of G and since U is topologically transverse to leaves of the foliation $\mathcal{F}$, for any point $z \in W$ and any $t \in [0,1]$ there exists a neighbourhood $W_1 \Subset W$ of z such that the closure of the image $G(W_1 \times \{t\})$ lies inside a (smooth) flow-box $\Phi : W_2 \times (0,1)^r \to \Sigma$, and moreover the map $\varphi_t := \pi \circ \Phi^{-1} \circ G : W_1 \times \{t\} \to W_2$ is injective (here $\pi : W_2 \times (0,1)^r \to W_2$ is the natural projection, as before). Then, by the Invariance of Domain theorem, φ_t is a homeomorphism onto the open image $W_3 := \varphi_t(W_1) \subset W_2$. This induces a smooth (resp. symplectic) structure on $W_1 \times \{t\}$. Moreover, since $W_1 \Subset W$ and $G(W_1 \times \{t\}) \Subset \Phi(W_2 \times (0,1)^r)$, it follows that we also have $G(W_1 \times \{t'\}) \Subset \Phi(W_2 \times (0,1)^r)$ as well and moreover $\varphi_{t'}(z,t') = \varphi_t(z,t)$ for every $z \in W_1$, whenever $t' \in [0,1]$ is sufficiently close to t (here $\varphi_{t'} = \pi \circ \Phi^{-1} \circ G : W_1 \times \{t'\} \to W_2$). Hence the induced smooth (resp. symplectic) structures on $W_1 \times \{t\}$ and on $W_1 \times \{t'\}$ coincide, when t' is sufficiently close to t.

The induced map Let $h : \Sigma \to \Sigma'$ be a homeomorphism between smooth manifolds Σ and Σ' which are endowed with foliations $\mathcal{F}$ and $\mathcal{F}'$, such that h maps $\mathcal{F}$ to $\mathcal{F}'$. Then h defines a natural map $\hat{h} : \operatorname{Red}(\Sigma, \mathcal{F}) \to \operatorname{Red}(\Sigma', \mathcal{F}')$ by $\hat{h}([U]) := [h(U)] \subset \Sigma'$. Clearly, the definition does not depend on the representative U of $[U]$. Recall that by a theorem of Humilière, Leclercq and Seyfaddini [HLS15], if $h : M \to M'$ is a symplectic homeomorphism and if h maps a smooth coisotropic submanifold Σ onto a smooth submanifold Σ', then Σ' is coisotropic, and the restriction $h_{|\Sigma} : \Sigma \to \Sigma'$ preserves the characteristic foliation, and hence we get a natural induced map $\hat{h} : \operatorname{Red}(\Sigma) \to \operatorname{Red}(\Sigma')$.

4.2 Proof of theorem 2

Let us adopt the following notation for the sake of fluidity. Given two submanifolds A, B with boundaries, we say that A is knotted with ∂B, relative to ∂A, if any homotopy of A relative to ∂A that avoids ∂B intersects B. Notice that here, the filling B of ∂B is fixed, and not made explicit in the sentence. In the following still, we will be interested in a knotting with $\partial Z(A,R)$, whose filling is implicitely meant by $Z(A,R)$.

Let h be a symplectic homeomorphism defined in a neighbourhood U of $B^{2k}(1) \times [-1,1]^l$ in $\mathbb{C}^{k+l} = \mathbb{C}^n$, with values in $\mathbb{C}^n$, such that $h\big(B^{2k}(1) \times [-1,1]^l\big) \subset \mathbb{C}^k \times \mathbb{R}^l$. We also assume

for later convenience that U is contractible. We need to prove that for δ small enough, for any $a < \delta$, if $h(B^{2k}(a) \times [-1,1]^l) \subset Z(A) \times \mathbb{R}^l$, then $A \geq a$.

Since h is a symplectic homeomorphism, we know by [HLS15] that it takes the characteristic leaves $\{*\} \times (-1,1)^l$ into other characteristic leaves $\{*\} \times \mathbb{R}^l$. In other terms, there exists a continuous function $\hat{h} : B^{2k}(1) \to \mathbb{C}^k$ and an open subset $\Omega(z) \subset \mathbb{R}^l$ for each point $z \in B^{2k}(1)$ such that

$$h(\{z\} \times (-1,1)^l) = \{\hat{h}(z)\} \times \Omega(z).$$

Since $Z(A)$ is invariant by translation along the z'-axis, we can assume that $\hat{h}(0) = (z_1, 0)$, which we do henceforth (recall that we split $\mathbb{C}^k = \mathbb{C} \times \mathbb{C}^{k-1}$, with (z_1, z') the split coordinates). We divide this proof in four steps:

<u>Step 1: Adjusting $\hat{h}$ and Ω.</u> As already observed in section 4.1, the map $\hat{h}$ is a local homeomorphism. Thus, its restriction to $B^{2k}(R)$ for R small enough is injective, so after maybe restricting h to $B^{2k}(R) \times (-1,1)^l$, we can assume that $\hat{h}$ is injective, which we do henceforth. Moreover, since h is a homeomorphism, the map $z \mapsto \Omega(z)$ is continuous, so by a further restriction, we can also ensure that

$$\Omega(z) \supset \frac{1}{2}\Omega(0) \qquad \forall z \in B^{2k}(R).$$

Finally, since $\Omega(0)$ is homeomorphic to $(-1,1)^l$, there exists a compactly supported diffeomorphism $\varphi : \mathbb{R}^l \to \mathbb{R}^l$ such that $\varphi(\Omega(0))$ is arbitrarily close to $(-4,4)^l$. Letting $\Phi : T^*\mathbb{R}^l \to T^*\mathbb{R}^l$ be the natural lift of φ and considering $(\mathrm{Id} \otimes \Phi) \circ h$ instead of h, we can therefore also assume that $\Omega(0)$ is close to $(-4,4)^l$, and that $\Omega(z) \supset (-2,2)^l$ for all z. Summarizing this discussion, we see that we can assume without loss of generality that

- h is a symplectic homeomorphism defined in a neighbourhood of $B^{2k}(1) \times [-1,1]^l$ with values in $\mathbb{C}^n$, that takes $B^{2k}(1) \times (-1,1)^l$ to a subset
$$\{(z,t) \in W \times \mathbb{R}^l,\ t \in \Omega(z)\},$$
- $\hat{h} : B^{2k}(1) \to \mathbb{C}^k$ is injective,
- $\Omega(z) \supset (-2,2)^l$ for all z.

Obviously, W is open, and if we fix any compact exhaustion W_n of W, there exists therefore $\delta_n > 0$ such that

$$\operatorname{Im} h \supset Q_n := W_n \times (-1,1)^l \times (-\delta_n, \delta_n)^l.$$

Moreover, since h is a homeomorphism, if δ_n is chosen small enough,

$$\operatorname{Im} h \cap Q_n \cap \mathbb{C}^k \times \mathbb{R}^l = h(B^{2k}(1) \times (-1,1)^l).$$

In other terms, $\operatorname{Im}(h)$ intersects the coisotropic subset $\mathbb{C}^k \times \mathbb{R}^l$ in Q_n exactly along $B^{2k}(1) \times (-1,1)^l$.

<u>Step 2: Knotting $\hat{h}$.</u> Let n_0 be such that $\hat{h}(0) \notin \partial W_{n_0}$, and put $R_0 := \frac{1}{2} d(\hat{h}(0), \partial W_{n_0})$ (measured in the infinity norm), so that $W_{n_0} \supset B_\infty(\hat{h}(0), 2R_0)$. For $\delta_1(h) \ll 1$, $d\big(\hat{h}(\partial B(\delta_1(h))), \hat{h}(0)\big) <$

R_0. Since $\hat{h}(0)$ has the form $(z_1, 0)$, if $a \leq \delta_1(h)$ and $\hat{h}(B(a)) \subset Z(A)$, we have $\hat{h}(B(a)) \subset D(A) \times B_\infty(R_0)$. Moreover, the first step above guarantees that

$$h(\{0\} \times (-1,1)^l) \cap D(A) \times B_\infty(R_0) \times B_\infty(\delta_{n_0}) \cap Q_{n_0} = \{\hat{h}(0)\} \times (-1,1)^l$$

Thus, provided that $R < R_0$ and $Z(A,R) \subset Q_{n_0}$, $h(B(a) \times (-1,1)^l)$ is knotted with $\partial Z(A,R)$ in the sense given at the beginning of this section. This condition can be written:

$$R < R_0, \quad A \leq \pi R_0^2, \quad R < \delta_{n_0}. \tag{4.2.1}$$

Indeed, under these conditions, if $p = (z, z', x) \in Z(A,R)$, we have $\|x\|_\infty < R < \delta_{n_0}$ and

$$\begin{aligned} d_\infty((z,z'), \hat{h}(0)) = d_\infty((z,z'),(z_1,0)) \quad &= \max(|z - z_1|, \|z'\|_\infty) \\ &\leq \max(|z - z_1|, R) \\ &\leq \max(|z - z_1|, R_0) \\ &\leq \max(2\sqrt{\tfrac{A}{\pi}}, R_0) \text{ because } z, z_1 \in D(A) \\ &\leq 2R_0. \end{aligned}$$

Thus

$$p = (z, z', x) \in B(\hat{h}(0), 2R_0) \times [-\delta_{n_0}, \delta_{n_0}] \subset W_{n_0} \times [-\delta_{n_0}, \delta_{n_0}] \subset Q_{n_0},$$

so $Z(A,R) \subset Q_{n_0}$.

Notice also that provided A is chosen small enough compared with R, $h\big(B(a) \times (-1,1)^l\big)$ is even knotted with $\partial Z(A,R) \cup F(A,R,CA)$, where the filling of this last set is still by $Z(A,R)$. For this to hold, it is necessary and sufficient that $F(A,R,CA) \cap \mathbb{C}^k \times \mathbb{R}^l = \emptyset$, which holds as soon as

$$R > CA, \tag{4.2.2}$$

and $F(A,R,CA) \subset Q_{n_0}$, which needs in top of (4.2.1)

$$CA < \delta_{n_0}. \tag{4.2.3}$$

To conclude this step, notice that the inequalities (4.2.1)-(4.2.3) can be achieved as soon as

$$A < \delta_2(h) := \min(1, \frac{1}{C}) \min(R_0, \delta_{n_0}, \pi R_0^2)$$

because then R can be chosen between CA and $\min(R_0, \delta_{n_0})$ and the inequalities are then all satisfied.

Step 3: from h to a symplectomorphism. Let now assume that $A < \delta(h) := \min(\delta_1(h), \delta_2(h))$. Since h is a symplectic homeomorphism, it can be approached in the uniform norm by symplectic diffeomorphisms $f_n : U \to \mathbb{C}^n$. Using the contractibility assumption on U, it is easy to see that f_n can be extended to a compactly supported Hamiltonian diffeomorphism of $\mathbb{C}^n$. Moreover, since f_n is close to h in U, the following two points are satisfied for large enough n:

- $f_n\big(B(a) \times (-1,1)^l\big)$ is knotted with $Z(A,R) \cup F(A,R,CA)$ in Q_{n_0},

- $f_n(U) \supset Q_{n_0}$, so $f_n\big(B(a) \times \mathbb{R}^l \backslash (-1,1)^l\big) \cap Q_{n_0} = \emptyset$.

As a result, and since $Z(A,R) \cup F(A,R,CA) \subset Q_{n_0}$, $f_n\big(B(a) \times \mathbb{R}^l\big)$ is knotted with $Z(A,R) \cup F(A,R,CA)$ so corollary 3.3 guarantees that $A \geq a$.

Step 4: from A to a. We have therefore got our non-squeezing inequality when $A < \delta(h)$. It only remains to get it for $a < \delta(h)$ in order to establish our statement. But this is obvious. Indeed, if $a < \delta(h)$, either $A \geq \delta(h)$, and then $A > a$, or $A < \delta(h)$ and then the previous analysis implies that $a \geq A$. □

4.3 Proof of theorem 3

Theorem 3 can be seen as a corollary of theorem 2. Indeed, what we have to prove is that if Σ, Σ' are two $n+1$-dimensional coisotropic submanifolds of $2n$-dimensional symplectic manifolds, if there exists a symplectic homeomorphism $h : (\mathrm{Op}\,(\Sigma), \Sigma) \longrightarrow (\mathrm{Op}\,(\Sigma'), \Sigma')$, the reduction $\hat{h}$ of h is area preserving. Since area preservation is a local property, it is enough to check it for small elements of the reduction. We can therefore assume that both Σ and Σ' are $D^2(1) \times \mathbb{R}^{n-1} \subset \mathbb{C}^n$. We now argue by contradiction. Assume that $\hat{h}$ is not area preserving, and considering maybe h^{-1} instead of h, we can assume that it expands the area of some elements of the reduction. Then there exists a subdisc $D(R_0) \Subset D(1)$ whose area is increased by $\hat{h}$. Moreover, there must exist a point $z \in D(R_0)$ and a decreasing sequence $\varepsilon_n > 0$ that tends to 0 such that the area of $\hat{h}(D(z, \varepsilon_n))$ exceeds ε_n for all n. But this is precisely ruled out by theorem 2. □

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