$\mathcal{C}^0$-rigidity of characteristics in symplectic geometry.

Emmanuel Opshtein.

April 14, 2009

Abstract

The paper concerns a C^0-rigidity result for the characteristic foliations in symplectic geometry. A symplectic homeomorphism (in the sense of Eliashberg-Gromov) which preserves a smooth hypersurface also preserves its characteristic foliation.

Introduction

Gromov and Eliashberg showed that a $\mathcal{C}^0$-limit of symplectic diffeomorphisms which is itself a diffeomorphism is symplectic ([8, 6], see also [9]). This rigidity result leads to the definition of symplectic homeomorphisms (the $\mathcal{C}^0$-limits of symplectic diffeomorphisms which are homeomorphisms), and shows that they define a proper subset of volume preserving homeomorphisms in dimension at least 4. It also raises the question of the survival of the symplectic invariants to this limit process. Which classical invariants of symplectic geometry remain invariants of this maybe softer $\mathcal{C}^0$-symplectic geometry? This paper shows that the characteristic foliation is one of them.

Theorem 1. *Let S and S' be smooth hypersurfaces of some symplectic manifolds (M,ω), (M',ω'). Any symplectic homeomorphism between M and M' which sends S to S' transports the characteristic foliation of S to that of S'.*

The characteristic foliation is a symplectic invariant of a given hypersurface S, which can be defined as the integral foliation of the (one dimensional) null space of the restriction of the symplectic form to S. This definition is intrinsically smooth since it involves the tangent spaces of S. But the roles of this foliation in symplectic geometry are many. In particular, one of its rather folkloric property concerns non-removable intersection : if two smoothly bounded open sets intersect exactly on their boundaries, and if no symplectic perturbation can separate them, then the boundaries share a common closed invariant subset of the characteristic foliation [10, 11, 13, 14]. This paper proceeds from the remark that this property has a meaning also in the continuous category, so defining this foliation in continuous terms is conceivable.

An application of this theorem is a weak answer to a question by Eliashberg and Hofer about the symplectic characterization of a hypersurface by the open set it bounds : *Under which conditions the existence of a symplectomorphism between two smoothly bounded open sets in symplectic manifolds imply that their boundaries are symplectomorphic also [5] ?* Previous works show that the only realistic constraints on the domains and their boundaries alone must be very restrictive [4, 2, 1]. In contrast, theorem 1 can allow to get rid of these conditions on the expense of only considering a special class of symplectomorphisms.

Theorem 2. *Let U be a smoothly bounded open set in $\mathbb{R}^4$. Assume that there is a symplectomorphism between $B^4(1)$ and U which extends continuously to a homeomorphism between S^3 and ∂U. Then ∂U is symplectomorphic to S^3.*

The paper is organized as follows. We first define symplectic hammers and explain their roles : theorem 1 proceeds from a localization of their actions along characteristics (section 1). This localization is proved in section 2. We then present the application in the last section.

Aknowledgements. I wish to thank Leonid Polterovich for making me aware of a serious mistake in the first version of the proof.

1 Symplectic hammers.

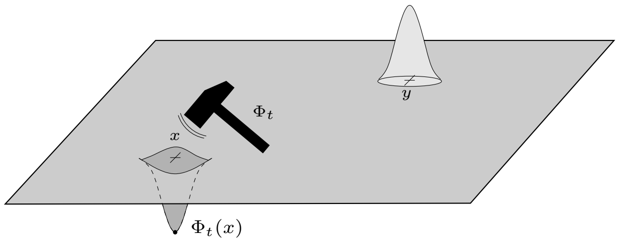
Let S be a hypersurface in a symplectic manifold M. We say that B is a small ball centered on S if it is a symplectic embedding of an euclidean ball centered at the origin into M which sends $\mathbb{R}^{2n-1} := \mathbb{R} \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$ to S. Such a ball is disconnected by S into two components denoted by S_+ and S_-. By a classical result, any point of S is the center of such a ball. Fix also a metric on M in order to refer to small sets.

Definition 1.1. *Given two points x, y on $S \cap B$ and a (small) positive real ε, an ε-symplectic hammer between x and y with support in B is a continuous path of symplectic homeomorphisms Φ_t ($t \in [0,1]$) with common supports in B, and for which there exist two open sets $U_\varepsilon(x)$ and $U_\varepsilon(y)$ contained in the ε-balls around x and y respectively such that :*

1. $\Phi_0 = Id$,
2. $\Phi_t(z) \in S_+$ *for all* $t \in]0,1]$ *and* $z \in S \cap U_\varepsilon(x)$,
3. $\Phi_t(z) \in S_-$ *for all* $t \in]0,1]$ *and* $z \in S \cap U_\varepsilon(y)$,
4. $\Phi_t(z) \in S$ *for all* $t \in [0,1]$ *and* $z \in S \backslash \big(U_\varepsilon(x) \cup U_\varepsilon(y)\big)$.

A smooth hammer will refer to a smooth isotopy of smooth symplectomorphisms verifying the four conditions above.

In other terms, Φ_t preserves the hypersurface S except for two bumps in opposite sides (a symmetry is necessary in view of the volume preservation).



One can easily construct examples of symplectic hammers.

Proposition 1.2. *If $x, y \in B \cap S$ lie in the same characteristic, there exist ε-symplectic hammers between x and y for all $\varepsilon > 0$.*

Proof : Since all hypersurfaces are locally symplectically the same, it is enough to produce a symplectic hammer for $\mathbb{R}^{2n-1} = \{\operatorname{Im} z_1 = 0\} \subset \mathbb{C}^n$ between the points $p = 0$ and $q = (1/2, 0, \ldots, 0)$. Putting $x_1 = \operatorname{Re} z_1$, $y_1 = \operatorname{Im} z_1$ and $r_i = |z_i|$, consider a Hamiltonian of the following type.

$$H(z_1, \ldots, z_n) := \chi(y_1)\rho(x_1)\Pi_{i=2}^n f(r_i).$$

If χ, ρ and f are the bell functions represented in figure 1, and maybe multiplying H by a small constant in order to slow the flow down produces a symplectic hammer between x and y. □

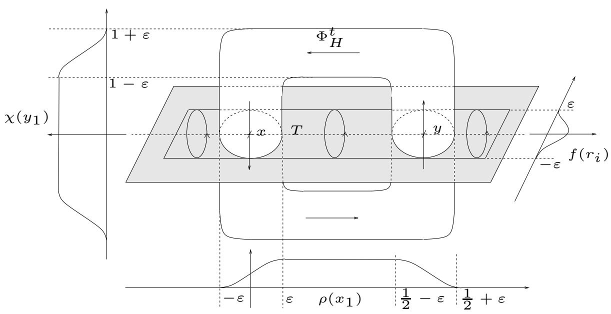


Figure 1: The Hamiltonian flow of H in the proof of proposition 1.2.

Proposition 1.2 can easily be reversed in the smooth category : two points lie in the same characteristic leaf of $S \cap B$ if and only if there exist smooth symplectic hammers between them. It is less obvious, but still true that *all* the symplectic hammers also meet this constraint. Theorem 1 obviously follows because the class of symplectic hammers is preserved by symplectic homeomorphisms.

Proposition 1.3. *A hypersurface S and a small ball B centered on S being given, there exists an ε-hammer between $x, y \in B \cap S$ with support in B for all small ε if and only if x and y are on the same characteristic leaf.*

Proof of theorem 1 (assuming proposition 1.3) : Let M, M', S, S' and Φ be as in theorem 1, and put any metric on M and M'. Consider two points $x, y \in S$ which lie in the same characteristic. Consider a covering $\mathcal{B} = \{B_\alpha\}$ of S by small balls (in the above sense) whose images by Φ are contained in small balls B'_α centered on S'. Let $(x_i)_{i\leq N}$ be a chain between x and y (that is $x_0 = x$, $x_N = y$) such that x_i and x_{i+1} are always in a same ball B_i. Then there exist ε-hammers $\Phi_t^{(\varepsilon)}$ with supports in B_i between x_i and x_{i+1} for all ε. The isotopies $\Phi \circ \Phi_t^{(\varepsilon)} \circ \Phi^{-1}$ define continuous $\delta(\varepsilon)$-symplectic hammers with support in B'_i between $\Phi(x_i)$ and $\Phi(x_{i+1})$, where $\delta(\varepsilon)$ goes to zero with ε. Therefore by proposition 1.3, $\Phi(x_i)$ and $\Phi(x_{i+1})$ are on the same characteristic, so $\Phi(x)$ and $\Phi(y)$ are also on the same characteristic. $\square$

2 Proof of proposition 1.3

The idea is the following. Since preserving a foliation is a local property, and since all hypersurfaces are locally the same in the symplectic world, we could translate the non-preservation of *one* characteristic by a symplectic homeomorphism to the existence of a local, hence universal object (a hammer between points on distinct characteristics) which would exist on all hypersurfaces. These continuous hammers would allow to break intersections between open sets as long as these intersections only consist of one characteristic. But some such intersections are known to be non-removable : the most famous one being the intersection between the complement of the cylinder $Z(1)$ and the closed ball $B^{2n}(1)$.

Lemma 2.1. *If proposition 1.3 does not hold, then for any point x of the euclidean sphere $S^{2n-1} \subset \mathbb{C}^n$ and for any positive (small) ε, there exists a continuous ε-symplectic hammer between x and a point y which does not lie in the characteristic circle passing through x.*

Proof : Assume that proposition 1.3 does not hold. Then there exists a small ball centered on a hypersurface S, two points $p, q \in S \cap B$ which are not in the same characteristic of $S \cap B$ and a family $\Phi^{(\varepsilon)} := (\Phi_t^{(\varepsilon)})_{t\in[0,1]}$ of ε-symplectic hammers with supports in B between p and q. By definition of a small ball, there is a symplectic diffeomorphism Ψ_1 between B and an euclidean ball $B_1 \subset \mathbb{C}^n$ around the origin with $\Psi(S \cap B) = \mathbb{R}^{2n-1} \cap B_1$. Then Ψ_1 takes $\Phi(\varepsilon)$ to an ε-hammer between $\Psi_1(p)$ and $\Psi_1(q)$ which are not on the same characteristic. By use of translation and rescaling, we can assume that $\Psi_1(p)$ is the origin and B_1 is as small a neighbourhood of 0 as wished.

Now given the point $x \in S^{2n-1}$, and if B_1 is small enough, there exists a symplectic diffeomorphism $\Psi_2 : B_1 \longrightarrow \mathbb{C}^n$ with $\Psi_2(B_1 \cap \mathbb{R}^{2n-1}) \subset S^{2n-1}$, $\Psi_2(0) = x$ and such that different characteristics of $\mathbb{R}^{2n-1} \cap B_1$ are sent by Ψ_2 not only to different characteristics of $S^{2n-1} \cap \Psi_2(B_1)$ but even of S^{2n-1} (this means that we do not allow Ψ_2 to "bend" B_1 so as to take two different characteristics to two different segments of a same characteristic circle of S^{2n-1}). The continuous symplectic isotopies obtained by transporting $\Phi^{(\varepsilon)}$ by $\Psi_2 \circ \Psi_1$ are ε-hammers between x and the point $y := \Psi_2(\Psi_1(q))$ which is not on the characteristic through x. □

Lemma 2.2. *Any bounded open set $U \subset Z(1) := B^2(1) \times \mathbb{C}^{n-1}$ whose boundary ∂U does not contain a characteristic circle $S^1 \times \{\cdot\}$ of $\partial Z(1)$ can be symplectically displaced from $\partial Z(1)$ to the interior of $Z(1)$.*

Proof : Recall that the characteristic flow of $\partial Z(1)$ can be oriented by the vector field JN where the vector N is the outward normal vector field to $\partial Z(1)$ and J is the standard complex structure on $\mathbb{C}^n$. Now observe that if the compact set $K := \partial U \cap \partial Z(1)$ does not contain any characteristic circle, there exists a smooth function H on $\mathbb{C}^n$ which decreases along the characteristic flow on a neighbourhood of K [15]. The corresponding Hamiltonian vector field points toward the inside of the cylinder on this neighbourhood of K because

$$g(X_H(x), N(x)) = \omega(X_H(x), JN(x)) = dH(JN(x)) < 0,$$

so U is driven inside $Z(1)$ by the flow of H for small enough times. □

Let us come back to proposition 1.3. Consider $B^{2n}(1)$ as an open set lying in $Z(1)$. Its boundary S^{2n-1} meets $Z(1)$ along precisely one characteristic circle of $\partial Z(1)$:

$$S^{2n-1} \cap \partial Z(1) = \{|z_1| = 1,\ z' = 0\} \subset B^2_{z_1}(1) \times \mathbb{C}^{n-1}_{z'}.$$

Assume then by contradiction that proposition 1.3 does not hold. Then by lemma 2.1, there exists ε-hammers Φ_t between the point $(1,0) \in S^{2n-1} \cap \partial Z(1)$ and an interior point $y \in S^{2n-1} \cap Z(1)$ for arbitrarily small ε. If ε is small enough, and slowing the flow of the hammer down enough (considering Φ_{at} in place of Φ_t), the image U of $B(1)$ by Φ_1 is therefore an open set of $Z(1)$ whose boundary intersection with $\partial Z(1)$ is the circle $S^1 \times \{0\}$ minus a small neighbourhood of $(1,0)$ which was taken inside the ball - hence inside the cylinder - by our hammer. By lemma 2.2, there exists therefore a smooth Hamiltonian K such that $\Phi^1_K(U)$ is relatively compact in $Z(1)$. Taking a good enough smooth approximation $\widetilde{\Phi}_t$ of Φ_t, $\Phi^1_K \circ \widetilde{\Phi}_1(B(1))$ is still relatively compact in $Z(1)$. But this is in contradiction with Gromov's non-squeezing theorem. □

3 Symplectic geometry from the inside.

In this section, we prove theorem 2. We first prove that the characteristics of the sphere are sent to the characteristics of the boundary of U. The next point is to see that the action on these characteristics coincide on both hypersurfaces. The result follows.

3.1 A one-sided version of theorem 1.

In fact, theorem 1 holds also in a slightly more general framework.

Theorem 3. *Let U and U' be smoothly bounded open sets in symplectic manifolds. Any symplectic homeomorphism between U and U' which extends continuously to a homeomorphism of their closures transports the characteristic foliation of ∂U to that of $\partial U'$.*

The proof below is rather quick because everything has already been explained. Exactly as for theorem 1, the point is to define a convenient notion of symplectic hammer which is invariant by one-sided symplectic homeomorphisms. Note that definition 1.1 *has* to be modified since it involves both sides of the hypersurface through the bumps. The solution is simply to forget about the part of the hammer which goes outside U.

Definition 3.1 (One-sided hammers)**.** *Let U be a smoothly bounded open set in M, B a small ball centered in ∂U, $x, y \in B \cap \partial U$. A one-sided ε-symplectic hammer between x and y is a continuous isotopy of homeomorphisms $\Phi_t : \overline{U} \backslash B_\varepsilon(y) \longrightarrow \overline{U}$ which can be uniformly approximated in $U \backslash B_\varepsilon(y)$ by smooth symplectic isotopies with common supports in B, and which verifies also properties 1), 2) and 4) of definition 1.1.*

This definition actually provides a one-sided definition of the characteristics because of the following.

Proposition 3.2. *There exists one-sided ε-hammers between x and y for all small ε if and only if x and y belong to the same characteristic of ∂U.*

Proof : The proof is very similar to the proof of proposition 1.3. On one hand, since the definition of a one-sided hammer is a local one, and since smooth hypersurfaces have no local invariants, the existence of a one-sided hammer between two points not in the same characteristic of $\partial U \cap B$ ensures the existence of such a hammer on the ellipsoid

$$\mathcal{E}(1,2) := \{|z_1|^2 + \frac{|z'|^2}{4} \leq 1\} \subset Z(1) \subset \mathbb{C}_z \times \mathbb{C}^{n-1}_{z'}$$

between the point $(1,0)$ of the "least action" characteristic C_0 and another point y not in this characteristic. This hammer isotopes $\mathcal{E}(1,2) \backslash B_\varepsilon(y)$ to an open set $U \subset Z(1)$ whose boundary contains no characteristic circle of $\partial Z(1)$. By a Hamiltonian flow, U can be symplectically displaced from $\partial Z(1)$ inside $Z(1)$. For ε small enough, and since y does not belong to C_0, $\mathcal{E}(1,2)\backslash B_\varepsilon(y)$ contains the ball of radius 1, contradicting again Gromov's non-squeezing theorem. □

3.2 Proof of theorem 2.

Let us fix the notations. On $S^3 = \partial B^4(1)$, the characteristic foliation defines the Hopf fibration $\pi : S^3 \longrightarrow \mathbb{P}^1$. Fix a circle $C := \pi^{-1}(N)$ where N is any point in $\mathbb{P}^1$. Then $\mathbb{P}^1\backslash\{N\}$ is a disc and there is a smooth section s of π over this disc, with boundary on C and which is transverse to the fibers of π. Also, if ω_0 is the standard area form on $\mathbb{P}^1$ with total area π, the restriction of the symplectic form ω on $\mathbb{R}^4$ to S^3 is $\pi^*\omega_0$.

Let U be a smoothly bounded domain in $\mathbb{R}^4$, f a symplectic diffeomorphism from U to $B^4(1)$ which extends continuously to a homeomorphism of the boundaries. Our first task is to produce a *diffeomorphism* $\Psi : \partial U \longrightarrow S^3$ which takes the characteristics of ∂U to the Hopf circles of S^3. Observe that by theorem 3, f sends the characteristics of ∂U to those of S^3, so the leaves of the characteristic foliation of ∂U are the fibers of the *topological* Hopf fibration $\widetilde{\pi} := \pi \circ f : \partial U \longrightarrow \mathbb{P}^1$. Constructing Ψ is therefore equivalent to finding a smooth section $\widetilde{s}$ of $\widetilde{\pi}$ over the disc $\mathbb{P}^1\backslash\{N\}$ transverse to the fibration, and such that $\widetilde{s}$ is a smooth embedding of the closed disc with boundary on $\widetilde{\pi}^{-1}(N)$ ($f \circ s$ only provides a continuous such section). The existence of this section in dimension 3 is precisely the point of the work of Epstein [7].

Since the restriction of $\Psi_*\omega$ to S^3 vanishes along the Hopf circles but never vanishes, $\Psi_*\omega$ is also the pull-back of an area form ω_1 on $\mathbb{P}^1$: $\Psi_*\omega = \pi^*\omega_1$. We claim that the ω_1-area of $\mathbb{P}^1$ is the expected one :

$$\left|\int_{\mathbb{P}^1} \omega_1\right| = \pi. \tag{1}$$

Let us accept this as a fact for a moment. Then, after a possible orientation-reversing change of coordinates on $\mathbb{P}^1$, there exists an area-preserving diffeomorphism $\varphi : (\mathbb{P}^1, \omega_1) \longrightarrow (\mathbb{P}^1, \omega_0)$. Any lift Φ of φ through π is a self-diffeomorphism of S^3 which pulls back $\pi^*\omega_0 = \omega_{|S^3}$ to $\Psi_*\omega$.

$$\begin{array}{ccccc}
(\partial U, \omega) & \xrightarrow{\Psi} & (S^3, \Psi_*\omega) & \xrightarrow{\Phi} & (S^3, \pi^*\omega_0) \\
 & & \downarrow \pi & & \downarrow \pi \\
 & & (\mathbb{P}^1, \omega_1) & \xrightarrow{\varphi} & (\mathbb{P}^1, \omega_0)
\end{array}$$

Hence, $(\Phi \circ \Psi)^*\omega_{|S^3} = \omega_{|\partial U}$ and theorem 2 follows by a classical argument of standard neighbourhood (see [12]).

In order to prove (1), observe that putting

$$\left| \int_{\mathbb{P}^1} \omega_1 \right| = \pi r^2,$$

the above argument shows the existence of a symplectic diffeomorphism Φ between a neighbourhood of the euclidean sphere of radius r and a neighbourhood of ∂U which takes $S^3(r)$ to ∂U. The map $g = f \circ \Phi$ defines therefore a symplectomorphism between two one-sided neighbourhoods V_r, V_1 of the euclidean spheres. The following lemma finally ensures that $r = 1$. $\square$

Lemma 3.3. *There exists a symplectomorphism between two open one-sided neighbourhoods of euclidean spheres $S^3(r)$ and $S^3(R)$ if and only if $r = R$.*

It is straightforward in view of [3]. We give however the argument for the convenience of the reader. Notice that it would be obvious, should the map g extend smoothly to $S^3(r)$ (which is precisely not the case) because the action of the characteristics on $S^3(r)$ and S^3 should coincide.
Proof : Assume that $r \leq R$, call V_r and V_R the one-sided open neighbourhoods of the two spheres and $g : V_R \longrightarrow V_r$ a symplectomorphism. Notice that since V_R is symplectically convex with respect to $S^3(R)$, so is V_r with respect to $S^3(r)$ (meaning that there is a contracting vector field on V_r flowing away from $S^3(r)$), so both V_r and V_R are contained inside the corresponding euclidean balls. On $B^4(R)$, the contracting vector field $X = -\sum r_i \partial/\partial r_i$ is ω-dual to the form $\lambda_0 = -\sum r_i^2 d\theta_i$. Its image $g_* X$ is also a contracting vector field on V_r, ω-dual to $g_*\lambda_0$. Since $H^1(V_r) = 0$, $g_*\lambda_0$ extends to a primitive of $-\omega$ on $B^4(r)$. This extension provides $B^4(r)$ with a contracting vector field X' which coincides with $g_* X$ on V_r and which is forward complete (its flow is defined for all positive time) because it points inside $B^4(r)$ near its boundary. Therefore g can be extended to a symplectic embedding $\widetilde{g} : B^4(R) \longrightarrow B^4(r)$ by the formula :

$$\widetilde{g}(p) = \Phi^t_{X'} \circ g\big(\Phi^{-t}_X(p)\big),\ \forall p \in B^4(R),\ \forall t \text{ such that } \Phi^{-t}_X(p) \in V_R.$$

But such an embedding is only possible if $R \leq r$ by volume considerations. $\square$

References

[1] D. Burns and R. Hind. Symplectic rigidity for Anosov hypersurfaces. *Ergodic Theory Dynam. Systems*, 26(5):1399–1416, 2006.

[2] K. Cieliebak. Symplectic boundaries: creating and destroying closed characteristics. *Geom. Funct. Anal.*, 7(2):269–321, 1997.

[3] Y. Eliashberg and M. Gromov. Convex symplectic manifolds. In *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, volume 52 of *Proc. Sympos. Pure Math.*, pages 135–162. Amer. Math. Soc., Providence, RI, 1991.

[4] Y. Eliashberg and H. Hofer. Towards the definition of symplectic boundary. *Geom. Funct. Anal.*, 2(2):211–220, 1992.

[5] Y. Eliashberg and H. Hofer. Unseen symplectic boundaries. In *Manifolds and geometry (Pisa, 1993)*, Sympos. Math., XXXVI, pages 178–189. Cambridge Univ. Press, Cambridge, 1996.

[6] Y. M. Eliashberg. A theorem on the structure of wave fronts and its application in symplectic topology. *Funktsional. Anal. i Prilozhen.*, 21(3):65–72, 96, 1987.

[7] D. B. A. Epstein. Periodic flows on three-manifolds. *Ann. of Math. (2)*, 95:66–82, 1972.

[8] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[9] H. Hofer and E. Zehnder. *Symplectic invariants and Hamiltonian dynamics.* Birkhäuser Advanced Texts: Basler Lehrbücher. 1994.

[10] F. Lalonde and D. McDuff. Local non-squeezing theorems and stability. *Geom. Funct. Anal.*, 5(2):364–386, 1995.

[11] F. Laudenbach and J.-C. Sikorav. Hamiltonian disjunction and limits of Lagrangian submanifolds. *Internat. Math. Res. Notices*, (4):161 ff., approx. 8 pp. (electronic), 1994.

[12] D. McDuff and D. Salamon. *Introduction to symplectic topology.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[13] E. Opshtein. Maximal symplectic packings in $\mathbb{P}^2$. *Compos. Math.*, 143(6):1558–1575, 2007.

[14] G. Paternain, L. Polterovich, and K. Siburg. Boundary rigidity for Lagrangian submanifolds, non-removable intersections, and Aubry-Mather theory. *Mosc. Math. J.*, 3(2):593–619, 745, 2003.

[15] D. Sullivan. Cycles for the dynamical study of foliated manifolds and complex manifolds. *Invent. Math.*, 36:225–255, 1976.

Emmanuel Opshtein,
Université de Strasbourg,
IRMA, UMR 7501
7 rue René-Descartes
67084 Strasbourg Cedex
opshtein@math.u-strasbg.fr