Space-only hyperbolic approximation of the Vlasov equation

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1. Plasma mathematical model

We consider the dimensionless Vlasov-Poisson system.

1.1 Vlasov-Poisson 1D model

The model reads

$$\begin{cases}\partial_t f + v\partial_x f + E\partial_v f = 0\\ \partial_x E = -1 + \int_v f dv\end{cases} \quad (1)$$

where

- $f(x,v,t)$: the distribution function
- $E(x,t)$: the electric field-solution of the Poisson equation.

1.2 The Vlasov-Fourier 1D model

We consider a Fourier transformation with respect to the velocity variable (we denote by $I=\sqrt{-1}$)

$$\phi(x,\eta,t) = \int_{v=-\infty}^{+\infty} f(x,v,t)\exp(-I\eta v)dv. \quad (2)$$

The Vlasov-Poisson system becomes

$$\begin{cases}\partial_t\phi + I\partial_x\partial_\eta\phi + IE\eta\phi = 0,\\ \partial_x E(x,t) = -1 + \phi(x,0,t).\end{cases} \quad (3)$$

1.3 Vlasov-Poisson 2D model

The Vlasov-Poisson system in 2D reads

$$\begin{cases}\partial_t f + \mathsf{v}\cdot\nabla_{\mathsf{x}} f + E\cdot\nabla_{\mathsf{v}} f = 0\\ \nabla_{\mathsf{x}}\cdot E = -1 + \int_{\mathsf{v}} f d\mathsf{v}\end{cases} \quad (4)$$

with $\mathsf{x}=(x,y)\in\mathbb{R}^2$ and $\mathsf{v}=(v_x,v_y)\in\mathbb{R}^2$.

2. The numerical method

We consider the Vlasov-Poisson 1D system.

2.1 Velocity basis expansion

The distribution function $f(x,v,t)$ is written:

$$f(x,v,t) = \sum_{j=1}^{P} w_j(x,t)\varphi_j(v) \quad (5)$$

where the functions $(\varphi_i)_{i=1\ldots P}$ are classical finite element interpolation d^{th} order Lagrange basis functions. The Vlasov equation becomes

$$M\partial_t w + A\partial_x w + EB(E)w = 0, \quad (6)$$

where the variable is

$$w = (w_1, w_2, \ldots, w_P)^T,$$

The matrices M, A, B are defined by

$$M_{ij} = \int_v \varphi_i\varphi_j, \quad A_{ij} = \int_v v\varphi_i\varphi_j, \quad (7)$$

and

$$(B(E))_{ij} = \frac{E^+}{2E}\varphi_j(-V)\varphi_i(-V) - \frac{E^-}{2E}\varphi_j(V)\varphi_i(V) + \int_v \varphi_i\varphi_j', \quad (8)$$

with

$$E^+ = \max(0,E), \quad E^- = \min(0,E). \quad (9)$$

Remark:

- We have replaced the original Vlasov equation, written in the (x,v)-space by an **hyperbolic system** written in the x-space only. We can solve this system by any efficient numerical method for hyperbolic systems.
- The model conserves the first d moments of the distribution function (in case $E=0$.)
- The methods easily extends to higher dimensions and/or to the Vlasov-Fourier model.

2.2 Finite volume schemes in space

If we denote $S=-EB(E)w$ the source term, the equation (6) becomes

$$M\partial_t w = -A\partial_x w + S. \quad (10)$$

The semi-discrete scheme in space reads

$$M\partial_t w_i = -\frac{F(w_i,w_{i+1}) - F(w_{i-1},w_i)}{\triangle x} + S(w_i), \quad (11)$$

where F is the numerical flux (the centered or upwind numerical flux).

2.3 Time discretization

We can consider

- the classical explicit Euler method, which is of order 1. It reads

$$M\frac{w_i^{n+1}-w_i^n}{\triangle t} = -\frac{F(w_i^n,w_{i+1}^n) - F(w_{i-1}^n,w_i^n)}{\triangle x} + S(w_i^n) \quad (12)$$

or

- a time second order scheme (the Runge-Kutta second order):

$$M\frac{w_i^{n+1/2}-w_i^n}{\triangle t/2} = -\frac{F(w_i^n,w_{i+1}^n) - F(w_{i-1}^n,w_i^n)}{\triangle x} + S(w_i^n), \quad (13)$$

$$M\frac{w_i^{n+1}-w_i^n}{\triangle t} = -\frac{F(w_i^{n+1/2},w_{i+1}^{n+1/2}) - F(w_{i-1}^{n+1/2},w_i^{n+1/2})}{\triangle x} + S(w_i^{n+1/2}).$$

3. Implementation

3.1 Computation of the electric field

We compute the electric field by solving the Poisson equation. As in many other works, we use the FFT (Fast Fourier Transform) algorithm.

3.2 Application to the Vlasov-Fourier model

We use the two following steps

- The first step: Discretization of the Vlasov-Fourier equation with respect to the Fourier velocity variable.
- The second step: Solve the hyperbolic system with the finite volume method.

The distribution function
The inverse Fourier transform reads

$$f(x,v,t) \approx \frac{1}{2\pi}\int_{-\eta_{max}}^{\eta_{max}} \phi(x,\eta,t)e^{I\eta v}d\eta. \quad (14)$$

We apply the rectangle method with oversampling for computing (14), in order to avoid Shannon aliasing. For this computation we can use a naive DFT computation instead of the FFT algorithm, because this step is applied only at the beginning and the end of the simulation.

3.3 Parallel implementation in 2D

We have written a parallel implementation with MPI in SeLaLib (INRIA library). We also use the Runge-Kutta fourth order scheme. We verify our program with the transport test case and landau damping test case. We also study the numerical convergence rates of our method with different schemes (Euler, Runge-Kutta second order and Runge-Kutta fourth order).

4. Test cases

We consider the following test cases: 1D Landau damping, 1D two-stream instability and 2D Landau damping.

4.1 Landau damping 1D

- The initial distribution function

$$f_0(x,v) = (1+\varepsilon\cos(kx))\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}},$$

where $\varepsilon > 0$ and $k\in\mathbb{N}^*$.
- The domain size is $L = 2\pi/k$.

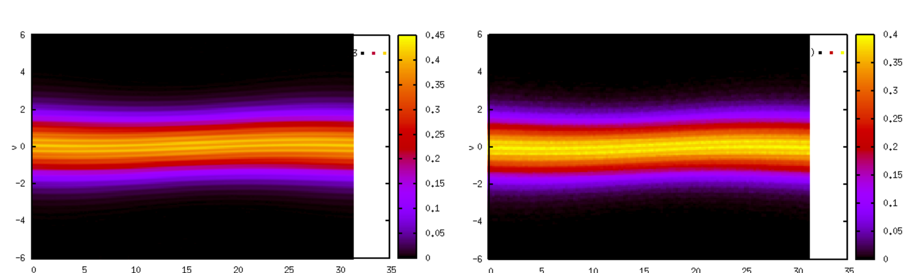

FIG 1 - The distribution function of the Landau damping test case at time $t=100$. Left: Vlasov-Fourier method. Right: the PIC method.

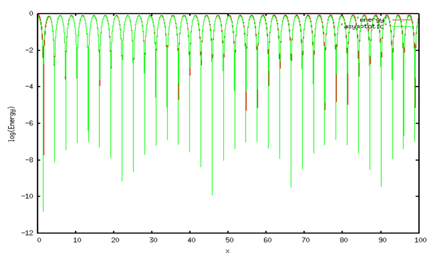

FIG 2 -The electric energy of the Landau damping test case up to time $t=100$, the green curve "asymptotic" is the analytical solution and the red curve "energy" is computed with the Vlasov-Fourier method.

4.2 Two-stream instability 1D

The initial distribution function reads

$$f_0(x,v) = (1+\epsilon\cos(kx))\frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{(v-v_0)^2}{2}} + e^{-\frac{(v+v_0)^2}{2}}\right).$$

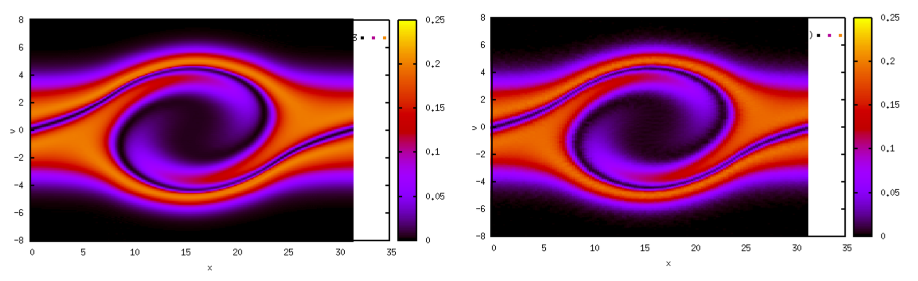

FIG 3 - The distribution function of the two-stream test case at time $t=25$. Left: Vlasov-Fourier method (with the centered flux). Right: the PIC method.

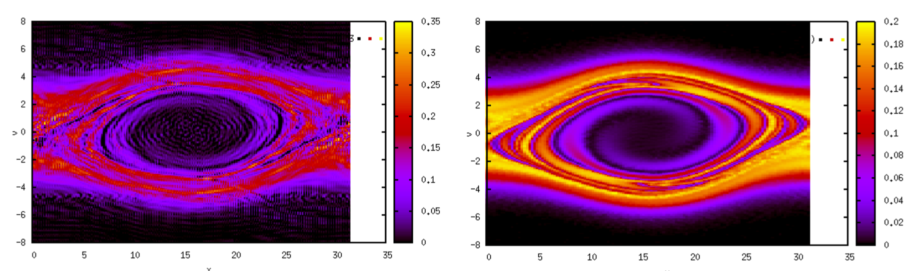

FIG 4 -The distribution function of the two-stream test case at time $t=50$. Left: Vlasov-Fourier method (with the centered flux). Right: the PIC method.

Remark: There are oscillations at time $t=50$.
Corrected flux: slight upwind
In order to remove these oscillations, we use the slightly upwinded flux:

$$F(w_L,w_R) = A\frac{w_L+w_R}{2} + \frac{\delta}{2}(w_R - w_L). \quad (15)$$

and we obtain the result

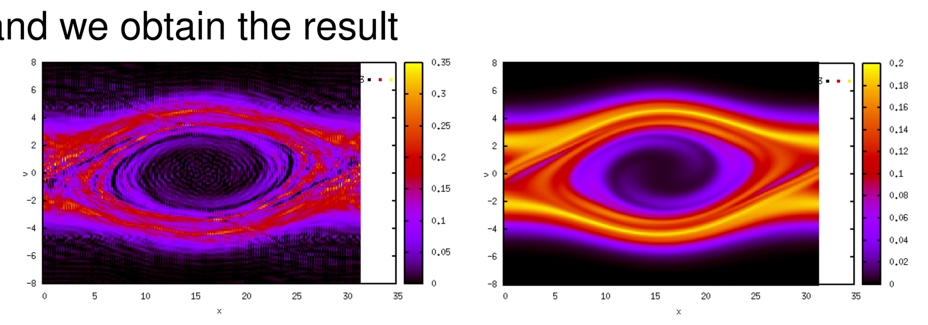

FIG 5 - The distribution function of the two-stream test case at time $t=50$ computed with the reduced Vlasov-Fourier method. Left: with the centered flux. Right: with the slightly upwinded flux ($\delta=0.05$).

4.3 Landau damping 2D

The initial distribution function reads

$$f_0(x,y,v_x,v_y) = \frac{1}{2\pi}(1+\epsilon\cos(k_x x)\cos(k_y y))\left(e^{-\frac{(v_x)^2+(v_y)^2}{2}}\right),$$

with $\epsilon=0.05$. The velocity space is truncated at $V=6$; the wave numbers are $k_x=k_y=0.5$ and the length of the periodic box in the physical space is $L_x=L_y=4\pi$.

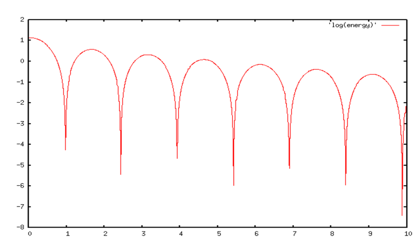

FIG 6 -The electric energy of the Landau damping 2D test case up to time $t=10$.

References

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