Compact hyperkähler manifolds: an introduction

Kieran G. O'Grady*

"Sapienza" Università di Roma

March 1 2013

Contents

1 Introduction **1**

2 Examples **2**
- 2.1 Hilbert schemes of smooth surfaces . . . 3
- 2.2 Beauville's examples . . . 6
- 2.3 Moduli of sheaves . . . 7
- 2.4 Two more examples . . . 12

3 Yau's Theorem and its implications **13**
- 3.1 Holonomy . . . 14
- 3.2 Twistor families . . . 15
- 3.3 Deformations are unobstructed . . . 16

4 The local period map and the B-B quadratic form **17**
- 4.1 The local period map . . . 17
- 4.2 The Bogomolov-Beauville quadratic form . . . 18
- 4.3 Marked pairs . . . 20
- 4.4 Matsushita's Theorem . . . 21

5 The Kähler cone **22**

6 Global Torelli **23**
- 6.1 Hausdorffization of the moduli space of marked pairs . . . 23
- 6.2 The descended period map is a topological covering . . . 24
- 6.3 Conclusion . . . 25

1 Introduction

A compact Kähler manifold is *hyperkähler* (HK) if it is simply connected and the space of its global holomorphic two-forms is spanned by a symplectic form. A 2-dimensional HK manifold is nothing else but a $K3$ surface. $K3$ surfaces were known classically as complex smooth projective surfaces whose generic hyperplane section is a canonically embedded curve (an example is provided by a smooth quartic surface in $\mathbb{P}^3$) and they have proved to have a very rich geometry. Beauville [2] was the first to provide examples of HK manifolds in each even dimension[1] greater than 2. The first series of examples is constructed out of a projective $K3$ surface S: in fact Beauville [2] proved

*Supported by PRIN 2007

[1] The dimension of a HK manifold is even because it carries a holomorphic symplectic form.

(see **Subsection 2.2**) that the Hilbert scheme $S^{[n]}$ parametrizing length-n subschemes of S is a HK variety[2]. Since $S^{[n]}$ has dimension $2n$ we get examples in each even dimension. Another series of examples is constructed out of an abelian surface T as follows. The Hilbert scheme $T^{[n+1]}$ carries a holomorphic symplectic form but it is not HK: in fact the fibration

$$\begin{array}{ccc} T^{[n+1]} & \xrightarrow{s_{n+1}} & T \\ Z & \mapsto & \sum_{p\in T} \ell(\mathcal{O}_{Z,p})p \end{array} \tag{1.0.1}$$

(the sum is in the group T and $\ell(\mathcal{O}_{Z,p})$ is equal to the dimension of $\mathcal{O}_{Z,p}$ as $\mathbb{C}$-vector space) shows that $H_1(T^{[n+1]};\mathbb{Q}) \neq 0$ and also that $T^{[n+1]}$ carries non-zero holomorphic 2-forms which are not symplectic if $n \geq 1$. Beauville proved that

$$K^{[n]}(T) := s_{n+1}^{-1}(0) \tag{1.0.2}$$

is a HK variety (of dimension $2n$). For $n = 1$ we get the (desingularized) Kummer surface of T, for that reason $K^{[n]}(T)$ is known as a *generalized Kummer manifold*. Let S be a projective $K3$ surface and T be an abelian surface. Let $n \geq 2$: then $b_2(S^{[n]}) = 23$ and $b_2(K^{[n]}(T)) = 7$ (see **Subsection 2.2**) and hence $S^{[n]}$ is not a deformation of $K^{[n]}(T)$. The definition of a HK manifold is motivated by the following result.

Theorem 1.1 (Beauville-Bogomolov decomposition [2])**.** *Let X be a compact Kähler manifold with $c_1(X) = 0$. There exists an étale finite cover $\prod_{i=1}^d M_i \longrightarrow X$ where each of the factors M_i is either a compact complex torus, a Calabi-Yau variety*[3] *or a HK.*

The above result follows from: (a) Yau's Theorem (formerly Calabi's conjecture) on the existence of Ricci-flat metrics on compact Käbler manifolds with $c_1 = 0$, (b) De Rham's decomposition Theorem for simply-connected complete riemannian manifolds, and (c) Berger's classification of holonomy groups of complete riemannian manifolds, see [2]. A general theory of HK manifolds was first developed thirty years ago by Bogomolov, Fujiki and Beauville. Roughly ten years ago Huybrechts made huge steps ahead in the general theory, and quite recently Verbitsky[4] added a global Torelli Theorem. A key ingredient in these developments is the existence of twistor families of HK manifolds: these are families parametrized by $\mathbb{P}^1$, and the generic element of the family is not projective. Existence of the twistor families is a consequence of Yau's solution of the Calabi conjecture. On the other hand HK varieties are very interesting from an algebro-geometric point of view. The known HK manifolds which are not deformations of Beauville's examples have been constructed by the author as desingularizations of moduli spaces of sheaves on projective $K3$ and abelian surfaces. Markman characterized the monodromy action of deformations of $K3^{[n]}$ by studying moduli of sheaves on $K3$'s, and the results on monodromy allow to translate Verbitsky's general global Torelli into a classical global Torelli for deformations of $K3^{[n]}$. Beauville and Voisin have formulated (and partially verified) some intriguing conjectures on the Chow ring of HK varieties.

We will try to present these two points of view: the general one, based on pure existence results, and the algebro-geometric one in which there is a very rich geometry and beautiful examples.

2 Examples

We will prove that the Hilbert scheme $S^{[n]}$ of a projective $K3$ surface S and the generalized Kummer $K^{[n]}(T)$ associated to an abelian surface T are HK varieties. We will also recall that moduli spaces of (semi)stable sheaves on projective surfaces with trivial canonical class (i.e. either $K3$ or abelian surfaces) provide examples of HK varieties (Mukai). That will lead us to the HK varieties of dimensions 6 and 10 constructed by the author. All the known HK manifolds are deformations of one of the Beauville examples or of one of our two examples.

[2] A HK variety is a projective HK manifold.

[3] A CalabiYau variety is a compact Kähler manifold M of dimension $n \geq 3$ with trivial canonical bundle and such that $h^0(\Omega^p_M) = 0$ for $0 < p < n$.

[4] A water-tight proof was provided by Huybrechts.

2.1 Hilbert schemes of smooth surfaces

Let S be a smooth complex projective surface and $S^{[n]}$ be the Hilbert scheme parametrizing length-n subschemes of S. A point of $S^{[n]}$ is a subscheme $Z \subset S$ such that $H^0(\mathcal{O}_Z)$ is finite-dimensional of dimension n. It is known that the generic such Z is reduced i.e. it consists of n distinct points and that $S^{[n]}$ is a smooth complex projective variety[5] of dimension $2n$. Let $S^{(n)}$ be the *symmetric* n-th power of S i.e. the quotient of S^n by the natural action of the symmetric group on n elements Σ_n. An element of $S^{(n)}$ may be written as a finite formal sum $\sum_i m_i p_i$ where $m_i \in \mathbb{N}$ for each i and $\sum_i m_i = n$. There is a regular *Hilbert-Chow map*

$$\begin{array}{ccc} S^{[n]} & \xrightarrow{\gamma} & S^{(n)} \\ Z & \mapsto & \sum_{p\in S} \ell(\mathcal{O}_{Z,p})p. \end{array} \tag{2.1.1}$$

Here the sum is a *formal* sum. Let $\sum_i m_i p_i \in S^{(n)}$ where the points p_i are *pairwise distinct*: by a result of Iarrobino [20]

$$\dim \gamma^{-1}(\sum_i m_i p_i) = \sum_i (m_i - 1). \tag{2.1.2}$$

Let $\Delta_n \subset S^{[n]}$ be the subset of *non-reduced* subschemes. By (2.1.2) Δ_n is the exceptional set of γ. Since $S^{(n)}$ is $\mathbb{Q}$-factorial it follows that Δ_n has pure codimension 1: by (2.1.2) we get that Δ_n is irreducible. It will be useful to have an explicit description of a large open subset of $S^{[n]}$. Let $U_n \subset S^{[n]}$ be

$$U_n := \{Z \in S^{[n]} \mid |\operatorname{supp} Z| \ge (n-1)\}. \tag{2.1.3}$$

In other words $Z \in U_n$ if either Z is reduced or it is the disjoint union of $(n-2)$ reduced points and a subscheme of length 2. Since $\gamma(U_n)$ is open and $U_n = \gamma^{-1}(\gamma(U_n))$ the subset U_n is open. Equation (2.1.2) gives that

$$\dim(S^{[n]} \setminus U_n, S^{[n]}) = 2. \tag{2.1.4}$$

An explicit description of U_n goes as follows. Let $V_n \subset S^n$ be defined by

$$V_n := \{(x_1, \ldots, x_n) \in S^n \mid |\{x_1, \ldots, x_n\}| \ge (n-1)\}. \tag{2.1.5}$$

In other words $(x_1, \ldots, x_n) \in V_n$ if there exists at most one couple $1 \le i < j \le n$ such that $x_i = x_j$. The tautological closed subset $\mathcal{W} \subset V_n \times S$ consisting of couples $((x_1, \ldots, x_n), y)$ such that $y = x_i$ for some $1 \le i \le n$ is *not* a flat family of length-n subschemes of S, in order to get a flat family we must blow-up the large diagonal. More precisely for $1 \le i < j \le n$ let

$$D_{ij} := \{(x_1, \ldots, x_n) \in S^n \mid x_i = x_j\}. \tag{2.1.6}$$

The *large diagonal* $D_n \subset S^n$ is the union of all the D_{ij}. We let

$$f \colon \widetilde{V}_n \to V_n \tag{2.1.7}$$

be the blow-up of $D_n \cap V_n$. There is a subscheme $\mathcal{Z} \subset \widetilde{V_n} \times S$ flat over $\widetilde{V_n}$ such that $Z_t := \mathcal{Z} \cap (\{t\} \times S)$ is a length-n subscheme for every $t \in \widetilde{V_n}$, moreover if $f(t) = (x_1, \ldots, x_n)$ where the x_i's are pairwise distinct then Z_t is the reduced scheme $\{x_1, \ldots, x_n\}$. Thus $\mathcal{Z}$ induces a regular surjective map

$$g \colon \widetilde{V_n} \longrightarrow U_n. \tag{2.1.8}$$

The group Σ_n on n elements acts on $\widetilde{V}_n$ and on $\mathcal{Z}$: it follows that g is invariant under the action of Σ_n on $\widetilde{V}_n$ and hence it descends to a regular map

$$h \colon \Sigma_n \backslash \widetilde{V}_n \longrightarrow U_n. \tag{2.1.9}$$

Since h is injective and $S^{[n]}$ is smooth it follows that h is an isomorphism.

[5] A variety is integral i.e. reduced and irreducible.

2.1.1 Topology

We will study the topology of $S^{[n]}$ for $n \geq 2$. We start by analyzing the fundamental group. Let $p_1, \ldots, p_{n-1} \in S$ be pairwise distinct and

$$\begin{array}{ccc} S \setminus \{p_1, \ldots, p_{n-1}\} & \overset{h}{\hookrightarrow} & S^{[n]} \\ p & \mapsto & \{p_1, \ldots, p_{n-1}, p\}. \end{array} \tag{2.1.10}$$

One proves that the homomorphism $h_{\#}: \pi_1(S \setminus \{p_1, \ldots, p_{n-1}\}) \to \pi_1(S^{[n]})$ is trivial on commutators (this is where we need to assume that $n \geq 2$) and surjective. Since $H_1(S \setminus \{p_1, \ldots, p_{n-1}\}) \cong H_1(S; \mathbb{Z})$ we get that $h_{\#}$ induces a surjective homomorphism

$$H_1(S; \mathbb{Z}) \longrightarrow \pi_1(S^{[n]}). \tag{2.1.11}$$

Proposition 2.1. *Keep assumptions and notation as above, in particular $n \geq 2$. Then* (2.1.11) *is an isomorphism.*

Proof. Let $\mathrm{Alb}(S) = H^0(\Omega^1_S)^\vee / H_1(S; \mathbb{Z})$ be the Albanese variety of S. Choose a base point $p_0 \in S$ and let

$$\begin{array}{ccc} S & \overset{u}{\longrightarrow} & \mathrm{Alb}(S) \\ p & \mapsto & (\omega \mapsto \int_{p_0}^{p} \omega) \end{array} \tag{2.1.12}$$

be the Albanese map. Let

$$\begin{array}{ccc} S^{[n]} & \overset{s_n}{\longrightarrow} & \mathrm{Alb}(S) \\ Z & \mapsto & \sum_{p \in S} \ell(\mathcal{O}_{Z,p}) u(p) \end{array} \tag{2.1.13}$$

where the sum is *not* a formal sum, it is the sum in the group $\mathrm{Alb}(S)$ (the notation is consistent with that of (1.0.1)). Composing (2.1.11) and the map $s_{n,\sharp}: \pi_1(S^{[n]}) \to \pi_1(\mathrm{Alb}(S))$ we get a homomorphism $H_1(S; \mathbb{Z}) \to \pi_1(Alb(S)) \cong H_1(S; \mathbb{Z})$ which is the identity. It follows that (2.1.11) is injective; since it is surjective it follows that it is an isomorphism. □

Next we will describe the low-dimensional cohomology groups of $S^{[n]}$. The rational cohomology of $S^{(n)}$ is naturally identified with the Σ_n-invariant summand of the rational cohomology of S^n i.e.

$$H^p(S^{(n)}; \mathbb{Q}) \cong H^p(S^n; \mathbb{Q})^{\Sigma_n}. \tag{2.1.14}$$

Thus Poincarè duality for S^n gives that

$$\begin{array}{ccc} H^p(S^{(n)}; \mathbb{Q}) \times H^{2n-p}(S^{(n)}; \mathbb{Q}) & \longrightarrow & \mathbb{Q} \\ (\alpha, \beta) & \mapsto & \int_{S^{(n)}} \alpha \cup \beta \end{array} \tag{2.1.15}$$

is a perfect pairing for all p. It follows that $H^p(\gamma): H^p(S^{(n)}; \mathbb{Q}) \to H^p(S^{[n]}; \mathbb{Q})$ is injective for all p. Since γ is an isomorphism outside the irreducible divisor Δ_n we get that $H^p(\gamma)$ is an isomorphism for $p \leq 1$ and that

$$H^2(S^{[n]}; \mathbb{Q}) \cong H^2(S^{(n)}; \mathbb{Q}) \oplus \mathbb{Q} c_1(\mathcal{O}_{S^{[n]}}(\Delta_n)). \tag{2.1.16}$$

Let's pass to integral cohomology. It is not difficult to prove that for $p \leq 2$ every integral Σ_n-invariant p-cohomology class on S^n descends to an integral cohomology class on $S^{(n)}$. More precisely there exists a symmetrization homomorphism

$$t_p: H^p(S; \mathbb{Z}) \longrightarrow H^p(S^{(n)}; \mathbb{Z}), \qquad p \leq 2 \tag{2.1.17}$$

characterized as follows. Let $q: S^n \to S^{(n)}$ be the quotient map and $\pi_i: S^n \to S$ the projection to the i-th factor: then

$$q^* \circ t_p^*(\alpha) = \pi_1^* \alpha + \ldots + \pi_n^* \alpha, \qquad \alpha \in H^p(S; \mathbb{Z}). \tag{2.1.18}$$

For simplicity we will assume from now on that $H^*(S; \mathbb{Z})$ has *no torsion.* It follows by Künneth's decomposition that $H^*(S^n; \mathbb{Z})$ has no torsion and that we have an isomorphism

$$H^p(S^{(n)}; \mathbb{Z}) \cong H^p(S^n; \mathbb{Z})^{\Sigma_n}, \qquad p \leq 2. \tag{2.1.19}$$

Thus we have a series of isomorphisms

$$H^p(S;\mathbb{Z}) \xrightarrow[\sim]{t_p} H^p(S^{(n)};\mathbb{Z}) \xrightarrow{\sim} H^p(S^{[n]};\mathbb{Z}), \qquad p \le 1. \tag{2.1.20}$$

For $p = 1$ this is the same isomorphism which one gets from **Proposition 2.1**. In order to describe integral 2-cohomology we must analyze $c_1(\mathcal{O}_{S^{[n]}}(\Delta_n))$. Let $\widetilde{V}_n$ be as in (2.1.7). The alternating group $A_n < \Sigma_n$ acts on $\widetilde{V}_n$, let $W_n := A_n\backslash\widetilde{V}_n$. Let $U_n \subset S^{n]}$ be the open subset given by (2.1.9). The natural map $\rho\colon W_n \longrightarrow U_n$ is a double cover ramified over $\Delta_n \cap U_n$. The action of $\mathbb{Z}/(2)$ on $\rho_*\mathcal{O}_{W_n}$ gives an eigenspace decomposition

$$\rho_*\mathcal{O}_{W_n} = \mathcal{O}_{U_n} \oplus \mathcal{L}_n^- \tag{2.1.21}$$

where $\mathcal{L}_n^-$ is the (-1)-eigensheaf - an invertible sheaf. By (2.1.4) there is an invertible sheaf $\mathcal{L}_n$ on $S^{[n]}$, unique up to isomorphism, extending $\mathcal{L}_n^-$. Let $\xi_n := c_1(\mathcal{L}_n^{-1})$. By construction

$$2\xi_n = c_1(\mathcal{O}_{S^{[n]}}(\Delta_n)). \tag{2.1.22}$$

Proposition 2.2. *Let S be a smooth complex projective surface. Assume that $H^*(S;\mathbb{Z})$ has no torsion. Then*

$$H^2(S^{[n]};\mathbb{Z}) \cong H^2(S;\mathbb{Z}) \bigoplus \bigwedge^2 H^1(S;\mathbb{Z}) \bigoplus \mathbb{Z}\xi_n \tag{2.1.23}$$

where $H^2(S;\mathbb{Z})$ stands for $\operatorname{Im} t_2$ and $H^1(S;\mathbb{Z})$ stands for $\operatorname{Im} t_1$. Moreover the embeddings

$$H^2(S;\mathbb{C}) \hookrightarrow H^2(S^{[n]};\mathbb{C}), \qquad \bigwedge^2 H^1(S;\mathbb{C}) \hookrightarrow H^2(S^{[n]};\mathbb{C}) \tag{2.1.24}$$

are morphisms of Hodge structures.

Proof. By **Proposition 2.1** and the hypothesis that that $H^*(S;\mathbb{Z})$ has no torsion we get that $H_1(S^{[n]};\mathbb{Z})$ has no torsion; by the Universal coefficients Theorem it follows that $H^2(S^{[n]};\mathbb{Z})$ has no torsion. Let $\{\alpha_1,\ldots,\alpha_m\}$ be an integral basis of the right-hand side of (2.1.23). Since we have Isomorphism (2.1.16) and $H^2(S^{[n]};\mathbb{Z})$ has no torsion it suffices to find classes $\beta_1,\ldots,\beta_m \in H_2(S^{[n]};\mathbb{Z})$ such that the intersection matrix with entries $\int_{\beta_i}\alpha_j$ is unimodular. We leave this as an exercise except for one point. Let $p_1,\ldots,p_{n-1} \in S$ be pairwise distinct and

$$\Gamma_n := \gamma^{-1}(2p_1 + p_2 + \ldots + p_{n-1}) \subset S^{[n]}. \tag{2.1.25}$$

Then Γ_n is isomorphic to $\mathbb{P}^1$. In fact let $f\colon \widetilde{V}_n \to V_n$ be as in (2.1.7) and

$$\widetilde{\Gamma}_n := f^{-1}(2p_1 + p_2 + \ldots + p_{n-1}). \tag{2.1.26}$$

Then $\widetilde{\Gamma}_n$ is isomorphic to $\mathbb{P}^1$ (it is the typical fiber of the map from the exceptional divisor of f to the big diagonal D_n) and the restriction of g (see (2.1.8)) to $\widetilde{\Gamma}_n$ defines an isomorphism $\widetilde{\Gamma}_n \xrightarrow{\sim} \Gamma_n$. Since g is simply ramified along D_n we get that

$$\Delta_n \cdot \Gamma_n = (h^*\Delta_n)\cdot\widetilde{\Gamma}_n = 2D_n \cdot \widetilde{\Gamma}_n = -2. \tag{2.1.27}$$

Thus $\int_{\Gamma_n}\xi_n = -1$. The statement about the embeddings given by (2.1.24) being morphisms of Hodge structures follows directly from their definition. □

Remark 2.3. The Betti numbers of $S^{[n]}$ have been computed by Göttsche [9], the Hodge numbers have been computed by Göttsche-Soergel [11]. See also the paper of J. Cheah [7].

2.1.2 Regular 2-forms

Let $\varphi \in H^0(\Omega_S^2)$: we will associate to φ a regular 2-form on $S^{[n]}$. For $1 \le i \le n$ let $\rho_i\colon \widetilde{V}_n \to S$ be the composition of the blow-down map (2.1.7) and projection to the i-th factor. The regular 2-form $\sum_{i=1}^{n}\rho_i^*\varphi$ is Σ_n-invariant. Let h be quotient map (2.1.9). A local computation shows that

$\sum_{i=1}^{n} \rho_i^* \varphi$ descends to a regular 2-form on U_n. By (2.1.4) we get that the descended 2-form extends to a regular 2-form $\varphi^{[n]}$ on $S^{[n]}$. We have defined a homomorphism

$$\begin{array}{ccc} H^0(\Omega_S^2) & \hookrightarrow & H^0(\Omega^2_{S^{[n]}}) \\ \varphi & \mapsto & \varphi^{[n]} \end{array} \tag{2.1.28}$$

In order to describe $\operatorname{div}(\bigwedge^n \varphi^{[n]})$ we introduce a piece of notation. Let D be an integral curve on S: we let Σ_D be the prime divisor on $S^{(n)}$ given by

$$\Sigma_D^{(n)} := \{A \in S^{(n)} \mid A \cap D \neq \varnothing\}. \tag{2.1.29}$$

Extending by linearity we get a map

$$\begin{array}{ccc} \operatorname{Div}(S) & \longrightarrow & \operatorname{Div}(S^{(n)}) \\ D & \mapsto & \Sigma_D^{(n)} \end{array} \tag{2.1.30}$$

Proposition 2.4. *Keep notation as above and let $0 \neq \varphi \in H^0(\Omega_S^2)$. Let $D = \operatorname{div}(\varphi) \in |K_S|$. Then*

$$\operatorname{div}(\varphi^{[n]}) = \gamma^*(\Sigma_D^{(n)}). \tag{2.1.31}$$

Proof. By (2.1.4) it suffices to prove that Equality (2.1.31) holds on the open subset U_n. This is clear away from $\Delta_n \cap U_n$. Thus $\operatorname{div}(\bigwedge^n \varphi^{[n]}) = a\Delta_n + \gamma^*(\Sigma_D^{(n)})$ for some $a \geq 0$. Let ω_{Δ_n} be the dualizing sheaf of Δ_n. By adjunction we get that $\omega_{\Delta_n} \cong \mathcal{O}_{\Delta_n}((a+1)\Delta_n)$. The restriction of the Hilbert-Chow map to the open dense $\gamma^{-1}(V_n \cap \Delta_n) \subset \Delta_n$ is a $\mathbb{P}^1$-fibration; the fibers are the Γ_n given by (2.1.25). It follows that $\omega_{\Delta_n}|_{\Gamma_n} \cong \mathcal{O}_{\Gamma_n}(-2)$. By (2.1.27) we get that $a = 0$. □

2.2 Beauville's examples

Throughout the present subsection $n \geq 2$. Let S be a projective $K3$ surface. We will prove that $S^{[n]}$ is a HK variety and that

$$b_2(S^{[n]}) = 23. \tag{2.2.1}$$

First $S^{[n]}$ is simply-connected by **Proposition 2.1**. Let $\varphi \in H^0(\Omega_S^2)$ be non-zero. Then φ is symplectic because S is a $K3$ surface and hence $\varphi^{[n]} \in H^0(\Omega^2_{S^{[n]}})$ is symplectic by **Proposition 2.4**. Lastly **Proposition 2.2** gives that $h^{2,0}(S^{[n]}) = 1$ and that (2.2.1) holds (the second Betti number of a $K3$ surface equals 22 by Noether's formula). Now let T be an abelian surface. We will prove that $K^{[n]}(T)$ is a HK variety and that

$$b_2(K^{[n]}(T)) = 7. \tag{2.2.2}$$

First let's prove that $K^{[n]}(T)$ is simply-connected. The long exact sequence of homotopy groups associated to Fibration (1.0.1) gives an exact sequence

$$\pi_2(T) \longrightarrow \pi_1(K^{[n]}(T)) \longrightarrow \pi_1(T^{[n+1]}) \xrightarrow{s_{n+1,\sharp}} \pi_1(T) \tag{2.2.3}$$

The map $s_{n+1,\sharp}$ is an isomorphism, see the proof of **Proposition 2.1**. Since $\pi_2(T)$ is trivial it follows that $K^{[n]}(T)$ is simply-connected. Next one shows that restriction gives a surjection

$$H^2(T^{[n+1]};\mathbb{Q}) \twoheadrightarrow H^2(K^{[n]}(T);\mathbb{Q}). \tag{2.2.4}$$

The assertion about surjectivity follows from irreducibility of $\Delta_{n+1}|_{K^{[n]}(T)}$ (that is why we need to assume that $n \geq 2$) and from a surjectivity statement involving

$$V^0_{n+1} := \{(x_1, \ldots, x_{n+1}) \in V_{n+1} \mid x_1 + \ldots + x_{n+1} = 0\}, \tag{2.2.5}$$

namely that restriction gives a surjection $H^2(V_{n+1};\mathbb{Q}) \to H^2(V^0_{n+1};\mathbb{Q})$. This proves surjectivity of (2.2.4). Now look at Equation (2.1.23) for $S = T$ with n replaced by $(n+1)$. Since $K^{[n]}(T)$ is simply connected we get that restriction defines a surjection

$$H^2(T;\mathbb{Q}) \bigoplus \mathbb{Q}\xi_{n+1} \twoheadrightarrow H^2(K^{[n]}(T);\mathbb{Q}). \tag{2.2.6}$$

In order to prove that the above map is an isomorphism we consider the regular map

$$\begin{array}{ccc} K^{[n]}(T)\times T & \xrightarrow{f} & T^{[n+1]} \\ (Z,a) & \mapsto & \tau_a(Z) \end{array} \tag{2.2.7}$$

where $\tau_a{:}T\to T$ is translation by a. The map f is Galois with group $T[n+1]$ (the group of $(n+1)$-torsion points of T). Since $K^{[n]}(T)$ is simply connected the Künneth decomposition gives an isomorphism $H^2(K^{[n]}(T)\times T;\mathbb{Q})\cong H^2(K^{[n]}(T);\mathbb{Q})\oplus H^2(T;\mathbb{Q})$. Thus $H^2(f)$ defines an injection $H^2(T^{[n+1]};\mathbb{Q})\hookrightarrow H^2(K^{[n]}(T);\mathbb{Q})\oplus H^2(T;\mathbb{Q})$; keeping in mind (2.2.6) we get that

$$2b_2(T)+1=b_2(T^{[n+1]})\le b_2(K^{[n]}(T))+b_2(T)\le 2b_2(T)+1. \tag{2.2.8}$$

Thus the inequalities above are equalities and hence (2.2.6) is an isomorphism. This proves (2.2.2) and hence also that $h^{2,0}(K^{[n]}(T))=1$ (see the claim about Hodge structures in the statement of **Proposition 2.2**). It remains to prove that there exists a holomorphic 2-form on $K^{[n]}(T)$. Let $0\ne\varphi\in H^0(\Omega^2_T)$. By **Proposition 2.4** the holomorphic 2-form $\varphi^{[n+1]}$ on $T^{[n+1]}$ is symplectic. Since f is śtale the pull-back $f^*\varphi^{[n+1]}$ is a symplectic form on $K^{[n]}(T)\times T$. Since $K^{[n]}(T)$ is simply-connected there exist a holomorphic 2-form α on $K^{[n]}(T)$ and a holomorphic 2-form β on T such that $f^*\varphi^{[n+1]}=p^*\alpha+q^*\beta$ where p,q are the projections of $K^{[n]}(T)\times T$ onto the first and second factor respectively. Since $f^*\varphi^{[n+1]}$ is symplectic both α and β are simplectic; thus α is a holomorphic symplectic form on $K^{[n]}(T)$.

Remark 2.5. By **Proposition 2.4** the variety $T^{[n+1]}$ has trivial first Chern class: Map (2.2.7) is its Beauville-Bogomolov decomposition of $T^{[n+1]}$.

Remark 2.6. The Hodge numbers of generalized Kummer varieties have been computed by Göttsche and Soergel [11].

2.3 Moduli of sheaves

2.3.1 Moduli of semistable sheaves

A general reference for moduli of semistable sheaves is [18]. Let X be a complex projective variety and H an ample Cartier divisor on S. We let $\mathcal{O}_X(1):=\mathcal{O}_X(H)$. Let F be a coherent sheaf on X (unless we state the contrary sheaves are always assumed to be coherent). Let $\mathrm{Ann}(F)\subset\mathcal{O}_X$ be the annihilator of F. Thus $\mathrm{Ann}(F)$ is an ideal sheaf; the *support of* F is the subscheme of X defined by $\mathrm{supp}(F):=V(\mathrm{Ann}(F))$. The *dimension of* F is equal to the dimension of $\mathrm{supp}(F)$; we denote it by $\dim(F)$. The sheaf F is *pure* if any non-zero subsheaf $G\subset F$ has dimension equal to $\dim(F)$.

Example 2.7. If $\dim X=1$ then a sheaf is pure if and only if it is torsion-free. If $\dim X=2$ a sheaf is pure of dimension 2 if and only if it is torsion-free. An example of a pure sheaf of dimension 1 on a surface X is given by $F:=\iota_*V$ where $\iota{:}C\hookrightarrow X$ is the inclusion of an irreducible curve and V is a torsion-free sheaf on C.

Given a sheaf F on X we let $F(n):=F\otimes\mathcal{O}_X(n)$. Suppose that F is non-zero and let $d:=\dim(F)\ge 0$. The Hilbert polynomial $\chi(F(n))$ is integer-valued, it follows that there exists a unique sequence of integers a_i for $0\le i\le d$ such that

$$\chi(F(n))=\sum_{i=0}^{d}a_i\binom{n}{i}\qquad\forall n\in\mathbb{Z}. \tag{2.3.1}$$

Furthermore $a_d(F)>0$; the *multiplicity of* F is equal to $a_d(F)$.

Example 2.8. Suppose that $\dim(F)=\dim X$. Then F is locally-free on an open dense subset $X_0\subset X$ and the rank of F, denoted by $\mathrm{rk}(F)$, is equal to the rank of the vector-bundle $F|_{X_0}$. Then $a_d(F)=\mathrm{rk}(F)\int_X c_1(H)^d$.

Definition 2.9. Let F be a sheaf on X. Let $d:=\dim(F)$. The *reduced Hilbert polynomial of* F, denoted by p_F is defined by

$$p_F(n):=\frac{\chi(F(n))}{a_d(F)}. \tag{2.3.2}$$

The set of isomorphism classes of sheaves on X with fixed Hilbert polynomial does not have a natural structure of quasi-projective variety except in special cases. The largest family of sheaves having a good moduli space is that of pure semistable sheaves.

Definition 2.10. Let X be a smooth irreducible projective variety equipped with an ample divisor H. A non-zero pure sheaf F on X is H-semistable if for every non-zero subsheaf $E \subset F$ we have

$$p_E(n) \le p_F(n) \qquad \forall n \gg 0. \tag{2.3.3}$$

If strict inequality holds whenever $E \ne F$ then F is H-stable.

Example 2.11. If $\dim F = \dim X$ and F has rank 1 then F is stable for arbitrary H. Suppose that $F = F_1 \oplus F_2$ with $F_i \ne 0$; then F is H-semistable if and only if each F_i is semistable and $p_{F_1} = p_{F_2}$.

We notice that in general (semi)stability does depend on the choice of H.

Claim 2.12. *Let X be a complex projective variety with ample Cartier divisor H and F a pure H-stable sheaf on X. Then F is* simple *i.e.* $\mathrm{Hom}(F,F) = \mathbb{C}\,\mathrm{Id}_F$.

Proof. Let $d := \dim F$. Assume that $\varphi\colon F \to F$ is a non-zero morphism of sheaves. We claim that φ is an isomorphism. In fact assume that $E := \ker\varphi \ne 0$ and let $G := \mathrm{Im}\,\varphi$. We have an exact sequence of pure d-dimensional sheaves

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0. \tag{2.3.4}$$

In particular $a_d(F) = a_d(E) + a_d(G)$. It follows that

$$p_F(n) = \frac{a_d(E)}{a_d(E)+a_d(G)} p_E(n) + \frac{a_d(G)}{a_d(E)+a_d(G)} p_G(n) \tag{2.3.5}$$

i.e. $p_F(n)$ lies in the segment spanned by $p_E(n)$ and $p_G(n)$. By stability of F we have that $p_E(n) < p_F(n)$ for $n \gg 0$. It follows that $p_F(n) < p_G(n)$ for $n \gg 0$: that is a contradiction because G is a subsheaf of F. We have proved that φ is injective. Thus φ is an injection $F \hookrightarrow F$. By stability we get that $\varphi(F) = F$. This proves that an endomorphism $\varphi\colon F \to F$ is either zero or an isomorphism. Thus $\mathrm{Hom}(F,F)$ is a finitely generated division $\mathbb{C}$-algebra: since $\mathbb{C}$ is algebraically closed it follows[6] that $\mathrm{Hom}(F,F) = \mathbb{C}\,\mathrm{Id}_F$. □

For pure sheaves of dimension equal to $\dim X$ there is the notion of μ(*slope*)-semistability: one replaces the reduced Hilbert polynomial by the slope. The slope of a sheaf F of dimension equal to $\dim X$ is

$$\mu(F) := \frac{1}{\mathrm{rk}(F)} \int_X c_1(F) \cdot c_1(H)^{\dim X - 1}. \tag{2.3.6}$$

F is μ-semistable (with respect to H) if for every non-zero subsheaf $E \subset F$ we have

$$\mu(E) \le \mu(F). \tag{2.3.7}$$

If strict inequality holds whenever $E \ne F$ then F is μ-stable. Notice that for a pure sheaf of dimension equal to $\dim X$ we have the following implications:

$$F \text{ is semistable} \Longrightarrow \quad F \text{ is } \mu\text{-semistable} \tag{2.3.8}$$

$$F \text{ is } \mu\text{-stable} \Longrightarrow \quad F \text{ is stable} \tag{2.3.9}$$

In order to obtain a separated moduli space we need to consider an equivalence relation which is weaker than isomorphism. Let F be a pure H-semistable sheaf on X. There exists (see [18]) a *Jordan-Hölder (J-H) filtration of* F

$$0 = F_0 \subset F_1 \subset \cdots \subset F_\ell = F \tag{2.3.10}$$

[6]Suppose that $\varphi \in (\mathrm{Hom}(F,F) \setminus \mathbb{C}\,\mathrm{Id}_F)$: then φ generates a non-trivial algebraic field extension of $\mathbb{C}$, that is a contradiction.

with the property that each quotient F_i/F_{i-1} is pure, H-stable with reduced Hilbert polynomial equal to P_F. A trivial example: if F is H-stable then a J-H filtration of F is necessarily trivial. Another example: $F = L \otimes_{\mathbb{C}} V$ where L is a line-bundle on X and V is a vector-space of dimension r. In this case the set of J-H filtrations of F is in bijective correspondence with the set of complete flags on V. As we see from the last example a J-H filtration is not unique. One proves that although the J-H filtration is not unique the associated graded sum

$$gr^{JH}(F) := \oplus_{i=1}^{\ell} F_i/F_{i-1} \tag{2.3.11}$$

is unique up to isomorphism.

Definition 2.13. Let F and G be pure H-semistable sheaves on X. Then F is S-equivalent to G if $gr^{JH}(F) \cong gr^{JH}(G)$.

If F is H-stable then F is S-equivalent to G if and only if $F \cong G$. On the other hand assume that F fits into the exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0 \tag{2.3.12}$$

where E, G are pure with $p_E = p_F = p_G$. Then F is S-equivalent to $E \oplus G$. Let P be an integer-valued polynomial of degree at most $\dim X$; we let

$$\mathcal{M}_X(P) := \{F \text{ pure } H\text{-s.s. sheaf on } X \mid \chi(F(n)) = P(n)\}/S\text{-equivalence} \tag{2.3.13}$$

(We do not include H in the notation although the isomorphism class of the moduli space does depend on H in general.) The main general theorem on moduli of pure sheaves is the result of research that was done during several years. Among the main constributors we quote Mumford, Seshadri, Narasimhan, Gieseker, Maruyama, Simpson. The most general result is due to Simpson [45].

Theorem 2.14. *There is a structure of projective scheme on $\mathcal{M}_X(P)$ such that the following hold:*

(1) Let T be a scheme and $\mathcal{F}$ be a sheaf on $X \times T$ which is $\mathcal{O}_T$-flat and such that for all $t \in T$ the restriction $\mathcal{F}|_{X \times \{t\}}$ is pure H-semistable with Hilbert polynomial P. Then there exists a regular map $T \to \mathcal{M}_X(P)$ which associates to closed points their S-equivalence class.

(2) $\mathcal{M}_X(P)$ "dominates"any other scheme satisfying Item (1).

Given a pure H-semistable sheaf with Hilbert polynomial P we let $[F] \in \mathcal{M}_X(P)$ be the point corresponding to the S-equivalence class of F. Let $\mathcal{M}_X(P)^s \subset \mathcal{M}_X(P)$ be the subset parametrizing stable sheaves: then $\mathcal{M}_X(P)^s$ is open. Let $[F] \in \mathcal{M}_X(P)^s$: there is a natural identification between the germ of $\mathcal{M}_X(P)$ at $[F]$ and the universal deformation space of F. In particular we get the following result.

Proposition 2.15. *Let $[F] \in \mathcal{M}_X(P)^s$. There is a natural isomorphism*

$$\Theta_{[F]}\mathcal{M}_X(P) \cong \operatorname{Ext}^1(F, F). \tag{2.3.14}$$

Let F be a (coherent) sheaf on X; one can define a *trace map*

$$\operatorname{Tr}^i \colon \operatorname{Ext}^i(F, F) \to H^i(\mathcal{O}_X) \tag{2.3.15}$$

which is the obvious map for $i = 0$ - see [18]. (If F is locally-free then Tr^i is induced by the sheaf map $End F \xrightarrow{Tr} \mathcal{O}_X$.) We let

$$\operatorname{Ext}^i(F, F)_0 := \ker \operatorname{Tr}^i. \tag{2.3.16}$$

Theorem 2.16 (Mukai [33], Artamkin [1])**.** *Suppose that $[F] \in \mathcal{M}_X(P)^s$ and that $\operatorname{Ext}^2(F, F)_0 = 0$. Then $\mathcal{M}_X(P)$ is smooth at $[F]$ and its tangent space is canonical identified with $\operatorname{Ext}^1(F, F)$.*

2.3.2 Semistable sheaves on symplectic surfaces

Let S be a symplectic projective surface i.e. either a $K3$ or an abelian surface. We let

$$\widetilde{H}(S) := H^0(S) \oplus H^2(S) \oplus H^4(S) \tag{2.3.17}$$

One gives $\widetilde{H}(S)$ a Hodge structure of weight 2 as follows:

$$\widetilde{H}(S)^{2,0} = H^{2,0}(S), \quad \widetilde{H}(S)^{0,2} = H^{0,2}(S), \quad \widetilde{H}(S)^{1,1} = H^0(S) \oplus H^{1,1}(S) \oplus H^4(S). \tag{2.3.18}$$

Thus $\widetilde{H}(S)$ has an integral Hodge structure - the integral structure coming from $\widetilde{H}(S;\mathbb{Z})$. The *Mukai lattice* [32] of S is the group $\widetilde{H}(S;\mathbb{Z})$ equipped with the symmetric bilinear form

$$\langle \sum_{i=0}^{2} \alpha_i, \sum_{i=0}^{2} \beta_i \rangle := \int_S (-\alpha_0\beta_4 - \alpha_4\beta_0 + \alpha_2 \wedge \beta_2) \tag{2.3.19}$$

where $\alpha_i, \beta_i \in H^{2i}(S;\mathbb{Z})$. Notice that $\langle , \rangle$ is even unimodular of signature $(4, 20)$. Let F be a coherent sheaf on S; following Mukai [32] one sets

$$v(F) := ch(F)\sqrt{Td(S)} = ch(F)(1 + \epsilon\eta), \tag{2.3.20}$$

where $\eta \in H^4(S;\mathbb{Z})$ is the orientation class and ϵ is equal to 1 if S is a $K3$ surface and is equal to 0 if S is an abelian surface. Notice that $v(F) \in \widetilde{H}^{1,1}_{\mathbb{Z}}(S)$. By Hirzebruch-Riemann-Roch we have

$$\langle v(E), v(F) \rangle = -\chi(E,F) := -\sum_{i=0}^{2} (-1)^i \dim \mathrm{Ext}^i(E,F). \tag{2.3.21}$$

By Serre duality we have $\mathrm{Ext}^2(F,F) \cong \mathrm{Hom}(F,F)^\vee$ and hence (2.3.21) gives

$$\dim \mathrm{Ext}^1(F,F) = 2\dim \mathrm{Hom}(F,F) + \langle v(F), v(F) \rangle. \tag{2.3.22}$$

Remark 2.17. Let S be a symplectic projective surface with an ample divisor H. Suppose that F is an H-stable sheaf on S. Then $\mathrm{Hom}(F,F) = \mathbb{C}\,\mathrm{Id}_F$ (see **Claim 2.12**) and hence (2.3.22) gives that

$$-2 \le v(F)^2. \tag{2.3.23}$$

Definition 2.18. A *Mukai vector* is a

$$\mathbf{v} = r + \ell + s\eta \in \widetilde{H}^{1,1}_{\mathbb{Z}}(S) \tag{2.3.24}$$

such that $r \ge 0$ and such that ℓ is effective if $r = 0$.

Notice that if F is a pure sheaf of dimension 2 or 1 then $v(F)$ is a Mukai vector. One indicizes moduli spaces of semistable pure sheaves on symplectic surfaces by Mukai vectors. Let $\mathbf{v} \in \widetilde{H}^{1,1}$ be a Mukai vector: the Hilbert polynomial $\chi(F(n))$ of a sheaf F such that $v(F) = \mathbf{v}$ is independent of F, call it P. Let

$$\mathcal{M}_S(\mathbf{v}) := \{[F] \in \mathcal{M}_S(P) | \ v(F) = \mathbf{v}\}. \tag{2.3.25}$$

We claim that $\mathcal{M}_S(\mathbf{v})$ is open and closed in $\mathcal{M}_S(P)$. In fact the rank of sheaves parametrized by $\mathcal{M}_S(P)$ is constant and the Chern classes of sheaves parametrized by $\mathcal{M}_S(P)$ are locally constant.

Proposition 2.19. *Let S be a projective symplectic surface and H an ample divisor on S. Let $\mathbf{v}$ be a Mukai vector. Then*

(1) $\mathcal{M}_S(\mathbf{v})$ is a projective scheme.

(2) Suppose that $[F] \in \mathcal{M}_S(\mathbf{v})^s$. Then $\mathcal{M}_S(\mathbf{v})$ is smooth at $[F]$ and

$$\dim_{[F]} \mathcal{M}_S(\mathbf{v}) = 2 + \langle \mathbf{v}, \mathbf{v} \rangle. \tag{2.3.26}$$

Proof. (1): This is because $\mathcal{M}_S(\mathbf{v})$ is open and closed in $\mathcal{M}_S(P)$. (2): Since F is stable **Claim 2.12** gives that $\mathrm{Hom}(F,F) = \mathbb{C}\,\mathrm{Id}_F$. By Serre duality it follows that $\mathrm{Ext}^2(F,F)_0 = 0$. By **Theorem 2.16** we get that $\mathcal{M}_S(\mathbf{v})$ is smooth at $[F]$ with tangent space canonically identified with $\mathrm{Ext}^1(F,F)$. Thus Equation (2.3.26) follows from (2.3.22) and $\mathrm{Hom}(F,F) = \mathbb{C}\,\mathrm{Id}_F$. □

Example 2.20. Let S be a $K3$ surface. We have an isomorphism

$$\begin{array}{ccc} S^{[n]} & \xrightarrow{\sim} & \mathcal{M}_S(1-(n-1)\eta) \\ [Z] & \mapsto & [I_Z]. \end{array} \tag{2.3.27}$$

Let $\varphi \in \Gamma(\Omega_S^2)$. Following Mukai [33] one defines a 2-form $\tau(\varphi)$ on $\mathcal{M}_S(\mathbf{v})^s$ by setting

$$\tau(\varphi)(\alpha,\beta) := \int_S \varphi \wedge Tr^2(\alpha \cup \beta), \tag{2.3.28}$$

where $\cup$ denotes Yoneda product. If F is locally-free $\cup$ is induced by the map of sheaves

$$\begin{array}{ccc} EndF \otimes EndF & \longrightarrow & EndF \\ (\phi,\psi) & \mapsto & \phi \circ \psi \end{array} \tag{2.3.29}$$

Proposition 2.21 (Mukai [33])**.** *Keep notation and hypothese as above. Then $\tau(\varphi)$ is holomorphic and closed. If φ is non-zero then $\tau(\varphi)$ is symplectic at each point of $\mathcal{M}_S(\mathbf{v})^s$.*

(Actually closedness of $\tau(\varphi)$ without the assumption that $\mathcal{M}_S(\mathbf{v})^s$ is closed is proved elsewhere - see for example [34].) Notice that non-degeneracy of $\tau(\varphi)$ follows immediately from Serre-duality.

Remark 2.22. Let S be a $K3$ surface and $\varphi \in H^0(\Omega_S^2)$. The Hilbert scheme $S^{[n]}$ is identified with the moduli space $\mathcal{M}_S(1-(n-1)\eta)$, see **Example 2.20**, and hence we have the holomorphic 2-forms $\varphi^{[n]}$ and $\tau(\varphi)$. The relation between the forms is the following:

$$\tau(\varphi) = -4\pi^2\varphi^{[n]}. \tag{2.3.30}$$

Before stating a general result about moduli of semistable sheaves on $K3$ and abelian surfaces we must discuss the notion of generic polarization. Let S be a smooth projective surface. Let $\mathrm{NS}(S)$ be the Néron-Severi group of S i.e. $H^{1,1}_{\mathbb{Z}}(S)$ and $\mathrm{NS}(S)_{\mathbb{R}} := \mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{R}(S)$. Let $A(S) \subset \mathrm{NS}(S)$ be the ample cone and $A(S)_{\mathbb{R}} \subset \mathrm{NS}(S)_{\mathbb{R}}$ be its tensor product with $\mathbb{R}$. A *wall* consist of $W_D := D^{\perp} \cap A(S)_{\mathbb{R}}$ where D is a divisor on S with *strictly negative* self-intersection.

Proposition 2.23. *Let S be a projective symplectic surface and $\mathbf{v}$ a Mukai vector for S. There exists a union of walls $\mathcal{W} = \bigcup_{D \in \mathcal{C}} W_D$ with the following properties:*

1. *$\mathcal{W}$ is locally finite and hence the complement in $A(S)_{\mathbb{R}}$ is a dense open subset.*
2. *Let $H \in (A(S) \setminus \mathcal{W})$ and F be a strictly H-semistable (i.e. semistable but* not *stable) sheaf with $v(F) = \mathbf{v}$. Then $gr^{JH}(F) = \bigoplus_i E_i$ where for each i we have $v(E_i) = a_i\mathbf{v}$ with $a_i \in \mathbb{Q}$.*

Definition 2.24. Let S be a projective symplectic surface and $\mathbf{v}$ a Mukai vector for S. An ample divisor H on$S$ is $\mathbf{v}$*-generic* if it lies outside the minimal union of walls $\mathcal{W}$ for which the conclusions of **Proposition 2.23** hold.

The following result is an immediate consequence of **Proposition 2.23**.

Corollary 2.25. *Let S be a projective symplectic surface and $\mathbf{v}$ be an* indivisible *Mukai vector for S. Let H be an ample $\mathbf{v}$-generic divisor. Then $\mathcal{M}_S(\mathbf{v})^{st} = \mathcal{M}_S(\mathbf{v})$.*

Remark 2.26. Let S be a projective symplectic surface and $\mathbf{v} = (r + \ell + s\eta)$ a Mukai vector for S with $-2 \leq \mathbf{v}^2$ (see **Remark 2.17**). Suppose that $r > 0$ and let

$$k := \begin{cases} \frac{r^2}{4}\mathbf{v}^2 + \frac{r^4}{2} & \text{if } S \text{ is a } K3, \\ \frac{r^2}{4}\mathbf{v}^2 & \text{if } S \text{ is an abelian surface.} \end{cases} \tag{2.3.31}$$

Let $\mathcal{W} \subset A(S)_{\mathbb{R}}$ be the union of the walls W_D where D runs through the set of divisors such that $-k \leq D \cdot D < 0$. Then the conclusions of **Proposition 2.23** hold for the above $\mathcal{W}$.

The following result was proved under more restrictive hypotheses in [32, 10, 35] and more in general by Yoshioka [48, 49].

Theorem 2.27 (Mukai, Göttsche - Huybrechts, O'Grady, Yoshioka)**.** *Let S be a projective $K3$ surface. Let $\mathbf{v}$ be Mukai vector as in* (2.3.24) *and suppose that*

(1) $\mathbf{v}$ *is indivisible,*

(2) $-2 \leq \langle \mathbf{v}, \mathbf{v} \rangle$,

(3) $(r, s) \neq (0, 0)$.

Let H be a $\mathbf{v}$-generic ample divisor on S. Then $\mathcal{M}_S(\mathbf{v})$ is an irreducible symplectic variety deformation equivalent to $S^{[n]}$ where $2n = 2 + \langle \mathbf{v}, \mathbf{v} \rangle$.

Mukai proved the result when $\dim \mathcal{M}_S(\mathbf{v}) = 2$. Göttsche and Huybrechts proved the result for rank 2 and c_1 is indivisible. O'Grady assumed that the rank is non-zero and c_1 is indivisible. Moreover the statements in [10, 35] are that $\mathcal{M}_S(\mathbf{v})$ is an irreducible symplectic variety which deforms to a variety birational to $(K3)^{[n]}$; the stronger statement follows by applying a general theorem of Huybrechts which states that birational HK manifolds are actually deformation equivalent.

Remark 2.28. Let S, $\mathbf{v}$ and H be as in **Theorem 2.27**. In general the moduli space $\mathcal{M}_S(\mathbf{v})$ is not isomorphic to a Hilbert scheme $F^{[n]}$, not even birational. Thus **Theorem 2.27** provides explicit examples of HK deformations of $K3^{[n]}$ which are not isomorphic to a Hilbert scheme of a $K3$ surface.

There is a result valid for moduli spaces of semistable sheaves on an abelian surface T which is analogous to **Theorem 2.27**. Given a 0-cycle $Z = \sum_i n_i(p_i)$ on T we let $\sigma(Z) \in T$ be given by $\sigma(Z) := \sum_i n_i p_i$ (we take the sum in the group T); if Z' is rationally equivalent to Z then $\sigma(Z') = \sigma(Z)$ and hence we have a well-defined homomorphism $\sigma\colon \mathrm{CH}_0(T) \to T$. Let $\mathbf{v}$ be a Mukai vector on T and

$$\begin{array}{ccc} \mathcal{M}_T(\mathbf{v}) & \xrightarrow{A_{\mathbf{v}}} & T \times Pic(T) \\ [F] & \mapsto & (\sum c_2^{CH}(F), [c_1^{CH}(F)]) \end{array} \tag{2.3.32}$$

where $c_i^{CH}(F)$ denotes the i-th Chern class in the Chow group of T. The map $A_{\mathbf{v}}$ is regular. Choose $[F_0] \in \mathcal{M}_S(\mathbf{v})$ (assuming $\mathcal{M}_S(\mathbf{v})$ is non-empty) and let $\alpha_0 := c_1^{CH}(F_0)$. We let

$$\mathcal{M}_T(\mathbf{v})^0 := A_{\mathbf{v}}^{-1}(0, \alpha_0). \tag{2.3.33}$$

The isomorphism class of $\mathcal{M}_T(\mathbf{v})^0$ is independent of the choice of F_0 as soon as $4 \leq \langle \mathbf{v}, \mathbf{v} \rangle$.

Theorem 2.29 (Mukai, Yoshioka)**.** *Let T be an abelian surface. Let $\mathbf{v}$ be a Mukai vector as in* (2.3.24) *and assume that*

(1) $\mathbf{v}$ *is indivisible,*

(2) $4 \leq \langle \mathbf{v}, \mathbf{v} \rangle$,

(3) $(r, s) \neq (0, 0)$.

Let H be a $\mathbf{v}$-generic ample divisor on T. Then $\mathcal{M}_T^0(\mathbf{v})$ is an irreducible symplectic variety deformation equivalent to $K^{[n]}(T)$ where $2n = \langle \mathbf{v}, \mathbf{v} \rangle - 2$.

2.4 Two more examples

Let S be a projective symplectic surface and $\mathbf{v}$ a Mukai vector for S which is *divisible.* Thus

$$\mathbf{v} = m\mathbf{v}_0, \qquad \mathbf{v}_0 \in \widetilde{H}^{1,1}_{\mathbb{Z}}(S) \text{ indivisible, } m \in \mathbb{N},\ m \geq 2. \tag{2.4.1}$$

Let H be a $\mathbf{v}$-generic ample divisor on S. Suppose that $\mathcal{M}_S(\mathbf{v})^{st}$ and $\mathcal{M}_S(\mathbf{v}_0)^{st}$ are non-empty. Let $F := \bigoplus_{i=1}^m E_i$ where E_i is a stable sheaf such that $v(E_i) = \mathbf{v}_0$. Then F is a strictly semistable sheaf parametrized by a point of $\mathcal{M}_S(\mathbf{v})$. We expect that $\mathcal{M}_S(\mathbf{v})$ is singular at these points - this is true except for special choices of $\mathbf{v}$. Assuming that this is the case one may ask: does there exist a desingularization $\widetilde{\mathcal{M}}_S(\mathbf{v}) \to \mathcal{M}_S(\mathbf{v})$ such that the holomorphic symplectic form on $\mathcal{M}_S(\mathbf{v})^{st}$ (under our hypothesis $\mathcal{M}_S(\mathbf{v})^{st}$ is the smooth locus of $\mathcal{M}_S(\mathbf{v})$) extends to a holomorphic symplectic form on $\widetilde{\mathcal{M}}_S(\mathbf{v})$? Such a desingularization is a *symplectic desingularization*. The following result, summarizing the work of many mathematicians, answers the above question.

Theorem 2.30 (O'Grady, Kiem, Rapagnetta, Kaledin, Lehn, Sorger, Perego)**.** *Let S be a symplectic projective surface. Let $\mathbf{v}$ be a divisible Mukai vector as in (2.4.1). Suppose that $\mathbf{v}_0^2 \geq 2$ and that $(r,s) \neq (0,0)$. Let H be a $\mathbf{v}$-generic ample divisor on S. Then $\mathcal{M}_S(\mathbf{v})$ is non-empty, irreducible of dimension $(2+\mathbf{v}^2)$ and its smooth locus is equal to $\mathcal{M}_S(\mathbf{v})^{st}$. There exits a symplectic desingularization $f: \widetilde{\mathcal{M}}_S(\mathbf{v}) \to \mathcal{M}_S(\mathbf{v})$ if and only if $m = 2$ and $\mathbf{v}_0^2 = 2$. Now suppose that $\mathbf{v}_0^2 = 2$. Then the following hold:*

1. *If S is a $K3$ surface then $\widetilde{\mathcal{M}}_S(2\mathbf{v}_0)$ is a 10-dimensional HK variety and $b_2(\widetilde{\mathcal{M}}_S(\mathbf{v}_0)) = 24$.*
2. *If S is an abelian surface let $\widetilde{\mathcal{M}}_S(2\mathbf{v}_0)^0 := f^{-1}(\mathcal{M}_S(2\mathbf{v}_0)^0)$. Then $\widetilde{\mathcal{M}}_S(2\mathbf{v}_0)^0$ is a 6-dimensional HK variety and $b_2(\widetilde{\mathcal{M}}_S(2\mathbf{v}_0)^0) = 8$.*
3. *Let S and S' be $K3$ surfaces, $\mathbf{v}_0$ and $\mathbf{v}'_0$ Mukai vectors for S and S' with $2 = \mathbf{v}_0^2 = (\mathbf{v}'_0)^2$ and H, H' ample divisors on S and S' respectively which are $2\mathbf{v}_0$ and $2\mathbf{v}'_0$ generic respectively. Then $\widetilde{\mathcal{M}}_S(2\mathbf{v}_0)$ is deformation equivalent to $\widetilde{\mathcal{M}}_{S'}(2\mathbf{v}'_0)$. A similar statement holds for abelian surfaces.*

Let $\widetilde{\mathcal{M}}_S(2\mathbf{v}_0)$ and $\widetilde{\mathcal{M}}_S(2\mathbf{v}_0)^0$ be as in Items (1) and (2) of **Theorem 2.30**. Since $\widetilde{\mathcal{M}}_S(2\mathbf{v}_0)$ has second Betti number different from that of $(K3)^{[n]}$ and of a generalized Kummer it is not not a deformation of the Beauville examples. A similar statement holds for $\widetilde{\mathcal{M}}_S(2\mathbf{v}_0)^0$. Thus we get two new deformation classes of HK manifolds. A word about the contributions of the mathematicians quoted in the statement of **Theorem 2.30**. Suppose that $\mathbf{v}_0 = (1-\eta)$: sheaf F parametrized by $\mathcal{M}_S(2\mathbf{v}_0)$ has rank 2, $c_1 = 0$ and c_2 equal to 4 if S is a $K3$ and 2 if S is an abelian surface. O'Grady [36, 37] proved that $\mathcal{M}_S(2(1-\eta))$ has a symplectic desingularization $\widetilde{\mathcal{M}}_S(2(1-\eta))$ and that if S is a $K3$ then $\widetilde{\mathcal{M}}_S(2(1-\eta))$ is a 10-dimensional HK variety with $b_2 \geq 24$ (and hence not a deformation of Beauville's examples), and that if S is an abelian surface then $\widetilde{\mathcal{M}}_S(2(1-\eta))^0$ is a 6-dimensional HK variety with $b_2 = 8$ (and hence not a deformation of Beauville's examples). Kiem [22] proved non-existence of a symplectic desingularization of $\mathcal{M}_S(2\mathbf{v}_0)$ for some choices of $\mathbf{v}_0$. Rapagnetta [40] proved that if S is a $K3$ surface then $b_2(\widetilde{\mathcal{M}}_S(2(1-\eta))) = 24$. Suppose that $\mathbf{v}_0^2 = 2$: Lehn and Sorger [24] showed that the symplectic desingularization $f: \widetilde{\mathcal{M}}_S(2\mathbf{v}_0) \to \mathcal{M}_S(2\mathbf{v}_0)$ can be obtained by a single blow-up, namely the blow-up of the singular locus of $\mathcal{M}_S(2\mathbf{v}_0)$. Kaledin, Lehn and Sorger [21] proved non-existence of a symplectic desingularization for all $\mathbf{v}$ with $m > 2$ or $\mathbf{v}_0^2 > 2$. Perego and Rapagnetta [38] proved Item (3) of **Theorem 2.30**.

Remark 2.31. We do not know all the Betti numbers of the 6 and 10 dimensional HK varieties appearing in **Theorem 2.30** - that is in contrast with the case of Hilbert schemes of $K3$ surfaces (or any surface) or of generalized Kummers. Rapagnetta [39] proved that the topological Euler characterstic of the 6-dimensional variety is equal to 1920.

3 Yau's Theorem and its implications

Let X be a compact Kähler manifold with $c_1^{\mathbb{R}}(X) = 0$ where $c_1^{\mathbb{R}}(X)$ is the first Chern class in De Rham cohomology - equivalently the integral first Chern class $c_1(X) \in H^2(X;\mathbb{Z})$ is torsion. A *Calabi-Yau metric* on X is a Kähler Hermitian metric h such that the unique connection ∇ on K_X compatible with the holomorphic structure and the metric h (see [12]) is flat i.e. its curvature F_∇ vanishes. Vanishing of F_∇ is equivalent to vanishing of the Ricci curvature of the riemannian metric associated to h, for that reason a Calabi-Yau metric is also called a Ricci-flat metric. Below is Yau's celebrated Theorem on existence of Calabi-Yau metrics.

Theorem 3.1 (Yau [47]). *Let (X,ω) be a compact Kähler manifold with $c_1^{\mathbb{R}}(X)=0$. There exists a unique Calabi-Yau metric h such that the Kähler form ω_h of h is cohomologous to ω.*

Example 3.2. Let $X=\mathbb{C}^n/L$ be a compact torus. In this case the statement of **Theorem 3.1** follows from the fact that every cohomology class is represented by a form on $\mathbb{C}^n$ with constant coefficients, moreover the connection on the tangent space itself is flat.

In general Yau's theorem is a pure existence result - as far as I know no one ever wrote down a Calabi-Yau metric of a single $K3$ surface. Yau's Theorem has some very strong consequences - we will go over these results for HK manifolds.

3.1 Holonomy

Let (M,g) be a Riemannian manifold. Let $p,q\in M$ and $\gamma:[a,b]\to M$ a piecewise-smooth path from p to q: parallel transport with respect to the Levi-Civita connection defines an isometry

$$\varphi_\gamma: T_pM \longrightarrow T_pM.$$

(We let T_pM be the real tangent space to M at p.) The *holonomy group* H_p at p is defined to be

$$H_p := \{\varphi_\gamma \mid \gamma(a)=\gamma(b)=p\} < O(T_pM).$$

Forgetting about the point p we may view the holonomy group as a subgroup $H<O(n)$ (here $n:=\dim M$) well-defined modulo conjugation (we assume that M is connected).

Example 3.3. Let X be a compact Kähler manifold with Kähler metric h. One identifies Θ_pX (the holomorphic tangent space to X at p) and T_pX by mapping $v\in\Theta_pX$ to $(v+\overline{v})/2$. Given the above identification multiplication by $-\sqrt{-1}$ on Θ_pX gets identified with an endomorphism $I:T_pX\to T_pX$ whose square is $-\mathrm{Id}$. Since h is Kähler the endomorphism I is invariant under the holonomy group [23]: it follows that H_p is a subgroup of $\mathrm{U}(\Theta_pX,h_p)$ (again we identify T_pX with Θ_pX).

The group H_p and its representation on T_pM encodes information on the geometry of M as follows. Let $\Gamma_{par}(M;TM^{\otimes a}\otimes T^\vee M^{\otimes b})$ be the space of parallel tensors.

Holonomy Principle 3.4. *Let (M,g) be a connected riemannian manifold and $p\in M$. Evaluation at p*

$$\Gamma_{par}(M;TM^{\otimes a}\otimes T^\vee M^{\otimes b}) \longrightarrow T_pM^{\otimes a}\otimes T_p^\vee M^{\otimes b}$$

is an injective map with image the subspace of tensors invariant under the action of H_p.

Next we recall Bochner's principle [2].

Bochner's Principle 3.5. *Let X be a compact Kähler manifold and suppose that h is a Calabi-Yau metric. Let σ be a* holomorphic *tensor i.e. a global holomorphic section of $\Theta_X^{\otimes a}\otimes\Omega_X^{\otimes b}$. Then σ is parallel.*

Example 3.6. Let X be a compact Kähler manifold with $c_1^{\mathbb{R}}(X)=0$ and ω be a Kähler class on X. By Yau's Theorem there exists a unique CY metric h such that ω_h is in the class of ω. Let $0\neq\alpha\in H^0(K_X)$. By Bochner's principle α is parallel: it follows that $H_p<\mathrm{SU}(\Theta_pX)$ (recall that $H_p<\mathrm{U}(\Theta_pX)$ by **Example 3.3**).

Example 3.7. Let X be a HK manifold and ω be a Kähler class on X. Let h be the unique CY metric such that ω_h is in the class of ω. Let σ be a holomorphic symplectic form. By Bochner's principle we get that $H_p<(U(\Theta_pX,h_p)\cap\mathrm{Sp}(\Theta_pX,\sigma_p))$ where $\mathrm{Sp}(\Theta_pX,\sigma_p)$ is the symplectic group of $\mathbb{C}$-linear automorphisms preserving the symplectic form σ_p on Θ_pX. Actually [2]

$$H_p = U(\Theta_pX,h_p)\cap\mathrm{Sp}(\Theta_pX,\sigma_p). \tag{3.1.1}$$

Theorem 3.8. *Let X be a HK manifold of dimension $2n$ and σ a holomorphic 2-form. Then*

$$H^0(\Omega_X^q) = \begin{cases} \mathbb{C}\sigma^i & \text{if } q = 2i \text{ for } 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.2}$$

Proof. Let $p \in X$. Every holomorphic global form on X is parallel by Bochner's principle: it follows that we have an isomorphism

$$\begin{array}{ccc} H^0(\Omega_X^q) & \xrightarrow{\sim} & (\bigwedge^q \Omega_p X)^{H_p} \\ \varphi & \mapsto & \varphi_p \end{array} \tag{3.1.3}$$

where $(\bigwedge^q \Omega_p X)^{H_p}$ is the space of H_p-invariant elements of $\bigwedge^q \Omega_p X$. The right-hand side of (3.1.3) is generated by $\bigwedge^m \sigma(p)$ if $q = 2m$ and is zero otherwise: the theorem follows. □

Corollary 3.9. *Let X be a HK manifold of dimension $2n$. Then $\chi(\mathcal{O}_X) = n + 1$.*

3.2 Twistor families

Let X be a HK manifold of dimension $2n$. Let ω be a Käbler class on X. Let h be the unique CY metric such that ω_h is in the class of ω and let g be the associated riemannian metric (the real part of h). One identifies the holonomy group $H_p = U(\Theta_p X, h_p) \cap \mathrm{Sp}(\Theta_p X, \sigma_p)$ with a group acting on $\mathbb{H}^n$ (here $\mathbb{H}$ is the algebra of quaternions) as follows. Recall that $\mathbb{H}$ is the associative real algebra with $\mathbb{R}$-basis $\{1, i, j, k\}$ such that

$$-1 = i^2 = j^2 = k^2, \qquad ij = k, \quad ji = -k, \quad jk = i, \quad kj = -i.$$

The *conjugate* of $x = x_1 + x_2 i + x_3 j + x_4 k$ is $x = x_1 - x_2 i - x_3 j - x_4 k$; notice that $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$. Multiplication on the *right* gives $\mathbb{H}^n$ the structure of an $\mathbb{H}$-module. Let $w, z \in \mathbb{H}^n$: for $s = 1, \ldots, n$ we write $w_s = a_s + j b_s$ and $z_s = c_s + j d_s$ where $a_s, b_s, c_s, d_s \in \mathbb{C}$. The *standard* hermitian quaternionic product on $\mathbb{H}^n$ is given by

$$\langle w, z \rangle := \sum_{s=1}^n \overline{w}_s z_s = \sum_{s=1}^n (\overline{a}_s c_s + \overline{b}_s d_s) + j \sum_{s=1}^n (a_s d_s - b_s c_s) = h_0(z, w) + j\sigma_0(z, w) \tag{3.2.1}$$

where h_0 and σ_0 are the standard hermitian and symplectic form on $\mathbb{H}^n$ viewed as complex vector-space (multiplication on the right). Notice that for every $z, w \in \mathbb{H}^n$ we have

$$h_0(z, wj) = \sigma_0(z, w). \tag{3.2.2}$$

(Notice the analogy with the decomposition of hermitian positive definite form on a complex vector space $\langle , \rangle$ as $(g_0 - \sqrt{-1}\omega_0)$ where g_0 is a euclidean product and ω_0 is a symplectic real form such that $g_0(iv, w) = \sigma_0(v, w)$). Let $\mathrm{U}(n, \mathbb{H})$ be the group of $\mathbb{H}$-linear automorphisms $f \colon \mathbb{H}^n \to \mathbb{H}^n$ which preserve $\langle , \rangle$. Then

$$\mathrm{U}(n, \mathbb{H}) = \mathrm{U}(2n) \cap \mathrm{Sp}(2n). \tag{3.2.3}$$

In fact the left-hand side is clearly contained in the right-hand side. In order to prove that the right-hand side is contained in the left-hand side it suffices to prove that if $T \in \mathrm{U}(2n) \cap \mathrm{Sp}(2n)$ then $T(vj) = (Tv)j$ for all $v \in \mathbb{H}^n$: that follows easily from (3.2.2). Now suppose that $\mu \in \mathbb{H}$ and that $\mu^2 = -1$. Then *right* multiplication by μ, call it R_μ, defines a complex structure on $\mathbb{H}^n$. Thus we have a family of complex structures on $\mathbb{H}^n$ parametrized by

$$\{(x_1 i + x_2 j + x_3 k) \mid (x_1, x_2, x_3) \in \mathbb{R}^3, \quad x_1^2 + x_2^2 + x_3^2 = 1\} \cong S^2. \tag{3.2.4}$$

Notice that R_μ commutes with $\mathrm{U}(n, \mathbb{H})$ and that it is an isometry for the euclidean product on $\mathbb{H}^n$ defined by

$$(w, z) := \Re\langle w, z \rangle = \sum_{s=1}^n (\overline{w}_s z_s + \overline{z}_s w_s). \tag{3.2.5}$$

Moreover every complex structure on $\mathbb{H}^n$ is equal to R_μ for some $\mu \in \mathbb{H}$ such that $\mu^2 = -1$. Now let $p \in X$ and let h be the Calabi-Yau metric. There exists an h_p-orthonormal basis of Θ_p such that the symplectic form σ_p is in standard form, i.e. we may identify h_p and σ_p with h_0 and σ_0 of (3.2.1). By (3.1.1) and (3.2.3) we may identify H_p with the unitary quaternionic group. It follows that there is a well-defined S^2 parametrizing complex structures on $\Theta_p X$ which commute with H_p. Each such complex structure μ is an isometry and is parallel for the Levi-Civita connection of g (because it commutes with H_p), it follows that it defines an integrable complex structure X_μ and g is the real part of a (unique) Kähler hermitian metric for that complex structure. Of course the complex structure X_i is the one we started from (and the corresponding Káhler metric is h), the others are new complex structures. The complex manifolds X_μ fit together: there exist a complex manifold $\mathcal{X}(\omega)$ (diffeomorphic to $X \times S^2$) and a holomorphic map

$$\pi: \mathcal{X}(\omega) \to \mathbb{P}^1_{\mathbb{C}} \tag{3.2.6}$$

such thet the fiber of π over μ is isomorphic to X_μ (we identify S^2 with $\mathbb{P}^1_{\mathbb{C}}$ by the obvious procedure), see [41]. The family (3.2.6) is the *twistor fibration* associated to (X, ω) and $\mathcal{X}(\omega)$ is the *twistor space*. The remarkable feature is that we get a global deformation of X starting from the datum of a Kähler class. Given $\mu \in \mathbb{P}^1_{\mathbb{C}}$ the complex manifold has the Kähler form $\omega_\mu(v, w) = g(\mu v, w)$. Since $\mathcal{X}(\omega)$ is diffeomorphic to $X \times S^2$ it makes sense to consider the cohomology class $[\omega_\mu] \in H^2(X; \mathbb{R})$ of ω_μ: as μ varies these classes span a 3-dimensional subspace $H^2_+(X; \mathbb{R})$ and they belong to an $S^2 \subset H^2_+(X; \mathbb{R})$. Similarly we may consider a holomorphic symplectic form σ_μ on X_μ, it is well-defined up to rescaling. Their cohomology classes in $H^2(X; \mathbb{R})$ span $H^2_+(X; \mathbb{R}) \otimes \mathbb{C}$ and the image in the projectivization $\mathbb{P}H^2_+(X; \mathbb{R}) \otimes \mathbb{C}$ is a conic.

3.3 Deformations are unobstructed

Let X be a HK manifold. Let σ be a symplectic holomorphic form on X. Contraction of tangent vectors with σ defines an isomorphism of vector-bundles

$$\begin{array}{ccc} \Theta_X & \xrightarrow[\sim]{L_\sigma} & \Omega^1_X \\ v & \mapsto & v \lrcorner \sigma. \end{array} \tag{3.3.1}$$

Thus $H^0(\Theta_X) \cong H^0(\Omega^1_X)$ and the latter space vanishes because by definition X is simpy connnected. Thus deformation theory gives that there exists a universal deformation space $\mathrm{Def}(X)$ of X.

Theorem 3.10 (Bogomolov [4])**.** *The deformation space of a HK manifold X is unobstructed.*

Explicitely Theorem **Theorem 3.10** asserts the following: There exist a submersive map $f: \mathcal{X} \to U$ of complex manifolds and a point $0 \in U$ such that U is a polydisc, $F^{-1}(0) \cong X$ and the Kodaira-Spencer map $\Theta_0 U \to H^1(\Theta_X)$ is an isomorphism.

Remark 3.11. Deformation theory gives that a representative of $\mathrm{Def}(X)$ is the zero-locus of an analytic obstruction map $\Phi: B \to H^2(\Theta_X)$ where B is a polydisc of dimension $h^1(\Theta_X)$. Thus Bogomolov's Theorem follows from general deformation theory if $H^2(\Theta_X) = 0$. Notice that by (3.3.1) we have $H^2(\Theta_X) \cong H^2(\Omega^1_X) \cong H^{1,2}(X)$. Thus $H^2(\Theta_X) = 0$ if and only if $b_3(X) = 0$ (recall that $h^{3,0}(X) = 0$ by **Theorem 3.8**). This is the case if X is a deformation of $(K3)^{[n]}$ but not if it is a deformation of a generalized Kummer. Similarly we expect that $H^2(\Theta_X) \neq 0$ if X is our 6-dimensional example of **Theorem 2.30**.

Proof of **Theorem 3.10** *according to Fujiki [8].* We must prove tha the obstruction map $\Phi: B \to H^2(\Theta_X)$ vanishes. By (3.3.1) we have an isomorphism $H^1(\Theta_X) \cong H^1(\Omega^1_X) \cong H^{1,1}(X)$. Thus we may view $H^{1,1}_{\mathbb{R}}(X)$ as a subspace of $H^1(\Theta_X)$ and $H^1(\Theta_X)$ as the complexification of $H^{1,1}_{\mathbb{R}}(X)$: since Φ is analytic it will suffice to show that the restriction of Φ to $H^{1,1}_{\mathbb{R}}(X)$ vanishes. Let $\mathcal{K}_X \subset H^{1,1}_{\mathbb{R}}(X)$ be the Kähler cone. If $\omega \in \mathcal{K}_X$ then there is a 1-parameter deformation of X whose associated class is equal to ω: in fact this is trivial if $\omega = 0$ and if $\omega \neq 0$ such a family is provided by the twistor family (of course this needs to be proved: follow the variation of a holomorphic symplectic form on the fibers of the twistor family). It follows that the restriction of Φ to $\mathcal{K}_X$ vanishes; since $\mathcal{K}_X$ is open in $H^{1,1}_{\mathbb{R}}(X)$ we get that Φ vanishes on $H^{1,1}_{\mathbb{R}}(X)$. $\square$

Corollary 3.12. *The deformation space of a HK manifold X has dimension equal to $(b_2(X)-2)$.*

Proof. By **Theorem 3.10** the deformation space of X has dimension $h^1(\Theta_X)$ and the latter equals $h^1(\Omega_X)$ by (3.3.1). Now $h^1(\Omega_X) = h^{1,1}(X)$ and by Hodge Theory $b_2(X) = 2h^{2,0}(X)+h^{1,1}(X)$, thus the corollary follows from $h^{2,0}(X) = 1$. □

Remark 3.13. **Corollary 3.12** shows that if $n \geq 2$ then the generic deformation of $K(3)^{[n]}$ is not isomorphic to $K(3)^{[n]}$. In fact **Corollary 3.12** gives that a $K3$ surface has 20 moduli (the second Betti number of a $K3$ surface equals 22 by Noether's formula) while $K(3)^{[n]}$ has 21 moduli by **Proposition 2.2** and **Corollary 3.12**. Similar considerations apply to the other examples of higher-dimensional (meaning of dimension greater than 2) HK varieties described in **Section 2**: they are all obtained starting from a (projective) $K3$ or an abelian surface but the generic deformation cannot be obtained by deforming the surface (there is a notion of stability for Kähler surfaces....).

4 The local period map and the B-B quadratic form

4.1 The local period map

Let $\pi: \mathcal{X} \to B$ be a holomorphic submersive map of analytic spaces such that each fiber $X_b := \pi^{-1}(b)$ is a HK manifold. We assume that B is connected and hence all the X_b are deformation equivalent. In particular there exists a finitely generated torsion-free abelian group Λ such that $H^2(X_b;\mathbb{Z})$ is isomorphic to Λ for every $b \in B$. Suppose that the local system $R^2\pi_*\mathbb{Z}$ is trivial, this is the case if B is simply connected. Choose a trivialization of $F: R^2\pi_*\mathbb{Z} \xrightarrow{\sim} B \times \Lambda$; it defines an isomorphism $f_b: H^2(X_b;\mathbb{Z}) \xrightarrow{\sim} \Lambda$ for each $b \in B$. Let $\Lambda_{\mathbb{C}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$; abusing notation we denote by f_b also the map $H^2(X_b;\mathbb{C}) \xrightarrow{\sim} \Lambda_{\mathbb{C}}$ obtained by extension of scalars. The period map is defined by

$$\begin{array}{ccc} B & \xrightarrow{\mathcal{P}_\pi} & \mathbb{P}(\Lambda_{\mathbb{C}}) \\ b & \mapsto & f_b(H^{2,0}(X_b)). \end{array} \tag{4.1.1}$$

(Of course $\mathcal{P}_\pi$ depends on the trivialization chosen, our notation is somewhat imprecise.) Fundamental results of Griffiths [51] (valid for arbitrary families of Kähler manifolds) asserts that the period map is holomorphic and computes its differential as follows. Let $0 \in B$. The differential of $\mathcal{P}_\pi$ at 0 has codomain

$$\mathrm{Hom}(\mathcal{P}_\pi(0),\Lambda_{\mathbb{C}}/\mathcal{P}_\pi(0)) \cong \mathrm{Hom}(H^{2,0}(X_0),H^2(X_0)/H^{2,0}(X_0)) = \mathrm{Hom}(H^{2,0}(X_0),H^{1,1}(X_0)) \oplus \mathrm{Hom}(H^{2,0}(X_0),H^{0,2}(X_0)). \tag{4.1.2}$$

Griffiths' first result is that the image of the differential lies in $\mathrm{Hom}(H^{2,0}(X_0), H^{1,1}(X_0))$. The second result expresses the differential in terms of the Kodaira-Spencer map

$$\Theta_0 B \xrightarrow{\kappa} H^1(X_0;\Theta_{X_0}) \tag{4.1.3}$$

associated to the family $\mathcal{X}$. Let σ be a holomorphic symplectic form on X_0 and L_σ be Isomorphism (3.3.1): then

$$\langle d\mathcal{P}_\pi(v), \sigma\rangle = H^1(L_\sigma)(\kappa(v)). \tag{4.1.4}$$

Theorem 4.1 (Infinitesimal Torelli). *Let X be a HK manifold and $\pi: \mathcal{X} \to B$ be a representative of $\mathrm{Def}(X)$ with $0 \in B$ the point such that $X_0 \cong X$ - thus B is smooth by* **Theorem 3.10**. *Suppose in addition that $R^2\pi_*\mathbb{Z}$ is trivial and let $F: R^2\pi_*\mathbb{Z} \xrightarrow{\sim} \Lambda$ be a trivialization of $R^2\pi_*\mathbb{Z}$. Then $\mathcal{P}_\pi: B \to \mathbb{P}(\Lambda_{\mathbb{C}})$ is an isomorphism of a neighborhood of 0 onto a smooth analytic hypersurface in a neighborhood of $\mathcal{P}_\pi(0)$.*

Proof. By **Corollary 3.12** we get that $\dim_0 B = b_2(X) - 2 = \dim \mathbb{P}(\Lambda_{\mathbb{C}}) - 1$. Thus it suffices to prove that the differential $d\mathcal{P}_\pi(0)$ is injective. That follows immediately from (4.1.4). □

Remark 4.2. **Theorem 4.1** was proved first for $K3$ surfaces by Andreotti and Weil. It suggests that one might reconstruct the isomorphism class of a HK manifold from the Hodge structure on its second cohomology i.e. that a Global Torelli might hold. For $K3$ surfaces such a statement was proved by Pjateckii-Shapiro and Safarevic in 1971. The precise statement involves the intersection form on H^2 of a $K3$ surface. In the next subsection we will show that there is a natural quadratic form on the H^2 of a HK manifold of arbitrary dimension - the Beauville-Bogomolov (BB) form. In order to give a reasonable formulation of Global Torelli for arbitrary HK's one needs to take into account the BB quadratic form.

4.2 The Bogomolov-Beauville quadratic form

Let X be a HK-manifold of dimension $2n$. Beauville [2] and Fujiki [8] proved that there exist an integral indivisible quadratic form

$$q_X : H^2(X) \to \mathbb{C} \tag{4.2.1}$$

(cohomology is with complex coefficients) and $c_X \in \mathbb{Q}_+$ such that

$$\int_X \alpha^{2n} = c_X \frac{(2n)!}{n!2^n} q_X(\alpha)^n, \qquad \alpha \in H^2(X). \tag{4.2.2}$$

The above equation determines c_X and q_X with no ambiguity unless n is even. If n is even then q_X is determined up to ± 1: one singles out one of the two choices by imposing the inequality

$$q_X(\sigma + \overline{\sigma}) > 0, \qquad 0 \neq \sigma \in H^{2,0}(X). \tag{4.2.3}$$

The *Beauville-Bogomolov* form and the *Fujiki constant* of X are q_X and c_X respectively. We notice that the equation in (4.2.2) is equivalent (by polarization) to

$$\int_X \alpha_1 \wedge \ldots \wedge \alpha_{2n} = c_X \sum_{\sigma \in \mathcal{R}_{2n}} (\alpha_{\sigma(1)}, \alpha_{\sigma(2)})_X \cdot (\alpha_{\sigma(3)}, \alpha_{\sigma(4)})_X \cdot \ldots \cdot (\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)})_X \tag{4.2.4}$$

where $(\cdot,\cdot)_X$ is the symmetric bilinear form associated to q_X and $\mathcal{R}_{2n}$ is a set of representatives for the left cosets of the subgroup $\mathcal{G}_{2n} < \mathcal{S}_{2n}$ of permutations of $\{1, \ldots, 2n\}$ generated by transpositions $(2i-1, 2i)$ and by products of transpositions $(2i-1, 2j-1)(2i, 2j)$ - in other words in the right-hand side of (4.2.4) we avoid repeating addends which are equal. In defining c_X we have introduced a normalization which is not standard in order to avoid a combinatorial factor in Equation (4.2.4).

Proof of existence of q_X and c_X. Let $\pi: \mathcal{X} \to B$ be a deformation of X representing $Def(X)$ with $0 \in B$ and $X_0 \xrightarrow{\sim} X$. By **Theorem 3.10** we know that B is smooth at 0. We may assume that B is contractible and hence there exists a trivialization $F: R^2\pi_*\mathbb{Z} \xrightarrow{\sim} B \times \Lambda$ where Λ is a finitely generated torsion-free abelian group. Let $\mathcal{P}_\pi$ be the period map (4.1.1). By Infintesimal Torelli, see **Theorem 4.1**, $\operatorname{Im}\mathcal{P}_\pi$ is an analytic hypersurface in an open (classical topology) neighborhood of $\mathcal{P}_\pi(0)$ and hence its Zariski closure $V = \overline{Im\pi}$ is either all of $\mathbb{P}(H^2(X))$ or a hypersurface. One shows that the latter holds by considering the (non-zero) degree-$2n$ homogeneous polynomial

$$\begin{array}{ccc} H^2(X) & \xrightarrow{G} & \mathbb{C} \\ \alpha & \mapsto & \int_X \alpha^{2n} \end{array} \tag{4.2.5}$$

In fact if $\sigma_t \in H^{2,0}(X_t)$ then

$$\int_{X_t} \sigma_t^{2n} = 0 \tag{4.2.6}$$

by type consideration. It follows by Gauss-Manin parallel transport that G vanishes on V. Thus $I(V) = (F)$ where F is an irreducible homogeneous polynomial. By considering the derivative of the period map (4.1.1) one checks easily that V is not a hyperplane and hence $\deg F \geq 2$. On the other hand type consideration gives something stronger than (4.2.6), namely

$$\int_{X_t} \sigma_t^{n+1} \wedge \alpha_1 \cdots \wedge \alpha_{n-1} = 0 \qquad \alpha_1, \ldots, \alpha_{n-1} \in H^2(X_t). \tag{4.2.7}$$

It follows that all the derivatives of G up to order $(n-1)$ included vanish on V. Since $\deg G = 2n$ and $\deg F \geq 2$ it follows that $G = c \cdot F^n$ and $\deg F = 2$. By integrality of G there exists $\lambda \in \mathbb{C}^*$ such that $c_X := \lambda c$ is rational positive, $q_X := \lambda \cdot F$ is integral indivisible and (4.2.2) is satisfied. □

Remark 4.3. Let X be a HK manifold of dimension $2n$ and $\omega \in H^{1,1}_{\mathbb{R}}(X)$ be a Kähler class.

(1) Equation (4.2.2) gives that with respect to $(,)_X$ we have

$$H^{p,q}(X) \perp H^{p',q'}(X) \text{ unless } (p',q') = (2-p, 2-q). \tag{4.2.8}$$

(2) $q_X(\omega) > 0$. In fact let σ be generator of $H^{2,0}(X)$; by Equation (4.2.4) and Item (1) above we have

$$0 < \int_X \sigma^{n-1} \wedge \overline{\sigma}^{n-1} \wedge \omega^2 = c_X (n-1)! (\sigma, \overline{\sigma})_X q_X(\omega). \tag{4.2.9}$$

Since $c_X > 0$ and $(\sigma, \overline{\sigma})_X > 0$ we get that $q_X(\omega) > 0$ as claimed.

(3) The index of q_X is $(3, b_2(X) - 3)$ (i.e. that is the index of its restriction to $H^2(X;\mathbb{R})$). In fact applying Equation (4.2.4) to $\alpha_1 = \ldots = \alpha_{2n-1} = \omega$ and arbitrary α_{2n} we get that $\omega^\perp$ is equal to the primitive cohomology $H^2_{pr}(X)$ (primitive with respect to ω). On the other hand Equation (4.2.4) with $\alpha_1 = \ldots = \alpha_{2n-2} = \omega$ and $\alpha_{2n-1}, \alpha_{2n} \in \omega^\perp$ gives that a positive multiple of $q_X|_{\omega^\perp}$ is equal to the standard quadratic form on $H^2_{pr}(X)$. By the Hodge index Theorem it follows that the restriction of q_X to $\omega^\perp \cap H^2(X;\mathbb{R})$ has index $(2, b_2(X) - 3)$. Since $q_X(\omega) > 0$ it follows that q_X has index $(3, b_2(X) - 3)$.

(4) Let D be an effective divisor on X; then $(\omega, D)_X > 0$. In fact the inequality follows from the inequality $\int_D \omega^{2n-1} > 0$ together with (4.2.4) and Item (2) above.

(5) Let $f \colon X \dashrightarrow Y$ be a birational map where Y is a HK manifold. Since X and Y have trivial canonical bundle f defines an isomorphism $U \xrightarrow{\sim} V$ where $U \subset X$ and $V \subset Y$ are open sets with complements of codimension at least 2. It follows that f induces an isomorphism $f^* \colon H^2(Y;\mathbb{Z}) \xrightarrow{\sim} H^2(X;\mathbb{Z})$; f^* is an isometry of lattices, see Lemma 2.6 of [14].

Of course if X is a $K3$ then q_X is the intersection form of X (and $c_X = 1$). In general q_X gives $H^2(X;\mathbb{Z})$ a structure of lattice just as in the well-known case of $K3$ surfaces. Suppose that X and Y are deformation equivalent HK-manifolds: it follows from (4.2.2) that $c_X = c_Y$ and the lattices $H^2(X;\mathbb{Z}), H^2(Y;\mathbb{Z})$ are isometric (see the comment following (4.2.2) if n is even). The Fujiki constant and BB quadratic form of the known HK manifolds of dimension greater than 2 are given in Table (1). A word about notation: H is the hyperbolic lattice i.e. $H \cong \mathbb{Z}^2$ with a basis e, f such that $0 = (e,e) = (f,f)$ and $(e,f) = 1$, $E_8(-1)$ is the unique negative definite even unimodular lattice of rank 8 (a root system of type E_8 gives a basis of $E_8(-1)$, provided we change sign to every product of roots), $A_2(-1)$ is given by the root system A_2 with signs changed, and for $d \in \mathbb{Z}$ we let (d) be the rank-1 lattice with generator of square d. For $(K3)^{[n]}$ and $K^{[n]}(T)$ the result is folklore, for the 6 and 10-dimensional examples $\mathcal{M}^0_v$ and $\mathcal{M}_v$ of **Theorem 2.30** the computations were done by Rapagnetta [40].

Remark 4.4. Let X be a HK manifold of dimension $2n$. Existence of the BB quadratic form and Fujiki constant is a rather strong topological condition. Salamon [42] proved the following relation between Betti numbers of X:

$$n b_{2n}(X) = 2 \sum_{i=1}^{2n} (-1)^i (3i^2 - n) b_{2n-i}(X). \tag{4.2.10}$$

Is it possible to obtain other topological constraints on HK manifolds ? In particular: can we bound rank, discriminant of the BB quadratic form and Fujiki constant in a given dimension ? That would give that the number of deformation classes of a given dimension is finite, see [16] for related work. Salamon's relation (4.2.10) gives (Beauville (unpublished) and Guan [13]) that if X is a HK 4-fold then $b_2(X) \leq 23$ (notice that $b_2(K3^{[2]}) = 23$) and that if equality holds then cup-product defines an isomorphism $\mathrm{S}^2 H^2(X;\mathbb{Q}) \xrightarrow{\sim} H^4(X;\mathbb{Q})$. Guan [13] has obtained other restrictions on $b_2(X)$ for a HK four-fold X: for example either $b_2(X) \leq 8$ or $b_2(X) = 23$.

Table 1: Fujiki constant and BB form of the known examples.

X	$\dim(X)$	$b_2(X)$	c_X	$H^2(X,\mathbb{Z})$
$(K3)^{[n]}$	$2n$	23	1	$H^3 \oplus E_8(-1)^2 \oplus (-2(n-1))$
$K^{[n]}(T)$	$2n$	7	$(n+1)$	$H^3 \oplus (-2(n+1))$
$\widetilde{\mathcal{M}}_S(2\mathbf{v}_0)$	10	24	1	$H^3 \oplus E_8(-1)^2 \oplus A_2(-1)$
$\widetilde{\mathcal{M}}^0_T(2\mathbf{v}_0)$	6	8	4	$H^3 \oplus (-2)^2$

The proof of existence of q_X and c_X may be adapted to prove the following useful generalization of (4.2.2).

Proposition 4.5. *Let X be a HK manifold of dimension $2n$. Let $\mathcal{X} \to T$ be a representative of the deformation space of X. Suppose that $0 \neq \gamma \in H^{p,p}(X)$ is a class which remains of type (p,p) under Gauss-Manin parallel transport (e. g. the Chern class $c_p(X)$). Then p is even and moreover there exists $c_\gamma \in \mathbb{R}$ such that*

$$\int_X \gamma \wedge \alpha^{2n-p} = c_\gamma q_X(\alpha)^{n-p/2}. \tag{4.2.11}$$

4.3 Marked pairs

Let Λ be a lattice i.e. a finitely generated torsion-free abelian group equipped with a non-degenerate symmetric bilinear form $(,)_\Lambda$. We let q_Λ be the associated quadratic form. For a commutative ring R let $\Lambda_R := \Lambda \otimes_{\mathbb{Z}} R$. Then $(,)_\Lambda$ extends to an R-valued bilinear symmetric form on Λ_R; abusing notation we will denote it by $(\alpha,\beta)_\Lambda$ or simply (α,β) if there is no risk of confusion. We let q_Λ be the associated quadratic form on Λ_R. Now assume that $\Lambda_{\mathbb{R}}$ has signature $(3, \operatorname{rk}\Lambda - 3)$. We let

$$\Omega_\Lambda := \{[\alpha] \in \mathbb{P}(\Lambda_{\mathbb{C}}) | \quad q_\Lambda(\alpha) = 0, \quad q_\Lambda(\alpha + \bar{\alpha}) > 0\}. \tag{4.3.1}$$

Thus Ω_Λ is an open subset (in the classical topology) of a smooth quadric of dimension $(\operatorname{rk}\Lambda - 2)$ and hence it is naturally a complex manifold: it is the *period domain* associated to Λ. Up to isomorphism Ω_L depends only on the rank of L - on the other hand the orthogonal group $O(L)$ acts naturally on Ω_L and the different realizations of Ω_L correspond to different group actions. Now let X_0 be a HK manifold and Λ a lattice isometric to $H^2(X;\mathbb{Z})$ equipped with the BB quadratic form. By **Remark 4.3** the signature of the BB quadratic form on $H^2(X;\mathbb{R})$ is $(3, b_2(X_0) - 3)$ the period domain Ω_Λ is defined.

Definition 4.6. Let X be a HK manifold deformation equivalent to X_0. A *marking* of X consists of an isometry $f: H^2(X;\mathbb{Z}) \xrightarrow{\sim} \Lambda$. A *marked pair* is a couple (X,f) where X is a HK manifold (deformation equivalent to X_0) and f is a marking of X. An isomorphism between marked pairs (X_1,f_1) and (X_2,f_2) is an isomorphism $\varphi: X_1 \xrightarrow{\sim} X_2$ such that $f_1 \circ H^2(\varphi) = f_2$.

Let (X,f) be a marked pair. We denote by f the linear map $H^2(X;\mathbb{C}) \xrightarrow{\sim} \Lambda_{\mathbb{C}}$ obtained by extension of scalars. Then $f(H^{2,0}(X)) \in \mathbb{P}(\Lambda_{\mathbb{C}})$. By (4.2.2) we have that $H^{2,0}(X)$ is an isotropic line for the BB quadratic form and moreover $(\sigma,\overline{\sigma})_X > 0$ for $0 \neq \sigma \in H^{2,0}(X)$ by (4.2.3). Since f is an isometry it follows that

$$P(X,f) := f(H^{2,0}(X)) \in \Omega_\Lambda. \tag{4.3.2}$$

The point $P(X,f)$ is the *period point* associated to the marked pair (X,f). Now let $\pi: \mathcal{X} \to B$ be a holomorphic submersive map of analytic spaces such that each fiber $X_b := \pi^{-1}(b)$ is a HK manifold deformation equivalent to X_0. Suppose that the local system $R^2\pi_*\mathbb{Z}$ is trivial. In order to define the period map (4.1.1) we choose a trivialization $F: R^2\pi_*\mathbb{Z} \xrightarrow{\sim} B \times \Lambda$ defining an isometry $f_b: H^2(X_b;\mathbb{Z}) \xrightarrow{\sim} \Lambda$ for each $b \in B$. Then $\mathcal{P}_\pi(B) \subset \Omega_\Lambda$. **Theorem 4.1** gives the following result.

Theorem 4.7 (Infinitesimal Torelli + Local surjectivity)**.** *Keep notation as above and suppose that* $\pi: \mathcal{X} \to B$ *is a representative of* $\mathrm{Def}(X)$ *with* $0 \in B$ *the point such that* $X_0 \cong X$ *- thus* B *is smooth by* **Theorem 3.10**. *Then* $\mathcal{P}_\pi: B \to \Omega_\Lambda$ *is an isomorphism of a neighborhood of* 0 *onto an open neighborhood of* $\mathcal{P}_\pi(0)$ *in* Ω_Λ.

Corollary 4.8. *A HK manifold is deformation equivalent to a HK variety.*

Proof. Let X be a HK manifold and $\pi: \mathcal{X} \to B$ be a representative of $\mathrm{Def}(X)$ with $0 \in B$ the point such that $X_0 \cong X$. There exist points $[\sigma] \in \Omega_\Lambda$ arbitrarily close to $\mathcal{P}_\pi(0)$ such that the span of $\{\sigma, \overline{\sigma}\}$ in $\Lambda_\mathbb{C}$ is defined over $\mathbb{Q}$ i.e. it is spanned by its intersection with $\Lambda_\mathbb{Q}$. By **Theorem 4.7** there exists $b \in B$ such that $\mathcal{P}_\pi(b)$ is such a $[\sigma]$. Then $H^{1,1}(X_b;\mathbb{R})$ is also defined over $\mathbb{Q}$ and hence $H^{1,1}(X_b;\mathbb{Q})$ is dense in $H^{1,1}(X_b;\mathbb{R})$; since the Kähler cone is open in $H^{1,1}(X_b;\mathbb{R})$ it follows that there exists a Káhler integral class and hence X_b is projective by Kodaira. □

Let X_0 be a HK manifold. One constructs a moduli space $\mathfrak{M}$ of marked pairs (X, f) where X is a deformation of X_0. A point of $\mathfrak{M}$ is an isomorphism class of a marked pair (X, f) as above, a structure of analytic space is provided by deformation theory. Since the deformation space of a HK manifold is unobstructed $\mathfrak{M}$ is a complex manifold: a neighborhood containing (X, f) is given by a simply-connected representative of $\mathrm{Def}(X)$, say B, with family $\pi: \mathcal{X} \to B$ and trivialization $F: R^2\pi_*\mathbb{Z} \xrightarrow{\sim} B \times \Lambda$ extending f. The complex manifold $\mathfrak{M}$ is *not* Hausdorff: this is true already for $K3$ surfaces. In fact let S be a $K3$ surface containing a smooth rational curve with Poincarè dual δ and let $r: H^2(S;\mathbb{Z}) \to H^2(S;\mathbb{Z})$ be the reflection

$$r(\alpha) := \alpha + (\alpha, \delta)\delta.$$

Given any marking $f: H^2(S;\mathbb{Z}) \to H^3 \oplus E_8(-1)^2$ the points $[(S, f)]$ and $[(S, r \circ f)]$ cannot be separated. **Theorem 4.7** may be restated as saying that the period map defines a local isomorphism

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\mathfrak{p}} & \Omega_\Lambda \\ (X, f) & \mapsto & P(X, f) \end{array} \tag{4.3.3}$$

4.4 Matsushita's Theorem

Let X be a HK manifold. A subvariety $Y \subset X$ is *lagrangian* if $2\dim Y = \dim X$ and the restriction to the smooth locus of Y of a holomorphic symplectic form on X is zero.

Theorem 4.9 (Matsushita [30, 31])**.** *Suppose that* X *is a HK manifold and that* $f: X \to B$ *is a surjective map with connected fibers to a Kähler manifold* B *such that* $0 < \dim B < \dim X$. *Then* $2\dim B = \dim X$ *and the generic fiber of* f *is a lagrangian torus in* X. *Moreover* $b_2(B) = 1$.

Proof. Suppose that $0 \neq \alpha \in H^0(\Omega_B^2)$. Then $f^*\alpha$ is a non-zero degenerate holomorphic 2-forms on X, that contradicts the definition of HK manifold. Thus $H^{2,0}(B) = 0$ and hence B is a smooth projective variety. Let $\dim X = 2n$ and $\dim B = m$. If $\alpha \in H^2(B)$ then $\int_X f^*\alpha^{2n} = 0$ because $m < 2n$ and hence $q_X(f^*\alpha) = 0$ by (4.2.2). Let H be an ample divisor on B and $\alpha := c_1(\mathcal{O}_B(H))$. Let ω be a Kähler form on X: then

$$\int_X (f^*\alpha)^m \wedge \omega^{2n-m} > 0. \tag{4.4.1}$$

Suppose that $m > n$: since $q_X(\alpha) = 0$ Equation (4.2.2) gives that the left-hand side of (4.4.1) vanishes, that contradicts (4.4.1). This proves that $m \leq n$. Next notice that $f^*\alpha \neq 0$ because H is ample (and $m > 0$). Since the BB form is non-degenerate there exists $\beta \in H^2(X)$ such that $0 \neq (f^*\alpha, \beta)_X$. Since $q_X(f^*\alpha) = 0$ Equation (4.2.2) gives that

$$\int_X (f^*\alpha)^n \wedge \omega^n = n!(f^*\alpha, \beta)_X^n \neq 0. \tag{4.4.2}$$

It follows that $m \geq n$. Thus $m = n$. Let $b \in B$ be generic and $X_b := f^{-1}(b)$. Then $2\dim X_b = \dim X$ because $2\dim B = \dim X$. Let's prove that X_b is lagrangian. Letσ be a holomorphic symplectic form on X: it suffices to prove that

$$\int_{X_b} \sigma \wedge \overline{\sigma} \wedge \omega^{n-2} = 0 \tag{4.4.3}$$

where ω is a Kähler form on X. Let H be an ample divisor on B and $\alpha := c_1(\mathcal{O}_B(H))$: then

$$\int_X (f^*\alpha)^n \wedge \sigma \wedge \overline{\sigma} \wedge \omega^{n-2} = \deg(\underbrace{H \cdot \ldots \cdot H}_{n}) \int_{X_b} \sigma \wedge \overline{\sigma} \wedge \omega^{n-2}. \tag{4.4.4}$$

On the other hand Equation (4.2.2) gives that the left-hand side of (4.4.4) vanishes because $q_X(f^*\alpha) = 0$ and $0 = (f^*\alpha, \sigma)_X = (f^*\alpha, \overline{\sigma})_X$ (see **Remark 4.3**). This proves (4.4.3), thus X_b is lagrangian. Since X_b is lagrangian the symplectic form defines an isomorphism between the tangent bundle of X_b and the conormal of X_b in X: the latter is trivial because X_b is a regular fiber of f. Hence the tangent bundle of X_b is trivial and therefore X_b is a torus. Lastly we notice that since B is a smooth projective variety the pull-back $H^2_{\mathbb{R}}(f): H^2(B;\mathbb{R}) \to H^2(X;\mathbb{R})$ is injective. We have proved that the image is isotropic for the BB quadratic form. The image of $H^2_{\mathbb{R}}(f)$ lies in $H^{1,1}_{\mathbb{R}}(X)$. By **Remark 4.3** the restriction of q_X to $H^{1,1}_{\mathbb{R}}(X)$ has signature $(1, b_2(X) - 3)$ and hence the maximum dimension of an isotropic subspace is 1: it follows that $b_2(B) = 1$. □

Example 4.10. Let S be a $K3$ surface and $\mathbf{v} \in \widetilde{H}(S)$ be a Mukai vector as in (2.3.24) with $r = 0$, i.e. $\mathbf{v} = \ell + \eta$. Assume that $\mathcal{M}_S(\mathbf{v})$ is not empty and that $\mathcal{M}_S(\mathbf{v}) = \mathcal{M}_S(\mathbf{v})^{st}$ so that $\mathcal{M}_S(\mathbf{v})$ is a HK variety. Let's suppose also that $\ell = c_1(\mathcal{O}_S(D))$ where D is an ample divisor on S. Then $\mathcal{M}_S(\mathbf{v})$ has a lagrangian fibration $f: \mathcal{M}_S(\mathbf{v}) \to |D|$ defined by mapping $[E] \in \mathcal{M}_S(\mathbf{v})$ to the schematic support of E. Let us check that $2\dim|D| = \dim \mathcal{M}_S(\mathbf{v})$. Let g be the arithmetic genus of curves in $|D|$. Then $\dim|D| = g$. On the other hand if $C \in |D|$ is smooth the fiber of f over C is identified with $\mathrm{Pic}^d(C)$ where d is determined by the Mukai vector $\mathbf{v}$. Thus we see that $\dim f^{-1}(C) = g = \dim|D|$.

Remark 4.11. In the examples above the base of the lagrangian fibration is a projective space. Hwang [19] has proved that if X is projective then B is a projective space. It is conjectured (e.g. [43]) that if X is a HK manifold and $0 \neq \gamma \in H^{1,1}_{\mathbb{Z}}(X)$ has square 0 and is nef then there exists a lagrangian fibration $X \to B$ such that $\gamma \in f^*H^2(B)$. See [29, 44, 28] for work on this conjecture.

5 The Kähler cone

We recall that the Kähler cone of a $K3$ surface S is described as follows. Let $\omega \in H^{1,1}_{\mathbb{R}}(S)$ be one Kähler class and $\mathcal{N}_S$ be the set of nodal classes

$$\mathcal{N}_S := \{\delta \in H^{1,1}_{\mathbb{Z}}(X) \mid q_S(\delta) = -2, \quad (\delta, \omega)_S > 0\} \tag{5.0.1}$$

The Kähler cone $\mathcal{K}_S$ is given by

$$\mathcal{K}_S := \{\alpha \in H^{1,1}_{\mathbb{R}}(S) \mid q_S(\alpha) > 0, \quad (\alpha, \delta)_S > 0 \quad \forall \delta \in \mathcal{N}_S\}. \tag{5.0.2}$$

In other words we have a Hodge-theoretic description of $\mathcal{K}_S$. In fact Hodge isometries of $H^2(X)$ act transitively on the set of connected components of the complement of the union of walls $\delta^{\perp} \cap H^{1,1}_{\mathbb{R}}(X)$ where $\delta \in H^{1,1}_{\mathbb{Z}}(X)$ has square -2, hence the choice of ω is needed only to pin-down which is the open chamber containing Kähler classes. We do not have (for the moment being) a purely Hodge-theoretic characterization of the Kähler cone $\mathcal{K}_X$ of a HK manifold. Below is a description of $\mathcal{K}_X$ which involves geometry (in general, it becomes purely Hodge-theoretical if $H^{1,1}_{\mathbb{Z}}(X) = 0$).

Theorem 5.1. [Huybrechts [15]+Boucksom [5]] *Let X be a HK manifold. A class $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ is Kähler if and only if it belongs to the positive cone $\mathcal{C}_X$ and moreover $\int_C \alpha > 0$ for every rational curve C*[7].

Now let X_0 be a HK manifold and $\mathfrak{M}$ the moduli space of marked pairs (X, f) where X is a deformation of X_0.

Theorem 5.2 (Theorem 4.3 of [14])**.** *Keep notation as above and assume that $[(X, f)]$, $[(Y, g)]$ are non-separated points of $\mathfrak{M}$. Then X is birational to Y.*

[7] A curve is rational if it is irreducible and its normalization is rational

The above result was first proved for $K3$ surfaces by Burns-Rapaport [6], the proof is a generalization of their proof. The surprising result is the following converse.

Theorem 5.3. *Let X and Y be bimeromorphic HK manifolds. Then X and Y are deformation equivalent.*

6 Global Torelli

We will state Verbitsky's Global Torelli for HK manifolds [50] and we will give a very brief sketch of the proof following the papers by Huybrechts [17] and Markman [27]. Let X_0 be a HK manifold with $H^2(X;\mathbb{Z})$ isometric to a lattice Λ. Let $\mathfrak{M}$ be the moduli space of marked pairs (X,f) where X is a deformation of X_0. By **Theorem 4.7** the period map $\mathfrak{p}\colon\mathfrak{M}\to\Omega_\Lambda$ is a local isomorphism. Below is a version of Verbitsky's Global Torelli (we have included Huybrechts' global surjectivity of the period map [14]).

Theorem 6.1 (Verbitsky [50], Huybrechts [17])**.** *Keep notation as above. Let $\mathfrak{M}^0$ be a connected component of $\mathfrak{M}$. The restriction of $\mathfrak{p}$ to $\mathfrak{M}^0$ is surjective. Suppose that $(X_1,f_1),(X_2,f_2)\in\mathfrak{M}^0$ and that $\mathfrak{p}(X_1,f_1)=\mathfrak{p}(X_2,f_2)$; then X_1 is birational to X_2.*

Remark 6.2. A birational map $\phi\colon X\dashrightarrow X'$ between HK manifolds induces an integral Hodge isometry $f^*\colon H^2(X';\mathbb{Z})\xrightarrow{\sim}H^2(X;\mathbb{Z})$.

Remark 6.3. There are Hodge-theoretic conditions on a HK X which ensure that any birational[8] map $X\dashrightarrow X'$ to another HK is regular (and hence an isomorphism). One such condition (valid for any HK) is that $H^{1,1}_{\mathbb{Z}}(X)$ is either 0 or spanned by a class of strictly positive square. Since the condition is Hodge-theoretic we get an explicit (dense) subset of Ω_Λ with the property that the fibers of $\mathfrak{p}|_{\mathfrak{M}^0}$ over the points of that subset are singletons.

We recall that for $K3$ surfaces we have the following Torelli Theorem.

Theorem 6.4 (Shafarevich and Pjateckii-Shapiro, Burns and Rapaport)**.** *Let X_1,X_2 be $K3$ surfaces. Then X_1 is isomorphic to X_2 if and only if there exists an integral Hodge isometry $\varphi\colon H^2(X_1)\longrightarrow H^2(X_2)$.*

A HK manifold is of *type* $K3^{[n]}$ if it is a deformation of $K3^{[n]}$. Markman's results [25] on monodromy of HK's of type $K3^{[n]}$ give the following analogue of **Theorem 6.4**.

Theorem 6.5 (Verbitsky+Markman)**.** *Let $n=1+p^r$ where p is a prime and $r\geq 0$. Let X_1,X_2 be HK's of type $K3^{[n]}$. Then X_1 is birational to X_2 if and only if there exists an integral Hodge isometry $\varphi\colon H^2(X_1)\longrightarrow H^2(X_2)$.*

6.1 Hausdorffization of the moduli space of marked pairs

Let Z be a topological space. We say that $x,y\in Z$ are *inseparable* if given open sets $x\in U\subset Z$ and $y\in V\subset Z$ the intersection $U\cap V$ is not empty: in symbols $x\sim y$ and we name $\sim$ the *Hausdorff* relation on Z. Let $\Delta_Z\subset Z\times Z$ be the diagonal: we have

$$\overline{\Delta}_Z=\{(x,y)\mid x\sim y\}. \tag{6.1.1}$$

Clearly $\sim$ is reflexive and symmetric. The example below shows that $\sim$ is not necessarily transitive i.e. it need not be an equivalence relation.

Example 6.6 (Verbitsky [50]). Let $\mathcal{R}$ be the equivalence relation on $\mathbb{R}\coprod\mathbb{R}\coprod\mathbb{R}$ defined as follows. We denote a point of $\mathbb{R}\coprod\mathbb{R}\coprod\mathbb{R}$ as $a_i\in\mathbb{R}$ for $i=1,2,3$ meaning that it belongs to the i-th copy of $\mathbb{R}$. Then $\mathcal{R}$ is generated by the relations $a_1\mathcal{R}b_2$ if $a_1<0$ and $a_1=b_2$ and $b_2\mathcal{R}c_3$ if $b_2>0$ and $b_2=c_3$. Let $X:=(\mathbb{R}\coprod\mathbb{R}\coprod\mathbb{R})/\mathcal{R}$. The points $[0_1],[0_2],[0_3]\in X$ are distinct and $0_1\sim 0_2$, $0_2\sim 0_3$ but $0_1\not\sim 0_3$.

[8]We use the word birational as synonimous of "bimeromorphic".

Proposition 6.7. *Keep notation as above and suppose that the following hold:*

(1) The Hausdorff relation $\sim$ is an equivalence relation and hence the quotient topological space $\overline{Z} := Z/\sim$ exists.

(2) The Hausdorff relation is open i.e. the quotient map $\pi: Z \to \overline{Z}$ is open.

Then $\overline{Z}$ is Hausdorff.

Proof. It suffices to prove that the diagonal $\Delta_{\overline{Z}} \subset \overline{Z} \times \overline{Z}$ is closed. Let $\pi: Z \to \overline{Z}$ be the quotient map. Let

$$\begin{array}{ccc} Z \times Z & \xrightarrow{\varphi} & \overline{Z} \times \overline{Z} \\ (z_1, z_2) & \mapsto & (\pi(z_1), \pi(z_2)) \end{array}$$

The map φ is the set-theoretic quotient map for the equivalence relation $\mathcal{R}$ defined by declaring $(z_1, z_2)\mathcal{R}(z_1', z_2')$ if $z_1 \sim z_1'$ and $z_2 \sim z_2'$. Moreover φ is continuous. We claim that φ is the quotient map in the category of topological spaces i.e. that if $\mathfrak{U} \subset \overline{Z} \times \overline{Z}$ is such that $\varphi^{-1}\mathfrak{U}$ is open then $\mathfrak{U}$ is open: this is where the hypothesis that $\sim$ is open is needed. Since φ is the quotient map in the category of topological spaces $\Delta_{\overline{Z}} \subset \overline{Z} \times \overline{Z}$ is closed if $\varphi^{-1}(\Delta_{\overline{Z}} \subset \overline{Z} \times \overline{Z})$ is closed in $Z \times Z$: the latter is closed by (6.1.1). □

Suppose that the hypotheses of **Proposition 6.7** are satisfied: then $\pi: Z \to \overline{Z}$ has the following universal property. Let W be a Hausdorff topological space and $f: Z \to W$ a (continuous) map: there exists a unique continuous map $\overline{f}: \overline{Z} \to W$ such that $f = \overline{f} \circ \pi$. One shows that the Hausdorff relation $\sim$ on $\mathfrak{M}$ satisfies the hypothesis of **Proposition 6.7**: the key ingredients are Huybrechts' results on non-separated points of $\mathfrak{M}$. Thus we have the hausdorffization $\overline{\mathfrak{M}}$ and the period map $\mathfrak{p}: \mathfrak{M} \to \Omega_\Lambda$ decsends to a local isomorphism

$$\overline{\mathfrak{p}}: \overline{\mathfrak{M}} \longrightarrow \Omega_\Lambda. \tag{6.1.2}$$

6.2 The descended period map is a topological covering

In order to prove that $\overline{\mathfrak{p}}$ is a topological covering one applies a result in general topology proved by Verbitsky (we will follow Markman's proof). Let us start by recalling the following result.

Lemma 6.8. *Let $f: M \to N$ be a local homeomorphism of topological spaces and suppose that M is Hausdorff. Let X be a connected topological space and $x_0 \in X$. Suppose that $\sigma, \tau: X \to M$ are continuous maps such that $\sigma(x_0) = \tau(x_0)$ and $f \circ \sigma = f \circ \tau$. Then $\sigma = \tau$.*

In order to state the result we give two definitions Let M be a topological manifold. A *closed ball* in M is a closed $D \subset M$ contained in a coordinate chart (U, φ) (here $\varphi: U \xrightarrow{\sim} \mathbb{R}^n$ is a homeomorphism) such that $\varphi(D)$ is a closed ball $\overline{D}_R(a)$ of strictly positive radius R (and center a). We let $B = D_R(a)$ be the interior of D (an open ball) and we denote D by $\overline{B}$. If M is a smooth manifold a *smooth closed ball* in M is defined as above - of course (U, φ) must belong to C^∞-atlas.

Proposition 6.9 (Verbistky [50])**.** *Let $f: M \to N$ be a local homeomorphism of topological (smooth) manifolds and suppose that M is Hausdorff. Then f is a topological covering if and only if the following holds for each closed ball (respectively smooth closed ball) $\overline{B} \subset N$: if C is a connected component of $f^{-1}\overline{B}$ then $f(C) = \overline{B}$.*

Proof. It is clear that the condition is necessary, the point is to prove that it is sufficient. Since N is covered by open sets of an atlas we may assume that $N = \mathbb{R}^n$. We will prove that if M is connected then $f: M \to \mathbb{R}^n$ is a homeomorphism: the proposition follows by restricting f to the connected components of the domain. Let $m \in M$ and $a := f(m)$. Let $I \subset [0, +\infty)$ be the set of R such that there exists a continuous section $s_R: \overline{D}_R(a) \to M$ through m i.e. $s_R(a) = m$ and $f \circ s_R = \mathrm{Id}_{\overline{D}_R(a)}$. Clearly $0 \in I$ and I is an interval. An easy compactness argument shows that I is open (use **Lemma 6.8**) - here we do not use our hypothesis ($f(C) = \overline{B}$), one only needs that f is a local homeomorphism. Thus it suffices to prove that $\sup I = +\infty$. Suppose the contrary and let $R_0 := \sup I$. There is a section $t_0: D_{R_0}(a) \to M$ through m. Let $C_0 := \overline{\mathrm{Im}\, t_0} \cap f^{-1}\overline{D}_{R_0}$. We claim that $f|_{C_0}$ is injective and that C_0 is open in $f^{-1}\overline{D}_{R_0}$. In fact let $x, y \in C_0$ such that $f(x) = f(y)$. If $f(x) = f(y) \in D_{R_0}(a)$ then $x = t_0(f(x)) = t_0(f(y)) = y$ because M is Hausdorff. Next suppose that If $f(x) = f(y) \in \partial D_{R_0}(a)$. Let $x \in U \subset M$ and $y \in V \subset M$ be open connected neighborhoods such that $f|_U$ and $f|_V$ are homeomorphisms onto their images. Let $\{z_n\}$ be a sequence in $f(U) \cap f(V) \cap D_{R_0}(a)$ converging to $f(x) = f(y)$. Then $t_0(z_n) \in U \cap V$ and $t_0(z_n) \to x$, $t_0(z_n) \to y$, since M is Hausdorff $x = y$. Moreover the inverse of $f|_{U\cap V}$ gives a section which coincides with t_0 on $U \cap V \cap D_{R_0}(a)$ by **Lemma 6.8**: this proves that C_0 is open. Since C_0 is closed by construction (and non-empty) it is a connected component of $f^{-1}\overline{D}_{R_0}$. By hypothesis $f(C_0) = \overline{D}_{R_0}$, i.e. $f|_{C_0}$ is bijective. The argument given above shows that $(f|_{C_0})^{-1}$ is continuous and hence it is the desired section $s_{R_0}: \overline{D}_{R_0} \to M$. □

How does one prove that the hypotheses of **Proposition 6.9** are satisfied by $\mathfrak{p}$? The key element in the proof is the existence of the twistor families !

6.3 Conclusion

Proof of **Theorem 6.1** By **Subsection 6.2** the restriction of $\mathfrak{p}$ to $\mathfrak{M}^0$ is a topological covering: it follows that it is surjective. As is easily proved Ω_Λ is simply-connected: by **Subsection 6.2** it follows that the restriction of $\mathfrak{p}$ to $\mathfrak{M}^0$ is a homeomorphism. Suppose that $(X_1, f_1), (X_2, f_2) \in \mathfrak{M}^0$ and that $\mathfrak{p}(X_1, f_1) = \mathfrak{p}(X_2, f_2)$; then (X_1, f_1) cannot be separated from (X_2, f_2) and hence X_1 is birational to X_2 by **Theorem 5.2**.

Proof of **Theorem 6.5** Let X be a HK. Following Markman we let $\mathrm{Mon}(X) < \mathrm{O}(H^2(X;\mathbb{Z}))$ be the subgroup of isometries obtained by monodromy. Now let X be of type $K3^{[n]}$ for $n = 1 + p^r$ where p is a prime. By results of Markman [25] one has that $\mathrm{O}(H^2(X;\mathbb{Z}))$ is generated by $\mathrm{Mon}(X)$ and multiplication by -1. It follows that $\mathfrak{M}^0$ is connected and that proves **Theorem 6.5**.

References

[1] I. V. Artamkin, *On the deformation of sheaves*, Math. USSR-Izv. 32 (1989), pp. 663-668.

[2] A. Beauville, *Variétes Kähleriennes dont la premiére classe de Chern est nulle*, J. Differential geometry 18 (1983), pp. 755-782.

[3] A. Beauville, R. Donagi, *La variétés des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. 301 (1985), pp. 703-706.

[4] F. Bogomolov, *Hamiltonian Kähler manifolds*, Soviet Math. Dokl. 19, 1978, pp. 14621465.

[5] S. Boucksom, *Le cône kählérien d'une variété hyperkählérienne*, C. R. Acad. Sci. Paris Sér. I Math. 333, 2001, pp. 935-938.

[6] D. Burns - M. Rapoport, *On the Torelli problem for Kählerian* $K3$ *surfaces*, Ann. scient. Éc. Norm. Sup. 8, 1975, pp. 235-274.

[7] J. Cheah, *On the cohomology of Hilbert schemes of points*, J. Algebraic Geom. 5 (1996), pp. 479-511.

[8] A. Fujiki, *On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold*, Adv. Studies in Pure Math. 10, Algebraic Geometry, Sendai 1985, 1987, pp. 105-165.

[9] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. 286 (1990), pp. 193-207.

[10] L. Göttsche, D. Huybrechts, *Hodge numbers of moduli spaces of stable bundles on* $K3$ *surfaces*, Internat. J. Math. 7, 1996, pp. 359-372.

[11] L. Göttsche, W. Soergel *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann. 296 (1993), pp. 235-245.

[12] P. Griffiths, J. Harris *Principles of algebraic geometry*, Wiley-Interscience (1978).

[13] D. Guan, *On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four*, Math Research Letters 8, 2001, pp. 663-669.

[14] D. Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. 135, 1999, pp. 63-113. *Erratum*, Invent. Math. 152, 2003, pp. 209-212.

[15] D. Huybrechts, *The Kähler cone of a compact hyperkähler manifold*, Math. Ann. 326 (2003), pp. 499-513.

[16] D. Huybrechts, *Finiteness results for compact hyperkähler manifolds*, J. Reine Angew. Math. 558 (2003), pp. 15-22.

[17] D. Huybrechts, *A global Torelli theorem for hyperkähler manifolds (after Verbitsky)*, arXiv:1106.5573, Bourbaki talk June 2011.

[18] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of shaves*, Aspects of Mathematics E 31, Vieweg (1997).

[19] J.M. Hwang, *Base manifolds for fibrations of projective irreducible symplectic manifolds*, Invent. Math. 174 (2008), 625644.

[20] A. Iarrobino, *Punctual Hilbert schemes*, Bull. Amer. Math. Soc. 78 (1972), pp. 819-823.

[21] D. Kaledin - M. Lehn - Ch. Sorger, *Singular symplectic moduli spaces*, Invent. Math. 164, 2006, pp. 591-614.

[22] Y-H. Kiem, *On the existence of a symplectic desingularization of some moduli spaces of sheaves on a K3 surface*, Compos. Math. 141 (2005), pp. 902-906.

[23] S. Kobayashi, K. Nomizu, *Foundations of differential geometry. Vol. II*, Interscience Tracts in Pure and Applied Mathematics, No. 15 John Wiley & Sons, (1969).

[24] M. Lehn - Ch. Sorger, *La singularité de O'Grady*, J. Algebraic Geom. 15, 2006, pp. 753-770.

[25] E. Markman. *On the monodromy of moduli spaces of sheaves on K3 surfaces*, J. Algebraic Geom. 17 (2008), pp. 29 - 99.

[26] E. Markman. *Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a* $K3$ *surface*, Internat. J. Math. 21 (2010), pp. 169-223.

[27] E. Markman. *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, Complex and Differential Geometry Conference, Springer Proceedings in Mathematics Vol. 8 (2011), pp. 257-322.

[28] E. Markman. *Lagrangian fibrations of holomorphic-symplectic varieties of* $K3^{[n]}$*-type*, arXiv:1301.6584 [mat.AG].

[29] D. Markushevich. *Rational Lagrangian fibrations on punctual Hilbert schemes of K3 surfaces*, Manuscripta Math. 120 (2006), pp. 131-150.

[30] D. Matsushita, *On fibre space structures of a projective irreducible symplectic manifold*, Topology 38, 1999, pp. 79-83.

[31] D. Matsushita, *Addendum: "On fibre space structures of a projective irreducible symplectic manifold"*, Topology 40, 2001, pp. 431-432.

[32] S. Mukai, *On the moduli space of bundles on* $K3$ *surfaces, I*, Vector Bundles on Algebraic Varieties, TIFR, Bombay, O.U.P. (1987), pp. 341-413.

[33] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or* $K3$ *surface*, Invent. math 77 (1984), pp. 101-116.

[34] K. G. O'Grady, *Moduli of vector-bundles on surfaces*, Proc. Sympos. Pure Math., 62, AMS (1997), pp. 101-126.

[35] K. O'Grady, *The weight-two Hodge structure of moduli spaces of sheaves on a $K3$ surface*, J. of Alg. Geom. 6, 1997, pp. 599-644.

[36] K. G. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. fur die reine und angew. Math. 512, 1999, pp. 49-117.

[37] K. G. O'Grady, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. 12 (2003), pp. 435-505.

[38] A. Perego - A. Rapagnetta, *Deformation of the O'Grady moduli space*, arXiv:1008.0190 [math.AG].

[39] A. Rapagnetta, *Topological invariants of O'Grady's six dimensional irreducible symplectic variety*, Math. Z. 256 (2007), pp. 1-34.

[40] A. Rapagnetta, *On the Beauville form of the known irreducible symplectic varieties*, Math. Ann. 340 (2008), pp. 77-95.

[41] S. Salamon, *Quaternionic Khler manifolds*, Invent. Math. 67 (1982), pp. 143-171.

[42] S. M. Salamon, *On the cohomology of Kähler and hyper-Kähler manifolds*, Topology 35, 1996, pp. 137-155.

[43] J. Sawon, *Abelian fibred holomorphic symplectic manifolds*, Turkish J. Math. 27, 2003, pp. 197-230.

[44] J. Sawon, *Lagrangian fibrations on Hilbert schemes of points on K3 surfaces*, J. Algebraic Geom. 16 (2007), pp. 477-497.

[45] C. T. Simpson *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. No. 79 (1994), pp. 47-129.

[46] M. Verbitsky, *A global Torelli theorem for hyperkähler manifolds*, arXiv:0908.4121.

[47] S. T. Yau, *On the Ricci curvature of a compact Khler manifold and the complex Monge-Ampre equation. I*, Comm. Pure Appl. Math. 31 (1978), pp. 339-411.

[48] K. Yoshioka, *Some examples of Mukai's reflections on $K3$ surfaces*, J. Reine Angew. Math. 515 (1999), pp. 97-123.

[49] K. Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. 321 (2001), pp. 817-884.

[50] M. Verbitsky, *A global Torelli theorem for hyperkähler manifolds*, arXiv:0908.4121.

[51] C. Voisin, *Hodge theory and complex algebraic geometry. II*, Cambridge Studies in Advanced Mathematics, 77, 2003, C.U.P.