# A Wong-Rosay type theorem for the iterations of a proper holomorphic self-map.

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**Abstract**

In this short paper, we show that the only proper holomorphic self-maps of bounded domains in $\mathbb{C}^k$ whose dynamics escape to a strictly pseudoconvex point of the boundary are automorphisms of the euclidean ball. This Wong-Rosay type theorem, in a dynamical situation is non-trivial since the degrees of the considered maps are a priori unbounded.

## 1 Introduction.

In 1977, Wong proved that the only strictly pseudoconvex domain with non-compact group of automorphisms is the ball [17]. This result was generalized by Rosay [13] (see also [12]).

**Theorem (Wong-Rosay).** *Let $\Omega$ be a bounded domain in $\mathbb{C}^k$ and $(f^n)$ a sequence of its automorphisms. Assume that the orbit of a point of $\Omega$ under $(f_n)$ accumulates a smooth strictly pseudoconvex point of $b\Omega$. Then $\Omega$ is biholomorphic to the euclidean ball.*

Ourimi showed that this version of Wong's theorem extends to proper maps, or even correspondences provided that the degrees of the maps $f_n$ remain bounded [10, 11]. In this paper, we prove that the theorem above holds true when replacing the sequence of automorphisms by the iterates of a proper holomorphic self-map.

**Theorem 1.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^k$ with a proper holomorphic self-map $f$ which extends smoothly to $b\Omega$. If there is a point $y$ of $\Omega$ whose orbit under the iterates of $f$ accumulates a smooth strictly pseudoconvex point $a$ of $b\Omega$ (that is $f^{n_k}(y) \longrightarrow a$), then $\Omega$ is biholomorphic to the euclidean ball and $f$ is an automorphism.*

The hypothesis of this theorem may seem weaker than it is actually. It may be useful to have in mind the following classical fact from the begining.

**Remark 1.1.** *Under the hypothesis of theorem 1, the subsequence $f^{n_k}$ converges locally uniformly to $a$ on $\Omega$. Also, $a$ is a fixed point of $f$.*

The main result on which relies the proof of theorem 1 is a local version of Wong-Rosay's theorem concerning sequences of CR-maps. It was first obtained by Webster in the wake of Chern-Moser's theory of strictly pseudoconvex hypersurfaces [16].

**Theorem 2 (Webster).** *Let $(\Sigma, a)$ be a germ of strictly pseudoconvex hypersurface. Assume there is a sequence of CR-automorphisms of $\Sigma$ whose images converge to $a$. Then $S$ is spherical, i.e. locally CR-diffeomorphic to the euclidean sphere.*

The idea to prove theorem 1 is to consider the CR-map induced by $f^n$ on the boundary rather than $f^n$ itself. Using techniques developped in [9, 8], we study the way these CR-maps degenerate and check that theorem 2 applies. The local information it gives then propagates to the whole of $\Omega$ thanks to the dynamical situation. Notice that the same techniques were used in [8] to get an elementary proof of Webster's theorem.

The paper is organised as follows. We first prove theorem 1 under the additionnal assumption that the domain is smooth and convex. This part serves as a model proof since it exhibits the two basic properties we need to prove the general case of theorem 1. One is a local information on the sphericity of $b\Omega$ around $a$. The other is a dynamical information : we need to explain that $(f^n)$ and not only a subsequence converges to $a$ on $\Omega$. In sections three and four we prove these two properties respectively, thus fixing the proof of theorem 1.

## 2 The convex case : a model proof.

In this section, we prove theorem 1 under the extra-assumption that $\Omega$ is a convex domain with smooth boundary. Almost all the technicity disappears in this context, so this section should help the reader to get a clear picture of the structure of the proof. The simplification comes from the fact that we understand the dynamics of proper holomorphic self-maps of these special domains. In [9, 7], the following was indeed proved :

**Proposition 2.1.** *Let $\Omega$ be a smoothly bounded convex domain in $\mathbb{C}^k$ and $f : \Omega \longrightarrow \Omega$ be a proper holomorphic self-map (then $f$ extends smoothly to $b\Omega$, see [2]). Assume the $f$-orbit of a point in $\Omega$ accumulates a point $a$ of $\mathcal{SC}(b\Omega)$ ($\mathcal{SC}(b\Omega)$ denotes the strongly convex part of $b\Omega$). Then the whole sequence $(f^n)_{n\in\mathbb{N}}$ converges locally uniformly to $a$ on $\Omega \cup \mathcal{SC}(b\Omega)$.*

Using this proposition we now proceed to the proof of theorem 1 when $\Omega$ is smooth and convex. Let us first fix the notation. By a complex affine change of coordinate, we can take $a$ to the origin in $\mathbb{C}^k$, and arrange that $T_a b\Omega = \{\operatorname{Re} z_1 = 0\}$ and $\Omega \subset \{\operatorname{Re} z_1 > 0\}$. We then denote $U_\alpha := b\Omega \cap \{\operatorname{Re} z_1 < \alpha\}$ and $\Omega_\alpha := \Omega \cap \{\operatorname{Re} z_1 < \alpha\}$.

The first step of the proof shows that $b\Omega$ is spherical around $a$. It should be noticed that it is the only point where we use the fact that the maps $f^n$ converge to $a$ on $\mathcal{SC}(b\Omega)$.

**Lemma 2.2.** *If $(f^n)$ converges to $a$ on $\mathcal{SC}(b\Omega)$ then a neighbourhood of $a$ in $b\Omega$ is spherical.*

*Proof :* Theorem 2 explains that it is enough to find a contracting sequence of CR-automorphisms on a neighbourhood of $a$. Proposition 2.1 asserts that $(f^n)$ is a sequence of contracting CR-maps on $\mathcal{SC}(b\Omega)$, and $f$ is in fact a local diffeomorphism (see [4]). We thus only need to prove that there is a fixed neighbourhood of $a$ on which all $f^n$ are injective. To see this, fix a neighbourhood $U_{\varepsilon_0}$ of $a$ on which $f$ is injective. Since $f^n$ converges locally uniformly to $a$ on $\mathcal{SC}(b\Omega)$, $f^n(U_{\varepsilon_0}) \subset U_{\varepsilon_0}$ for all large enough integers $n \geq n_0$. Consider now a neighbourhood $U_\varepsilon$ of $a$ in $U_{\varepsilon_0}$ whose images $U_\varepsilon, f(U_\varepsilon), \ldots, f^{n_0}(U_\varepsilon)$ are all contained in $U_{\varepsilon_0}$. Such a set exists because $f$ is continuous and $a$ is a fixed point of $f$. By construction $f^n(U_\varepsilon) \subset U_{\varepsilon_0}$ for all $n \in \mathbb{N}$, and the restriction of $f^n$ to $U_\varepsilon$ is injective as a composition of injective maps. □

The second step of the proof consists in using the dynamics of $f$ to propagate the local sphericity around $a$ to produce a biholomorphism from $\Omega$ to $B$. We have just established the existence of a CR-diffeomorphism $\Phi : U_\varepsilon \longrightarrow V \subset bB$. A classical extension theorem shows that $\Phi$ extends to a biholomorphism $\Phi : \Omega_\varepsilon \longrightarrow D$ where $D$ is an open set of $B$ whose boundary contains $V$ (see [3]). Transporting $f$ by this biholomorphism gives a local automorphism of $B$, defined by

$$\begin{array}{rcccl} g & : & \Phi(\Omega_\varepsilon \cap f^{-1}(\Omega_\varepsilon)) & \longrightarrow & \Phi(\Omega_\varepsilon) \\ & & x & \longmapsto & \Phi \circ f \circ \Phi^{-1}(x). \end{array}$$

The key point of the whole proof is an extension phenomena discovered by Alexander [1] (see also [12, 15] for the form of the result we use here). The local biholomorphism $g$ uniquely extends to a global automorphism of the ball. This extension of $g$ allows to extend $\Phi$ to the union of the sets $O_i := f^{-i}(\Omega_\varepsilon)$ :

$$\begin{array}{rcccl} \Phi & : & \cup_{i\in\mathbb{N}} O_i & \longrightarrow & B \\ & & z \in O_i & \longmapsto & g^{-i} \circ \Phi \circ f^i(z). \end{array}$$

Since $f^n$ tends to $a$ on $\Omega$ this formula defines in fact $\Phi$ on $\Omega$, and we have the obvious functional equality : $\Phi \circ f = g \circ \Phi$.

**Lemma 2.3.** *The map $\Phi$ is a proper map from $\Omega$ to $B$.*

*Proof :* Recall that in the proof of lemma 2.2 we saw that a certain neighbourhood of $a$ remains in $U_\varepsilon$ under iteration and eventually ends to $a$. By conjugacy, the map $g$ inherits of this local behaviour around $S := \Phi(a)$. The dynamics of ball's automorphisms being well-known (see proposition 4.1), this property is enough to guarantee that $g$ is hyperbolic. It has exactly two fixed points $S$ and $N$ which attract all orbits of $\overline{B}\backslash\{N, S\}$ respectively positively and negatively. The basic consequence of this fact is that $\Phi(O_n)\backslash O_{n-1}$ goes to $bB$ with $n$. Indeed, the $f$-orbit of a point $z$ in this set reaches $O_0$ only at time $n$, so the $g$-orbit of $\Phi(z)$ only reaches $D$ after the same time. If $n$ is large, $\Phi(z)$ has to be very close to $N$ which is on the boundary of $B$.

For an arbitrary sequence $(z_i)_{i\in\mathbb{N}} \in \Omega$ converging to $b\Omega$, we must show that $\Phi(z_i)$ tends to the boundary of $B$. For this, fix a positive real number $\varepsilon$ and an integer $n_0$ such that $d(\Phi(O_n\backslash O_{n-1}), bB) \leq \varepsilon$ for all $n \geq n_0$. Split then $(z_i)$ in two subsequences :

$$\begin{array}{rcl} (z_i^1) & := & \{z_{n_i} \in \{z_n\} \mid z_{n_i} \notin O_{n_0}\} \\ \text{and } (z_i^2) & := & \{z_{n_i} \in \{z_n\} \mid z_{n_i} \in O_{n_0}\}. \end{array}$$

By construction, $d(\Phi(z_i^1), bB) \leq \varepsilon$. Since $f^{n_0}(z_i^2) \subset O_0$ and since $f^{n_0}$, $\Phi_{|O_0}$ and $g$ are proper maps, $\Phi(z_i^2) = g^{-n_0} \circ \Phi_{|O_0} \circ f^{n_0}(z_i^2)$ is also $\varepsilon$-close to $bB$ for $i$ large enough. □

Finally, we need to show that $\Phi$ is a biholomorphism. It is not yet clear since there obviously exist holomorphic coverings of the ball. Anyway we know that any proper map to a bounded domain has a finite degree (see [14], chap. 15). In particular, there is an integer $d$ which bounds the numbers of preimages by $\Phi$ :

$$\#\Phi^{-1}(z) \leq d, \qquad \forall z \in B.$$

Notice now that the degree of $\Phi$ bounds this of $f^n$ for all $n$ because $\Phi = g^{-n} \circ \Phi \circ f^n$. The degree of $f^n$ is thus bounded on one hand and equal to $(\deg f)^n$ on the other. So $f$ is an automorphism of $\Omega$. The injectivity of $\Phi$ is now immediate because $\Phi_{|O_i} = g^{-n} \circ \Phi_{|O_0} \circ f^n$ is a composition of injective maps for all $i$. □

Let us notice that the convexity of $\Omega$ was only used through proposition 2.1. Moreover, the convergence of $f^n$ to $a$ on $\mathcal{SC}(b\Omega)$ only served to get the sphericity near $a$.

**Remark 2.4.** *If $\Omega$ is a bounded domain in $\mathbb{C}^n$ whose boundary is spherical in a neighbourhood of a point $a$, and $f$ is a proper holomorphic self-map of $\overline{\Omega}$ whose dynamics converge to $a$ inside $\Omega$ then $f$ is an automorphism and $\Omega$ is the ball.*

The strategy for the general result is thus to prove the sphericity near $a$ and the convergence of $f^n$ to $a$ on $\Omega$.

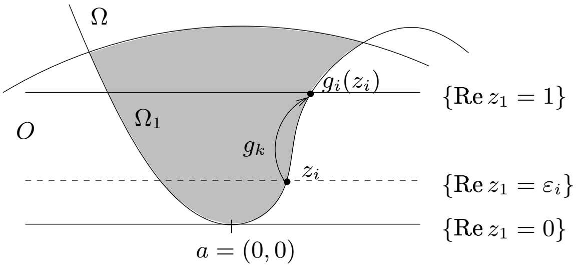
# 3 Local sphericity near the attractive point.

In this section we come back to the general situation of theorem 1 and we prove that a neighbourhood of $a$ in $b\Omega$ is spherical. The idea behind this technical part of the proof is based on previous results concerning behaviours of sequences of CR-maps (see [9, 7]). Unformally speaking, they explain that non-equicontinuous sequences of CR-maps on strictly pseudoconvex hypersurface dilate a certain (anisotropic) distance. The proof of the sphericity then goes as follows. Either $f^{n_k}$ converges to $a$ on $\mathcal{SPC}(b\Omega)$ and the situation is very similar to that of lemma 2.2 (see proposition 3.1). Or $f^{n_k}$ is not equicontinuous on $\mathcal{SPC}(b\Omega)$ and it is dilating. Then the inverse branches of $f^{n_k}$ are contracting CR-diffeomorphisms and theorem 2 gives the sphericity. Let us first fix the easy situation where $f^{n_k}$ converges to $a$ on $\mathcal{SPC}(b\Omega)$.

**Proposition 3.1.** *Assume $f^{n_k}$ converges locally uniformly to $a$ on a neighbourhood of $a$ in $b\Omega$. Then $b\Omega$ is spherical near $a$.*

*Proof :* As a matter of fact, the convergence of the subsequence $f^{n_k}$ to $a$ implies the convergence of the whole dynamics of an iterate $g = f^p$ to $a$. To see this, pick a small neighbourhood $U$ of $a$ in $\mathcal{SPC}(b\Omega)$ and an integer $p = n_{k_0}$ such that $f^p(U) \subset U$. The map $g := f^p$ restricts to $U$ to a local diffeomorphism from $U$ to itself, whose sequence of images $g^n(U)$ is obviously decreasing (*i.e.* $g^i(U) \supset g^{i+1}(U)$). Observe then that the subsequence $(g^{n'_k})$ defined by $n'_k := E(n_k/p) + 1$ converges uniformly to $a$ on $U$. Indeed, $g^{n'_k} = f^{pn'_k} = f^{n_k+i}$ with $i < p$, so $g^{n'_k}(U) \subset \cup_{i \leq p} f^i(f^{n_k}(U))$. Since $f^{n_k}(U)$ is close to $a$ by hypothesis (for $k$ large enough) and $a$ is a fixed point of $f$, the continuity of $f$ implies that $g^{n'_k}(U)$ is also close to $a$. Since the sequence $g^n(U)$ decreases, it thus converges to $a$. Replacing $f$ by $g$, we are therefore in the condition of application of lemma 2.2 and a neighbourhood of $a$ is spherical. □

Consider now the situation when $f^{n_k}$ does not converge to $a$ on a neighbourhood of $a$. Let us first describe the figure and notation. Since $a$ is a strictly pseudoconvex point, it has a neighbourhood which is strongly convex when considered through a polynomial change of coordinates. Henceforth, we assume that $\Omega$ is strongly convex in a neighbourhood $O$ of $a$, that $a$ is the origin and that $\Omega \cap O$ is contained in $\{\operatorname{Re} z_1 \geq 0\}$. Also, since all the arguments to come are purely local, we focus on $\Omega$ in $O$. We put $\Omega_\varepsilon := \Omega \cap O \cap \{\operatorname{Re} z_1 \leq \varepsilon\}$ and $U_\varepsilon := b\Omega \cap O \cap \{\operatorname{Re} z_1 \leq \varepsilon\}$. Assume without loss of generality that $\Omega_1 \Subset O$. The non-convergence of $f^{n_k}$ means the existence of a sequence of points $z_i \in b\Omega$ tending to $a$, and integers $k_i$ such that the points $f^{n_{k_i}}(z_i)$ lay out of a fixed neighbourhood of $a$, say $U_1$. Since $a$ is fixed by $f^{n_{k_i}}$, we can even assume that $f^{n_{k_i}}(z_i) \in bU_1 = b\Omega \cap O \cap \{\operatorname{Re} z_1 = 1\}$ by moving $z_i$ closer to $a$. Finally, put $g_i := f^{n_{k_i}}$ and define $\varepsilon_i$ by $z_i \in \{\operatorname{Re} z_1 = \varepsilon_i\}$.




The main point of this section is that $f^{n_k}$ has a strong expanding behaviour.

**Proposition 3.2.** *(see also [8]) For all $\varepsilon$ there exists an integer $k = k(\varepsilon)$ such that $g_k(U_\varepsilon) \supset U_1 \backslash U_\varepsilon$.*

The sphericity near $a$ is a direct consequence of this proposition :

**Corollary 3.3.** *If $(f^{n_k})$ does not converge to $a$ in a neighbourhood of $a$ then $b\Omega$ is spherical near $a$.*

*Proof :* Fix an open contractible set $V$ compact in $U_1 \backslash \{a\}$. For $\varepsilon$ small enough, $V \subset U_1 \backslash U_\varepsilon$ and there is an integer $k_\varepsilon$ such that $g_{k_\varepsilon}(U_\varepsilon) \supset V$. Moreover, there are no critical value of $g_{k_\varepsilon |U_\varepsilon}$ inside $V$ because both $U_\varepsilon$ and $V$ are strictly pseudoconvex (see [4]). Since $V$ is simply connected, there exists an inverse branch of $g_{k_\varepsilon |U_\varepsilon}$ on $V$, which means a CR-diffeomorphism $h_\varepsilon : V \longrightarrow U_\varepsilon$ with $g_{k_\varepsilon} \circ h_\varepsilon =$Id. The sequence $h_\varepsilon$ is therefore contracting on $V$, and theorem 2 implies that $V$ is spherical. We have thus proved the local sphericity of $U_1 \backslash \{a\}$, which even proves the sphericity of $U_1$ because $a$ is a strictly pseudoconvex point. Indeed, Chern-Moser's theory expresses the sphericity of an open strictly pseudoconvex hypersurface by the vanishing of a continuous invariant tensor. Since this tensor vanishes on $U_1 \backslash \{a\}$, it also vanishes on the whole of $U_1$ so $U_1$ itself is spherical. In the spirit of [8], It would be pleasant to get a more down-to-earth argument for this last point. □

The proof of proposition 3.2 relies on the following lemma.

**Lemma 3.4.** *There exists a diverging sequence $c_i \longrightarrow +\infty$ such that for all $p \in U_1$ with $g_i(p) \notin U_\varepsilon$ we have :*

$$\|g_i'(p)u\| \geq c_i \|u\| \quad \forall u \in T_p^{\mathbb{C}} b\Omega.$$

*Proof :* The idea is that Hopf's lemma gives estimates on the normal derivative of $g_i$, which transfer automatically to complex tangential estimates in strictly pseudoconvex geometry. For $p \in U_1$, let $\vec{N}(p)$ be the unit vector normal to $b\Omega$ pointing inside $\Omega$ and

$$B_\delta^+(p) := B(p + \delta \vec{N}(p), \delta) \cap \{\langle \vec{N}(p), \cdot \rangle \geq \delta\}.$$

When $\delta$ is small enough but fixed, $B_\delta^+(p)$ is in $\Omega$ and its image by $g_i$ for $i$ large is in $\Omega_{\frac{\varepsilon}{2}}$. Thus if $g_i(p) \notin U_\varepsilon$, the non-positive $p.s.h$ function $\varphi := -\langle \vec{N}(g_i(p)), g_i(y) - g_i(p) \rangle$ vanishes at $p$ while it is less than $-c\varepsilon^2$ on $B_\varepsilon^+(p)$ ($c$ is a constant depending only on the curvature of $b\Omega$ at $a$). Hopf's lemma then asserts that

$$n_i(p) := \langle g_i'(p)\vec{N}(p), \vec{N}(g_i(p)) \rangle = \|\vec{\nabla}\varphi(p)\| \geq \frac{c'\varepsilon^2}{\delta}.$$

Since $\delta$ was arbitrary, we could take it much smaller than $\varepsilon^2$, so that the radial escape rate $n_i(p)$ is large. To transfer this radial estimate on the derivatives of $g_i$ to complex tangential ones, consider the Levi form $\mathcal{L}$ of $b\Omega$ defined by

$$\mathcal{L}(p, u) := \langle [u, iu], i\vec{N}(p) \rangle, \qquad u \in T_p^{\mathbb{C}} b\Omega,$$

where $u$ stands for the vector in $T_p^{\mathbb{C}} b\Omega$ as well as any extension of it to a vector field of $T^{\mathbb{C}} b\Omega$. The smoothness and strict pseudoconvexity of $U_1$ implies the existence of geometric constants $c_1$, $c_2$ such that

$$c_1 \|u\|^2 \leq \mathcal{L}(p, u) \leq c_2 \|u\|^2 \quad \forall p \in U_1, \ \forall u \in T_p^{\mathbb{C}} b\Omega.$$

Easy computations show that :

$$c_2\|g_i'(p)u\|^2 \geq \mathcal{L}(g_i(p), g_i'(p)u) = n_i(p)\mathcal{L}(p,u) \geq c_1 n_i(p)\|u\|^2.$$

Since $n_i(p)$ is large when $i$ is, this serie of inequalities imply lemma 3.4. $\square$

The previous lemma asserts that $g_i$ dilates the complex tangential directions of $b\Omega$ if $g_i(p)$ is not close to $a$. The last observation we need to make in order to prove proposition 3.2 is that this "complex tangential dilation" property implies a genuine dilation.

A path $\gamma$ in $b\Omega$ will be called a complex path if $\dot\gamma(t) \in T^{\mathbb{C}}_{\gamma(t)}b\Omega$ for all $t$. Its euclidean length will be denoted by $\ell(\gamma)$. For $x, y \in U_1$, define the CR-distance $d^{\mathrm{CR}}(x,y)$ between $x$ and $y$ as the infimum of the lengths of complex paths joining $x$ to $y$. The point is that the strict pseudoconvexity condition means that the complex tangential distribution is contact so complex paths can join any two points. Even more, the open set $U_1 \backslash U_\varepsilon$ is $d^{\mathrm{CR}}$-bounded (see theorem 4 of [6], or [9]).

*Proof of proposition 3.2 :* Fix $\tau > 0$ such that $B^{\mathrm{CR}}(z_i,\tau) \subset U_\varepsilon$ for all $i$ large enough. Since $U_1\backslash U_\varepsilon$ is $d^{\mathrm{CR}}$-bounded, it is enough to prove that

$$bg_i(B^{\mathrm{CR}}(z_i,\tau)) \cap B^{\mathrm{CR}}(g_i(z_i), c_i\tau) \cap \big(U_1\backslash U_\varepsilon\big) = \emptyset$$

because $c_i\tau$ can be made greater than the CR-diameter of $U_1\backslash U_\varepsilon$. Take a point $x \in bg_i(B^{\mathrm{CR}}(z_i,\tau)) \cap U_1\backslash U_\varepsilon$ and let us prove that

$$d^{\mathrm{CR}}(g_i(z_i), x) \geq c_i\tau. \tag{1}$$

Consider an arc-length parameterized complex path $\gamma$ in $U_1\backslash U_\varepsilon$ joining $g_i(z_i)$ to $x$. Since $g_i$ is a local CR-diffeomorphism at each point of $B^{\mathrm{CR}}(z_i,\tau)$ whose image lies in the strictly pseudoconvex part of $b\Omega$, the connected component of $g_i(z_i)$ in $\gamma \cap g_i(B^{\mathrm{CR}}(z_i,\tau))$ can be lifted to a complex path $\widetilde\gamma$ through $g_i$. Thus there exists $l \leq \ell(\gamma)$ and $\widetilde\gamma : [0,l] \longrightarrow B^{\mathrm{CR}}(z_i,\tau)$ joining $z_i$ to $bB^{\mathrm{CR}}(z_i,\tau)$ such that $g_i \circ \widetilde\gamma(t) = \gamma(t)$ for all $t \in [0,l]$. Since $\widetilde\gamma(t) \in U_1$ and $g_i(\widetilde\gamma(t)) \in U_1\backslash U_\varepsilon$ for all $t$, the estimates obtained in lemma 3.4 yield :

$$\ell(\gamma) \geq l = \int_0^l \|\dot\gamma(t)\|dt = \int_0^l \|g_i'(\widetilde\gamma(t))\dot{\widetilde\gamma}(t)\|dt \geq c_i \int_0^l \|\dot{\widetilde\gamma}(t)\|dt \geq c_i\ell(\widetilde\gamma).$$

This proves (1) since $\widetilde\gamma$ joins $z_i$ to $bB^{\mathrm{CR}}(z_i,\tau)$ (so $\ell(\widetilde\gamma) \geq \tau$) and $\gamma$ is any complex path joining $g_i(z_i)$ to $x$. $\square$

# 4 Dynamics of $f$ inside $\Omega$.

In this section we explain why the maps $f^n$ - and not only $f^{n_k}$ - converge to $a$ inside $\Omega$. By remark 2.4, and since we already know that $b\Omega$ is spherical near $a$, it is exactly what remains at this point to conclude the proof of theorem 1. We get this information by conjugating locally $f$ to an automorphism of the ball, whose dynamics is easily understood thanks to the following classical result (see [14]).

**Proposition 4.1.** *The dynamics of a ball automorphism $g$ is necessarily of one of the following types.*

1. *Hyperbolic (or North-South) : There exist exactly two fixed points $N, S \in bB$ of $g$ and $g^n$ converges locally uniformly to $S$ on $\overline{B}\backslash\{N\}$.*

2. *Parabolic (or South-South) : There exists a unique fixed point $S \in bB$ of $g$ and $g^n$ converges locally uniformly to $S$ on $\overline{B}\backslash\{S\}$.*

3. *Recurrent (or compact) : The $g$-orbits remain at fixed distance from $bB$. If $g$ has a fixed point on $bB$ then it has a whole complex line through this point, pointwise fixed by $g$ (see also [5]).*

We keep the same notation as in the previous part. Consider a CR-diffeomorphism $\Phi : U_\varepsilon \longrightarrow V \subset bB$ and extend it to a biholomorphism $\Phi : \Omega_\varepsilon \longrightarrow D \subset B$ exactly as in the convex case. Define the $\Phi$-conjugate of $f$

$$\begin{array}{cccc} g & : & \Phi(\Omega_\varepsilon \cap f^{-1}(\Omega_\varepsilon)) & \longrightarrow & \Phi(\Omega_\varepsilon) \\ & & y & \longmapsto & \Phi \circ f \circ \Phi^{-1}(y), \end{array}$$

and extend it to a global automorphism of the ball.

**Lemma 4.2.** *The dynamics of $g$ is either parabolic or hyperbolic with south pole $\Phi(a)$.*

*Proof :* Since $f$ has no fixed point in $\Omega$, $g$ cannot have any fixed point in $\Phi(\Omega_\varepsilon)$, so $\Phi(a)$ is an isolated fixed point and the dynamics of $g$ cannot be recurrent. We therefore only need to check that if $g$ is hyperbolic then $\Phi(a)$ is the south pole of the dynamics. Assume by contradiction it is not : $\Phi(a)$ is a repulsive fixed point of $g$, so $f$ is also repulsive at $a$. After shrinking $\Omega_\varepsilon$ if necessary, we can therefore assume that $f^{-1}$ is defined on $\Omega_\varepsilon$, with values in $\Omega_\varepsilon$ and even that $d(f^{-1}_{|\Omega_\varepsilon}(y), a) < d(y, a)$ for any $y \in \Omega_\varepsilon$. By assumption, there is a point $y_0 \in \Omega$ such that $f^{n_k}(y_0) \in \Omega_\varepsilon$ as soon as $k$ is large enough. Define then

$$n'_k := \min\{n \mid f^i(y_0) \in \Omega_\varepsilon, \quad \forall i \in [n, n_k]\},$$

so that $f^{n'_k-1}(y_0) \notin \Omega_\varepsilon$. Since $f^{-1}_{|\Omega_\varepsilon}$ is contracting, the point $f^{n'_k}(y_0)$ is closer to $a$ than $f^{n_k}(y_0)$, so it tends to $a$ (in particular $(n'_k)$ is an extraction). Equivalently $f^{n'_k-1}(f(y_0))$ tends to $a$, so $f^{n'_k-1}$ converges locally uniformly to $a$ by remark 1.1. This is in contradiction with $f^{n'_k-1}(y_0) \notin \Omega_\varepsilon$. □

This lemma implies the existence of a point $x \in \Phi(\Omega_\varepsilon)$ whose orbit remains in $\Phi(\Omega_\varepsilon)$ and tends to $\Phi(a)$. The orbit of $y = \Phi^{-1}(x)$ is thus the preimage of the $g$-orbit of $x$ by $\Phi$ so $f^n(y)$ tends to $a$. By remark 1.1, we thus get the

**Corollary 4.3.** *Under the assumption of theorem 1, the sequence $(f^n)$ converges to $a$ on $\Omega$.*

**Proof of theorem 1.** Theorem 1 is now a formal consequence of remark 2.4, proposition 3.1 and corollaries 3.3 and 4.3.

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