Theoretical and Numerical aspects for incompressible fluids
Part II: Shape optimization for fluids

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Outlines of the lectures

Session 1: tools for shape optimization in Fluid Mechanics and existence of optimal shapes

Session 2: Existence and optimality condition (shape derivative)

Session 3: Application in Fluid Mechanics and algorithms
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Grading: 1 written test (2H) + 1 oral examination
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Outlines of the lesson

1. Several examples of shape optimization problems

2. Tools for shape optimization
   - Generalities
   - The perimeter functional in shape optimization
   - Existence results in shape optimization
   - Derivation with respect to the domain
   - Shape derivatives using Eulerian and material derivatives: the rigorous ‘difficult’ way
   - A formal, easier way to compute shape derivatives: Céa’s method

3. Numerics and applications to Fluid Mechanics
   - Reminders in Fluid Mechanics
   - Shape optimization in Fluid Mechanics
   - Numerical aspects
Several examples of shape optimization problems

Isoperimetric problems

The problem of Dido

Dido was the legendary founder of Carthage (Tunisia). When she arrived in 814 B.C. on the coast of Tunisia, she asked for a piece of land. Her request was satisfied provided that the land could be encompassed by an ox-hide. With a remarkable mathematical intuition, she cut the ox-hide into a long thin strip and used it to encircle the land. This land became Carthage and Dido became the Queen.

Question : What is the closed curve which has the maximum area for a given perimeter ?
Isoperimetric problems

Mathematically, this problem is very close to the standard isoperimetric one:

Find $\Omega \subset \mathbb{R}^2$ solution of the problem

\[
\begin{cases}
\text{maximize } \text{area}(\Omega) \\
\text{such that } \text{Per}(\Omega) = 4 \text{ km}.
\end{cases}
\]

Back to the problem of Dido. A naive formulation writes: find the plane curve enclosing with the segment joining its extremity the subdomain having a maximal area. On other words, one has to solve for $b > a \geq 0$,

\[
\sup_{y \in \mathcal{E}} \int_a^b y(x) \, dx
\]

where

\[
\mathcal{E} = \{ y \in Y \mid \int_a^b \sqrt{1+y'^2(x)} \, dx = \ell \text{ et } y(a) = y(b) = 0 \}
\]

with $Y$, a given functional space (chosen e.g. so that the problem has a solution).
Isoperimetric problems

The proof

- **Zénodore** (2\textsuperscript{nd} century B.C.) proves the isoperimetric inequality in the particular case where $\Omega =$ polygon.
- Until the 20\textsuperscript{th} century: this result is conjectured but not proved.
- **Steiner** (Swiss mathematician of the 19\textsuperscript{th} century) publishes a proof, but ... this proof is erroneous!

- **Weierstrass** (German mathematician of the 19\textsuperscript{th} century) concludes the proof, by using modern tools of calculus of variations.
- Generalization in $\mathbb{R}^3$ or $\mathbb{R}^N$ only known since 1960 (geometric measure theory)
Aerodynamic optimization of airplane
Aerodynamic shape optimization of transonic wings

The model

It requires the use of Navier-Stokes modeling due to the strong nonlinear coupling between airfoil shape, wave drag, and viscous effects.

Criterion, unknown

- **Goal**: reduce the drag (reaction of the flow on the wing; its component in the direction of flight is the drag proper and the rest is the lift)
- A few percent of drag optimization means a great saving on commercial airplanes;
- **Unknown**: the shape of the wing
Several examples of shape optimization problems

Aerodynamic optimization of airplane
Aerodynamic shape optimization of transonic wings

- For viscous drag the **Navier-Stokes equations** must be used: for a given shape $S$ of the wing, it yields the velocity $u$ and pressure $p$ of the fluid at every point.

- For a wing with boundary $S$ moving at constant speed $u_\infty$, the force acting on the wing is
  \[ \vec{F} = \int_S \left( \mu (\nabla u + [\nabla u]^\top) - \frac{2}{3} \nabla \cdot u \right) n \, ds - \int_S p n \, ds, \]
  where $n$ is the normal to $S$ pointing outside the domain occupied by the fluid, and $\mu$ the viscosity of the fluid.

- **Shape optimization problem.** The drag is the component of $\vec{F}$ parallel to the velocity at infinity. The optimal design problem writes
  \[ \inf_{S \in \mathcal{O}_{ad}} J(S) \quad \text{where} \quad J(\Omega) = \vec{F} \cdot u_\infty. \]

**Choice of $\mathcal{O}_{ad}$ (admissible set of shapes)?**
- $S$ is an open connected bounded subset of $\mathbb{R}^d$ ($d = 2$ or $d = 3$)
- A geometrical constraint such as the volume being greater than a given value in order to avoid that the wing collapses ($|S| \geq S_0$)
- An aerodynamic constraint: the lift must be greater than a given value. Else, the wing will not fly.
Optimal shape of a duct

Problem and modeling

Consider a fluid of viscosity $\mu$ flowing inside a cannula-shaped pipe/duct. For instance, we look for the optimal shape of a pipeline.

The optimal design problem writes

$$\inf_{\Omega \in O_{ad}} J(\Omega) \quad \text{where} \quad J(\Omega) = 2\mu \int_{\Omega} |\varepsilon(u)|^2 \, dx,$$

with

- $\mu$, the viscosity of the fluid, $u$ the velocity of the fluid at every point (e.g. given by the Navier-Stokes equations),
- $O_{ad}$ is the set of admissible shapes, for instance, $E$ (inlet) and $S$ (outlet) are fixed and we look for the lateral boundary such that $\Omega$ open connected subset of $\mathbb{R}^d$ with $|\Omega| = V_0$ (given).
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A shape optimization problem writes as the minimization of a cost (or objective) function $J$ of the domain $\Omega$:

$$\inf_{\Omega \in \mathcal{O}_{ad}} J(\Omega),$$

where $\mathcal{O}_{ad}$ is a set of admissible shapes (e.g. that satisfy constraints).

In most mechanical or physical applications, the relevant objective functions $J(\Omega)$ depend on $\Omega$ via a state $u_\Omega$, which arises as the solution to a PDE posed on $\Omega$ (e.g. the linear elasticity system, or Stokes equations).
Various settings for shape optimization (I)

1. Parametric optimization

The considered shapes are described by means of a set of physical parameters \( \{ p_i \}_{i=1,\ldots,N} \), typically thicknesses, curvature radii, etc...

Description of a wing by NURBS; the parameters of the representation are the control points \( p_i \).

A plate with fixed cross-section \( S \) is parametrized by its thickness function \( h : \Omega \rightarrow \mathbb{R} \).
Various settings for shape optimization (II)

The parameters describing shapes are the only optimization variables, and the shape optimization problem rewrites:

$$\min_{\{p_i\} \in \mathcal{P}_{ad}} J(p_1, \ldots, p_N),$$

where $\mathcal{P}_{ad}$ is a set of admissible parameters.

Parametric shape optimization is eased by the fact that it is straightforward to account for variations of a shape $\{p_i\}_{i=1,\ldots,N}$:

$$\{p_i\}_{i=1,\ldots,N} \rightarrow \{p_i + \delta p_i\}_{i=1,\ldots,N}.$$ 

However, the variety of possible designs is severely restricted, and the use of such a method implies an a priori knowledge of the sought optimal design.
Various settings for shape optimization (III)

II. Geometric shape optimization

- The topology (i.e. the number of holes in 2d) of the considered shapes is fixed.
- The boundary $\partial \Omega$ of the shapes $\Omega$ itself is the optimization variable.
- Geometric optimization allows more freedom than parametric optimization, since no a priori knowledge of the relevant regions of shapes to act on is required.

Optimization of a shape by performing ‘free’ perturbations of its boundary.
In some applications, the suitable topology of shapes is unknown, and also subject to optimization.

In this context, it is often preferred not to describe the boundaries of shapes, but to resort to different representations which allow for a more natural account of topological changes.

For instance: Describing shapes $\Omega$ as characteristic functions $\chi_\Omega : D \to \{0, 1\}$. Optimizing a shape by acting on its topology.
A shape optimization process is a combination of:

- A **physical model**, most often based on PDE (e.g. the linear elasticity equations, Stokes system, etc...) for describing the mechanical behavior of shapes,
- A **description** of shapes and their variations (e.g. as sets of parameters, density functions, etc...),
- A **numerical description** of shapes (e.g. by a mesh, a spline representation, etc...)

These choices are strongly inter-dependent and influenced by the sought application.

However very different in essence, all these different methods for shape optimization share a lot of common features.

We are going to focus on **geometric shape optimization methods**.
Disclaimer

- This course is very introductory, and by no means exhaustive, as well for theoretical as for numerical purposes.

- See the (non exhaustive) References section to go further.
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General definition of perimeter

**Definition : Perimeter in the sense of De Giorgi**

Let $\Omega$ be a (Lebesgue) measurable set in $\mathbb{R}^d$. The *perimeter of $\Omega$* is defined by

$$\text{Per}(\Omega) = \sup\left\{ \int_{\Omega} \text{div}(\varphi) \, dx \mid \varphi \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\},$$

where $\mathcal{D}(\mathbb{R}^d, \mathbb{R}^d)$ is the set of $\mathcal{C}^\infty$ functions $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ having a compact support.

- Definition introduced by Ennio De Giorgi (1928-1996)
- Note that, for a function $\varphi \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d)$, there holds

$$\int_{\Omega} \text{div}(\varphi) \, dx = \int_{\mathbb{R}^d} \text{div}(\varphi) \chi_\Omega(x) \, dx = -\langle \nabla \chi_\Omega, \varphi \rangle_{\mathcal{D}', \mathcal{D}},$$

where $\chi_\Omega$ denotes the *characteristic function of $\Omega$*, in other words

$$\chi_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{else.} \end{cases}$$
General definition of perimeter

Reminder on the notion of “regular set”

1. An open set $\Omega$ of $\mathbb{R}^d$ is said to have a **Lipschitz boundary** if, for all $x_0 \in \partial \Omega$, there exist a cylinder $K = K' \times ]-a, a[$ in a local cartesian basis with origin $x_0 = 0$, with $K'$ an open ball in $\mathbb{R}^{d-1}$ with radius $r$, and a Lipschitz function $\varphi : K' \to ]-a, a[$ such that $\varphi(0) = 0$ and

$$\Omega \cap K = \{(x', y) \in K \mid y > \varphi(x')\} \quad \text{et} \quad \partial \Omega \cap K = \{(x', \varphi(x')) \mid x' \in K'\}.$$

2. An open set is said of class $\mathcal{C}^1$ by replacing the word ”Lipschitz” by $\mathcal{C}^1$ in the definition above.

Proposition (Link with the usual definition of perimeter for regular sets)

Let $\Omega$, be a bounded open set of class $\mathcal{C}^1$. Then, $\text{Per}(\Omega) = \int_{\partial \Omega} d\sigma$, where $d\sigma$ denotes the surface element on $\partial \Omega$. 
General definition of perimeter

Proposition

Let \( \Omega \), be a bounded open set of class \( \mathcal{C}^1 \). Then, \( \text{Per}(\Omega) = \int_{\partial \Omega} d\sigma \), where \( d\sigma \) denotes the surface element on \( \partial \Omega \).

Sketch of the proof: let \( \varphi \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d) \) such that \( \|\varphi\|_\infty \leq 1 \). According to Green’s formula, one has

\[
\int_{\Omega} \text{div}(\varphi) \, dx = \sum_{i=1}^{d} \int_{\Omega} \frac{\partial \varphi_i}{\partial x_i} \, dx = \int_{\partial \Omega} \varphi_i n_i \, d\sigma = \int_{\partial \Omega} \varphi \cdot \nu \, d\sigma,
\]

where \( \nu \) is the outward unit normal vector. Using the triangle inequality, one has for every smooth function \( \varphi \),

\[
\int_{\Omega} \text{div}(\varphi) \, dx \leq \int_{\partial \Omega} |\varphi| |\nu| \, d\sigma \leq \|\varphi\|_\infty \int_{\partial \Omega} \, d\sigma \leq \int_{\partial \Omega} \, d\sigma.
\]

Passing to the sup, one gets \( \text{Per}(\Omega) \leq \int_{\partial \Omega} d\sigma \).
General definition of perimeter

Proposition

Let \( \Omega \) be a bounded open set of class \( C^1 \). Then, \( \text{Per}(\Omega) = \int_{\partial \Omega} d\sigma \), where \( d\sigma \) denotes the surface element on \( \partial \Omega \).

Proof (converse sense) : One would like to choose \( \varphi = \nu \) on \( \partial \Omega \).

Idea : find a sequence of functions \((\varphi_n)_{n \in \mathbb{N}} \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d)\mathbb{N}\) to approximate \( \nu \). Introduce \( \mathcal{N} \), a continuous extension of the normal vector \( \nu \). One regularizes \( \mathcal{N} \) by defining \( \varphi_n = \mathcal{N} \ast \rho_n \), where \((\rho_n)_{n \in \mathbb{N}} \) is a mollifier.

One thus construct a sequence of functions in \( \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d) \) converging uniformly to \( \mathcal{N} \) and s.t.

\[
\int_{\Omega} \text{div} \varphi_n \, dx = \int_{\partial \Omega} \varphi_n \cdot \nu \, d\sigma \xrightarrow{n \to +\infty} \int_{\partial \Omega} \nu \cdot \nu \, d\sigma = \int_{\partial \Omega} d\sigma.
\]
General definition of perimeter

- Reminder on Radon measures: recall that if \( f \in L^1(\mathbb{R}^d, \mathbb{R}^d) \), then by duality

\[
\| f \|_{L^1} = \int_{\mathbb{R}^d} |f(x)| \, dx = \sup \left\{ \int_{\mathbb{R}^d} f(x) \cdot \varphi(x) \, dx, \varphi \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d) \text{ and } \|\varphi\|_\infty \leq 1 \right\}.
\]

More generally, this formula makes sense for every Radon measure \( \mu \) having a finite mass (i.e. every linear continuous form on the space \( C^0_0(\mathbb{R}^d) \) of continuous functions having a compact support in \( \mathbb{R}^d \)).

- Moreover, a Radon measure \( \mu \) has a finite mass on \( \mathbb{R}^d \) iff the quantity

\[
\| \mu \|_1 = \sup \{ \langle \mu, \varphi \rangle_{\mathcal{D}', \mathcal{D}}, \varphi \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d) \text{ et } \|\varphi\|_\infty \leq 1 \}
\]

is finite.

- Recall that

\[
\text{Per}(\Omega) = \sup \{ -\langle \nabla \chi_\Omega, \varphi \rangle_{\mathcal{D}', \mathcal{D}}, \varphi \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \}
\]

**Proposition**

Let \( \Omega \) be a measurable subset of \( \mathbb{R}^d \). Then, the quantity \( \text{Per}(\Omega) \) is finite iff \( \nabla \chi_\Omega \) is a Radon measure having a finite mass and in that case, \( \text{Per}(\Omega) = \| \nabla \chi_\Omega \|_1 \).
General definition of perimeter

**Theorem (compactness properties of the perimeter)**

Let \((\Omega_n)_{n \in \mathbb{N}}\) be a sequence of measurable sets in \(\mathbb{R}^d\). Let us assume the existence of \(C > 0\) s.t.

\[
\forall n \in \mathbb{N}, \quad |\Omega_n| + \text{Per}(\Omega_n) \leq C.
\]

Then, there exist a measurable set \(\Omega \subset \mathbb{R}^d\) and a subsequence \((\Omega_{n_k})_{k \in \mathbb{N}}\) s.t.

\[
\chi_{\Omega_{n_k}} \to \chi_{\Omega} \text{ dans } L^1_{loc}(\mathbb{R}^d) \quad \text{et} \quad \langle \nabla \chi_{\Omega_{n_k}}, \varphi \rangle \to \langle \nabla \chi_{\Omega}, \varphi \rangle, \quad \forall \varphi \in C^0_0(\mathbb{R}^d).
\]

Moreover, if all the elements of the sequence \((\Omega_n)_{n \in \mathbb{N}}\) belong to an open set \(D\) having a finite measure, then the sequence \((\chi_{\Omega_{n_k}})_{k \in \mathbb{N}}\) converges to \(\chi_{\Omega}\) in \(L^1(D)\).

- The convergence of the gradients of the characteristic functions coincides with the weak-\(*\) convergence of the bounded Radon measures.
- The proof of this theorem rests upon the compactness of the injection \(BV(\mathbb{R}^d) \hookrightarrow L^1_{loc}(\mathbb{R}^d)\).
General definition of perimeter

Definition: convergence in the sense of characteristic functions.

Let \((\Omega_n)_{n \in \mathbb{N}}\) and \(\Omega\) denote respectively a sequence of measurable sets and a measurable set of \(\mathbb{R}^d\). The sequence \((\Omega_n)_{n \in \mathbb{N}}\) is said to converge in the sense of characteristic functions if \(\chi_{\Omega_n} \to \chi_{\Omega}\) in \(L^1_{loc}(\mathbb{R}^d)\).

- According to the previous theorem, a sequence of domains contained in a bounded open set \(D\) and whose perimeter is uniformly bounded converges (up to a subsequence) in the sense of characteristic functions.
- Let \((\Omega_n)_{n \in \mathbb{N}}\) be a sequence of measurable subsets of \(\mathbb{R}^d\). Without additional assumption, one cannot conclude that \((\chi_{\Omega_n})_{n \in \mathbb{N}}\) converges up to a subsequence to \(\chi_{\Omega}\). Nevertheless, since \(L^\infty(\mathbb{R}^d; \{0, 1\})\) is the topological dual of the space \(L^1(\mathbb{R}^d; \{0, 1\})\), there exist a subsequence \((\chi_{\Omega_{n_k}})_{k \in \mathbb{N}}\) converging weakly-\(\star\) in \(L^\infty\) to a function \(a \in L^\infty(\mathbb{R}^d, [0, 1])\) (Banach-Alaoglu theorem).
- Consider \(d = 1\) and the sequence \((\omega_n)_{n \in \mathbb{N}}\) of subsets of \([0, \pi]\) defined by \(\omega_n = \bigcup_{k=1}^n \left(\frac{k\pi}{n+1} - \frac{\pi}{4n}, \frac{k\pi}{n+1} + \frac{\pi}{4n}\right)\), for every \(n \in \mathbb{N}^*\). One can prove (exercise): \(|\omega_n| = \frac{\pi}{2}\), and the sequence \((\chi_{\omega_n})_{n \in \mathbb{N}}\) converges weakly-\(\star\) in \(L^\infty\) to the constant function \(a(\cdot) = \frac{1}{2}\).
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Existence of solutions in shape optimization

General issues

Let us consider the general optimal design problem

\[ \inf_{\Omega \in \mathcal{O}_{\text{ad}}} J(\Omega) \quad \text{with} \quad \mathcal{O}_{\text{ad}} \subset \mathcal{A}(D) = \{ \Omega \subset D, \ \Omega \text{ open} \}, \]

where \( D \) is an open set of \( \mathbb{R}^d \).

Study plan for existence issues

1. is the quantity \( \inf_{\Omega \in \mathcal{O}_{\text{ad}}} J(\Omega) \) finite?
   In that case, introduce a minimizing sequence \((\Omega_n)_{n \in \mathbb{N}}\) of elements of \( \mathcal{O}_{\text{ad}} \).
   Moreover, the sequence \((J(\Omega_n))_{n \in \mathbb{N}}\) is bounded.

2. what topology to choose on the admissible set of domains \( \mathcal{O}_{\text{ad}} \) to guarantee:
   - that one can find a subsequence of \((\Omega_n)_{n \in \mathbb{N}}\) converging to a certain \( \Omega \in \mathcal{O}_{\text{ad}} \) for this topology?
   - that the functional \( J \) be lower semi-continuous for this topology, in other words so that
     \[ \Omega_n \to \Omega \quad \Rightarrow \quad J(\Omega) \leq \liminf_{n \to +\infty} J(\Omega_n) \]

Consequence: the problem above has a solution.

In the context of fluid mechanics: we have to deal with PDE constraints!!
Existence of solutions in shape optimization

A toy problem for PDE constraints

Let $D$ a bounded connected open set and $\mathcal{O}_{\text{ad}}$ be a subset of

$$\mathcal{A}(D) = \{\Omega \text{ open, } \Omega \subset D\}.$$ 

We consider the shape optimization problem

$$\inf_{\Omega \in \mathcal{O}_{\text{ad}}} J(\Omega) \text{ where } J(\Omega) = \int_{\Omega} (u_{\Omega}(x) - u_0(x))^2 \, dx,$$

with $u_0 \in L^2(D)$ given, and $u_{\Omega}$ the unique solution of the P.D.E.

$$\begin{cases}
-\Delta u_{\Omega} = f & x \in \Omega \\
u_{\Omega} = 0 & x \in \partial \Omega,
\end{cases}$$

with $f \in H^{-1}(D)$.

To obtain the existence of a solution, we will have, in addition to the convergence of the minimizing sequence $(\Omega_n)_{n \in \mathbb{N}}$ in a sense to make precise, to obtain the convergence of $(u_{\Omega_n})_{n \in \mathbb{N}}$ in $L^2(D)$ for example.
Existence of solutions in shape optimization

Definition (Hausdorff complement topology)

- Let \( x \in \mathbb{R}^d \) and \( E \) a closed subset of \( \mathbb{R}^d \). We denote
  \[
d_E(x) = \text{dist}(x, E) := \inf \{|x - y|, \ y \in E\}.
  \]

- **Hausdorff distance between two closed sets**: for \( A, B \), two closed sets of \( \mathbb{R}^d \),
  \[
d_H(A, B) = \max\{\sup_{x \in A} d_B(x), \sup_{x \in B} d_A(x)\}.
  \]

- **Convergence of open sets in the sense of Hausdorff (Hausdorff complementary topology)**: for \( A \) and \( B \) in \( \mathcal{A}(D) \),
  \[
d_{Hc}(A, B) = d_H(D \setminus A, D \setminus B).
  \]

As a consequence, we will say that \((\Omega_n)_{n \in \mathbb{N}}\) (sequence of \( \mathcal{A}(D) \)) converges to \( \Omega \) in the sense of Hausdorff if \( d_{Hc}(\Omega_n, \Omega) \to 0 \) as \( n \to +\infty \).
Existence of solutions in shape optimization

Remarks on Hausdorff convergence

Very important property

For every $A, B$ closed subsets of $D \subset \mathbb{R}^d$,

$$d_H(A, B) = \|d_A - d_B\|_{C^0(D)} := \sup_{x \in D} |d_A(x) - d_B(x)|.$$

**Proof:** there holds

$$\|d_A - d_B\|_{C^0(D)} \geq \max\{\|d_A - d_B\|_{C^0(A)}, \|d_A - d_B\|_{C^0(B)}\} \geq d_H(A, B).$$

Let us show the converse inequality. Let $x \in \overline{B}$ and $y \in A$. By compactness, there exist $x_B \in \overline{B}$ s.t. $d_B(x) = |x - x_B|$. Hence,

$$d_A(x) - d_B(x) \leq |y - x| - |x - x_B| \leq |y - x_B| \leq d_A(x_B) \leq \sup_{x \in B} d_A(x).$$

By inverting the roles played by $A$ and $B$, we infer the desired inequality. $\square$
Existence of solutions in shape optimization

Back to the Hausdorff complement topology

**Proposition**

1. Let $A$ and $B$ in $\overline{D}$. One has $d_{\mathcal{H}}(A, B) = 0$ iff $\overline{A} = \overline{B}$.

2. Introduce the space of "distance" functions of the subsets of $\overline{D}$

$$C_d(D) = \{d_{\Omega}, \text{ with } \Omega \neq \emptyset \text{ and } \Omega \subset \overline{D}\} \quad \text{with} \quad d_{\Omega} : x \mapsto d(x, \Omega).$$

Then, $C_d(D)$ is a compact subspace of $C^0(\overline{D})$.

✓ It is enough to restrict the set of admissible shapes to classes of open sets. Indeed,

$$\{d_{\Omega^c}, \ \Omega \subset \overline{D} \text{ and } \Omega \neq \emptyset\} = \{d_{\Omega^c}, \ \Omega \text{ open set included in } D\}.$$

To show this equality, consider $\Omega \subset \overline{D}$ s.t. $\Omega \neq \emptyset$.

Consider the open set $O = \overline{\Omega^c}^c$. Since $O^c = \overline{\Omega^c}$, there holds $d_{O^c} = d_{\overline{\Omega^c}} = d_{\Omega^c}$.

Therefore, the proposition above must be interpreted in the following sense: if $(\Omega_n)_{n \in \mathbb{N}}$ is a sequence of $\mathcal{A}(D)$ converging to $\tilde{\Omega}$ in the sense of Hausdorff, then the limits of the sequence $(\Omega_n)_{n \in \mathbb{N}}$ belong to the same equivalence class and there exists $\Omega \in \mathcal{A}(D)$ s.t. $(\Omega_n)_{n \in \mathbb{N}}$ converges to $\Omega$. 
Existence of solutions in shape optimization

Proposition (compactness and l.s.c. for the topology $H^c$)

1. The set $\mathcal{A}(D)$ endowed with the metric $d_{H^c}$ is compact.

2. Let $(\Omega_n)_{n \in \mathbb{N}}$, a sequence of elements in $\mathcal{A}(D)$ that converges in the sense of Hausdorff to $\Omega$. For every compact subset $K \subset \Omega$, there exists an integer $N(K) \in \mathbb{N}$ such that $K \subset \Omega_n$ for every $n \geq N(K)$.

3. The Lebesgue measure is lower semi-continuous for the topology $d_{H^c}$.

Examples (exercise)

- Consider an open set $D$ and $(x_n)_{n \in \mathbb{N}}$, a sequence of dense points in $D$. Set $\Omega_n = D \setminus \{x_k\}_{1 \leq k \leq n}$. Then, $(\Omega_n)_{n \in \mathbb{N}}$ converges in the sense of Hausdorff to $\emptyset$.

- $\Omega_n = ]0, 1[ \setminus \{\frac{1}{n}\}_{n \in \mathbb{N}}$ converges in the sense of Hausdorff to $]0, 1[$.
The uniform cone property

Definition

Let $y$, a point of $\mathbb{R}^d$, $\xi$, a unit vector and $\varepsilon > 0$.

1. Introduce the cone $C(y, \xi, \varepsilon)$ with apex $y$, direction $\xi$ and dimension $\varepsilon$, defined by

$$C(y, \xi, \varepsilon) = \{ z \in \mathbb{R}^d \mid \langle z - y, \xi \rangle_{\mathbb{R}^d} \geq \cos \varepsilon \| z - y \|_{\mathbb{R}^d} \text{ and } 0 < \| z - y \|_{\mathbb{R}^d} < \varepsilon \},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ is the euclidean scalar product of $\mathbb{R}^d$ and $\| \cdot \|_{\mathbb{R}^d}$, the associated euclidean norm.

2. An open set $\Omega$ verifies the $\varepsilon$-cone property if for every $x \in \Omega$, there exists $\xi_x$, a unit vector such that :

$$\forall y \in \overline{\Omega} \cap B(x, \varepsilon), \quad C(y, \xi_x, \varepsilon) \subset \Omega,$$

where $B(x, \varepsilon)$ denotes the open ball with center $x$ and radius $\varepsilon$.

✓ Note that, according to the definition of $C(y, \xi, \varepsilon)$, $y \notin C(y, \xi, \varepsilon)$.

✓ Theorem : an open bounded set $\Omega$ satisfies the $\varepsilon$-cone property iff it has a Lipschitz boundary.
The uniform cone property

Illustration of the $\varepsilon$-cone property

**Left:** set verifying the $\varepsilon$-cone property,
**Right:** set that does not verify the $\varepsilon$-cone property
The uniform cone property

Compactness property of $\varepsilon$-cone property

**Proposition**

Let $\varepsilon > 0$ and $(\Omega_n)_{n \in \mathbb{N}}$, a sequence of open sets contained in a bounded open set $D$ satisfying the $\varepsilon$-cone property for some $\varepsilon > 0$. Then, there exist an open set $\Omega$ and a subsequence $(\Omega_{n_k})_{k \in \mathbb{N}}$ converging to $\Omega$ in the sense of Hausdorff. Moreover,

- $\Omega$ has the $\varepsilon$-cone property,
- $\overline{\Omega_{n_k}}$ and $\partial \Omega_{n_k}$ converge in the sense of Hausdorff to $\overline{\Omega}$ and $\partial \Omega$.
- one may assume that $(\Omega_{n_k})_{k \in \mathbb{N}}$ also converges to $\Omega$ in the sense of characteristic functions.
Existence issues : two examples

Example 1 : an isoperimetric like problem

\[ \inf_{\Omega \in \mathcal{O}_{V_0}} \text{Per}(\Omega) \quad \text{avec} \quad \mathcal{O}_{V_0} = \{ \Omega \subset D \mid \int_{\Omega} f(x) \, dx = V_0 \} , \]

with \( D \), a bounded open set of \( \mathbb{R}^d \), \( f \in L^1_{loc}(\mathbb{R}^d) \) and \( V_0 > 0 \) given.

Theorem

The problem above has (at least) a solution.
Existence issues : two examples

Theorem

The problem above has (at least) a solution.

Proof : let \((\Omega_n)_{n \in \mathbb{N}}\) be a minimizing sequence in \(O_{V_0}\). Then, there exists \(c > 0\) s.t. \(\text{Per}(\Omega_n) \leq c\) for all \(n \in \mathbb{N}\).

Since \(\Omega_n \subset D\) for all \(n \in \mathbb{N}\), one infers by compactness of sets having a bounded perimeter that : \(\exists (\Omega_{n_k})_{k \in \mathbb{N}}\) and a measurable set \(\Omega\) s.t. \(\chi_{\Omega_{n_k}} \to \chi_\Omega\) in \(L^1_{\text{loc}}(\mathbb{R}^d)\). Therefore,

\[
\int_{\Omega_{n_k}} f(x) \, dx = \int_{\mathbb{R}^d} \chi_{\Omega_{n_k}}(x) f(x) \, dx \xrightarrow{n \to +\infty} \int_{\Omega} f(x) \, dx.
\]

Let \(\varphi \in D(\mathbb{R}^d, \mathbb{R}^d)\). Hence,

\[
\int_{\Omega} \text{div}(\varphi) \, dx = \lim_{n \to +\infty} \int_{\Omega_{n_k}} \text{div}(\varphi) \, dx \leq \liminf_{k \to +\infty} \sup_{\|\varphi\|_\infty \leq 1, \varphi \in D(\mathbb{R}^d, \mathbb{R}^d)} \int_{\Omega_{n_k}} \text{div}(\varphi) \, dx
\]

\[
= \liminf_{k \to +\infty} \text{Per}(\Omega_{n_k}).
\]

Passing to the sup in this inequality yields \(\text{Per}(\Omega) \leq \liminf_{k \to +\infty} \text{Per}(\Omega_{n_k})\). The conclusion follows.
Existence issues : two examples

Example 2 : back to the toy problem

Let $\varepsilon > 0$ and $D$ be a bounded open set of $\mathbb{R}^d$.

$$\inf_{\Omega \in \mathcal{O}_{\text{ad}, \varepsilon}} J(\Omega) \quad \text{where} \quad J(\Omega) = \int_{\Omega} (u_\Omega(x) - u_0(x))^2 \, dx,$$

where

$$\mathcal{O}_{\text{ad}, \varepsilon} = \{ \Omega \in \mathcal{A}(D) \mid |\Omega| \leq 1 \text{ and } \Omega \text{ satisfies the } \varepsilon\text{-cone property} \},$$

with $u_0 \in L^2(D)$ given, and $u_\Omega$ the unique solution of the P.D.E.

$$\begin{cases} 
-\Delta u_\Omega = f & x \in \Omega \\
 u_\Omega = 0 & x \in \partial \Omega,
\end{cases}$$

with $f \in H^{-1}(D)$.

Theorem

The problem above has (at least) a solution.
Existence issues: two examples

Theorem

The problem above has (at least) a solution.

Proof: let \((\Omega_n)_{n \in \mathbb{N}}\) be a minimizing sequence in \(\mathcal{O}_{\text{ad,} \varepsilon}\). Then, there exists a subsequence \((\Omega_{n_k})_{k \in \mathbb{N}}\) such that

- \((\Omega_{n_k})_{k \in \mathbb{N}}\) converges to \(\Omega \in \mathcal{A}(D)\) for the Hausdorff complement topology and in the sense of characteristic functions;
- \(\Omega\) satisfies the \(\varepsilon\)-cone property;
- \(|\Omega| \leq \liminf_{k \to +\infty} |\Omega_{n_k}| \leq 1\) (the measure is l.s.c. for the Hausdorff topology);
- Multiply \(-\Delta u_{\Omega_{n_k}} = f\) by \(u_{\Omega_{n_k}}\) and integrate by parts. It yields

\[
\|u_{\Omega_{n_k}}\|_{H^1_0(D)} \leq \langle f, u_{\Omega_{n_k}} \rangle_{H^{-1},H^1_0} \leq \|f\|_{H^{-1}(D)}\|u_{\Omega_{n_k}}\|_{H^1_0(D)}.
\]

Thus, \((u_{\Omega_{n_k}})_k\) is bounded in \(H^1(D)\) and according to Rellich-Kondratov theorem, it converges to \(u^* \in H^1_0(D)\) weakly in \(H^1\) and strongly in \(L^2\).
Existence issues: two examples

Theorem

The problem above has (at least) a solution.

Proof:

- \((u_{\Omega_{nk}})_{k \in \mathbb{N}}\) is bounded in \(H^1(D)\) and according to Rellich-Kondratov theorem, it converges to \(u^* \in H^1_0(D)\) weakly in \(H^1(D)\) and strongly in \(L^2(D)\).

- Passing to the limit in the variational formulation

\[
\forall \varphi \in H^1_0(\Omega), \quad \int_D \nabla u_{\Omega_{nk}} \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle_{H^{-1}, H^1_0}
\]

proves that \(u^*\) satisfies \(-\Delta u^* = f\) in a distributional sense.

- Writing \(u_{\Omega_{nk}}(x)(\chi_{\Omega_{nk}}(x) - \chi_D(x)) = 0\) for almost every \(x \in D\). Writing and passing to the limit yields \(u^* = 0\) almost everywhere in \(D \setminus \Omega^*\).

Consequently, \(u^* \in H^1_0(\Omega^*)\) and then \(u^*\) solves the P.D.E.

\[
\begin{cases}
-\Delta u^* = f & x \in \Omega \\
u^* = 0 & x \in \partial \Omega,
\end{cases}
\]

- Conclusion: \(\inf J = \lim_{k \to +\infty} J(\Omega_{nk}) = \int_\Omega (u^* - u_0)^2 \, dx\) whence the existence.
Contents

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Hadamard’s boundary variation method describes variations of a reference, Lipschitz domain $\Omega$ of the form:

$$\Omega \mapsto \Omega_\theta := (I + \theta)(\Omega),$$

for ‘small’ vector fields $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.

**Lemma**

*For $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ such that $\|\theta\|_{W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)} < 1$, the application $(I + \theta)$ is a Lipschitz diffeomorphism.*
Differentiation with respect to the domain: Hadamard’s method

Definition

Given a smooth domain $\Omega$, a scalar function $\Omega \mapsto J(\Omega) \in \mathbb{R}$ is said to be shape differentiable at $\Omega$ if the function

$$\mathcal{W}_{1,\infty} \left( \mathbb{R}^d, \mathbb{R}^d \right) \ni \theta \mapsto J(\Omega_\theta)$$

is Fréchet-differentiable at 0, i.e. the following expansion holds in the vicinity of 0:

$$J(\Omega_\theta) = J(\Omega) + J'(\Omega)(\theta) + o \left( \|\theta\| \mathcal{W}_{1,\infty} (\mathbb{R}^d, \mathbb{R}^d) \right).$$

The linear mapping $\theta \mapsto J'(\Omega)(\theta)$ is the shape derivative of $J$ at $\Omega$. 
Structure of shape derivatives (I)

Idea: The shape derivative $J'(\Omega)(\theta)$ of a ‘regular’ functional $J(\Omega)$ only depends on the normal component $\theta \cdot n$ of the vector field $\theta$.

Figure: At first order, a tangential vector field $\theta$, (i.e. $\theta \cdot n = 0$) only results in a convection of the shape $\Omega$, and it is expected that $J'(\Omega)(\theta) = 0$. 
Structure of shape derivatives (II)

Lemma

Let $\Omega$ be a domain of class $C^1$. Assume that the application

$$ C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \ni \theta \mapsto J(\Omega_\theta) \in \mathbb{R} $$

is of class $C^1$. Then, for any vector field $\theta \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ such that $\theta \cdot n = 0$ on $\partial \Omega$, one has:

$$ J'(\Omega)(\theta) = 0. $$

Corollary

Under the same hypotheses, if $\theta_1, \theta_2 \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ have the same normal component, i.e. $\theta_1 \cdot n = \theta_2 \cdot n$ on $\partial \Omega$, then:

$$ J'(\Omega)(\theta_1) = J'(\Omega)(\theta_2). $$
Actually, the shape derivatives of ‘many’ integral objective functionals $J(\Omega)$ can be put under the form:

$$J'(\Omega)(\theta) = \int_{\partial \Omega} v_\Omega (\theta \cdot n) \, ds,$$

where $v_\Omega : \partial \Omega \to \mathbb{R}$ is a scalar field which depends on $J$ and on the current shape $\Omega$.

This structure lends itself to the calculation of a descent direction: letting $\theta = -tv_\Omega n$, for a small enough descent step $t > 0$ yields:

$$J(\Omega_{t\theta}) = J(\Omega) - t \int_{\partial \Omega} v_\Omega^2 \, ds + o(t) < J(\Omega).$$
First examples of shape derivatives

**Theorem**

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and $f \in W^{1,1}(\mathbb{R}^d)$ be a fixed function. Consider the functional:

$$J(\Omega) = \int_{\Omega} f(x) \, dx;$$

then $J$ is shape differentiable at $\Omega$ and its shape derivative is:

$$\forall \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \quad J'(\Omega)(\theta) = \int_{\partial \Omega} f(\theta \cdot n) \, ds.$$
First examples of shape derivatives

**Figure:** Physical intuition: $J(\Omega_\theta)$ is obtained from $J(\Omega)$ by adding the blue area, where $\theta \cdot n > 0$, and removing the red area, where $\theta \cdot n < 0$. The process is ‘weighted’ by the integrand function $f$. 
First examples of shape derivatives

Remarks:

- This result is actually a particular case of the Transport (or Reynolds) theorem, used to derive the equations of conservation from conservation principles.

- It allows to calculate the shape derivative of the volume functional $\text{Vol} (\Omega) = \int_{\Omega} 1 \, dx$:

  $$\forall \theta \in W^{1,\infty} (\mathbb{R}^d, \mathbb{R}^d), \quad \text{Vol}' (\Omega) (\theta) = \int_{\partial \Omega} \theta \cdot n \, ds = \int_{\Omega} \text{div} (\theta) \, dx.$$  

  In particular, if $\text{div} (\theta) = 0$, the volume does not vary (at first order) when $\Omega$ is perturbed by $\theta$. 
First examples of shape derivatives

Proof: The formula proceeds from a change of variables:

\[ J(\Omega_\theta) = \int_{(I+\theta)(\Omega)} f(x) \, dx = \int_{\Omega} |\det(I + \nabla \theta)| \, f \circ (I + \theta) \, dx. \]

- The mapping \( \theta \mapsto \det(I + \nabla \theta) \) is Fréchet differentiable at 0, and:

\[ \det(I + \nabla \theta) = 1 + \text{div}(\theta) + o(\theta), \quad \frac{o(\theta)}{||\theta||_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)}} \rightarrow 0. \]

- If \( f \in W^{1,1}(\mathbb{R}^d) \), \( \theta \mapsto f \circ (I + \theta) \) is also Fréchet differentiable at 0 and:

\[ f \circ (I + \theta) = f + \nabla f \cdot \theta + o(\theta). \]

- Combining those three identities and Green’s formula lead to the result.
First examples of shape derivatives

**Theorem**

Let $\Omega_0 \subset \mathbb{R}^d$ be a bounded, regular enough domain, and $g \in W^{2,1}(\mathbb{R}^d)$ be a fixed function. Consider the functional:

$$J(\Omega) = \int_{\partial \Omega} g(x) \, ds;$$

then $J$ is shape differentiable at $\Omega_0$ and its shape derivative is:

$$J'(\Omega)(\theta) = \int_{\partial \Omega} \left( \frac{\partial g}{\partial n} + \kappa g \right) (\theta \cdot n) \, ds,$$

where $\kappa$ stands for the mean curvature of $\partial \Omega$.

**Example**:
The shape derivative of the perimeter $P(\Omega) = \int_{\partial \Omega} 1 \, ds$ is:

$$P'(\Omega)(\theta) = \int_{\partial \Omega} \kappa (\theta \cdot n) \, ds.$$
Towards more sophisticated examples

The examples of physical interest are those of PDE constrained shape optimization, i.e. one aims at minimizing functions which depend on $\Omega$ via the solution $u_\Omega$ of a PDE posed on $\Omega$, for instance (in most of the forthcoming examples):

$$J(\Omega) = \int_\Omega j(u_\Omega) \, dx + \int_{\partial \Omega} k(u_\Omega) \, ds,$$

where $u_\Omega$ is e.g. the solution to the linear elasticity system posed on $\Omega$, and $j, k$ are given functions.

Doing so borrows methods from optimal control theory (adjoint techniques, etc...)
The framework

- Henceforth, we rely on the model of the Laplace equation: the state $u_\Omega$ is solution to the system

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \quad \text{(Dirichlet B.C)} \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \quad \text{(Neumann B.C)} \end{cases}$$

where $\int_\Omega f \, dx = 0$ in the Neumann case.

- The associated variational formulation reads:

$$\forall v \in H_0^1(\Omega)/H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega fv \, dx = 0.$$

- We aim at calculating the shape derivative of $J(\Omega) = \int_\Omega j(u_\Omega) \, dx$, where $j : \mathbb{R} \to \mathbb{R}$ is a ‘smooth enough’ function.
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Eulerian and Lagrangian derivatives (I)

The rigorous way to address this problem requires a notion of differentiation of functions \( \Omega \mapsto u_\Omega \), which to a domain \( \Omega \) associate a function defined on \( \Omega \). One could think of two ways of doing so:

**The Eulerian point of view:** For a fixed \( x \in \Omega \), \( u'_\Omega(\theta)(x) \) is the derivative of the application \( \theta \mapsto u_{\Omega_\theta}(x) \).

**The Lagrangian point of view:** For a fixed \( x \in \Omega \), \( \dot{u}_\Omega(\theta)(x) \) is the derivative of the application \( \theta \mapsto u_{\Omega_\theta}((I + \theta)(x)) \).
Eulerian and Lagrangian derivatives (II)

- The Eulerian notion of shape derivative, however more intuitive, is more difficult to define rigorously. In particular, differentiating the boundary conditions satisfied by $u_\Omega$ is clumsy: even for $\theta$ ‘small’, $u_{\Omega (\theta)}(x)$ may not make any sense if $x \in \partial \Omega$!

- The Lagrangian notion of shape derivative can be rigorously defined, and lends itself to mathematical analysis.

- The Eulerian derivative will be defined after the Lagrangian derivative, from the formal use of chain rule over the expression $u_{(I+\theta)}(\Omega) \circ (I + \theta)$:

  $\forall x \in \Omega$, $u_\Omega(\theta)(x) = u'_\Omega(\theta)(x) + \nabla u_\Omega(x) \cdot \theta(x)$. 

Eulerian and Lagrangian derivatives (III)

Let $\Omega \mapsto u(\Omega) \in H^1(\Omega)$ be a function which to a domain, associates a function on the domain.

**Definition**

The function $u : \Omega \mapsto u(\Omega)$ admits a material, or Lagrangian derivative $\dot{u}(\Omega)$ at a given domain $\Omega$ provided the transported function

$$W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \ni \theta \mapsto \bar{u}(\theta) := u(\Omega_\theta) \circ (I + \theta) \in H^1(\Omega),$$

which is defined in the neighborhood of $0 \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, is differentiable at $\theta = 0$. 
Eulerian and Lagrangian derivatives (IV)

We are now in position to define the notion of Eulerian derivative.

**Definition**

The function $u : \Omega \mapsto u(\Omega)$ admits a **Eulerian derivative** $u'(\Omega)(\theta)$ at a given domain $\Omega$ in the direction $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ if it admits a material derivative $\dot{u}(\Omega)(\theta)$ at $\Omega$, and $\nabla u(\Omega) \cdot \theta \in H^1(\Omega)$. One defines then:

$$u'(\Omega)(\theta) = \dot{u}(\Omega)(\theta) - \nabla u(\Omega) \cdot \theta \in H^1(\Omega).$$ (1)
Eulerian and Lagrangian derivatives (V)

**Proposition**

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain, and suppose that $\Omega \mapsto u(\Omega)$ has a **Lagrangian derivative** $\dot{u}(\Omega)$ at $\Omega$. If $j : \mathbb{R} \to \mathbb{R}$ is regular enough, the function $J(\Omega) = \int_{\Omega} j(u(\Omega)) \, dx$ is then **shape differentiable** at $\Omega$, and:

$$
J'(\Omega)(\theta) = \int_{\Omega} (\dot{u}(\Omega)(\theta) j'(u(\Omega)) + j(u(\Omega)) \text{div}(\theta)) \, dx.
$$

for every $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.

If $u(\Omega)$ has a **Eulerian derivative** $u'(\Omega)$ at $\Omega$, one has the ‘chain rule’:

$$
J'(\Omega)(\theta) = \int_{\partial \Omega} j(u(\Omega)) \theta \cdot n \, ds + \int_{\Omega} j'(u(\Omega)) u'(\Omega)(\theta) \, dx.
$$

Derivative of $\Omega \mapsto \int_{\Omega} j(u_{\Omega})$ with respect to its first argument

Derivative of $\Omega \mapsto \int_{\Omega} j(u_{\Omega})$ with respect to its second argument

Practically speaking, one mainly uses this last expression of the **shape derivative**.
Eulerian and Lagrangian derivatives (VI)

Let us return to our problem of calculating the shape derivative of:

\[ J(\Omega) = \int_{\Omega} j(u_{\Omega}) \, dx, \text{ where } \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases}. \]

The following result characterizes the Lagrangian derivative of \( \Omega \mapsto u_{\Omega} \). Its proof can be adapted to many different PDE models:

**Theorem**

Let \( \Omega \subset \mathbb{R}^d \) be a smooth bounded domain. The application \( \Omega \mapsto u_{\Omega} \in H^1_0(\Omega) \) admits a Lagrangian derivative \( \dot{u}_{\Omega}(\theta) \), and for any \( \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \), \( \dot{u}_{\Omega}(\theta) \in H^1_0(\Omega) \) is the unique solution to:

\[ \begin{cases} -\Delta Y = -\Delta (\theta \cdot \nabla u_{\Omega}) & \text{in } \Omega \\ Y = 0 & \text{on } \partial \Omega \end{cases}. \]
Eulerian and Lagrangian derivatives (VII)

Idea of the proof: The variational problem satisfied by $u_{\Omega, \theta}$ is:

$$\forall v \in H^1_0(\Omega), \quad \int_{\Omega, \theta} \nabla u_{\Omega, \theta} \cdot \nabla v \, dx = \int_{\Omega, \theta} f v \, dx.$$ 

By a change of variables, the transported function $\bar{u}(\theta) := u_{\Omega, \theta} \circ (I + \theta)$ satisfies:

$$\forall v \in H^1_0(\Omega), \quad \int_{\Omega} A(\theta) \nabla \bar{u}(\theta) \cdot \nabla v \, dx = \int_{\Omega, \theta} |\det(I + \nabla \theta)| f v \, dx,$$

where $A(\theta) := |\det(I + \nabla \theta)| \nabla \theta \nabla \theta^T$.

This variational problem features a fixed domain and a fixed function space $H^1_0(\Omega)$, and only the coefficients of the formulation depend on $\theta$. 
Eulerian and Lagrangian derivatives (VIII)

- The problem can now be written as an equation for \( \bar{u}(\theta) \):
  \[
  \mathcal{F}(\theta, \bar{u}(\theta)) = \mathcal{G}(\theta),
  \]
  for appropriate definitions of the operators:
  - \( \mathcal{F} : W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \times H^1_0(\Omega) \to H^{-1}(\Omega) \),
  - \( \mathcal{G} : W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \to H^{-1}(\Omega) \).

- A use of the implicit function theorem provides the result.

- In particular, the Lagrangian derivative \( \dot{u}_\Omega(\theta) \) satisfies:
  \[
  \forall v \in H^1_0(\Omega), \quad \int_\Omega \nabla \dot{u}_\Omega(\theta) \cdot \nabla v \, dx = \int_\Omega \nabla (\theta \cdot \nabla u_\Omega) \cdot \nabla v \, dx.
  \]
Eulerian and Lagrangian derivatives (IX)

**Remark:** The Eulerian derivative of $u_\Omega$ can now be computed from its Lagrangian derivative. It satisfies:

$$
\begin{cases}
-\Delta U = 0 & \text{in } \Omega \\
U = - (\theta \cdot n) \frac{\partial u_\Omega}{\partial n} & \text{on } \partial \Omega,
\end{cases}
$$

or under variational form:

$$
\forall v \in H^1_0(\Omega), \quad \int_\Omega \nabla u'_\Omega(\theta) \cdot \nabla v \, dx = \int_{\partial \Omega} \frac{\partial u_\Omega}{\partial n} v \, \theta \cdot n \, ds.
$$

Using this formula in combination with:

$$
J'(\Omega)(\theta) = \int_{\partial \Omega} j(u_\Omega) \, \theta \cdot n \, ds + \int_\Omega j'(u_\Omega) u'_\Omega(\theta) \, dx
$$

will allow to express $J'(\Omega)(\theta)$ as a completely explicit expression of $\theta$: this is the adjoint method from optimal control theory.

**Goal of the adjoint method:** write $J'(\Omega)(\theta)$ under the form

$$
J'(\Omega)(\theta) = \int_{\partial \Omega} \text{something} \times (\theta \cdot n) \, ds
$$

where something does not depend on $\theta$. 
Eulerian and Lagrangian derivatives (X) : the adjoint method

**Idea** : ‘lift up’ the term of $J'(\Omega)(\theta)$ which features the Eulerian derivative of $u_\Omega$ by introducing an **adequate auxiliary problem**.

- Let $p_\Omega \in H^1_0(\Omega)$ be defined as the solution to the problem :
  \[
  \begin{cases}
    -\Delta p = -j'(u_\Omega) & \text{in } \Omega \\
    p = 0 & \text{on } \partial \Omega
  \end{cases}
  \]

- The variational formulation for $p_\Omega$ is :
  \[
  \forall v \in H^1_0(\Omega), \quad \int \nabla p_\Omega \cdot \nabla v \, dx = -\int j'(u_\Omega) v \, dx,
  \]

- ... to be compared with :
  \[
  J'(\Omega)(\theta) = \int_{\partial \Omega} j(u_\Omega) \theta \cdot n \, ds + \int_{\Omega} j'(u_\Omega) u'_\Omega(\theta) \, dx.
  \]
Eulerian and Lagrangian derivatives (XI) : the adjoint method

Thus, \( J'(\Omega)(\theta) = \int_{\partial \Omega} j(u_\Omega) \theta \cdot n \, ds + \int_{\Omega} j'(u_\Omega) u'_\Omega(\theta) \, dx \)

\(= \int_{\partial \Omega} j(u_\Omega) \theta \cdot n \, ds - \int_{\Omega} \nabla p_\Omega \cdot \nabla u'_\Omega(\theta) \, dx \)

\(= \int_{\partial \Omega} j(u_\Omega) \theta \cdot n \, ds - \int_{\partial \Omega} \frac{\partial p_\Omega}{\partial n} \frac{\partial u_\Omega}{\partial n} \theta \cdot n \, dx \)

where the variational problem for \( u'_\Omega \) :

\[ \forall v \in H^1_0(\Omega), \quad \int_{\Omega} \nabla u'_\Omega(\theta) \cdot \nabla v \, dx = \int_{\partial \Omega} \frac{\partial u_\Omega}{\partial n} v \theta \cdot n \, ds. \]

was used in the last line, with test function \( v = p_\Omega \).

\[ \forall \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \quad J'(\Omega)(\theta) = \int_{\partial \Omega} \left( j(u_\Omega) - \frac{\partial u_\Omega}{\partial n} \frac{\partial p_\Omega}{\partial n} \right) \theta \cdot n \, ds, \]
Eulerian and Lagrangian derivatives : summary

- Mathematically speaking, it is the rigorous way to assess the differentiability of shape functionals.

- Several techniques presented above (in particular the adjoint technique) exist in much more general frameworks than shape optimization, and pertain to the framework of optimal control theory.

- This way of obtaining shape derivatives is very involved in terms of calculations.

- In practice, a formal method, which is much simpler, allows to calculate shape derivatives : Céa’s method.
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Céa’s method

The philosophy of Céa’s method comes from optimization theory:

| Write the problem of minimizing $J(\Omega)$ as that of searching for the saddle points of a Lagrangian functional: |
| $\mathcal{L}(\Omega, u, p) = \int_{\Omega} j(u) \, dx + \int_{\Omega} (-\Delta u - f)p \, dx$, |
| where the variables $\Omega, u, p$ are independent. |

This method is formal: in particular, it assumes that we already know that $\Omega \mapsto u_\Omega$ is differentiable.
Céa’s method : the Neumann case (I)

Consider the following Lagrangian functional:

\[
\mathcal{L}(\Omega, v, q) = \underbrace{\int_{\Omega} j(v) \, dx}_{\text{Objective function}} + \underbrace{\int_{\Omega} \nabla v \cdot \nabla q \, dx - \int_{\Omega} f q \, dx}_{\text{Penalization of the 'constraint' } v = u_\Omega:} ,
\]

which is defined for any shape \( \Omega \in \mathcal{U}_{ad} \), and for any \( v, q \in H^1(\mathbb{R}^d) \), so that the variables \( \Omega, v \) and \( q \) are independent.

One observes that, evaluating \( \mathcal{L} \) with \( v = u_\Omega \), it comes:

\[
\forall q \in H^1(\mathbb{R}^d), \quad \mathcal{L}(\Omega, u_\Omega, q) = \int_{\Omega} j(u_\Omega) \, dx = J(\Omega).
\]
Céa’s method : the Neumann case (II)

For a fixed shape $\Omega$, we search for the saddle points $(u, p) \in \mathbb{R}^d \times \mathbb{R}^d$ of $\mathcal{L}(\Omega, \cdot, \cdot)$. The first-order necessary conditions read:

1. $\forall q \in H^1(\mathbb{R}^d), \quad \frac{\partial \mathcal{L}}{\partial q}(\Omega, u, p)(q) = \int_{\Omega} \nabla u \cdot \nabla q \, dx - \int_{\Omega} fq \, dx = 0.$

2. $\forall v \in H^1(\mathbb{R}^d), \quad \frac{\partial \mathcal{L}}{\partial v}(\Omega, u, p)(v) = \int_{\Omega} j'(u) \cdot v \, dx + \int_{\Omega} \nabla v \cdot \nabla p \, dx = 0.$
Céa's method: the Neumann case (III)

**Step 1**: Identification of $u$:

$$\forall q \in H^1(\mathbb{R}^d), \quad \int_{\Omega} \nabla u \cdot \nabla q \, dx - \int_{\Omega} f q \, dx = 0.$$

- Taking $q$ as any $C^\infty$ function $\psi$ with compact support in $\Omega$ yields:

$$\forall \psi \in C^\infty_c(\Omega), \quad \int_{\Omega} \nabla u \cdot \nabla \psi \, dx - \int_{\Omega} f \psi \, dx = 0 \Rightarrow -\Delta u = f \text{ in } \Omega.$$

- Now taking $q$ as a $C^\infty$ function $\psi$ and using Green's formula:

$$\forall \psi \in C^\infty_c(\mathbb{R}^d), \quad \int_{\partial \Omega} \frac{\partial u}{\partial n} \psi \, ds = 0 \Rightarrow \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega.$$

**Conclusion**: $u = u_\Omega$. 
Céa’s method: the Neumann case (IV)

**Step 2: Identification of \( p \):**

\[
\forall v \in H^1(\mathbb{R}^d), \quad \int_\Omega j'(u)v + \int_\Omega \nabla v \cdot \nabla p \, dx = 0.
\]

- Taking \( v \) as any \( C^\infty \) function \( \psi \) with compact support in \( \Omega \) yields:

\[
\forall \psi \in C^\infty_c(\Omega), \quad \int_\Omega \nabla p \cdot \nabla \psi \, dx + \int_\Omega j'(u)\psi \, dx = 0
\]

\[
\Rightarrow -\Delta u = -j'(u_\Omega) \quad \text{in} \ \Omega.
\]

- Now taking \( v \) as a \( C^\infty \) function \( \psi \) and using Green’s formula:

\[
\forall \psi \in C^\infty(\mathbb{R}^d), \quad \int_{\partial \Omega} \frac{\partial p}{\partial n} \varphi \, ds = 0 \Rightarrow \frac{\partial p}{\partial n} = 0 \quad \text{on} \ \partial \Omega.
\]

**Conclusion:** \( p = p_\Omega \), solution to

\[
\begin{cases}
-\Delta p = -j'(u_\Omega) & \text{in} \ \Omega \\
\frac{\partial p}{\partial n} = 0 & \text{on} \ \partial \Omega
\end{cases}
\]
Céa’s method : the Neumann case (V)

**Step 3 : Calculation of the shape derivative $J'(\Omega)(\theta)$ :**

- We go back to the fact that:
  $$\forall q \in H^1(\mathbb{R}^d), \quad \mathcal{L}(\Omega, u_\Omega, q) = \int_\Omega j(u_\Omega) \, dx.$$  

- Differentiating with respect to $\Omega$ yields:
  $$\forall \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \quad J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, u_\Omega, q)(\theta) + \frac{\partial \mathcal{L}}{\partial v}(\Omega, u_\Omega, q)(u'_\Omega(\theta)),$$

  where $u'_\Omega(\theta)$ is the Eulerian derivative of $\Omega \mapsto u_\Omega$ (assumed to exist).

- Now, choosing $q = p_\Omega$ produces, since $\frac{\partial \mathcal{L}}{\partial v}(\Omega, u_\Omega, p_\Omega) = 0$:
  $$J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, u_\Omega, p_\Omega)(\theta).$$
Céa’s method: the Neumann case (VI)

This last (partial) derivative amounts to the shape derivative of a functional of the form:

$$\Omega \mapsto \int_\Omega f(x) \, dx,$$

where $f$ is a fixed function.

Using the derivative of $\Omega \mapsto \int_\Omega f$, we end up with:

$$\forall \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d),$$

$$J'(\Omega)(\theta) = \int_{\partial \Omega} (j(u_\Omega) + \nabla u_\Omega \cdot \nabla p_\Omega - fp_\Omega) \theta \cdot n \, ds.$$
Céa’s method: the Dirichlet case (I)

- We now consider the problem of derivating:

\[ J(\Omega) = \int_\Omega j(u_\Omega) \, dx, \text{ where } \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \]

- **Warning**: When the state \( u_\Omega \) satisfies essential boundary conditions, i.e. boundary conditions that are tied to the definition space of functions (here, \( H^1_0(\Omega) \)), an additional difficulty arises.

- We can no longer use the Lagrangian

\[ \mathcal{L}(\Omega, v, q) = \int_\Omega j(v) \, dx + \int_\Omega \nabla v \cdot \nabla q \, dx - \int_\Omega fv \, dx, \]

since it would have to be defined for \( v, q \in H^1_0(\Omega) \).

- In this case, the variables \( \Omega, v, q \) would not be independent.
Céa’s method : the Dirichlet case (II)

**Solution:** Add an extra variable $\mu \in H^1(\mathbb{R}^d)$ to the Lagrangian to penalize the boundary condition: for all $v, q, \mu \in H^1(\mathbb{R}^d)$:

$$
\mathcal{L}(\Omega, v, q, \mu) = \int_{\Omega} j(v) \, dx + \int_{\Omega} (-\Delta v - f)q \, dx + \int_{\partial \Omega} \mu v \, ds.
$$

- Objective function where $u_\Omega$ is replaced by $v$
- Penalization of the ‘constraint’ $-\Delta v = f$
- Penalization of the ‘constraint’ $v = 0$ on $\partial \Omega$

By Green’s formula, $\mathcal{L}$ rewrites:

$$
\mathcal{L}(\Omega, v, q, \mu) = \int_{\Omega} j(v) \, dx + \int_{\Omega} \nabla v \cdot \nabla q \, dx - \int_{\Omega} f q \, dx + \int_{\partial \Omega} \left( \mu v - \frac{\partial v}{\partial n} q \right) \, ds.
$$

Of course, evaluating $\mathcal{L}$ with $v = u_\Omega$, it comes:

$$
\forall q, \mu \in H^1(\mathbb{R}^d), \quad \mathcal{L}(\Omega, u_\Omega, q) = \int_{\Omega} j(u_\Omega) \, dx.
$$
Céa’s method: the Dirichlet case (III)

For a fixed shape \( \Omega \), we look for the saddle points \( (u, p, \nu) \in (H^1(\mathbb{R}^d))^3 \) of the functional \( \mathcal{L}(\Omega, \cdot, \cdot, \cdot) \). The first-order necessary conditions are:

- For all \( q \in H^1(\mathbb{R}^d) \),
  \[
  \frac{\partial \mathcal{L}}{\partial q}(\Omega, u, p, \lambda)(q) = \int_\Omega \nabla u \cdot \nabla q \, dx - \int_\Omega f q \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial n} q \, ds = 0.
  \]

- For all \( v \in H^1(\mathbb{R}^d) \),
  \[
  \frac{\partial \mathcal{L}}{\partial v}(\Omega, u, p, \lambda)(v) = \int_\Omega j'(u) \cdot v \, dx + \int_\Omega \nabla v \cdot \nabla p \, dx + \int_{\partial \Omega} \left( \lambda v - \frac{\partial v}{\partial n} p \right) \, ds = 0.
  \]

- For all \( \mu \in H^1(\mathbb{R}^d) \),
  \[
  \frac{\partial \mathcal{L}}{\partial \mu}(\Omega, u, p, \lambda)(\mu) = \int_{\partial \Omega} \mu u \, ds = 0.
  \]
Céa’s method: the Dirichlet case (IV)

Step 1: Identification of $u$:

\[ \forall q \in H^1(\mathbb{R}^d), \quad \int_{\Omega} \nabla u \cdot \nabla q \, dx - \int_{\Omega} f q \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial n} q \, ds = 0. \]

- Taking $q$ as any $C^\infty$ function $\psi$ with compact support in $\Omega$ yields:

\[ \forall \psi \in C^\infty_c(\Omega), \quad \int_{\Omega} \nabla u \cdot \nabla \psi \, dx = \int_{\Omega} f \psi \, dx \Rightarrow -\Delta u = f \text{ in } \Omega. \]

- Using $\frac{\partial L}{\partial \mu}(\Omega, u, p, \lambda)(\mu) = 0$ for any $\mu = \psi \in C^\infty_c(\mathbb{R}^d)$ yields:

\[ \forall \psi \in C^\infty_c(\mathbb{R}^d), \quad \int_{\partial \Omega} \psi u \, dx = 0 \Rightarrow u = 0 \text{ on } \partial \Omega. \]

Conclusion: $u = u_\Omega$. 
Céa’s method : the Dirichlet case (V)

Step 2 : Identification of $p$ :

$$\forall v \in H^1(\mathbb{R}^d), \quad \int_{\Omega} j'(u) \cdot v \, dx + \int_{\Omega} \nabla v \cdot \nabla p \, dx + \int_{\partial\Omega} \left( \lambda v - \frac{\partial v}{\partial n} p \right) \, ds = 0.$$  

Taking $q$ as any $C^\infty$ function $\psi$ with compact support in $\Omega$ yields :

$$\forall \psi \in C^\infty_c(\Omega), \quad \int_{\Omega} \nabla p \cdot \nabla \psi \, dx + \int_{\Omega} j'(u) \cdot \psi \, dx = 0$$  

$$\Rightarrow -\Delta p = -j'(u_\Omega) \text{ in } \Omega.$$  

Now taking $v$ as a $C^\infty$ function $\psi$ and using Green’s formula :

$$\forall \psi \in C^\infty_c(\mathbb{R}^d), \quad \int_{\partial\Omega} \frac{\partial p}{\partial n} \psi \, ds + \int_{\partial\Omega} \left( \lambda \psi - \frac{\partial \psi}{\partial n} p \right) \, ds = 0.$$
Céa’s method: the Dirichlet case (VI)

Step 2 (continued):

- Varying the normal trace $\frac{\partial \psi}{\partial n}$ while imposing $\psi = 0$ on $\partial \Omega$, one gets:

$$p = 0 \text{ on } \partial \Omega.$$ 

**Conclusion**: $p = p_\Omega$, solution to

$$\begin{cases}
-\Delta p = -j'(u_\Omega) & \text{in } \Omega \\
p = 0 & \text{on } \partial \Omega
\end{cases}$$

- In addition, varying the trace of $\psi$ on $\partial \Omega$ while imposing $\frac{\partial \psi}{\partial n} = 0$:

$$\lambda_\Omega = -\frac{\partial p_\Omega}{\partial n} \text{ on } \partial \Omega.$$
Céa’s method : the Dirichlet case (VII)

Step 3 : Calculation of the shape derivative $J'(\Omega)(\theta)$ :

- We go back to the fact that:
  \[
  \forall q, \mu \in H^1(\mathbb{R}^d), \quad L(\Omega, u_\Omega, q, \mu) = \int_\Omega j(u_\Omega) \, dx.
  \]

- Differentiating with respect to $\Omega$ yields, for all $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$:
  \[
  J'(\Omega)(\theta) = \frac{\partial L}{\partial \Omega}(\Omega, u_\Omega, q, \mu)(\theta) + \frac{\partial L}{\partial v}(\Omega, u_\Omega, q, \mu)(u'_\Omega(\theta)),
  \]
  where $u'_\Omega(\theta)$ is the Eulerian derivative of $\Omega \mapsto u_\Omega$.

- Taking $q = p_\Omega$, $\mu = \lambda_\Omega$ produces, since $\frac{\partial L}{\partial v}(\Omega, u_\Omega, p_\Omega, \lambda_\Omega) = 0$:
  \[
  J'(\Omega)(\theta) = \frac{\partial L}{\partial \Omega}(\Omega, u_\Omega, p_\Omega, \lambda_\Omega)(\theta).
  \]
Céa’s method : the Dirichlet case (VIII)

Again, this \textit{(partial)} derivative amounts to the shape derivative of a functional of the form:

\[
\Omega \mapsto \int_{\Omega} f(x) \, dx,
\]

where \(f\) is a \textit{fixed} function.

Using the derivative of \(\Omega \mapsto \int_{\Omega} f\) (and after some calculation), we end up with:

\[
\forall \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \quad J'(\Omega)(\theta) = \int_{\partial \Omega} \left( j(u_\Omega) - \frac{\partial u_\Omega}{\partial n} \frac{\partial p_\Omega}{\partial n} \right) \theta \cdot n \, ds,
\]
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Reminders on the PDE in Fluid Mechanics

Definition

Let $\Omega$ be a bounded open set in $\mathbb{R}^d$ with a smooth boundary and let $\mathbf{u}$ and $\mathbf{v}$, two vector fields in $H^1(\Omega, \mathbb{R}^d)$.

- We define the stretching tensor $\varepsilon$ of the vector field $\mathbf{u}$ by

$$
\varepsilon(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) = \left( \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right)_{1 \leq i, j \leq d}.
$$

- We define the doubly contracted product of two stretching tensors $\varepsilon(\mathbf{u})$ and $\varepsilon(\mathbf{v})$ by

$$
\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) = \frac{1}{4} \sum_{i,j=1}^{d} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).
$$

- $|\varepsilon(\mathbf{u})|^2 = \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u})$.

- If $p \in L^2(\Omega)$ stands for the fluid pressure at every point of $\Omega$, if $\mathbf{u}$ denotes the fluid velocity at every point of $\Omega$, then one defines the stress tensor $\sigma(\mathbf{u}, p) \in L^2(\Omega, S^d(\mathbb{R}))$ by

$$
\sigma(\mathbf{u}, p) = -pl_d + 2\mu \varepsilon(\mathbf{u}).
$$
Reminders on the PDE in Fluid Mechanics

Properties of these tensors

Lemma

Let $\Omega$, an open set of $\mathbb{R}^d$. For every $u \in H^1(\Omega, \mathbb{R}^d)$, one has

$$
\|\varepsilon(u)\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} \quad \text{and} \quad \|\text{div}(u)\|_{L^2(\Omega)} \leq d\|\nabla u\|_{L^2(\Omega)}.
$$

Proof: using the well-known inequality $(a + b)^2 \leq 2(a^2 + b^2)$ valid for all real numbers $a$, $b$ and by definition $\varepsilon(u)$, there holds

$$
\|\varepsilon(u)\|_{L^2(\Omega)}^2 = \frac{1}{4} \int_{\Omega} \sum_{1 \leq i, j \leq d} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \, dx \leq \frac{1}{2} \sum_{1 \leq i, j \leq d} \int_{\Omega} \left( \left( \frac{\partial u_i}{\partial x_j} \right)^2 + \left( \frac{\partial u_j}{\partial x_i} \right)^2 \right) \, dx
$$

$$
= \frac{1}{2} \left( \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \right),
$$

whence the result.
### Lemma

Let $\Omega$, an open set of $\mathbb{R}^d$. For every $u \in H^1(\Omega, \mathbb{R}^d)$, one has

$$\|\varepsilon(u)\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} \quad \text{et} \quad \|\text{div}(u)\|_{L^2(\Omega)} \leq d\|\nabla u\|_{L^2(\Omega)}.$$

**Proof (continued):**

Similarly, one has

$$\|\text{div}(u)\|_{L^2(\Omega)}^2 = \int_{\Omega} \left( \sum_{1 \leq i \leq d} \frac{\partial u_i}{\partial x_i} \right)^2 \, dx = \sum_{1 \leq i,j \leq d} \int_{\Omega} \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} \, dx$$

$$\leq \frac{1}{2} \sum_{1 \leq i,j \leq d} \int_{\Omega} \left( \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \left( \frac{\partial u_j}{\partial x_j} \right)^2 \right) \, dx = \frac{2d}{2} \sum_{i=1}^{d} \int_{\Omega} \left( \frac{\partial u_i}{\partial x_i} \right)^2 \, dx$$

$$\leq d \sum_{1 \leq i,j \leq d} \int_{\Omega} \left( \frac{\partial u_i}{\partial x_j} \right)^2 \, dx = d\|\nabla u\|_{L^2(\Omega)}^2.$$
Reminders on the PDE in Fluid Mechanics

Theorem (Korn inequality)
Let \( \Omega \) be an open set in \( \mathbb{R}^d \) of class piecewise \( C^1 \). Then, there exists a constant \( C_\Omega > 0 \) such that for every function \( u \in H^1(\Omega, \mathbb{R}^d) \), one has

\[
\|u\|_{H^1} \leq C_\Omega (\|u\|_{L^2(\Omega)} + \|\varepsilon(u)\|_{L^2(\Omega)}).
\]

Introduce the functional spaces

\[
\begin{align*}
L^2_0(\Omega) &= \{ p \in L^2(\Omega) \mid \int_{\Omega} p \, dx = 0 \} \\
\mathcal{V}(\Omega) &= \{ \varphi \in \mathcal{D}(\Omega, \mathbb{R}^d) \mid \text{div}(\varphi) = 0 \} \\
V(\Omega) &= \{ v \in H^1_0(\Omega, \mathbb{R}^d) \mid \text{div}(v) = 0 \} = \overline{\mathcal{V}(\Omega)}^{H^1_0(\Omega, \mathbb{R}^d)}
\end{align*}
\]

Theorem (De Rham lemma)
Let \( \Omega \) be a Lipschitz bounded connected open set in \( \mathbb{R}^d \). Let \( f \in H^{-1}(\Omega, \mathbb{R}^d) \) s.t.

\[
\langle f, \varphi \rangle_{H^{-1}, H^1_0} = 0 \quad \forall \varphi \in \mathcal{V}(\Omega).
\]

Then, there exists a unique function \( p \) in \( L^2_0(\Omega) \) such that \( f = \nabla p \).
Reminders on the PDE in Fluid Mechanics

The Navier-Stokes system

\[
\begin{cases}
-\mu \nabla^2 u + \nabla p + (u \cdot \nabla) u = f & x \in \Omega, \\
\text{div } u = 0 & x \in \Omega \\
u = u_0 & x \in E \\
u = 0 & x \in \Gamma \\
-p n + 2\mu \varepsilon(u) \cdot n = -p_0 n & x \in \Gamma,
\end{cases}
\]

where \( p_0 \in \mathbb{R} \) and \( u_0 \in H^{1/2}(E, \mathbb{R}^d) \).

Theorem (Existence and uniqueness)

Assume that \( u_0 \in H^1_0(E, \mathbb{R}^d) \) and \( f \in L^{3/2}(\Omega, \mathbb{R}^d) \).
There exist \( \mu_0 > 0 \) and \( \varepsilon_0 > 0 \) such that if \( \mu \geq \mu_0 \) or if \( \|u_0\|_{H^1_0(E)} \leq \varepsilon_0 \), then the Navier-Stokes problem above has a unique solution \( (u, p) \in H^1(\Omega) \times L^2(\Omega) \). Moreover, there exists \( C > 0 \) depending only on \( \Omega \) s.t. the solution \( u \) verifies

\[
\|u\|_{H^1(\Omega)}^2 \leq C (\|u_0\|_{H^1_0(E)}^2 + \|u_0\|_{H^1_0(E)} + \|f\|_{L^{3/2}(\Omega)}^2).
\]
Reminders on the PDE in Fluid Mechanics

Variational formulation of this problem

Introduce the functional spaces:

\[ W_0(\Omega) \overset{\text{déf}}{=} \left\{ (v, q) \in H^1(\Omega, \mathbb{R}^d) \times L^2(\Omega) : v = 0 \text{ on } E \cup \Gamma \right\} \]

\[ Z_{u_0}(\Omega) \overset{\text{déf}}{=} \left\{ (v, q) \in H^1(\Omega, \mathbb{R}^d) \times L^2(\Omega) : v = u_0 \text{ on } E \text{ et } v = 0 \text{ sur } \Gamma \right\} , \]

endowed with the usual associated topologies.

Set \( \tilde{p} = p - p_0 \). The variational formulation of the previous Navier-Stokes system writes

\[
\left\{ \begin{array}{l}
\text{Find } (u, \tilde{p}) \in Z_{u_0}(\Omega) \text{ s.t. } \forall (w, \psi) \in W_0(\Omega), \\
\int_{\Omega} (2\mu \varepsilon(u) : \varepsilon(w) + (u \cdot \nabla)u \cdot w - \tilde{p} \text{div } w) \, dx = 0 \\
\int_{\Omega} \psi \text{div } u \, dx = 0.
\end{array} \right.
\]
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A toy problem: the optimal shape of a duct

Problem and modeling

Consider a fluid driven by the previous Navier-Stokes system, of viscosity $\mu$ flowing inside a cannula-shaped pipe/duct. For instance, we look for the optimal shape of a pipeline.

The optimal design problem writes

$$\inf_{\Omega \in \mathcal{O}_{ad}} J(\Omega) \quad \text{where} \quad J(\Omega) = 2\mu \int_{\Omega} |\varepsilon(u)|^2 \, dx,$$

with

- $\mu$, the viscosity of the fluid, $u$ the velocity of the fluid at every point (e.g. given by the Navier-Stokes equations),
- $\mathcal{O}_{ad}$ is the set of admissible shapes, for instance, $E$ (inlet) and $S$ (outlet) are fixed and we look for the lateral boundary such that $\Omega$ open connected subset of $\mathbb{R}^d$ with $|\Omega| = V_0$ (given).
A toy problem: the optimal shape of a duct

Computation of the shape derivative (I)

✓ Dirichlet boundary conditions on the lateral free boundary.
✓ Replacing $p$ by $p - p_0$ leads to assume that the right-hand side on $S$ is 0.

Let us use Céa’s method to compute the derivative of $J$. Introduce the Lagrangian function

$$\mathcal{L}(\Omega, u, p, v, q) = 2\mu \int_\Omega |\varepsilon(u)|^2 \, dx - P(\Omega, u, p, v, q),$$

where

$$P(\Omega, u, p, v, q) = \int_\Omega (2\mu \varepsilon(u) : \varepsilon(v) + [(u \cdot \nabla)u]v - p \text{div}(v)) \, dx - \int_\Omega \text{div}(u) q \, dx + \int_\Gamma \mu \cdot u \, ds$$
A toy problem: the optimal shape of a duct

Computation of the shape derivative (II)

For a fixed shape $\Omega$, we look for the saddle points of the functional $\mathcal{L}$. The first-order necessary conditions are:

$$\forall \delta p \in C_0^\infty(\Omega, \mathbb{R}^d), \quad \frac{\partial \mathcal{L}}{\partial p}(\Omega, u, p, \nu, q, \nu)(\delta p) = \int_\Omega \delta p \text{div}(\nu) = 0.$$ 

and

$$\forall \delta u \in C_0^\infty(\Omega, \mathbb{R}^d), \quad \frac{\partial \mathcal{L}}{\partial u}(\Omega, u, p, \nu, q, \nu)(\delta u) = 0.$$ 

This last equation rewrites

$$2\mu \int_\Omega \varepsilon(u) : \varepsilon(\delta u) \, dx - \int_\Omega (2\mu \varepsilon(\delta u) : \varepsilon(\nu) + [(\delta u \cdot \nabla)u + (u \cdot \nabla)(\delta u)]\nu) \, dx - \int_\Omega \text{div}(\delta u)q \, dx + \int_\Gamma \mu \cdot \delta u \, ds = 0.$$
A toy problem: the optimal shape of a duct

Computation of the shape derivative (III)

We will use the following useful integration by parts formula.

**Theorem (Existence and uniqueness)**

Let $y \in H^1(\Omega, \mathbb{R}^d)$ and $z \in H^2(\Omega, \mathbb{R}^d)$. Then,

$$2 \int_{\Omega} \varepsilon(z) : \varepsilon(y) \, dx = - \int_{\Omega} (\triangle z + \nabla \text{div} z) \cdot y \, dx + 2 \int_{\partial \Omega} \varepsilon(z) n \cdot y \, ds.$$

**Step 1:** Identification of $u$ : similar to what was done in the elliptic case. In particular, the function $\frac{\partial L}{\partial \mu}$ vanishes at the saddle point yielding to $u = 0$ on $\partial \Omega$. 
A toy problem : the optimal shape of a duct

**Computation of the shape derivative (IV)**

**Step 2 : Identification of the adjoint state :** We are now in position to identify the adjoint state.

\[
\forall \delta p \in C_0^\infty(\Omega, \mathbb{R}^d), \quad \frac{\partial L}{\partial p}(\Omega, u, p, v, q, \nu)(\delta p) = \int_{\Omega} \delta p \text{div}(v) = 0
\]

yields

\[
\text{div } v = 0 \quad \text{in } \Omega.
\]

Using

\[
\forall \delta u \in C_0^\infty(\Omega, \mathbb{R}^d), \quad \frac{\partial L}{\partial u}(\Omega, u, p, v, q, \nu)(\delta u) = 0.
\]

**Exercise :** integrating by parts this relation, and taking first \(\delta u\) with compact support in \(\Omega\) and then varying on \(\Gamma\) yields that \((v, q)\), solves the following PDE :

\[
\begin{cases}
- \mu \triangle v + (\nabla u)^\top, v - (\nabla v)u + \nabla q = -2\mu \triangle u & x \in \Omega \\
\text{div } v = 0 & x \in \Omega \\
v = 0 & x \in E \cup \Gamma \\
- qn + 2\mu \varepsilon(v) \cdot n + (u \cdot n)v - 4\mu \varepsilon(u) \cdot n = 0 & x \in S.
\end{cases}
\]

\(\rightarrow\) Boundary conditions obtained exactly as in the elliptic case.
A toy problem: the optimal shape of a duct

Computation of the shape derivative ($V$)

Assume for the moment that the adjoint system is well posed on the space

$$W_0(\Omega) = \left\{ (v, q) \in (H^1(\Omega))^3 \times L^2(\Omega) : v = 0 \text{ sur } E \cup \Gamma \right\}.$$ 

Recall that

$$J(\Omega) = 2\mu \int_{\Omega} |\varepsilon(u)|^2 \, dx$$

According to the chain's rule shape derivative formula, we claim that

$$J'(\Omega)(\theta) = 4\mu \int_{\Omega} \varepsilon(u) : \varepsilon(u') \, dx + 2\mu \int_{\Gamma} |\varepsilon(u)|^2 (\theta \cdot n) \, ds.$$ 

Proposition

Assume that $\Omega$ is of class $C^2$. One has

$$J'(\Omega)(\theta) = 2\mu \int_{\Gamma} \left( \varepsilon(u) : \varepsilon(v) - |\varepsilon(u)|^2 \right) (\theta \cdot n) \, ds.$$
A toy problem: the optimal shape of a duct

Proof: according to the "chain derivation rules", there holds

$$J'(\Omega)(\theta) = 4\mu \int_{\Omega} \varepsilon(u) : \varepsilon(u') \, dx + 2\mu \int_{\Gamma} |\varepsilon(u)|^2 (\theta \cdot n) \, ds$$

where $u'$ denotes the unique solution of the system

\[
\begin{cases}
-\mu \Delta u' + \nabla u \cdot u' + \nabla u' \cdot u + \nabla p' = 0 & x \in \Omega \\
\text{div } u' = 0 & x \in \Omega \\
u' = 0 & x \in E \\
u' = -\frac{\partial u}{\partial n}(\theta \cdot n) & x \in \Gamma \\
-p' n + 2\mu\varepsilon(u') \cdot n = 0 & x \in S.
\end{cases}
\]

Using the Green formula, we infer

\[
J'(\Omega)(\theta) = 4\mu \int_{\Omega} \varepsilon(u) : \varepsilon(u') \, dx + 2\mu \int_{\Gamma} |\varepsilon(u)|^2 (\theta \cdot n) \, ds
\]

\[
= -2\mu \int_{\Omega} ((\Delta u + \nabla \text{div } u) \cdot u') \, dx + 4\mu \int_{\partial\Omega} \varepsilon(u) n \cdot u' \, ds
\]

\[
+ 2\mu \int_{\partial\Omega} |\varepsilon(u)|^2 (\theta \cdot n) \, ds
\]
A toy problem: the optimal shape of a duct

Let us multiply the main equation of the adjoint state by $u'$ and then integrate on $\Omega$. One gets

$$
-\mu \int_\Omega \nabla v \cdot u' \, dx + \int_\Omega \nabla q \cdot u' \, dx + \int_\Omega (\nabla u)^T v \cdot u' \, dx
$$

$$
- \int_\Omega (\nabla v) u \cdot u' \, dx = -2\mu \int_\Omega \Delta u \cdot u' \, dx.
$$

Integrating by parts and using the boundary conditions on $u'$ and $v$ yields

$$
\int_\Omega (2\mu \varepsilon(u') : \varepsilon(v) - (\nabla v) u' \cdot u + (\nabla u') u \cdot v) \, dx - \int_S \sigma(v, q) n \cdot u' \, ds
$$

$$
+ \int_S ((u \cdot v)(u' \cdot n) - (u \cdot n)(u' \cdot v)) \, ds - \int_\Gamma \sigma(v, q) n \cdot u' \, ds = -2\mu \int_\Omega \Delta u \cdot u' \, dx.
$$
A toy problem: the optimal shape of a duct

In a similar way, let us multiply the main equation of the PDE satisfied by $u'$ by $v$ and then integrate on $\Omega$. One gets

$$-\mu \int_{\Omega} \Delta u' \cdot v \, dx + \int_{\Omega} \nabla \dot{p} \cdot v \, dx + \int_{\Omega} \nabla u' \cdot u \cdot v \, dx + \int_{\Omega} \nabla u \cdot u' \cdot v \, dx = 0.$$ 

Integrating by parts and using the boundary conditions on $u'$ and $v$ yields

$$\int_{\Omega} (2\mu \varepsilon(u') : \varepsilon(v) + (\nabla u')u \cdot v - (\nabla v)u' \cdot u) \, dx$$

$$+ \int_{S} (-\sigma(u', \dot{p})n \cdot v + (u \cdot v)(u' \cdot n)) \, ds = 0.$$
A toy problem: the optimal shape of a duct

Let us come back to the computation of the shape derivative:

\[ J'(\Omega)(\theta) = -2\mu \int_{\Omega} ((\nabla^2 u + \nabla \text{div} u) \cdot u') \, dx + 4\mu \int_{\partial \Omega} \varepsilon(u) n \cdot u' \, ds \]

\[ + 2\mu \int_{\partial \Omega} |\varepsilon(u)|^2 (\theta \cdot n) \, ds \]

\[ = A + 4\mu \int_{\partial \Omega} \varepsilon(u) n \cdot u' \, ds + 2\mu \int_{\partial \Omega} |\varepsilon(u)|^2 (\theta \cdot n) \, ds, \]

where we have set \( A := -2\mu \int_{\Omega} ((\nabla^2 u + \nabla \text{div} u) \cdot u') \, dx. \)

According to the previous computations, we claim that

\[ A = \int_{\Gamma \cup S} (q n - 2\mu \varepsilon(v) n) \cdot u' \, ds - \int_{S} (u \cdot n)(v \cdot u') \, ds. \]
A toy problem: the optimal shape of a duct

Consequently and since \((v, q)\) solves the adjoint state, one has

\[
J'(\Omega)(\theta) = \int_{\Gamma \cup S} (qn - 2\mu \varepsilon(v)n) \cdot \dot{u} \, ds - \int_{\mathcal{S}} (u \cdot n)(v \cdot \dot{u}) \, ds \\
+ 4\mu \int_{\mathcal{S} \cup \Gamma} \varepsilon(u)n \cdot \dot{u} \, ds + 2\mu \int_{\Gamma} |\varepsilon(u)|^2 (\theta \cdot n) \, ds \\
= \int_{\Gamma} (qn - 2\mu \varepsilon(v)n + 4\mu \varepsilon(u)n) \cdot \dot{u} \, ds + 2\mu \int_{\Gamma} |\varepsilon(u)|^2 (\theta \cdot n) \, ds \\
= -\int_{\Gamma} \left( (qn - 2\mu \varepsilon(v)n + 4\mu \varepsilon(u)n) \cdot \frac{\partial u}{\partial n} + 2\mu |\varepsilon(u)|^2 \right) (\theta \cdot n) \, ds
\]

This is not exactly the expected form. One can use a “cosmetic” lemma to conclude.

**Lemma**

Let \(x\) and \(y\), two regular vector fields of \(\mathbb{R}^3\) s.t. \(x|_{\Gamma} = y|_{\Gamma} = 0\) and \(\text{div } x = \text{div } y = 0\) in \(\Omega\). Let \(\theta \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)\). Thus, there holds on \(\Gamma\):

1. \((\nabla x)\theta \cdot n = ((\nabla x)n \cdot n)(\theta \cdot n) = 0\),
2. \(\varepsilon(x) : \varepsilon(y) = \varepsilon(x) : (\varepsilon(y)(n \otimes n)) = (\varepsilon(x)n) \cdot (\varepsilon(y)n)\),
3. \((\varepsilon(x)n) \cdot ((\nabla y)\theta) = (\varepsilon(x)n) \cdot ((\nabla y)n)(\theta \cdot n) = (\varepsilon(x)n) \cdot (\varepsilon(y)n)(\theta \cdot n)\).

With the convention that \(n \otimes n = \sum_{i,j=1}^3 n_i n_j\).
A toy problem: the optimal shape of a duct

The first equality proves that $q n \cdot \frac{\partial u}{\partial n} = 0$ sur $\Gamma$.

The first identity shows that $\Gamma$, $\varepsilon(u) n \cdot \frac{\partial u}{\partial n} = |\varepsilon(u)|^2$.

Finally, using the second and the third identity allows to show that on $\Gamma$,

$$(\varepsilon(v)n) \cdot \frac{\partial u}{\partial n} = \varepsilon(u) : \varepsilon(v).$$

As a consequence, the shape gradient of $J$ writes:

$$\nabla J(\Omega) = 2\mu \left( \varepsilon(u) : \varepsilon(v) - |\varepsilon(u)|^2 \right) n.$$
On the well-posed character of the adjoint problem

Recall that

\[ W_0(\Omega) = \left\{ (v, q) \in (H^1(\Omega))^3 \times L^2(\Omega) : v = 0 \text{ sur } E \cup \Gamma \right\}. \]

**Proposition**

The variational formulation of the adjoint state writes

\[
\begin{align*}
\text{Find } (v, q) \in W_0(\Omega) \text{ s.t. } & \forall (w, \psi) \in W_0(\Omega), \\
& \int_{\Omega} \left( 2\mu \varepsilon(v) : \varepsilon(w) + (\nabla w) u \cdot v + (\nabla u) w \cdot v - q \, \text{div} \, w \right) \, dx = 4\mu \int_{\Omega} \varepsilon(u) : \varepsilon(w) \, dx \\
& \int_{\Omega} \nabla \psi \cdot v \, dx = 0.
\end{align*}
\]
On the well-posed character of the adjoint problem

**Proof:** some hints.
Consider $w \in C_0^\infty(\Omega)$ s.t.

\[
\int_{\Omega} \left( -\mu \Delta v + (\nabla u)^\top v - (\nabla v)u + \nabla q + 2\mu \Delta u \right) \cdot w \, dx \\
+ \int_{S} (-qn + \mu \varepsilon(v)n + (u \cdot n)v - 2\mu \varepsilon(u)n) \cdot w \, ds = 0.
\]

Applying Green formula yields

\[
\int_{S} (w \cdot v)(u \cdot n) \, ds - \int_{\Omega} (\nabla v)u \cdot w \, dx = \int_{\Omega} (\nabla w)u \cdot v \, dx,
\]

and similarly,

\[
\int_{S} q(w \cdot n) \, ds - \int_{\Omega} \nabla q \cdot w \, dx = \int_{\Omega} q \text{div } w \, dx.
\]

Finally, a change of index shows that

\[
\int_{\Omega} (\nabla u)^\top v \cdot w \, dx = \int_{\Omega} (\nabla u w) \cdot v \, dx.
\]
On the well-posed character of the adjoint problem

Thus, the integration by parts formula allows to write

\[ -\mu \int_{\Omega} (\nabla v + \nabla \text{div } v) \cdot w \, dx + 2\mu \int_{\partial \Omega} \varepsilon(v) n \cdot w \, ds = 2\mu \int_{\Omega} \varepsilon(v) : \varepsilon(w) \, dx \]

and

\[ -\mu \int_{\Omega} \Delta u \cdot w \, dx + 2\mu \int_{S} \varepsilon(u) n \cdot w \, ds = 2\mu \int_{\Omega} \varepsilon(u) : \varepsilon(w) \, dx. \]

Using a standard density argument yields that for \((w, \psi) \in W_0(\Omega)\), there holds

\[
\begin{cases}
\int_{\Omega} (2\mu \varepsilon(v) : \varepsilon(w) + (\nabla w) u \cdot v + (\nabla u) w \cdot v - q \text{div } w) \, dx = 4\mu \int_{\Omega} \varepsilon(u) : \varepsilon(w) \, dx \\
\int_{\Omega} \nabla \psi \cdot v \, dx = 0,
\end{cases}
\]

which concludes the proof.
On the well-posed character of the adjoint problem

Theorem (Existence and uniqueness of the adjoint state)

Assume that $d = 2$ or $d = 3$ and that $\Omega$ is of class $C^2$. Introduce $(v, q)$, solution of the adjoint state

$$
\begin{align*}
- \mu \Delta v + (\nabla u)^\top v - \nabla v u + \nabla q &= -2\mu \Delta u & x \in \Omega \\
\text{div} \, v &= 0 & x \in \Omega \\
v &= 0 & x \in E \cup \Gamma \\
-qn + 2\mu \varepsilon(v) \cdot n + (u \cdot n)v - 4\mu \varepsilon(u) \cdot n &= 0 & x \in S.
\end{align*}
$$

If the viscosity $\mu$ is large enough, then the adjoint problem has a unique solution $(v, q) \in H^1(\Omega) \times L^2(\Omega)$. 
On the well-posed character of the adjoint problem

Proof: without loss of generality, assume that \( d = 3 \). To prove the existence and uniqueness of a solution, let us apply Lax-Milgram theorem.

Introduce the functional spaces:

\[
\mathcal{D}(\Omega) : \text{space of } C^\infty \text{ real valued functions having a compact support in } \Omega.
\]
\[
\mathcal{V}(\Omega) = \{ u \in \mathcal{D}(\Omega) \mid \text{div } u = 0 \}
\]
\[
\mathcal{V}(\Omega) = \text{completion of } \mathcal{V} \text{ in the space } \{ u \in H^1(\Omega) \mid u = 0 \text{ on } E \cup \Gamma \}.
\]

Remark: note that one has also

\[
\mathcal{V}(\Omega) = \{ u \in H^1(\Omega) : \text{div } u = 0 \text{ and } u = 0 \text{ on } E \cup \Gamma \}.
\]

According to De Rham lemma, let us work on the space \( \mathcal{V}(\Omega) \). Using the variational formulation of the adjoint state, \( v \) satisfies in a distributional sense

\[
\forall w \in \mathcal{V}(\Omega), \quad \alpha(v, w) = \langle \ell, w \rangle,
\]

where \( \alpha \) and \( \ell \) are respectively the bilinear and linear forms:

\[
\alpha(v, w) = \int_{\Omega} (2\mu \varepsilon(v) : \varepsilon(w) + (\nabla w)u \cdot v + (\nabla u)w \cdot v) \, dx
\]
\[
\langle \ell, w \rangle = 4\mu \int_{\Omega} \varepsilon(u) : \varepsilon(w) \, dx,
\]

whenever it exists.
On the well-posed character of the adjoint problem

Proof (continued) : To apply Lax–Milgram theorem, it remains to show that $\alpha$ is continuous, coercive on $V(\Omega)^2$, and that $\ell$ is continuous on $V(\Omega)$.

- $\alpha$ is continuous in $H^1$. The first integral term is dominated by the product of the $H^1$-norms of $v$ and $w$ using the equivalence of the norms $\|\nabla\|_{L^2}$ and $\|\varepsilon(\cdot)\|_{L^2}$ in $H^1_0$ (Korn inequality) and the Cauchy-Schwarz inequality. Let us show how to estimate the second and the third integral terms. There holds

$$\left|\int_{\Omega} (\nabla w) u \cdot v \, dx\right| \leq \sum_{1 \leq i,j \leq 3} \int_{\Omega} \left| \frac{\partial w_i}{\partial x_j} u_j v_i \right| \, dx \leq \frac{1}{2} \|\nabla w\|_{L^2}^2 + \frac{1}{2} \|u\|_{L^4}^2 \|v\|_{L^4}^2.$$  

The conclusion follows by using that the injection $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is continuous since $d \leq 3$.

- $\ell$ is continuous. It is enough to write that for all $v \in H^1(\Omega, \mathbb{R}^3)$ and according to the Cauchy-Schwarz inequality, there holds

$$|\langle \ell, v \rangle| \leq 4 \mu \|\varepsilon(v)\|_{L^2(\Omega, \mathbb{R}^3)} \|\varepsilon(u)\|_{L^2(\Omega, \mathbb{R}^3)}.$$  

We conclude by using the Korn inequality.
On the well-posed character of the adjoint problem

Proof (continued) :

- \( \alpha \) is coercive in \( H^1 \). Similar computations prove that for all \( \mathbf{v} \in H^1(\Omega, \mathbb{R}^3) \):

\[
\alpha(\mathbf{v}, \mathbf{v}) \geq 2\mu \int_{\Omega} |\varepsilon(\mathbf{v})|^2 \, dx - \|\mathbf{u}\|_{H^1(\Omega, \mathbb{R}^3)}^2 \|\mathbf{v}\|_{H^1(\Omega, \mathbb{R}^3)}^2.
\]

Now, using one more time the Korn inequality, there exists \( C > 0 \) (depending only on \( \Omega \)) s.t.

\[
\alpha(\mathbf{v}, \mathbf{v}) \geq \left( 2C\mu - \|\mathbf{u}\|_{H^1(\Omega, \mathbb{R}^3)}^2 \right) \|\mathbf{v}\|_{H^1(\Omega, \mathbb{R}^3)}^2.
\]

\( \alpha \) is then coercive provided that \( \mu \) be chosen large enough.

Lax-Milgram theorem shows the existence of \( \mathbf{v} \in H^1(\Omega, \mathbb{R}^3) \) satisfying the variational formulation in \( \mathcal{V}(\Omega) \) and then by density in \( \mathcal{V}(\Omega) \).

The use of De Rham lemma yields the existence of a distribution \( q \in L^2(\Omega) \) solution of the adjoint state.

The uniqueness of \( q \) is obvious, according to the boundary conditions on the outlet \( S \). indeed, it there would exist two solutions \( q_1 \) and \( q_2 \), and since the velocity \( \mathbf{v} \) is unique, then the function \( q_1 - q_2 \) would verify at the same time \( \nabla(q_1 - q_2) = 0 \) in \( \Omega \) and \( p_1 = p_2 \) on \( S \).
Contents

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The generic numerical algorithm

**Gradient algorithm**: Start from an initial shape $\Omega^0$, 
**For** $n = 0, \ldots$ **convergence,**

1. Compute the state $u_{\Omega^n}$ (and possibly the adjoint $p_{\Omega^n}$) of the considered PDE system on $\Omega^n$.
2. Compute the shape gradient $J'(\Omega^n)$ thanks to the previous formula, and infer a descent direction $\theta^n$ for the cost functional.
3. **Advec**t the shape $\Omega^n$ according to this displacement field for a small pseudo-time step $\tau^n$, so as to get

   $$\Omega^{n+1} = (I + \tau^n \theta^n)(\Omega^n).$$
One possible implementation

- Each shape $\Omega^n$ is represented by a simplicial mesh $\mathcal{T}^n$ (i.e. composed of triangles in 2d and of tetrahedra in 3d).

- The Finite Element method is used on the mesh $\mathcal{T}^n$ for computing $u_{\Omega^n}$ (and $p_{\Omega^n}$). The descent direction $\theta^n$ is then calculated using the theoretical formula for the shape derivative of $J(\Omega)$.

- The shape advection step $\Omega^n \xrightarrow{(I+\tau^n\theta^n)} \Omega^{n+1}$ is performed by pushing the nodes of $\mathcal{T}^n$ along $\tau^n\theta^n$, to obtain the new mesh $\mathcal{T}^{n+1}$.

Deformation of a mesh by relocating its nodes to a prescribed final position.
Numerical examples (I)

- In the context of linear elasticity, one aims at minimizing the compliance $C(\Omega)$ of a cantilever beam:
  \[ C(\Omega) = \int_{\Omega} A e(u_\Omega) : e(u_\Omega) \, dx. \]
- An equality constraint on the volume $\text{Vol}(\Omega)$ of shapes is imposed.

Minimization of the compliance of a cantilever (from [?]; code available on [?]).
Numerical examples (II)

- In the context of fluid mechanics (Stokes equations), one aims at minimizing the viscous dissipation $D(\Omega)$ in a pipe:

$$D(\Omega) = 2\mu \int_{\Omega} |\varepsilon(u_\Omega)|^2 \, dx.$$ 

- A volume constraint is imposed by a fixed penalization of the function $D(\Omega)$ - i.e. the minimized function is $D(\Omega) + \ell \text{Vol}(\Omega)$, where $\ell$ is a fixed Lagrange multiplier.

Minimization of the viscous dissipation inside a pipe.
Numerical examples (II)

- Still in **fluid mechanics**, minimization of the **viscous dissipation** $D(\Omega)$ in a double pipe.
- A **volume constraint** is imposed by a *fixed* penalization of the function $D(\Omega)$.

Minimization of the viscous dissipation inside a double pipe.
I - The difficulty of mesh deformation:

- When the shape is explicitly meshed, an update of the mesh is necessary at each step $\Omega^n \mapsto (I + \theta^n)(\Omega^n) = \Omega^{n+1}$: the new mesh $\mathcal{T}^{n+1}$ is obtained by relocating each node $x \in \mathcal{T}^n$ to $x + \tau^n \theta^n(x)$.

- This may prove difficult, partly because it may cause inversion of elements, resulting in an invalid mesh.

Pushing nodes according to the velocity field may result in an invalid configuration.

- For this reason, mesh deformation methods are generally preferred for accounting for ‘small displacements’.
Numerical issues and difficulties (II)

II - Velocity extension:

- A descent direction \( \theta = -v_\Omega n \) from a shape \( \Omega \) is inferred from the shape derivative of \( J(\Omega) \):

\[
J'(\Omega)(\theta) = \int_{\partial\Omega} v_\Omega(\theta \cdot n) \, ds.
\]

- The new shape \((I + \theta)(\Omega)\) only depends on these values of \(\theta\) on \(\partial\Omega\).

- For many reasons, in numerical practice, it is crucial to extend \(\theta\) to \(\Omega\) (or even \(\mathbb{R}^d\)) in a ‘clever’ way.

  (for instance, deforming a mesh of \(\Omega\) using a ‘nice’ vector field \(\theta\) defined on the whole \(\Omega\) may considerably ease the process)

- The ‘natural’ extension of the formula \(\theta = -v_\Omega n\), which is only legitimate on \(\partial\Omega\) may not be a ‘good’ choice.
Numerical issues and difficulties (III)

III - Velocity regularization:

- Taking $\theta = -\nu_n \Omega$ on $\partial \Omega$ may produce a very irregular descent direction, because of:
  - numerical artifacts arising during the finite element analyses.
  - an inherent lack of regularity of $J'(\Omega)$ for the problem at stake.

- In numerical practice, it is often necessary to smooth this descent direction so that the considered shapes stay regular.

Irregularity of the shape derivative in the very sensitive problem of drag minimization of an airfoil (Taken from [?]). In one iteration, using the unsmoothed shape derivative of $J(\Omega)$ produces large undesirable artifacts.
Numerical issues and difficulties (IV)

A popular idea: extend AND regularize the velocity field

Suppose we aim at extending the scalar field \( v_\Omega : \partial \Omega \to \mathbb{R} \) to \( \Omega \).

**Idea**: \((\approx\) Laplacian smoothing) Trade the ‘natural’ inner product over \( L^2(\partial \Omega) \) for a more regular inner product over functions on \( \Omega \).

**Example**: Search the extended / regularized scalar field \( V \) as:

Find \( V \in H^1(\Omega) \) s.t. \( \forall w \in H^1(\Omega) \),

\[
\alpha \int_\Omega \nabla V \cdot \nabla w \, dx + \int_\Omega Vw \, dx = \int_{\partial \Omega} v_\Omega w \, ds.
\]

The regularizing parameter \( \alpha \) controls the balance between the fidelity of \( V \) to \( v_\Omega \) and the intensity of smoothing.
Numerical issues and difficulties (IV)

- The resulting scalar field $V$ is inherently defined on $\Omega$ and more regular than $v_\Omega$.

- Multiple other regularizing problems are possible, associated to different inner products or different function spaces.

- A similar process allows:
  - to extend $v_\Omega$ to a large computational box $D$ (an inner product over functions defined on $D$ is used),
  - to extend the vector velocity $\theta = -v_\Omega n$ to $\Omega \setminus D$ (an inner product over vector functions is used, e.g. that of linear elasticity).
The choice of a numerical method for shape optimization has to reach a tradeoff between numerical accuracy and robustness:

- The more accurate the representation of the boundaries of shapes, the more accurate the mechanical analyses performed on shapes (computation of $u_{\Omega^n}$, $p_{\Omega^n}$, etc...), and the more accurate the computation of descent directions.
- ... But the more tedious and error-prone the advection step between shapes $\Omega^n \rightarrow \Omega^{n+1}$. 
Numerical results: optimal shape of a cannula