Singular polarizations and ellipsoid packings.

Emmanuel Opshtein. *

Abstract

We prove in this paper that any 4-dimensional symplectic manifold is essentially made of finitely many symplectic ellipsoids. The key tool is a singular analogue of Donaldson's symplectic hypersurfaces in irrational symplectic manifolds.

1 Introduction.

Donaldson proved in [6] that a symplectic manifold (M,ω) with $[\omega] \in H^2(M,\mathbb{Z})$ (so-called *rational*)[1] always admits a symplectic polarization of large enough degree k, that is a symplectic hypersurface Poincaré-dual to $k[\omega]$. In [5], Biran showed that these polarizations decompose the manifold into a standard "fat" part and a "thin" part which is isotropic in the Kähler case, and which has zero-volume in any case. In [15], it was noticed that the standard part of the decomposition is itself made of a standard ellipsoid and an object of codimension one. Put together, these results show that rational symplectic manifolds are always covered by *one* ellipsoidal Darboux chart up to a negligible set. This approach is rather satisfactory for $\mathbb{P}^2$ or $(S^2\times S^2,\omega\oplus\omega)$ where polarizations of low degrees can easily be found. However, as the degree of the polarization becomes larger, the ellipsoid gets more intricate and the codimension-one part more significant. This explosion of degree prevents from getting anything interesting on irrational manifolds. The present work shows however that an analogous result holds in the irrational setting.

Theorem 1. *Any closed* 4*-dimensional symplectic manifold has a full packing by a finite number of ellipsoids. This number can be bounded by a purely topological quantity : the dimension of* $H^2(M,\mathbb{R})$*.*

This theorem is not really about symplectic embeddings : it does not address the question of how flexible they might be, like for instance [8, 9, 11, 13, 19]. It rather gives a description of a symplectic manifold as a patchwork of euclidean pieces (ellipsoids) whose complexity - if only measured by the number N of pieces - does not really depend on the symplectic structure

*Partially supported by ANR projects "Floer Power" ANR-08-BLAN-0291-03 and "Symplexe" BLAN06-3-137237

[1] There is another use of the term *rational* in 4-dimensional symplectic topology, meaning that the manifold admits a fibration by pseudo-holomorphic spheres. It has nothing to do with our definition.

(see also [18] for a result in this spirit). The bound above is rather loose (for instance when the symplectic form is rational). It can be improved by a closer look at the proof. In fact,

$$N \leq \min\{\dim V,\ V \subset H^2(M, \mathbb{Q}),\ [\omega] \in \mathrm{Span}_{\mathbb{R}} V\}.$$

The theorem is a consequence of the following two results. First, Donaldson's construction of polarizations extends to irrational symplectic manifolds.

Theorem 2. *For any closed symplectic manifold (M, ω) there exist symplectic hypersurfaces $(\Sigma_1, \ldots, \Sigma_N)$ with transverse and positive intersections such that*

$$[\omega] = \sum_{i=1}^{N} a_i \mathrm{PD}(\Sigma_i), \qquad a_i \in \mathbb{R}^+. \tag{1}$$

A family of symplectic hypersurfaces that satisfies (1) will be called a *singular* polarization of M. In dimension higher than four, the meaning of "positive intersection" obviously has to be explained, and we refer the reader to section 5. As their classical analogues, singular polarizations can be used to embed ellipsoids, at least in dimension four.

Theorem 3. *Let (M^4, ω) be a closed symplectic manifold with*

$$[\omega] = \sum_{i=1}^{N} a_i \mathrm{PD}(\Sigma_i), \qquad a_i \in \mathbb{R}^+,$$

where Σ_i are symplectic curves whose pairwise intersections are all positive. Then M has a full packing by the ellipsoids $\mathcal{E}(A_i, a_i)$ where A_i denotes the symplectic area of Σ_i. Precisely, for all $\varepsilon > 0$, there exists an embedding

$$\Phi : \amalg\, \mathcal{E}(A_i - \varepsilon, a_i) \hookrightarrow M$$

which admits $(\Sigma_1, \ldots, \Sigma_N)$ as supporting surfaces, i.e. the image of the "horizontal" disc $\{z_2 = 0\}$ in $\mathcal{E}(A_i - \varepsilon, a_i)$ covers Σ_i up to an area ε.

Some remarks are in order. First, a simple computation shows that M is covered by the image of Φ up to a volume of order ε, hence the wording "full packing". Together with theorem 2, it obviously proves the basic assertion of this paper. Next, in theorem 3, the *embedded* curves Σ_i are allowed to have negative self-intersections (*i.e.* $\Sigma_i^2 < 0$) : the positivity condition only concerns intersections between different curves. As such, it applies for instance naturally in the context of blow-ups. It can therefore be used to understand what happens to the ellipsoid decomposition in rational sympletic manifolds equipped with polarizations with singularities. It allows in some sense to make the desingularization process compatible with Biran's decompositions. Another application concerns symplectic isotopies

: the proof of theorem 3 goes along the same lines as the proof of Biran's decomposition theorem given in [16], and it extends the range of the method of isotopy developed there. Finally, it may be worth pointing out that the dimension hypothesis seems mostly technical, and may be removed at least in some concrete situations[2]. Here is an illustration of how theorem 2 may be used :

Corollary 1.1. *The one-point blow-up* $\hat{\mathbb{P}}^2_1$ *of* $\mathbb{P}^2$ *obtained by blowing-up a ball of capacity* a *has full packing by* $\mathcal{E}(1-a,1) \sqcup \mathcal{E}(a,1-a)$.

Proof : Calling E the exceptional divisor and e its Poincaré-Dual, write $[\hat{\omega}] = l - ae = (l-e) + (1-a)e = \mathrm{PD}(L-E) + (1-a)\mathrm{PD}(E)$. □

The paper is organized as follows. We first discuss the main idea of the paper through the two easiest examples : the non-singular and the "flat" cases. In section 3, we give a local model for a neighbourhood of a singular polarization, as well as the main properties of this model in terms of Liouville forms. In section 4, we prove theorem 3. We then explain the small modifications to Donaldson's arguments needed to prove the existence of singular polarizations (theorem 2). We finally deal with the applications in the last two sections : Biran's decomposition associated to singular curves in section 6 and isotopies of balls in section 7.

Notations : We adopt the following (not so conventional) conventions throughout this paper :

- *All* angles will take value in $\mathbb{R}/\mathbb{Z}$. In other terms an angle 1 is a full turn in the plane, and the integral of the form $d\theta$ over a circle around the origin in the plane is 1.
- The standard symplectic form on $\mathbb{C}^n = \mathbb{R}^{2n}$ is $\omega_{\mathrm{st}} := \sum dr_i^2 \wedge d\theta_i$, where (r_i,θ_i) are polar coordinates on the plane factors. With this convention, the euclidean ball of radius 1 has capacity 1.
- A Liouville form λ of a symplectic structure ω is a one-form satisfying $\omega = -d\lambda$. The standard Liouville form on the plane is $\lambda_{\mathrm{st}} := -r^2 d\theta$.
- A symplectic ball or ellipsoid is the image of a Euclidean ball or ellipsoid in $\mathbb{C}^n$ by a symplectic embedding.
- The Hopf discs of a Euclidean ball in $\mathbb{C}^n$ are its intersections with complex lines.
- $\mathcal{E}(a,b)$ denotes the 4-dimensional ellipsoid $\{a^{-1}|z|^2 + b^{-1}|w|^2 < 1\} \subset \mathbb{C}^2(z,w)$. Because of our normalizations, its Gromov's width is $\min(a,b)$.

Aknowledgement : I wish to thank an anonymous referee for suggesting a much simpler proof of theorem 2.

[2]Since the paper was submitted, I have understood how to do (see section 8).

2 Two easy examples.

2.1 The non-singular case.

In this paragraph we review briefly for self-containedness the result of [15] in the setting of smooth polarizations. Let (M, ω) be a rational symplectic manifold with a polarization Σ of degree k. Biran's result states that there is an embedding of a symplectic disc bundle $\mathrm{SDB}(\Sigma, k)$ into M which has full volume. This disc bundle can be seen as the part of the normal line bundle of Σ - denoted by $\mathcal{N}_\Sigma$ - in M on which the closed 2-form ω_0 (to be defined soon) is symplectic. The line bundle $\mathcal{N}_\Sigma$ can be equiped with a hermitian metric and a connection form which allow to define a form α on $\mathcal{N}_\Sigma \backslash \mathcal{L}_0$ satisfying $\alpha_{|F} = d\theta$ and $d\alpha = -k\pi^*\omega_{|\Sigma}$. The form ω_0 is then simply given in these coordinates by :

$$\omega_0 := \pi^*\omega_{|\Sigma} + d(r^2\alpha) = (1 - kr^2)\pi^*\omega_{|\Sigma} + dr^2 \wedge \alpha.$$

It was proved in [15] that the restriction of this disc bundle to a disc of area A in the base is an ellipsoid $\mathcal{E}(A, 1/k)$.

Lemma 2.1. *Let $\pi : \mathrm{SDB}(\Sigma, k) \longrightarrow \Sigma$ be the symplectic disc bundle defined above and let D_A be a disc of area A in Σ. Then $(\pi^{-1}(D_A), \omega_0) \simeq (\mathcal{E}(A, 1/k), \omega_{\mathrm{st}})$.*

Let us mention an application that was not made completely explicit in [15]. It answers a question of McDuff [11].

Theorem 4. *There exists a symplectic embedding of $\mathcal{E}(2, \frac{1}{2})$ into $B^4(1)$.*

Proof : First notice that $\mathbb{P}^2$ has such a full packing because it has a polarization of degree 2, of area 2, namely a conic. Let us give now an explicit description of a prefered disc bundle over the conic $Q := \{z_0^2 + z_1^2 + z_2^2 = 0\} \subset \mathbb{P}^2$. Since Q is real, it is invariant by conjugation, and each real projective line (*i.e.* invariant by complex conjugation) intersects Q in exactly two distinct conjugate points. Moreover, $\mathbb{RP}^2$ splits each of these lines into two disks of equal area one-half, that contain one of these two points each. The fibers of the disc bundle over the points of Q are precisely these half real lines [5]. Fix now $z, \overline{z} \in Q$ and call $d_{z,\overline{z}}$ the (real) line passing through z and $\overline{z}$. Consider also a disc D_Q in Q of full area which misses z and $\overline{z}$. The restriction of this symplectic disc bundle to D_Q is an open ellipsoid $\mathcal{E}(\mathcal{A}_\omega(D_Q), 1/2) = \mathcal{E}(2, 1/2)$. By construction, this ellipsoid does not meet the fibers above $z, \overline{z}$, so it misses the projective line $d_{z,\overline{z}}$. Since $\mathbb{P}^2 \backslash d_{z,\overline{z}} = B^4(1)$, the ellipsoid $\mathcal{E}(2, \frac{1}{2})$ embeds in fact into $B^4(1)$. $\square$

Lemma 2.1 serves also to split an ellipsoid into smaller ones. As such, it proved useful to give a natural construction of a maximal symplectic packing of $\mathbb{P}^2$ by five balls [15]. Let us now mention a far less successful story : looking for such maximal symplectic packings of $\mathbb{P}^2$ by seven balls (known by [12] to be of radius $r^2 = 3/8$). Using the same idea, one can easily pack $\mathbb{P}^2$ with eight ellipsoids $\mathcal{E}(\frac{3}{8}, \frac{1}{3})$ using a smooth polarization of degree three. These ellipsoids fail to contain the desired balls because $\frac{1}{3} < \frac{3}{8}$. But there are eight of them instead of seven. Notice that one of these ellipsoids can then be split into eight ellipsoids $\mathcal{E}(\frac{3}{8}, \frac{1}{24})$. In this approach, the question would now to be able to glue seven of these eight thin ellipsoids with the seven bigger ones to get seven ellipsoids $\mathcal{E}(\frac{3}{8}, \frac{1}{3} + \frac{1}{24}) = \mathcal{E}(\frac{3}{8}, \frac{3}{8}) = B^4(\frac{3}{8})$. But this point seems rather hard.

2.2 The product case.

Let us discuss now the basic idea of the paper, in the easiest case of a product. Consider the symplectic manifold $M := (S^2 \times S^2, \omega \oplus \frac{p}{q}\omega)$, where p, q are relatively prime integer. This manifold has a symplectic polarization of degree q which is a smoothing of a curve

$$\begin{array}{rccc} \varphi & : & S^2 & \longrightarrow & S^2 \times S^2 \\ & & z & \longmapsto & (f_1(z), f_2(z)), \end{array}$$

where f_1, f_2 are self-maps of S^2 of degrees p and q respectively. Over this complicated polarization, there is a symplectic ellipsoid $\mathcal{E}(2p, 1/q)$ which cannot be very simple. For instance, when p/q tends to an irrational number, Gromov's capacity of the ellipsoid tends to zero, and nothing remains at the limit. By contrast, there is a much simpler *singular* polarization on the homological level given by $(S^2 \times \{*\}, \{*\} \times S^2)$, which provides a decomposition of M into two ellipsoids $\mathcal{E}(1, \frac{p}{q})$ in the following way.

Put coordinates $\big((r_1, \theta_1), (r_2, \theta_2)\big)$ on $S^2 \times S^2$ (remember that $\theta_i \in [0, 1[$) with the convention that $\{r_1^2 = 1\}$ and $\{r_2^2 = p/q\}$ is one point (S^2 is seen as the one point compactification of the disc of suitable radius). Denote also $\Sigma_1 := S^2 \times \{0\}$ and $\Sigma_2 := \{0\} \times S^2$. The symplectic form on M is

$$\begin{aligned} \omega = \;& dr_1^2 \wedge d\theta_1 + dr_2^2 \wedge d\theta_2 \\ & = -d\lambda, \qquad \text{where} \\ \lambda = \;& (1 - r_1^2) d\theta_1 + (\tfrac{p}{q} - r_2^2) d\theta_2. \end{aligned}$$

The Liouville form λ is defined on $M \backslash (\Sigma_1 \cup \Sigma_2)$ and gives rise to a forward complete Liouville vector field, which is easily seen to be

$$X = \frac{1 - r_1^2}{2r_1} \frac{\partial}{\partial r_1} + \frac{\frac{p}{q} - r_2^2}{2r_2} \frac{\partial}{\partial r_2}.$$

The flow lines of this vector field are best seen on the toric coordinates $(R_1, R_2) := (r_1^2, r_2^2)$ on M, and are shown in figure 1.

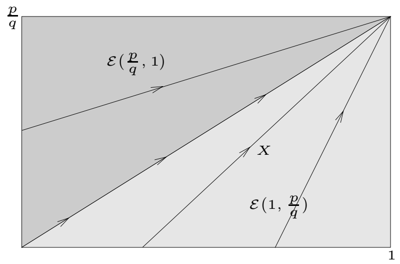

Figure 1: The vector field X in the toric coordinates on $(S^2 \times S^2, \omega \oplus \frac{p}{q}\omega)$.

We actually see that X is tangent to the line $R_2 := \frac{p}{q}R_1$, so the trajectories of X emanating from $\Sigma_1\backslash\{(0,0)\}$ and $\Sigma_2\backslash\{(0,0)\}$ are respectively $R_1 \geq \frac{p}{q}R_2$ and $R_2 \geq \frac{p}{q}R_1$. These triangles are well-known to be filled by the ellipsoid $\mathcal{E}(1, p/q)$. Thus we see that we get the toric decomposition of $S^2 \times S^2$ into two ellipsoids (up to zero volume) out of data consisting of a singular polarization (Σ_1, Σ_2) and a Liouville vector field X on the complement of $\Sigma_1 \cup \Sigma_2$. This approach provides much simpler objects (in a geometric sense) than the one giving only one ellipsoid. In particular, both the singular polarization and the embeddings survive the process of degenerating p/q to an irrational number. The aim of this note is to understand this simple picture in a general context.

3 Plumbed symplectic disc bundles.

Let $(M, \omega), \Sigma_1, \ldots, \Sigma_n$ be as in theorem 3, that is the Σ_i are symplectic smooth curves with

$$[\omega] = \sum_1^n a_i\sigma_i, \qquad \sigma_i := \mathrm{PD}(\Sigma_i),\ a_i > 0, \tag{2}$$

and all intersection points between any two of these curves is positive. Put $\Sigma_i \cap \Sigma_j =: \{(p_{ij}^k)_{k\in[1,l_{ij}]}\}$. With no loss of generality, we can assume that the curves are symplectically orthogonal with respect to ω at each intersection point (such a configuration can be achieved by small local perturbations).

3.1 Local model near the polarization.

Decompose first the area form on Σ_i as $\omega_{|\Sigma_i} = \tau_i + \sum_{j,k} \tau_{ij}^k$, where :

- the forms τ_{ij}^k have supports on small discs D_{ij}^k around p_{ij}^k, with total masses εa_j,

- the form τ_i has support on the complement $\Sigma_i \backslash (\cup D_{ij}^k{}')$ of smaller discs also centered in p_{ij}^k, with total mass

$$\mathcal{A}_i^\varepsilon := \mathcal{A}_{\tau_i}(\Sigma_i) = \mathcal{A}_\omega(\Sigma_i) - \varepsilon \sum_{j \neq i} \Sigma_i \cdot \Sigma_j a_j = a_i \Sigma_i \cdot \Sigma_i + (1-\varepsilon) \sum_{j \neq i} a_j \Sigma_i \cdot \Sigma_j .$$

  We can also assume that the area of τ_i on the complement of the discs D_{ij}^k is $\mathcal{A}_i^{\varepsilon'}$ for ε' slightly bigger than ε.

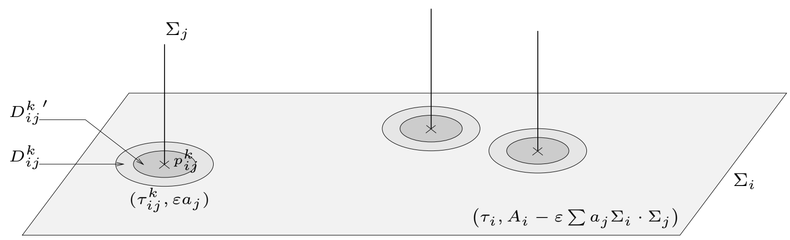

Figure 2: Local model near Σ_i.

Consider now the line bundle $\pi_i : \mathcal{L}_i \to \Sigma_i$ which is modeled on the (symplectic) normal bundle of Σ_i in M - *i.e.* they have the same Chern class. Endow this bundle with a hermitian metric, (local) coordinates (r_i, θ_i, z) and a connection with curvature $2i\pi\gamma_i a_i^{-1} \pi_i^* \tau_i$, where

$$\frac{1}{\gamma_i} := \frac{a_i^{-1} \mathcal{A}_{\tau_i}(\Sigma_i)}{\Sigma_i \cdot \Sigma_i} = 1 + (1-\varepsilon) \frac{\sum_{j \neq i} a_j \Sigma_j \cdot \Sigma_i}{a_i \Sigma_i \cdot \Sigma_i} \tag{3}$$

Notice that γ_i is negative when $\Sigma_i \cdot \Sigma_i < 0$, vanishes when $\Sigma_i \cdot \Sigma_i = 0$ and is never greater than 1. Defining the form α_i on $\mathcal{L}_i$ by asking that its restriction to the fiber is $a_i d\theta_i$ and that it vanishes on the horizontal planes of the connection, we get a form that satisfies :

$$\begin{cases} \alpha_{i|F} = a_i d\theta_i \quad (F \text{ is the fiber}), \\ d\alpha_i = -a_i \gamma_i a_i^{-1} \pi_i^* \tau_i = -\gamma_i \pi_i^* \tau_i. \end{cases}$$

We define now a closed two-form on $\mathcal{L}_i$ by

$$\begin{aligned} \omega_i := \quad & \pi_i^* \tau_i + d(r_i^2 \alpha_i) + \sum_{j,k} \pi_i^* \tau_{ij}^k, \\ = \quad & (1 - \gamma_i r_i^2) \pi_i^* \tau_i + dr_i^2 \wedge \alpha_i + \sum_{j,k} \pi_i^* \tau_{ij}^k. \end{aligned}$$

When γ_i is non-positive, this form is symplectic on $\mathcal{L}_i$. But in the positive situation, ω_i is only symplectic on the disc bundle with fibers of area γ_i^{-1} (on even larger discs over D_{ij}^k). We will denote in the sequel by $\mathrm{SDB}(\mathcal{L}_i)$ the symplectic part of the line bundle.

A standard Moser argument shows moreover that there are neighbourhoods $\mathcal{U}_i, \mathcal{V}_i$ of the zero-section $\mathcal{L}_i^0$ and Σ_i respectively which are symplectomorphic. In other terms, there exists an embedding

$$\varphi_i : (\mathcal{U}_i, \omega_i) \hookrightarrow (M, \omega), \qquad \varphi_i(\mathcal{U}_i) = \mathcal{V}_i,\ \varphi_i(\mathcal{L}_i^0) = \Sigma_i.$$

For simplicity, we henceforth assume that $\mathcal{V}_i$ is itself endowed with a fibration (given by $\pi_i \circ \varphi_i^{-1}$) and coordinates (r_i, θ_i). Moreover, since Σ_i and Σ_j are symplectically orthogonal at p_{ij}^k, a local symplectomorphism allows to make the fibration structures of $\mathcal{V}_i$ and $\mathcal{V}_j$ coincide on $\mathcal{V}_i \cap \mathcal{V}_j$, namely arranging that $(r_i, \theta_i, r_j, \theta_j)$ provide full coordinate charts in $\mathcal{V}_i \cap \mathcal{V}_j$, for which the two sets of fibers are given by the fibers of (r_i, θ_i) and (r_j, θ_j). With such normalization, we can finally assume that

$$\pi_i^* \tau_{ij}^k = a_j df_{ij}^k(r_j) \wedge d\theta_j, \tag{4}$$

where $f_{ij}^k \equiv \varepsilon$ outside D_{ij}^k and coincides with r_j^2 near p_{ij}^k. In some neighbourhood of this point, we therefore have :

$$\omega = a_i dr_i^2 \wedge d\theta_i + a_j dr_j^2 \wedge d\theta_j.$$

Let us sum up the above discussion:

Proposition 3.1 (Weinstein). *Let (M, ω, Σ_i) be a symplectic manifold with a singular polarization as in (2). There exist neighbourhoods $\mathcal{V}_i$ of Σ_i in M and $\mathcal{U}_i$ of the zero-section in $\mathcal{L}_i$ which are identified via diffeomorphisms $\varphi_i : \mathcal{U}_i \to \mathcal{V}_i$. In these coordinates, the symplectic form is given by*

$$\varphi_i^* \omega = \omega_i = \pi_i^* \tau_i + d(r_i^2 \alpha_i) + \sum \pi_i^* \tau_{ij}^k,$$

where τ_{ij}^k has support in $\mathcal{V}_i \cap \mathcal{V}_j$ and

$$\gamma_i = \frac{a_i \Sigma_i \cdot \Sigma_i}{\mathcal{A}_{\tau_i}(\Sigma_i)}, \quad \begin{cases} \alpha_{i|F} = a_i d\theta_i \\ d\alpha_i = -\gamma_i \pi^* \tau_i \end{cases}, \quad \begin{cases} \pi_i^* \tau_{ij}^k = a_j df_{ij}^k(r_j) \wedge d\theta_j, \\ f_{ij|^c D_{ij}^k}^k \equiv \varepsilon, \quad f_{ij}^k = r_j^2 \ \text{near } p_{ij}^k \end{cases}.$$

Finally, near p_{ij}^k, $\omega_i = a_i dr_i^2 \wedge d\theta_i + a_j dr_j^2 \wedge d\theta_j$.

In other words, a neighbourhood $\mathcal{V} := \cup \mathcal{V}_i$ of the whole polarization is a plumbing of the $\mathcal{U}_i$ along the bidiscs $D_{ij}^k \times D_{ji}^{k'}$ (where $p_{ij}^k = p_{ji}^{k'}$).

3.2 Liouville forms on the symplectic disc bundles.

The symplectic disc bundles $\mathrm{SDB}(\mathcal{L}_i)$ defined in the previous paragraph come naturally with Liouville forms (recall they are primitives of the *opposite* of the symplectic forms). A more careful analysis - that we perform now - shows that, as long as intersections are positive, it is possible to impose compatibility conditions on these forms in order to glue them to get a Liouville form on $\mathcal{V}$.

Lemma 3.2. *There is a Liouville form* λ_i *on* $\mathrm{SDB}(\mathcal{L}_i)\backslash\big(\mathcal{L}_i^0\cup\pi_i^{-1}(p_{ij}^k)\big)$ *such that* $\lambda_i = a_i(1-r_i^2)d\theta_i + a_j(1-r_j^2)d\theta_j$ *near* p_{ij}^k*. In fact,*

$$\lambda_i = (1-r_i^2)\alpha_i + (1-\gamma_i)\pi_i^*\lambda_i' + \sum \pi_i^*\lambda_{ij}^k \tag{5}$$

for well-chosen Liouville forms λ_i'*,* λ_{ij}^k *for* τ_i*,* τ_{ij}^k *in* $\Sigma_i\backslash\cup\{p_{ij}^k\}$*. The Liouville form* λ_i' *can however be chosen arbitrarily on any disc compactly supported in* $\Sigma_i\backslash\cup D_{ij}^k$*.*

Proof : Consider first any Liouville forms $\lambda_i', \lambda_{ij}^k$ for τ_i, τ_{ij}^k in $\Sigma_i\backslash\cup\{p_{ij}^k\}$. Then the one-form defined by (5) is a Liouville form for ω_i. Indeed,

$$\begin{aligned} d\lambda_i = &\ -dr_i^2\wedge\alpha_i - (1-r_i^2)\gamma_i\pi_i^*\tau_i - (1-\gamma_i)\pi_i^*\tau_i - \textstyle\sum\pi_i^*\tau_{ij}^k \\ = &\ -dr_i^2\wedge\alpha_i + (-\gamma_i+\gamma_i r_i^2 - 1 + \gamma_i)\pi_i^*\tau_i - \textstyle\sum\pi_i^*\tau_{ij}^k \\ = &\ -dr_i^2\wedge\alpha_i - (1-\gamma_i r_i^2)\pi_i^*\tau_i - \textstyle\sum\pi_i^*\tau_{ij}^k \\ = &\ -\omega_i. \end{aligned}$$

We now need to choose well the forms λ_i' and λ_{ij}^k. Define first λ_{ij}^k by

$$\lambda_{ij}^k := a_j(\varepsilon - f_{ij}^k(r_j))d\theta_j,$$

and recall that by definition of f_{ij}^k, it vanishes identically outside D_{ij}^k. In order to define λ_i', notice that τ_i has support in $\Sigma_i\backslash {D_{ij}^k}'$ and $(1-\gamma_i)\mathcal{A}_{\tau_i}(\Sigma_i) = (1-\varepsilon)\sum_{j\neq i} a_j\Sigma_i\cdot\Sigma_j$. Therefore, there exists a Liouville form λ_i' of τ_i such that

$$(1-\gamma_i)\lambda_i' = (1-\varepsilon)a_j d\theta_j \qquad \text{near } p_{ij}^k.$$

It is moreover obvious that this condition is compatible with any requirement on λ_i' on a disc compactly supported in $\Sigma_i\backslash\cup D_{ij}^k$. Putting all this together, we get the following expression for λ_i in a neighbourhood of p_{ij}^k :

$$\begin{aligned} \lambda_i = &\ (1-r_i^2)\alpha_i + (1-\varepsilon)a_j d\theta_j + a_j(\varepsilon - r_j^2)d\theta_j \\ = &\ a_i(1-r_i^2)d\theta_i + a_j(1-r_j^2)d\theta_j. \end{aligned} \qquad \square$$

Recall that a Liouville form λ gives rise to a vector field X_λ - called Liouville - by symplectic duality : $\iota_{X_\lambda}\omega = \lambda$. This vector field has the property of contracting the symplectic form : $\Phi_{X_\lambda}^{t*}\omega = e^{-t}\omega$. Thanks to the careful choices we made until now, both the sets of Liouville forms (λ_i) and vector fields (X_{λ_i}) glue together to well-defined objects on $\mathcal{V}\backslash(\cup\Sigma_i)$.

Lemma 3.3. *The formulas*

$$\begin{cases} \lambda_{|\mathcal{V}_i} & := \varphi_{i*}\lambda_i \\ X_{\lambda|\mathcal{V}_i} & := \varphi_{i*}X_{\lambda_i} \end{cases}$$

define a Liouville form and its associated Liouville vector fields on $\mathcal{V}\backslash\cup\Sigma_i$*. Moreover, the vector field* X_λ *points outside* $\mathcal{V}$ *if this neighbourhood is well-chosen.*

Proof : The first point is an obvious consequence of the previous lemma because $\lambda_i = \lambda_j$ near p_{ij}^k. The second statement is a straightforward consequence of the fact that each X_{λ_i} points outside the zero-section on $\mathcal{L}_i$, and this is a simple computation :

$$\begin{aligned}\omega_i\left(X_{\lambda_i}, \frac{\partial}{\partial \theta_i}\right) &= dr_i^2 \wedge \alpha_i\left(X_{\lambda_i}, \frac{\partial}{\partial \theta_i}\right) &&= a_i dr_i^2(X_{\lambda_i}) \\ &= \lambda_i\left(\frac{\partial}{\partial \theta_i}\right) &&= (1-r_i^2)a_i \text{ (see (5)).}\end{aligned}$$

Thus $dr_i^2(X_{\lambda_i}) = 1 - r_i^2 > 0$ near the zero-section $\{r_i = 0\}$. □

The following lemma gives a nice expression of the Liouville vector fields associated to the forms λ_i defined above. In the statement, the disc $\mathcal{D}_A$ should be thought of as a disc in $\Sigma_i \backslash \cup D_{ij}^k$ of approximately full area.

Lemma 3.4. *Consider the trivial disc bundle $\pi : \mathcal{D}_A \times \mathbb{D}_{\gamma^{-1}} \longrightarrow \mathcal{D}_A$ (or $\mathcal{D}_A \times \mathbb{C}$ if $\gamma < 0$) over a disc in $\mathbb{C}$, with polar coordinates (r, θ) and (ρ, ζ) on $\mathbb{D}_{\gamma^{-1}}$ and $\mathcal{D}_A$ respectively. Equip this bundle with the symplectic structure $\omega := \pi^*\omega_{\rm st} + d(r^2\alpha)$, where $\alpha_{|\{x\}\times\mathbb{D}} = ad\theta$ and $d\alpha = -\gamma\pi^*\omega_{\rm st}$. Let λ be the Liouville form for ω defined by*

$$\lambda = (1-r^2)\alpha + (1-\gamma)\pi^*\lambda_{\rm st}.$$

and X_λ its associated vector field. Then

i)
$$X_\lambda = \frac{1-r^2}{2r}\frac{\partial}{\partial r} - \frac{1-\gamma}{1-\gamma r^2}\frac{\rho}{2}\frac{\partial}{\partial \rho};$$

ii) there exists a smooth function $h : \mathcal{D}_A \longrightarrow \mathbb{R}$ such that the map

$$\begin{array}{rcccl}\Phi & : & (\mathcal{D}_A \times \mathbb{D}_{\gamma^{-1}}, \omega) & \longrightarrow & \big(\mathcal{E}(A, a\gamma^{-1}), \omega_{\rm st}\big) \\ & & (z, w) & \longmapsto & (z', w') = (\sqrt{1-\gamma|w|^2}z, \sqrt{a}e^{i\,h(z)}w)\end{array}$$

is a symplectomorphism (when γ is negative, $\mathcal{E}(A, a\gamma^{-1})$ is an hyperboloid rather than an ellipsoid);

iii) setting $R' := r'^2 = |w'|^2$ and $\mathcal{P}' := \rho'^2 = |z'|^2$,

$$\Phi_* X_\lambda = (a - R')\frac{\partial}{\partial R'} - \mathcal{P}'\frac{\partial}{\partial \mathcal{P}'} + *\frac{\partial}{\partial \theta'},$$

where $*$ stands for an arbitrary function, completely irrelevant for us.

Proof : The point ii) is the same statement as lemma 2.1 in [15]. It is an easy computation, which we do not repeat here. The point i) is a simple verification. Write $\omega = (1-\gamma r^2)d\rho^2 \wedge d\zeta + dr^2 \wedge \alpha$ and compute :

$$\begin{array}{rcl}\omega(\frac{1-r^2}{2r}\frac{\partial}{\partial r} - \frac{1-\gamma}{1-\gamma r^2}\frac{\rho}{2}\frac{\partial}{\partial \rho}, \cdot) = & (1-r^2)dr \wedge \alpha(\frac{\partial}{\partial r}, \cdot) - (1-\gamma)\rho^2 d\rho \wedge d\zeta(\frac{\partial}{\partial \rho}, \cdot) \\ = & (1-r^2)\alpha - (1-\gamma)\rho^2 d\zeta \\ = & \lambda. \end{array}$$

For iii), first express Φ in the good coordinates $\Phi(\mathcal{P}, \zeta, R, \theta) = (\mathcal{P}', \zeta', R', \theta')$:

$$\mathcal{P}' = (1-\gamma R)\mathcal{P}, \qquad R' = aR, \qquad \zeta' = \zeta \qquad \theta' = \theta + h(\mathcal{P}, \zeta). \tag{6}$$

Then,

$$\left\{\begin{array}{l} \Phi_* \dfrac{\partial}{\partial R} = -\gamma \mathcal{P}\dfrac{\partial}{\partial \mathcal{P}'} + a\dfrac{\partial}{\partial R'} \\ \Phi_* \dfrac{\partial}{\partial \mathcal{P}} = (1-\gamma R)\dfrac{\partial}{\partial \mathcal{P}'} + *\dfrac{\partial}{\partial \theta'}. \end{array}\right. \tag{7}$$

Taking (6) and (7) into account, we therefore get :

$$\begin{array}{rcl}\Phi_* X_\lambda = & \Phi_*\big((1-R)\dfrac{\partial}{\partial R} - \dfrac{1-\gamma}{1-\gamma R}\mathcal{P}\dfrac{\partial}{\partial \mathcal{P}}\big) \\ \overset{(7)}{=} & (1-R)\big[-\gamma\mathcal{P}\dfrac{\partial}{\partial \mathcal{P}'} + a\dfrac{\partial}{\partial R'}\big] - (1-\gamma)\mathcal{P}\dfrac{\partial}{\partial \mathcal{P}'} + *\dfrac{\partial}{\partial \theta'} \\ = & a(1-R)\dfrac{\partial}{\partial R'} - \big[-\gamma\mathcal{P} + \gamma\mathcal{P}R - \mathcal{P} + \gamma\mathcal{P}\big]\dfrac{\partial}{\partial \mathcal{P}'} + *\dfrac{\partial}{\partial \theta'} \\ \overset{(6)}{=} & (a-R')\dfrac{\partial}{\partial R'} - \mathcal{P}'\dfrac{\partial}{\partial \mathcal{P}'} + *\dfrac{\partial}{\partial \theta'}. \end{array}$$

□

3.3 Ellipsoids in the standard bundles.

The ellipsoids of theorem 3 naturally arise from $\mathrm{SDB}(\mathcal{L}_i)$ as the set of points that can be reached by flowing out of a disc in $\Sigma_i \backslash \cup D_{ij}^k$ along the Liouville vector field. Precisely,

Proposition 3.5. *Let $\mathcal{D}_{A_i-\delta}$ be a disc of symplectic area $A_i - \delta$ in $\Sigma_i \backslash \cup D_{ij}^k$, viewed as part of the zero-section of $\mathrm{SDB}(\mathcal{L}_i)$. Then, if the form λ_i is well-chosen on $\mathcal{D}_{A_i-\delta}$, the basin of attraction of this disc, defined as*

$$\mathcal{B}_i := \Big\{p \in \mathrm{SDB}(\mathcal{L}_i) \mid \exists t \in \mathbb{R}^+, \ \Phi_{X_{\lambda_i}}^{-t}(p) \in \mathcal{D}_{A_i-\delta}\Big\}$$

is symplectomorphic to the ellipsoid $\mathcal{E}(A_i - \delta, a_i)$.

Proof : Since $\mathcal{D}_{A_i-\delta}$ is contained in $\Sigma_i \backslash \cup D_{ij}^k$, the symplectic form on the restriction of $\mathrm{SDB}(\mathcal{L}_i)$ to $\mathcal{D}_{A_i-\delta}$ is exactly of the form of lemma 3.4 :

$$\left\{\begin{array}{l} \omega_i = \pi_i^*\tau_i + d(r_i^2\alpha_i) \\ \lambda_i = (1-r^2)\alpha_i + (1-\gamma_i)\pi_i^*\lambda_i', \quad d\lambda_i' = -\tau_i. \end{array}\right.$$

Provided λ'_i corresponds also to the Liouville form called "standard" in this lemma (which can always be achieved because λ'_i can be any Liouville form on $\mathcal{D}_A$ by lemma 3.2), it provides a symplectic embedding $\Phi : \left(\pi_i^{-1}(\mathcal{D}_{A_i-\delta}),\omega_i\right) \hookrightarrow \left(\mathbb{C}^2,\omega_{\mathrm{st}}\right)$. This map sends the set $\mathcal{B}_i$ to

$$\Phi(\mathcal{B}_i) = \left\{p \in \mathbb{C}^2 \mid \exists t \in \mathbb{R}^+,\ \Phi^{-t}_{\Phi_* X_{\lambda_i}}(p) \in \mathbb{D}_{A_i-\delta} \times \{0\}\right\}.$$

By lemma 3.4 iii), if (z,w) are coordinates on $\mathbb{C}^2$, $\mathcal{P} = |z|^2$ and $R = |w|^2$, the differential equation associated to $\Phi_* X_{\lambda_i}$ is

$$\left\{\begin{array}{l} \dot{R} = a_i - R \\ \dot{\mathcal{P}} = -\mathcal{P} \end{array}\right. ,\ \text{with solutions} \left\{\begin{array}{l} R(p,t) = a_i - c_1(p)e^{-t} \\ \mathcal{P}(p,t) = -c_2(p)e^{-t} \end{array}\right. .$$

Now $\Phi(\mathcal{B}_i)$ is the set of points $p \in \mathbb{C}^2$ that verify :

$$R(p,t_0) = 0 \Longrightarrow \mathcal{P}(p,t_0) < A_i - \delta. \qquad (*)$$

An easy computation shows that $\mathcal{P}(p,t_0) = \frac{c_2(p)}{c_1(p)}a_i$, so that $(*)$ writes $c_2(p)a_i < c_1(p)(A_i-\delta)$. This in turn means

$$\begin{aligned} R(p) = a_i - \frac{c_1(p)}{c_2(p)}\mathcal{P}(p) < a_i - \frac{a_i}{A_i-\delta}\mathcal{P}(p) &\Longleftrightarrow \frac{R(p)}{a_i} + \frac{\mathcal{P}(p)}{A_i-\delta} < 1 \\ &\Longleftrightarrow p \in \mathcal{E}(A_i-\delta, a_i). \qquad \square \end{aligned}$$

We conclude this paragraph by noting that this ellipsoid is contained in the part of the bundle above the disc $\mathcal{D}_{A_i-\delta}$ simply because of the formula i) of lemma 3.4. Indeed, since $\gamma_i < 1$ (see (3), p.7), the "horizontal" part

$$-\frac{1-\gamma_i}{1-\gamma_i r^2}\frac{\rho}{2}\frac{\partial}{\partial\rho}$$

of the vector field X_{λ_i} above $\mathcal{D}_{A_i-\delta}$ points inside $\mathcal{D}_{A_i-\delta}$.

Remark 3.6. *The set $\mathcal{B}_i$ lies inside $\pi_i^{-1}(\mathcal{D}_{A_i-\delta})$.*

3.4 Variations of the Liouville forms.

Liouville forms are never unique : they can always be modified by adding a closed one-form. In the previous paragraphs, we needed to impose several compatibility conditions for the Liouville forms, namely to fix them on discs $\mathcal{D}_{A_i-\delta}, (D^k_{ij})$. These requirements only rigidify slightly the situation but still leaves a lot of freedom, which will be fully needed in the proof of theorem 3. Precisely, we will need the following set of objects :

- A family $\vartheta := (\vartheta_i)$ of closed one-forms on Σ_i which *vanish identically* on all the D^k_{ij} and $\mathcal{D}_{A_i-\delta}$. Notice that all homological classes in $H^1_{dR}(\Sigma_i)$ have such representatives.

- A family $\lambda_\vartheta := (\lambda_i + \pi_i^*\vartheta_i)$ of Liouville forms on $\mathrm{SDB}(\mathcal{L}_i)$.

These forms obviously satisfy the same compatibility conditions as the (λ_i), *i.e.* they give rise to a well-defined Liouville form still denoted λ_ϑ on $\mathcal{V}\backslash\cup\Sigma_i$. Moreover, since $\lambda_\vartheta = \lambda$ in $\mathcal{D}_{A_i-\delta}$ (and therefore in $\pi_i^{-1}(\mathcal{D}_{A_i-\delta})$), the remark 3.6 ensures that proposition 3.5 holds when λ is replaced by λ_ϑ. Finally, since λ_ϑ differs from λ only by a pull-back by π_i, the radial component of its Liouville vector field does not change : it still moves away from the zero-section, so that lemma 3.3 also holds for λ_ϑ.

4 Proof of theorem 3.

We adopt in this paragraph all conventions, notations and results of section 3 . The core lemma is now the following :

Lemma 4.1. *There exists a family of one-forms (ϑ_i) on Σ_i which vanish identically on $\mathcal{D}_{A_i-\delta}$ and D_{ij}^k such that the form λ_ϑ defined on $\mathcal{V}\backslash\cup\Sigma_i$ extends to a Liouville form β on $M\backslash\cup\Sigma_i$.*

Let us first explain quickly why theorem 3 is a direct consequence of this lemma. Since M is compact and $X_\vartheta := X_{\lambda_\vartheta}$ points outside the Σ_i, it defines a forward-complete vector field on $M\backslash\cup\Sigma_i$. Therefore, since β is really an extension of λ_ϑ, the elementary dynamical procedure that consists in extending the local symplectic embeddings $\varphi_i : \mathcal{U}_i \hookrightarrow \mathcal{V}_i$ (given by proposition 3.1) by

$$\begin{array}{rcccl} \Phi_i & : & \mathcal{B}_i & \longrightarrow & M \\ & & x & \longmapsto & \left\{ \begin{array}{cl} \varphi_i(x) & \text{if } x \in \mathcal{U}_i \\ \Phi^{\tau}_{X_\beta}\circ\varphi_i\circ\Phi^{-\tau}_{X_{\vartheta_i}}(x) & \text{if } \Phi^{-\tau}_{X_{\vartheta_i}}(x) \in \mathcal{U}_i \end{array}\right. \end{array}$$

provides symplectic embeddings Φ_i which clearly do not overlap. We therefore have an embedding $\Phi : \cup\mathcal{B}_i \hookrightarrow M$ which is the desired ellipsoid packing by proposition 3.5. $\square$

Before proving lemma 4.1, we need the following :

Lemma 4.2. *There exists a Liouville form β on $M\backslash\cup\Sigma_i$ such that*

$$\beta(\gamma_\varepsilon^i) \underset{\varepsilon\to 0}{\longrightarrow} a_i,$$

where γ_ε^i is a small loop around Σ_i contained in a fiber of π_i and defined by the equation $r_i = \varepsilon$.

Proof : By definition of the curves Σ_i, the symplectic form ω vanishes on any cycle of $M\backslash\cup\Sigma_i$, so it is exact on $M\backslash\cup\Sigma_i$, and we can pick a Liouville form β' for ω on this set. The form $\beta' - \lambda$ is therefore closed, and since $\lambda(\gamma_\varepsilon^i)$ converges to a_i as ε goes to 0, $\beta(\gamma_\varepsilon^i)$ also has a limit, $a_i + f_i$. We will prove our lemma if we can construct a 1-form ϖ on $M \backslash \cup \Sigma_i$ such that

$$\left\{ \begin{array}{l} d\varpi = 0 \\ \varpi(\gamma_\varepsilon^i) := \int_{\gamma_\varepsilon^i} \varpi = f_i \end{array} \right. \tag{8}$$

(the form $\beta := \beta' - \varpi$ will work). To achieve this construction, consider a closed 2-form σ_i on M which represents $\mathrm{PD}(\Sigma_i)$. The form σ_i is exact on $M \backslash \Sigma_i$, so there is a 1-form α_i defined on $M \backslash \Sigma_i$ such that $\sigma_i = d\alpha_i$. In order to compute the value of $\lim \alpha_i(\gamma_\varepsilon^i)$, we need to distinguish two cases. If $\Sigma_i \cdot \Sigma_i = 0$, the normal bundle of Σ_i is trivial and we can easily impose that $\alpha_i = d\theta_i$ in a neighbourhood of Σ_i (where θ_i is the now well-defined angle function around Σ_i), and $\lim \alpha_i(\gamma_\varepsilon^i) = 1$. If $\Sigma_i \cdot \Sigma_i \neq 0$, we consider a perturbation Σ_i' of Σ_i whose intersections with Σ_i are all very close to γ_ε^i. Then,

$$\lim_{\varepsilon \to 0} \alpha_i(\gamma_i^\varepsilon)\Sigma_i' \cdot \Sigma_i = \int_{\Sigma_i} \sigma_i = \mathrm{PD}(\Sigma_i) \cdot \Sigma_i = \Sigma_i \cdot \Sigma_i = \Sigma_i' \cdot \Sigma_i,$$

so $\lim \alpha_i(\gamma_\varepsilon^i) = 1$ (because $\Sigma_i \cdot \Sigma_i \neq 0$). Of course, since α_i is defined on $M \backslash \Sigma_i$, $\lim \alpha_i(\gamma_\varepsilon^j) = 0$, so the form

$$\varpi' := \sum f_i \alpha_i \qquad \text{on } M \backslash \cup \Sigma_i$$

verifies :

$$\varpi'(\gamma_\varepsilon^i) \underset{\varepsilon \to 0}{\longrightarrow} f_i \quad \forall i.$$

Consider now a two-cycle C in M and perturb it so that it becomes transverse to the curves Σ_i. Notice that since $d\beta' = -\omega$, we have :

$$\begin{aligned} \int_C \omega = & \sum_i \lim \beta'(\gamma_\varepsilon^i) C \cdot \Sigma_i \\ = & \sum (a_i + f_i) C \cdot \Sigma_i \\ = & \ \omega([C]) + \sum f_i \Sigma_i \cdot C. \end{aligned}$$

Thus, $\sum f_i \mathrm{PD}(\Sigma_i)$ vanishes in $H^2(M, \mathbb{R})$. Thus, the form $d\varpi' = \sum f_i \sigma_i$ (defined on M) is an exact 1-form *on* M. So $d\varpi' = dh$, where h is a smooth 1-form on M, and $\varpi := \varpi' - h$ verifies (8). □

Proof of lemma 4.1 : Consider a Liouville form β on $M \backslash \cup \Sigma_i$ given by lemma 4.2. In $\mathcal{V} \backslash \cup \Sigma_i$, the difference $\beta - \lambda$ is closed. If it is moreover exact, the lemma follows because any extension of the function h defined by $\beta - \lambda = dh$ gives an extension $\beta - dh$ of λ to the whole of $M \backslash \cup \Sigma_i$. We explain now that although this difference may well not be exact, we can find a "correction" closed one-form ϑ as in paragraph 3.4 such that $\beta - \lambda_\vartheta$ is exact. To understand this point, consider a family $\{\gamma_\varepsilon^i, \gamma_l^i\}_{i,k}$ generating the one-dimensional homology of $\mathcal{V} \setminus \cup \Sigma_i$, where γ_ε^i is, as before, the small loop around Σ_i (contained in a fiber of π_i and defined by the equation $r_i = \varepsilon$) and the γ_l^i are π_i-lifts of simple closed loops $\gamma_l^{i\prime}$ in Σ_i which span $H_1(\Sigma_i)$. First, since $\beta - \lambda$ is closed and $\lim(\beta - \lambda) \cdot \gamma_\varepsilon^i = 0$ by lemma 4.2, $\beta - \lambda$ vanishes on the classes $[\gamma_\varepsilon^i]$ for all i. Define now ϑ by requiring that $\int_{\gamma_l^{i\prime}} \vartheta_i = \int_{\gamma_l^i} \beta - \lambda$. Provided that we were careful to take $\gamma_l^{i\prime}$ with no intersection with $\mathcal{D}_{A_i - \delta}$ and D_{ij}^k, we can even require ϑ_i to vanish on these discs. Then a simple computation (explicitly made in [16]) shows that $\beta - \lambda_\vartheta$ vanishes on each class $[\gamma_l^i] \in H_1(\mathcal{V} \setminus \cup \Sigma_i)$. Moreover, since $\lambda_\vartheta = \lambda + \pi^* \vartheta$, its values on the loops γ_ε^i remain unchanged, so that $[\lambda_\vartheta - \beta](\gamma_\varepsilon^i) = 0$ also. The form $\beta - \lambda_\vartheta$ has therefore no period in $\mathcal{V} \setminus \Sigma_i$, so it is exact. $\square$

5 Existence of singular polarizations.

We now prove theorem 2, which asserts that singular polarizations always exist. Let us fix a symplectic manifold (M, ω). We have to find a decomposition of the cohomolgy class of the symplectic form into a sum of Poincaré-duals of symplectic hypersurfaces Σ_i which intersect transversally and positively. Of course, positive intersection is well-defined only in dimension four. In higher dimensions, we model the definition on complex manifolds.

Definition 5.1. *Symplectic submanifolds $\Sigma_1, \dots, \Sigma_k$ of (M^{2n}, ω) are said to intersect transversely and positively if all intersections $\Sigma_{j_1,\dots,j_p} := \Sigma_{j_1} \cap \dots \cap \Sigma_{j_p}$ between p of these submanifolds are transverse, symplectic and if moreover, at each point $q \in \Sigma_{j_1,\dots,j_p}$, the basis $\mathcal{B}$ obtained by the concatenations*

$$\mathcal{B} := \mathcal{B}_{j_1,\dots,j_p} \vee \mathcal{B}'_{j_1} \vee \dots \vee \mathcal{B}'_{j_p}$$

is positive, where $\mathcal{B}_{j_1,\dots,j_p}$ is a positive basis of $T_q \Sigma_{j_1,\dots,j_p}$ and $\mathcal{B}'_{j_i}$ is such that $\mathcal{B}_{j_1,\dots,j_p} \vee \mathcal{B}'_{j_i}$ is a positive basis of $T_q \Sigma_{j_i}$.

5.1 Proof of theorem 2.

Let (M, ω) be our symplectic manifold and write $\omega = \sum_1^k b_i \sigma_i$, where $[\sigma_i] \in H^2(M, \mathbb{Z})$. The real vector $b = (b_i)$ is a convex combination of N rational vectors nearby (at most $\dim H^2(M, \mathbb{R}) + 1$), that is for any small $\varepsilon > 0$ we have

$$b = \sum_1^N \lambda_j b^j, \quad \sum_1^N \lambda_j = 1, \quad \|b - b^j\| < \varepsilon, \quad b^j \in \mathbb{Q}^N.$$

Thus,

$$\begin{aligned}\omega = & \sum_{i=1}^{k} \big(\sum_{j=1}^{N} \lambda_j b_i^j \big) \sigma_i \\ = & \sum_{j=1}^{N} \lambda_j \big(\sum_{i=1}^{k} b_i^j \sigma_i \big) \quad = \sum_{j=1}^{N} \lambda_j \omega_j,\end{aligned}$$

where $|\omega - \omega_j| < \varepsilon$ and $\omega_j \in H^2(M, \mathbb{Q})$. If ε is small enough, the forms ω_j are symplectic, so by a result of Donaldson, there are ω_j-symplectic hypersurfaces $(\Sigma_1, \ldots, \Sigma_N)$ and positive integers $k_1, \ldots, k_N$ such that $\mathrm{PD}(\Sigma_j) = k_j \omega_j$. Thus,

$$[\omega] = \sum_{j=1}^{N} a_j \mathrm{PD}(\Sigma_j), \qquad a_j = \frac{\lambda_j}{k_j} \in \mathbb{R}^+.$$

Recall at this point that the Σ_j are known to be ω_j-symplectic because they are almost J_j-holomorphic for an ω_j-compatible almost-complex structure. Now if the ω_j are close enough to ω, all the J_j tame ω, so the Σ_j are also ω-symplectic. What remains to show is that the Σ_j can be required to meet positively and transversely.

One way to get through, suggested by a referee, is to invoke a folkloric transversality result in Donaldson's approximately holomorphic techniques, due to Mohsen [14]. This result can be stated as follows :

Theorem (Mohsen)**.** *Let (M, ω, J) be a rational symplectic manifold with a compatible almost-complex structure, and N a J-holomorphic submanifold of M (thus symplectic). Then, there exists $\eta > 0$ such that for all k large enough, there exists a symplectic polarization of degree k Σ_k of M, which is $\eta/\sqrt{k}$-close to being J-holomorphic, and whose intersection with N is η-transverse.*

More precisely, the angle between $T_p\Sigma_k$ and the closest J-holomorphic tangent subspace of T_pM is less than $\eta/\sqrt{k}$ and the angle between $T_q\Sigma_k$ and T_qN is at least η. Since N is J-holomorphic, these two conditions together imply that Σ_k intersects N transversally and positively. In order to prove theorem 2, then proceed as follows. Choose rational symplectic forms with compatible almost complex structures $(\omega_1, J_1), \ldots, (\omega_N, J_N)$ ε-close to (ω, J) and such that ω is in the simplex spanned by the ω_j. Donaldson's construction gives hypersurfaces Σ_1^k Poincaré-dual to $k\omega_1$ which are $\eta_1/\sqrt{k}$-close to being J_1-holomorphic. For k large enough, $\Sigma_1 := \Sigma_1^k$ is therefore ε-close to be J_i holomorphic for all i (they are all ε-close one to another). Thus, J_2 can be ε-perturbed to make Σ_1 J_2-holomorphic. Apply Mohsen's theorem above to get a hypersurface Σ_2 Poincaré-Dual to $k_2\omega_2$, which intersects Σ_1 transversally and positively, and which is ε-close to being J_2-holomorphic, hence J_3-holomorphic. Again, after a small perturbation among ω_3-compatible structures, J_3 can be assumed to make Σ_1 and Σ_2 J_3-holomorphic. Inductively, we get N hypersurfaces Poincaré-Dual to $k_i\omega_i$, which intersect transversally and positively in the sense of definition 5.1. These hypersurfaces are ω-symplectic because they are ε-close to being J-holomorphic. For the sake of completeness, let us mention that one should add the following precision to Mohsen's theorem above in order to carry on the induction : the intersection of Σ with N in Mohsen's theorem above can be choosen as close as needed to some given polarization of N obtained within the same framework of Donaldson's techniques.

We give now a different proof of this transversality statement, that relies on theorem 5 below. It is longer and a bit harder because, unlike the above proof, it does not follow formally from well-known results in Donaldson's theory. But almost nothing (and certainly nothing deep) must be added to Donaldson's original paper, and the theorem has its own interest : it shows a (maybe) unexpected robustness of Donaldson's arguments with respect to small changes in the symplectic structure. Before we state theorem 5 - only in dimension 4 for simplicity - we need to recall the following facts (see for instance [13], p. 64) :

Lemma 5.2. *Let (M,ω,J,g) be a symplectic manifold with a compatible almost-complex structure and its associated metric. There exists a natural and smooth way to associate to each symplectic form ω' close to ω a compatible almost complex structure J'. The associated metrics $g' := \omega'(\cdot, J'\cdot)$ are close to g provided ω' is close enough to ω, i.e. :*

$$(1+\varepsilon)^{-1}d_g(x,y) \leq d_{g'}(x,y) \leq (1+\varepsilon)d_g(x,y). \qquad (*)$$

Notice that measuring distances or $\mathcal{C}^l$-norms with respect to kg or kg' yield results wich only differ by a universal multiplicative factor (say 2), provided ω' is close enough to ω.

Theorem 5. *Let (M^4,ω,J,g) be a symplectic manifold with a compatible almost-complex structure and its associated metric. Let ω_j be rational symplectic forms on M close to ω, and consider the natural pairs (J_j,g_j) associated by lemma 5.2 to (ω_j,g). Let $\mathcal{L}_j \to M$ be a hermitian line bundle endowed with a connection of curvature $2i\pi q\omega_j$ (q being such that $q\omega_j \in H^2(M,\mathbb{Z})$ for all j). Denoting $g_k := kqg$, there exist $\eta > 0$ and sequences of sections $s_j = (s_j^k)$ of $\mathcal{L}_j^{\otimes k}$ such that :*

i) s_j is approximately J_j-holomorphic, i.e. :

$$|s_j^k|_{g_k,\mathcal{C}^1} \leq C, \quad |\overline{\partial}_{J_j} s_j^k|_{\mathcal{C}^1,g_k} \leq C/\sqrt{k} \quad \textit{for large } k,$$

ii) s_j is η-transverse to 0, i.e. $|s_j^k| \leq \eta \Rightarrow |\partial_{J_j} s_j^k| \geq \eta$, *iii) for all (i,j), the sequence of sections (s_i, s_j) of $\mathcal{L}_i^{\otimes k} \oplus \mathcal{L}_j^{\otimes k} \to M$ is η-transverse to 0, i.e. :*

$$\forall p \in M, |(s_i^k, s_j^k)| < \eta \Longrightarrow \quad (\partial_{J_i} s_i^k, \partial_{J_j} s_j^k) \in \mathcal{L}(T_pM, \mathbb{C}^2) \textit{ has a right inverse of } g_k\textit{-norm less than } \eta^{-1},$$

iv) For all (i,j,l), the section (s_i, s_j, s_l) of $\mathcal{L}_i^{\otimes k} \oplus \mathcal{L}_j^{\otimes k} \oplus \mathcal{L}_l^{\otimes k} \to M$ is η-transverse, i.e. it has norm at least η.

It is important to notice that in the theorem above, everything concerns *sequences* of sections, the norm involving s_j^k is *always* $g_k := kqg$, and the constants C and η depend neither on k *nor on the choice of the symplectic structures* ω_j provided they are on a small neighbourhood of ω. In the next paragraph, we review Donaldson's technique (with Auroux's contributions) and we include in the discussion the small modifications we need to make in our setting. Before, let us explain why theorem 5 indeed implies theorem 2.

Proof of theorem 2 (assuming theorem 5) : As we already noticed, the vanishing sets of $s_j^{k_0}$ for $k_0 \gg 1$ (which we denote s_j in the sequel since k_0 is fixed) give ω-symplectic hypersurfaces $\Sigma_j \subset M$ such that

$$[\omega] = \sum a_j \mathrm{PD}(\Sigma_j).$$

We need to understand that the transversality conditions iii) and iv) imply transversality and positivity of the intersections between Σ_i and Σ_j. First, condition iv) obviously implies that the intersections are simple : they never involve more than two branches. Let now $p \in \Sigma_i \cap \Sigma_j$, that is $s_i(p) = s_j(p) = 0$. In order to show that the intersection between Σ_i and Σ_j is positive at p, we make the following two observations :

1. $T_p\Sigma_i$ and $T_p\Sigma_j$ are very close (for k_0 large enough) to $J_{i/j}$-holomorphic hyperplanes (=lines in the 4-dimensional situation) Π_i, Π_j in T_pM.

2. The angle between Π_i and Π_j is bounded from below by some constant $C(\eta)$ depending neither on k nor on the symplectic structures ω_j.

Taking the complex structures J_i, J_j close to J by an amount $\varepsilon \ll C(\eta)$, we therefore find that $T_p\Sigma_i$ and $T_p\Sigma_j$ are ε-close to J-holomorphic lines which form an angle approximately $C(\eta)$-large. Since two J-holomorphic lines intersect positively when they are different, we conclude that $T_p\Sigma_i \cap T_p\Sigma_j$ is a positive transverse intersection. Points (1) and (2) are classical and at the core of Donaldson and Auroux's proofs. Let us prove them anyway.

Write $ds_j(p) = u_j + \varepsilon_j$ where $u_j = \partial_{J_j} s_j(p)$, $\varepsilon_j = \overline{\partial}_{J_j} s_j(p)$. Then by i) $|\varepsilon_j| \ll 1$ if k_0 is large enough (recall that $|\cdot|$ means $|\cdot|_{g_{k_0}}$), while $|u_j| \geq \eta$ by ii) and

$$(u_i, u_j) : T_pM \longrightarrow \mathbb{C}^2$$

is invertible (recall that $\dim_{\mathbb{R}} T_pM = 4 = \dim \mathbb{C}^2$ so right-invertible means invertible) with inverse R of norm less than η^{-1} by iii).

To understand (1), notice that $T_p\Sigma_j = \ker ds_j(p) = \ker(u_j + \varepsilon_j)$, and consider a unitary vector $x \in T_p\Sigma_j$ decomposed as $x_0 + \tau$ with $x_0 \in \ker u_j$ and $\tau \perp_{g_{k_0}} \ker u_j$. Then,

$$(u_j + \varepsilon_j)(x) = 0 = u_j(\tau) + \varepsilon_j(x),$$

so $u_j(\tau) = -\varepsilon_j(x)$. Taking into account that $\tau \in (\ker u_j)^\perp$ we know that $|u_j(\tau)| = |u_j||\tau|$, so

$$|\tau| \leq \frac{|\varepsilon_j|}{|u_j|} \ll 1.$$

Therefore, x is close to a unitary vector in $\Pi_j := \ker u_j = \ker \partial s_j(p)$, so $T_p\Sigma_j$ is close (in the angle sense) to the J_j-holomorphic hyperplane Π_j. In order to estimate the angles between Π_i and Π_j, put

$$\begin{aligned} \kappa := & \min\{|\langle x, y\rangle|, x \in \ker u_i, y \in \ker u_j, |x| = |y| = 1\} \\ = & \min\{|\pi_i(y)|, y \in \ker u_j, |y| = 1\}, \end{aligned}$$

where π_i stands for the g_{k_0}-orthogonal projection on $\ker u_i$. Then, $\kappa = \cos\theta$ where θ is the angle between Π_i and Π_j, so bounding θ from below amounts to bounding κ away from 1. Now put $\kappa = |\pi_i(y)|$ for a unitary vector $y \in \Pi_j$. Then

$$|\pi_i(y)|^2 + |y - \pi_i(y)|^2 = 1,$$

and since $y - \pi_i(y) \perp \ker u_i$, we get

$$|u_i||y - \pi_i(y)| \geq |u_i(y - \pi_i(y))| = |u_i(y)|.$$

But since $u_j(y) = 0$ and $|y| = 1$ we have $|u_i(y)| \geq \eta$ by iii), so

$$\kappa^2 = 1 - |y - \pi_i(y)|^2 \leq 1 - \frac{|u_i(y)|^2}{|u_i|^2} \leq 1 - \frac{\eta^2}{|u_i|^2}.$$

Finally the uniform bound $|u_i| \leq 2C$ yields the desired estimate (recall that $|s_i|_{C^1} \leq C$, while $|\varepsilon_i| \ll 1$). $\square$

5.2 Proof of theorem 5.

In this paragraph, we review Donaldson's and Auroux's works [6, 7, 1, 2, 3] on the subject and indicate what must be changed to get theorem 5. Let us emphasize that our need for adapting these works mostly comes from the fact that the almost-complex structures J_j are not fixed. We must thus be very careful that the choices for (ω_j, J_j) - which depend on η as we saw above - do not affect the transversality estimates (*i.e.* η does not depend in turn on (ω_j, J_j)). This is not obvious because modifying ω_j changes the line bundles in consideration, twisting them more and more when getting closer to ω. We claim, however, that the decisive argument is already in Donaldson's original work : the estimates do not depend on the tensoring parameter k.

A brief explanation. Before getting to actual proofs, let us explain quickly this independence of η (the transversality parameter) with respect to the pairs (ω', J') close to (ω, J). Recall that Donaldson's method consists in finding approximately holomorphic *and* transverse sections in a neighbourhood of any approximately holomorphic section, by adding small linear combinations of very localized approximately holomorphic peak sections. The transversality estimates depend only on how approximately holomorphic the initial section is, and on the decay rates of these peak sections. Since the zero-section exists and is a very nice holomorphic section on any line bundle, the crux of the argument is to understand that the decay rates of these peak sections for $\mathcal{L}'$ (associated to (ω', J')) vary continuously with ω'. But the peak sections centered at p are compactly supported sections of the line bundles of curvature $k\omega$ *over a Darboux chart centered at* p. So although the global line bundle of curvature $k\omega$ does not exist because $k\omega$ is not an integral class, the line bundles over these symplectic balls are well-defined. Now these line bundles over Darboux charts indeed vary smoothly with respect to the symplectic form.

All of Donaldson's construction relies on the existence of heavily localized approximately holomorphic sections. Namely, given (M^{2n}, ω, J) with $[\omega] \in H^2(M, \mathbb{Z})$ and $\mathcal{L}$ a line-bundle on M with connection of curvature $2i\pi\omega$, Donaldson remarks :

Lemma 5.3. *For all $p \in M$, there exist (compactly supported) sections σ_p^k of $\mathcal{L}^{\otimes k}$ such that :*

i) $|\sigma_p^k(q)| \geq 1$ *if* $d_k(p,q) \leq 1$,

ii) $|\sigma_p^k(q)|_{\mathcal{C}^1} \leq C_1 e^{-C_2 d_k(p,q)^2}$,

iii) $|\overline{\partial}_J \sigma_p^k(q)|_{\mathcal{C}^1} \leq \frac{C_1}{\sqrt{k}} e^{-C_2 d_k(p,q)^2}$,

iv) the constants C_1, C_2 do neither depend on p nor k.

Following [6], the sections verifying estimates ii) and iii) above will be called approximately J-holomorphic. Usually the k will be implicit and we denote these sections by σ_p. We must first check that this lemma can be extended to give sections $\sigma_{p,j}$ of $\mathcal{L}_j^{\otimes k}$ with the same estimates, where the constants C_1, C_2 are independent of the (ω_j, J_j). Precisely,

Lemma 5.4. *For all $p \in M$, there exist sections $\sigma_{p,j}^k$ of $\mathcal{L}_j^{\otimes k}$ verifying the same estimates as in lemma 5.3, where norms and lengths are still measured by $g_k = kqg$ and C_1, C_2 are independent of ω_j (provided ω_j is chosen in some fixed sufficiently small neighbourhood of ω). More precisely, the estimates i),ii) and iv) are unchanged, while iii) is replaced by*

$$|\overline{\partial}_J \sigma_p^k(q)|_{\mathcal{C}^1} \leq \frac{C_1}{\sqrt{kq}} e^{-C_2 d_k(p,q)^2}. \tag{9}$$

Proof : These estimates are possible because the dependence of the complex structure J_j and of the Darboux balls with respect to the symplectic form ω_j are continuous in $\mathcal{C}^l$-sense. Indeed, the sections σ_p^k of lemma 5.3 are of the form $\widetilde{\chi}_k \circ f_k \circ \chi_k^{-1}(z)$, where

- $\chi_k : \sqrt{k}B \hookrightarrow M$ is the composition of the contraction of $\mathbb{C}^n$ $\delta_k : x \to x/\sqrt{k}$, and a Darboux chart $\chi_p : B(0,1) \hookrightarrow M$ such that $\chi_p(0) = p$ and $\chi_p^* J(p) = i$,
- $f_k : \sqrt{k}B \longrightarrow \mathbb{C}$ is (a far cut-off of) the map $f(z) = e^{-k|z|^2}$ viewed as a holomorphic section of the line bundle $\mathcal{L}_{\mathrm{st}}^{\otimes k}$ with curvature $2i\pi k\omega_{\mathrm{st}}$,
- $\widetilde{\chi}_k$ is a horizontal lift of χ_k to a bundle isomorphism between $(\mathcal{L}_{\mathrm{st}}^{\otimes k}, 2i\pi k\omega_{\mathrm{st}})$ and $(\mathcal{L}^{\otimes k}, 2i\pi k\omega)$ above $\sqrt{k}B$.

In fact, the constant C_2 in lemma 5.3 is universal, while C_1 depends only on the $\mathcal{C}^l$-norm of χ_p and of the Nijenhuis tensor of J :

$$C_2 = C(\sup_p \|\chi_p\|_{\mathcal{C}^l}, \|N(J)\|_{\mathcal{C}^l}) \qquad \text{for some } l \ (l = 3 \text{ is safe}).$$

Since (ω_j, J_j) may be chosen arbitrarily $\mathcal{C}^l$-close to (ω, J), it follows that $\sigma_{p,j}^k$ verify the estimates 5.3 i), ii), iii), iv), where k should be replaced by qk at each occurence and the constants may be altered by a multiplicative factor arbitrarily close to 1. Since g_j and g are equivalent with ratio close to 1 (lemma 5.2 $(*)$), we finally get these estimates for $\sigma_{p,j}^k$ with the metric $g_k = kqg$. □

The existence of a global approximately holomorphic *and* uniformly transverse section is guaranteed in the classical setting by the following proposition (see [6, 3]) :

Proposition 5.5. *Given an approximately holomorphic sequence (s_k) of sections of $\mathcal{L}^{\otimes k}$, there exist points $(p_1, \ldots, p_r)$ with $\cup B_k(p_i, 1) = M$ and vectors $(w^1, \ldots, w^r)$ in $\mathbb{C}^{n+1}$ with $|w^i| \leq \delta$ such that the sequence*

$$s_w^k = s_k + \sum_{i=1}^r (w_0^i + \sum_{l=1}^n w_l^i z_l)\sigma_{p_i}^k,$$

is approximately holomorphic and η-transverse, where η does not depend on k but only on δ and g (z_i denote the coordinates of the chart χ_{p_i}).

The number of points involved in the process depends on k and g but on nothing else. It corresponds to the number of points in a (g_k-) well-distributed network. Since we stated the estimates of lemma 5.4 in terms of the metric g_k, we can choose the same points in our perturbed setting. The previous proposition relies itself on the following result from [3] :

Theorem 6. *Let $B^+ := B(\frac{11}{10}) \subset \mathbb{C}^n$ and $f : B^+ \longrightarrow \mathbb{C}$ of class $\mathcal{C}^\infty$. There exists $p \in \mathbb{R}$ depending only on n such that if $|f|_{\mathcal{C}^1(B^+)} \leq 1$ and $|\overline{\partial} f|_{\mathcal{C}^1(B^+)} \leq \delta Q_p(\delta) := \delta |\ln \delta|^{-p}$, then there exists $w = (w_0, \ldots, w_n) \in \mathbb{C}^{n+1}$ with $|w| < \delta$ and $f - w_0 - \sum w_i z_i$ is $\delta Q_p(\delta)$-transverse to zero on $B(1)$.*

In the classical setting, the next observation is that $\sigma_p(q)$ is large on $B^+(p_i) := B_k(p_i, \frac{11}{10})$ by 5.3, i), σ_p and (s_k) are approximately holomorphic, and $\overline{\partial}_{\widetilde{J}} := \overline{\partial}_{\chi_i^* J}$ is $k^{-1/2}$-close to $\overline{\partial}$ on $\chi_i^{-1}(B^+(p_i)) = B^+(0) \subset \mathbb{C}^2$ ($\chi_i := \chi_{p_i}$) :

$$\left|\overline{\partial}_{\widetilde{J}} - \overline{\partial}\right| \leq C_3 k^{-1/2}, \qquad \text{where } C_3 = C_3(|\chi_p|_{\mathcal{C}^3}, |N(J)|_{\mathcal{C}^3}). \tag{10}$$

We can therefore apply the previous theorem to $f_i := {}^{s_k}/_{\sigma^k_{p_i}}$, and get a w for which $s_k - (w_0^i + \sum w_l^i z_l)\sigma^k_{p_i}$ is $\delta Q_p(\delta)$-transverse to zero on $B_k(p_i, 1)$. In the perturbed setting, the argument also applies because estimate (10) is essentially unchanged when J_j is sufficiently close to J.

The global construction then goes as follows. One can divide the r points into K classes $\{\{p_i\}_{i \in I_\alpha}, \alpha \in \{1, \ldots, K\}\}$ and find constants $1 > \delta_1 > \cdots > \delta_K$ with $\delta_{\alpha+1} = C\delta_\alpha Q_p(\delta_\alpha)$ such that :

- The contributions of the $\{\sigma^k_{p_i}\}_{i \in I_\alpha}$ do not affect subsequently the transversality at points in the same class. Precisely, points in a same class are sufficiently (g_k)-distant for the following to hold :

$$\left\{ \begin{array}{l} |w^i| \leq \delta_\alpha \\ s_k \text{ is } 2\eta_\alpha = \delta_\alpha Q_p(\delta_\alpha)\text{-transverse on } B(p_{i'}, 1) \end{array} \right\} \Longrightarrow$$
$$s_k + \sum_{\substack{i \in I_\alpha \\ i \neq i'}} (w_0^i + \sum w_l^i z_l)\sigma^k_{p_i} \text{ is } \eta_\alpha\text{-transverse on } B(p_{i'}, 1).$$

- C is a constant depending only on the constants C_1, C_2 of lemma 5.3 and on g, so small that the contributions of the $\{\sigma^k_{p_i}\}_{i \notin I_1 \cup \cdots \cup I_\alpha}$ does not affect the η_α-transversality on $V_\alpha := \cup_{i \in I_1 \cup \cdots \cup I_\alpha} B_k(p_i, 1)$. Precisely,

$$\left\{ \begin{array}{l} |w^i| \leq \delta_{\alpha+1} \\ s_k \ \eta_\alpha\text{-transverse on } V_\alpha \end{array} \right\} \Longrightarrow$$
$$s_k + \sum_{i \notin I_1 \cup \cdots \cup I_\alpha} (w_0^i + \sum w_l^i z_l)\sigma^k_{p_i} \text{ is } \frac{\eta_\alpha}{2}\text{-transverse on } V_\alpha.$$

- The number K depends only on C, thus not on k nor on (ω_j, J_j).

Putting all this together, and using theorem 6 inductively on $B_k(p_i, 1)$ for $i \in I_1, \dots, I_K$, we get proposition 5.5 and thus Donaldson's theorem (starting with $(s_k) \equiv 0$). Since the constants K, C, p above do not depend on (ω_j, J_j) in a neighbourhood of (ω, J), we conclude that we can achieve the η-transversality with fixed η $(= \eta_K/2)$ for the sections (s_j^k) of $\mathcal{L}_j^{\otimes k}$ independently of the approximation ω_j we fixed.

We now give the details for the adaptation of the higher rank result, because it is the main difference (although nothing deep happens). The overall strategy is the same, but theorem 6 must be replaced by the following (see [3]) :

Theorem 7. *Let $B^+ := B(\frac{11}{10}) \in \mathbb{C}^n$ and $f : B^+ \longrightarrow \mathbb{C}^m$, $m \leq n$. There exists $p \in \mathbb{R}$ depending only on n such that if $|f|_{\mathcal{C}^1(B^+)} \leq 1$ and $|\overline{\partial} f|_{\mathcal{C}^1(B^+)} \leq \delta Q_p(\delta)$, there exists $w = (w_0, \dots, w_n) \in \mathbb{C}^{m(n+1)}$ (each w_i is a vector in $\mathbb{C}^m$) with $|w| < \delta$ and $f - w_0 - \sum w_i z_i$ is $\delta Q_p(\delta)$-transverse to zero on $B(1)$.*

In order to apply it to our setting, decompose our section s_k of $\mathcal{L}_1^{\otimes k} \oplus \mathcal{L}_2^{\otimes k}$ on $B_k(p, 1)$ as $s_k = (s_1^k, s_2^k) = f_1 \sigma_{p,1}^k + f_2 \sigma_{p,2}^k$ (identifying $\sigma_{p,1}^k$ with $(\sigma_{p,1}^k, 0)$). The approximate holomorphicity of s_1 and s_2 means that $|(\overline{\partial}_{J_1} s_1^k, \overline{\partial}_{J_2} s_2^k)| < Ck^{-1/2}$, which implies in turn that

$$\left|(\overline{\partial}_{J_1} f_1, \overline{\partial}_{J_2} f_2)\right| < Ck^{-\frac{1}{2}} \qquad \text{on } B_k(p, 1)$$

because $\sigma_{p,1}^k$ and $\sigma_{p,2}^k$ are bounded below. In $\mathbb{C}^2$, putting $\widetilde{J}_j := \chi_j^* J_j$, we get

$$\left|(\overline{\partial}_{\widetilde{J}_1} f_1 \circ \chi_{p,1}, \overline{\partial}_{\widetilde{J}_2} f_2 \circ \chi_{p,2})\right| < Ck^{-\frac{1}{2}} \qquad \text{on } B(1).$$

But $\widetilde{J}_1, \widetilde{J}_2$ are $k^{-1/2}$-close to i (see estimate (10)) so $\overline{\partial}(f_1 \circ \chi_{p,1}, f_2 \circ \chi_{p,2})$ is small. By theorem 7, we get a perturbation $(\widetilde{f}_1, \widetilde{f}_2)$ of (f_1, f_2) which is α-transverse, *i.e.*

$$\left|(\partial \widetilde{f}_1 \circ \chi_{p,1}, \partial \widetilde{f}_2 \circ \chi_{p,2})^{-1}\right| < \alpha \text{ whenever } |(\widetilde{f}_1 \circ \chi_{p,1}, \widetilde{f}_2 \circ \chi_{p,2})| < \alpha.$$

But again, since both $\partial_{\widetilde{J}_1}, \partial_{\widetilde{J}_2}$ are $k^{-1/2}$-close to the usual ∂-operator, we get that for any point in $B_k(p, 1)$ where $|(\widetilde{f}_1, \widetilde{f}_2)| < \alpha'$, $(\partial_{\widetilde{J}_1} \widetilde{f}_1 \circ \chi_{p,1}, \partial_{\widetilde{J}_2} \widetilde{f}_2 \circ \chi_{p,2}) = (\partial_{J_1} \widetilde{f}_1, \partial_{J_2} \widetilde{f}_2)$ has inverse of norm at most ${\alpha'}^{-1}$, for α' slightly less than α. Finally, setting $\widetilde{s}_j^k := s_j^k + \widetilde{f}_j \sigma_{p,j}^k$ $(j = 1, 2)$,

$$(\partial_{J_1} \widetilde{s}_1^k, \partial_{J_2} \widetilde{s}_2^k) = (\sigma_{1,p}^k \partial_{J_1} \widetilde{f}_1, \sigma_{2,p}^k \partial_{J_2} \widetilde{f}_2) + (\widetilde{f}_1 \partial_{J_1} \sigma_{1,p}^k, \widetilde{f}_2 \partial_{J_2} \sigma_{2,p}^k).$$

Since σ_j^k is bounded from below, $|(\sigma_{1,p}^k \partial_{J_1} \widetilde{f}_1, \sigma_{2,p}^k \partial_{J_2} \widetilde{f}_2)^{-1}| < (C\alpha')^{-1}$ (where C is a universal constant), so for $|(\widetilde{f}_1, \widetilde{f}_2)| < \frac{C\alpha'}{2}$,

$$\left|(\partial_{J_1} \widetilde{s}_1^k, \partial_{J_2} \widetilde{s}_2^k)^{-1}\right| \leq \frac{C\alpha'}{2}.$$

This is the needed transversality for (s_1, s_2). Getting it for all pairs (s_{j_1}, s_{j_2}) is then achieved by induction over these pairs, considering much smaller perturbations at each step. This is possible because we never destroy the approximate holomorphicity during this induction. In order to get iv), or to prove theorem 5 in arbitrary dimension, one simply repeats the same analysis for the vector bundles of increasing dimension $\mathcal{L}_1^{\otimes k} \oplus \cdots \oplus \mathcal{L}_n^{\otimes k}$ inductively, each time perturing less so as not to destroy the transversality obtained at the previous step. $\square$

6 Desingularization and Biran decompositions.

The aim of this section is to use theorem 3 to give a generalization of Biran's decomposition's theorem to situations where the polarization is not smooth. Although nothing prevents a general study, I prefer discussing an easy and concrete example in order to illustrate this point.

Consider $(\mathbb{P}^2, \omega_{\text{FS}})$ normalized so that the symplectic area of a projective line is 1. Given our normalization of the standard form on $\mathbb{R}^{2n}$, this means that $\mathbb{P}^2$ is the compactification of the ball of radius 1. Any smooth cubic C of $\mathbb{P}^2$ is a polarization of degree 3, hence gives rise to an embedding of a standard disc bundle of radius $1/3$ over C by [5] and to a full packing of $\mathbb{P}^2$ by one ellipsoid $\mathcal{E}(3, 1/3)$. The question studied in this paragraph is : what can we say when C is a singular cubic of $\mathbb{P}^2$ instead of a smooth one ? As we shall see, although theorem 3 does not formally consider singular curves, it can be easily complemented by the classical desingularization techniques of algebraic geometry to provide a relevant answer to this question.

Theorem 8. *Let C be a singular cubic of $\mathbb{P}^2$ with self-intersection at a point p. There exists a full packing of $\mathbb{P}^2$ by*

$$B(\mu) \sqcup \mathcal{E}(3-2\mu, \frac{1}{3}) \sqcup \mathcal{E}(\mu, \frac{2}{3}-\mu) \qquad \textit{for all } \mu < \frac{2}{3}.$$

Moreover, the cubic is covered by $B(\mu)$ - which it intersects along two Hopf discs, of area μ - and $\mathcal{E}(3-2\mu, 1/3)$ - which it intersects along the big axis, of area $3-2\mu$. It does not intersect $\mathcal{E}(\mu, 2/3-\mu)$.

Proof : Assume for the moment that there exists a ball $B(\mu)$ centered at p and whose intersection with C is exactly two Hopf discs (this is certainly true for small μ). Blowing-up this ball, we get the symplectic manifold $(\hat{\mathbb{P}}^2_1, \hat{\omega})$, where $[\hat{\omega}] = l - \mu e$, endowed with a curve $\hat{C}$ (the proper transform of C) in the homology class of $3L - 2E$. The curves $\hat{C}$ and E are now smooth symplectic curves which intersect exactly twice, positively. They constitute a singular polarization of $(\hat{P}^2_1, \hat{\omega})$ when $\mu < 2/3$, with

$$[\hat{\omega}] = l - \mu e = \frac{1}{3}(3l - 2e) + (\frac{2}{3} - \mu)e.$$

By theorem 3, $(\hat{\mathbb{P}}^2_1;\hat\omega)$ has a full packing by

$$\mathcal{E}(3-2\mu, 1/3)\sqcup\mathcal{E}(\mu, 2/3-\mu).$$

Moreover, it is easy to see that the disc $\{z_1=0\}$ can be brought out of $\mathcal{E}(a,b)\subset\mathbb{C}^2$ by a symplectic isotopy with support in a small neighbourhood of $\mathcal{E}(a,b)$. Thus, since $E\cap\mathcal{E}(\mu-\varepsilon, 2/3-\mu)$ is a Hopf disc and $E\cap\mathcal{E}(3-2\mu-\varepsilon, 1/3)=\emptyset$, the manifold $\hat{\mathbb{P}}^2_1\backslash E,\hat\omega$ also has full packing by $\mathcal{E}(3-2\mu, 1/3)\sqcup\mathcal{E}(\mu, 2/3-\mu)$. Blowing-down the exceptional divisor E, we therefore get a full packing of $\mathbb{P}^2$ as announced. The result for any μ is now a consequence of the next lemma. □

Lemma 6.1. *For any singular cubic C and for any $\mu<1$, there exists a ball $B(\mu)$ of capacity μ centered at the self-intersection point of C and whose intersection with C consists exactly of two Hopf discs.*

Proof : We show in fact that for any ball $B(\mu)$ there exists a cubic whose intersection with the ball is two Hopf discs. Since any two singular cubics are symplectically isotopic in $\mathbb{P}^2$, the lemma follows [4]. The proof is based on the blow-up construction of McDuff, for which we refer to [12, 13], and Gromov's theory of pseudo-holomorphic curves. Let $B(\mu)$ be a one-parameter family of balls of capacity μ in $\mathbb{P}^2$ for $\mu\in]0,1[$, J_μ an almost-complex structure suited for blow-up and $p_1(\mu),\dots,p_6(\mu)$ six generic points outside $B(\mu)$. Calling p_0 the center of $B(\mu)$, genericity means here that no three of the points $p_0,\dots,p_6$ lie on a same J_μ-line and no six of them lie on a same J_μ-conic. Denote also by $(\hat{\mathbb{P}}^2_1,\omega_\mu,J_\mu)$ the symplectic blow-up of $\mathbb{P}^2$ along $B(\mu)$ endowed with the induced almost-complex structure (see [13]). Proving lemma 6.1 then amounts to prove that the moduli space

$$\mathcal{M}_\mu:=\{u:\mathbb{P}^1\longrightarrow\hat{\mathbb{P}}^2_1\mid du\circ i=J_\mu\circ du,\ [u]=3L-2E,\ (p_1,\dots,p_6)\in\operatorname{Im}u\},$$

is not empty for $\mu<1$. We can also assume that the path of almost-complex structures J_μ is generic since we can modify J_μ in a neighbourhood of p_1 and $3L-2E$ is primitive. For μ small enough, this moduli space is obviously non-empty and actually consists of exactly one point. If $\mathcal{M}_\mu$ is empty for some μ, there must be bubbling by Gromov's compactness theorem. This means that the class $3L-2E$ splits into a sum of classes $A_1+\dots+A_n$, where $A_i=k_iL-l_iE$ which are represented by J_μ-holomorphic curves. Since E is also represented by a J_μ-holomorphic curve, $A_i=-l_iE$ is not allowed, and we see by positivity of intersections with E that :

$$(l_1,\dots,l_n)\in\{(1,1,0\dots,0),(2,0,\dots,0)\}.$$

Since each A_i has a symplectic representative, the k_i are moreover positive. The decomposition can therefore only have two or three terms, and the full list of possibilities is the following :

$$(L-E, L-E, L),\ (L-2E, L, L),\ (L-E, 2L-E),\ (L-2E, 2L),\ (2L-2E, L).$$

Further, the decompositions where $L-2E$ appears can be discarded, because holomorphic lines do not have self-intersection points. We are therefore left with only three possibilities :

- The decompositions $(L-E, L-E, L)$ and $(2L-2E, L)$ correspond to situations where, blowing down back to $\mathbb{P}^2$, the seven points must lie in a configuration of three lines (indeed, $2L-2E$ is a conic with a self-intersection, *i.e.* in fact two lines). These three lines pass through the points $p_0, \dots, p_6$ so three of these points must belong to a same line, which is a contradiction.

- If the decomposition is $(2L-E, L-E)$, blowing down leads to a configuration of one conic and one line intersecting at p_0, passing through a total of seven points, again impossible. □

7 Application to symplectic isotopies.

In [16], I explain a construction for isotoping balls. The principle is the following. Given a symplectic ball $B \subset (M^4, \omega)$ (meaning that B is the symplectic image of a 4-dimensional euclidean ball), define a supporting polarization for B to be any *smooth* polarization Σ of M whose intersection with B is exactly a Hopf disc in B (the image of the intersection of $B^4 \subset \mathbb{C}^2$ with a complex line). Very roughly, when there is a supporting polarization of degree k for a ball of capacity less than k^{-1}, this ball can be brought into a standard position by symplectic isotopy. A precise statement is the following :

Theorem 9. *Let $B_1, B_2 \subset (M^4, \omega)$ be symplectic balls of a rational symplectic manifold. Assume that :*

- *B_1, B_2 have supporting polarizations Σ_1, Σ_2 of same degree k,*
- *B_1, B_2 have same capacity $c < k^{-1}$,*
- *Σ_1 and Σ_2 are symplectic isotopic.*

Then B_1, B_2 are symplectic isotopic.

The idea is that a given polarization allows to construct balls supported by this polarization in a very easy and flexible way. Conversely, any ball with this polarization as a supporting curve can be realized by such a construction. This theorem applies to some manifolds like $\mathbb{P}^2$ or $(S^2 \times S^2, \omega \oplus \omega)$, but it is useless for irrational symplectic manifolds, where there are no polarization at all. Even more unsatisfactory is the inaccuracy of the method for some very simple rational manifolds. For instance, when $\mu \in \mathbb{Q}\backslash\mathbb{Z}$, the smooth polarizations of $(S^2 \times S^2, \omega \oplus \mu\omega)$ have genus, and are therefore much more difficult to isotop, or even to bend to a supporting polarization, than spheres. As we will see below, this paper shows that singular polarizations are as good as smooth ones for the purpose of isotopies. This remark may be interesting in two respects. First, it sometimes allows to avoid using higher genus GW-invariants (for instance in the case of $S^2 \times S^2$ as explained above). The second point is that singular polarizations may in practice be more *stable* objects than smooth ones, because they may arise as degeneracy of smooth polarizations through bubbling for instance. In view of the way the supporting polarization are produced (using pseudo-holomorphic curves), this stability property can be useful. We illustrate here the first point by the following example (already well-understood, see [10]) :

Theorem 10. *Any two balls in* $(S^2 \times S^2, \omega \oplus \mu\omega)$ *are symplectically isotopic.*

Below is a sketch of the proof. For more details, see also [16] which is really devoted to the matter of isotopies. My aim here is only to explain how to use theorem 3 in an isotopy problem, when the method exposed in [16] does not apply.
Sketch of proof of theorem 10 : Let us assume without loss of generality that $\mu > 1$. Consider two symplectic balls B_1, B_2 in $M = (S^2 \times S^2, \omega \oplus \mu\omega)$ of same capacity c ($c < 1$ by the non-squeezing theorem). By standard SFT arguments (stretching the neck) or blowing-up, it is easy to find supporting curves Σ_i of B_i in the homology class of $[S^2 \times \{*\}]$ and symplectic curves Σ_i' in class $[\{*\} \times S^2]$ which do not meet B_i. Notice that (Σ_i, Σ_i'), $i = 1, 2$, are singular polarizations of M in the sense of the present paper. Now by standard arguments, and because Σ_i, Σ_i' are spheres, the two pairs of curves are isotopic. The two balls can therefore be assumed to share a common singular supporting polarization. Notice now that a singular polarization (Σ, Σ') gives rise to embeddings of an ellipsoid $\mathcal{E}(1, \mu)$, which contains of course a ball of capacity c, by paragraph 3.3. As in the smooth case, these embeddings are completely determined by the single data of a Liouville form on $M\backslash(\Sigma \cup \Sigma')$. The rest of the argument is now exactly the same as in [16] and we do not repeat it here : passing from B_1 to B_2 is only a matter of interpolating between two Liouville forms on $M\backslash(\Sigma \cup \Sigma')$, which is easy.$\square$

8 Ellipsoids in higher dimension.

We sketch here very briefly how to construct full packings of 6-dimensional symplectic manifolds by ellipsoids, and we leave the induction to the reader for higher dimensions. Start from a singular polarization $(\Sigma_1, \dots, \Sigma_N)$ which can be perturbed to $(\Sigma_i')_{i=1\dots N}$ such that Σ_i intersects Σ_i' positively and transversally (use [1]). Now

$$[\omega] = \sum a_i \mathrm{PD}(\Sigma_i),$$

but also

$$[\omega_{|\Sigma_i}] = a_i \mathrm{PD}(\Sigma_i' \cap \Sigma_i) + \sum_{j \neq i} a_j \mathrm{PD}(\Sigma_j \cap \Sigma_i),$$

and by positivity of intersections, $\big(\Sigma_i' \cap \Sigma_i, (\Sigma_j \cap \Sigma_i)_{j\neq i}\big)$ is a singular polarization of Σ_i (which is symplectic and 4-dimensional). As explained here, we thus get full packings of each Σ_i by N ellipsoids, which intersect the other Σ_j exactly along Hopf discs. Since a Hopf disc of an ellipsoid can be moved symplectically to the boundary of this ellipsoid, there also exist a full packing of $\Sigma_i \backslash (\cup \Sigma_j)_{j \neq i}$ by ellipsoids.

On the other hand, [17] explains how to construct a so-called *tame Liouville form* on $M \backslash \cup \Sigma_i$. Although less explicit than the Liouville forms produced here, tame Liouville forms can be used mostly the same, and the basins of attraction of the 4-dimensional ellipsoids that cover the hypersurfaces Σ_i (up to ε-volume) are disjoint ellipsoids, and they also cover M up to ε-volume.

References

[1] D. Auroux. Asymptotically holomorphic families of symplectic submanifolds. *Geom. Funct. Anal.*, 7(6):971–995, 1997.

[2] D. Auroux. Symplectic 4-manifolds as branched coverings of $\mathbf{CP}^2$. *Invent. Math.*, 139(3):551–602, 2000.

[3] D. Auroux. A remark about Donaldson's construction of symplectic submanifolds. *J. Symplectic Geom.*, 1(3):647–658, 2002.

[4] J.-F. Barraud. Nodal symplectic spheres in $\mathbf{C}\mathrm{P}^2$ with positive self-intersection. *Internat. Math. Res. Notices*, (9):495–508, 1999.

[5] P. Biran. Lagrangian barriers and symplectic embeddings. *Geom. Funct. Anal.*, 11(3):407–464, 2001.

[6] S. K. Donaldson. Symplectic submanifolds and almost-complex geometry. *J. Differential Geom.*, 44(4):666–705, 1996.

[7] S. K. Donaldson. Lefschetz pencils on symplectic manifolds. *J. Differential Geom.*, 53(2):205–236, 1999.

[8] L. Guth. Symplectic embeddings of polydisks. *Invent. Math.*, 172(3):477–489, 2008.

[9] R. Hind and E. Kerman. New obstructions to symplectic embeddings. *Prepublication http://arXiv.org/abs/math/0906.4296*, 2009.

[10] D. McDuff. From symplectic deformation to isotopy. In *Topics in symplectic* 4-*manifolds (Irvine, CA, 1996)*, First Int. Press Lect. Ser., I, pages 85–99. Internat. Press, Cambridge, MA, 1998.

[11] D. McDuff. Symplectic embeddings of 4-dimensional ellipsoids. *J. Topol.*, 2(1):1–22, 2009.

[12] D. McDuff and L. Polterovich. Symplectic packings and algebraic geometry. *Invent. Math.*, 115(3):405–434, 1994. With an appendix by Yael Karshon.

[13] D. McDuff and D. Salamon. *Introduction to symplectic topology.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[14] J.-P. Mohsen. *PhD thesis*, 2003.

[15] E. Opshtein. Maximal symplectic packings in $\mathbb{P}^2$. *Compos. Math.*, 143(6):1558–1575, 2007.

[16] E. Opshtein. Polarizations and symplectic isotopies. *J. Symplectic Geom.*, 2011.

[17] E. Opshtein. Symplectic packings in dimension 4 and singular curves. *Prepublication http://arXiv.org/abs/math/1110.2385*, 2011.

[18] Y. B. Rudyak and F. Schlenk. Minimal atlases of closed symplectic manifolds. *Commun. Contemp. Math.*, 9(6):811–855, 2007.

[19] F. Schlenk. Packing symplectic manifolds by hand. *J. Symplectic Geom.*, 3(3):313–340, 2005.

Emmanuel Opshtein,
Institut de Recherche Mathématique Avancée
UMR 7501, Université de Strasbourg et CNRS
7 rue René Descartes
67000 Strasbourg, France
opshtein@math.unistra.fr