SOME REMARKS ON A THEOREM OF M.J.GREENBERG.

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IHES/Sept 79/M/303

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The theorem we are alluding to is the main result in [G1], to wit :

(0.) Let $f = (f_1,\dots,f_r) \in (R[X_1,\dots, X_s])^r$, where R is an excellent henselian discrete valuation ring. Then there exist $0 \leq c$, $d \in \mathbb{R}$, $0 < d \leq 1$, depending on f, such that for all $x \in R^s$ and all real $0 < \alpha \leq 1$: if $|fx| < \alpha^c.d$, then there is $y \in R^s$ satisfying $fy = 0$, and $|x-y| < \alpha$.

Here $|\ |$ denotes the (multiplicative) valuation $R \longrightarrow \mathbb{R}$ as well as the max-norm on R^s.

Usually, of course, the result is stated in terms of an additive valuation, or even congruence mod. powers of the maximal ideal of R. And it is in this form that Greenberg's theorem is to be regarded as the first example of an "Approximation Theorem" (in M. Artin's sense) for a class of excellent noetherian local hensel rings; see [A] . But the particularly simple formula of (0.) , telling us how much better we have to approximate 0 by fx in order to find an actual root of f in a prescribed neighborhood of x, is not retained in the more general situations. (It should be noted, however, that G. Pfister and D.Popescu, following ideas of M. Artin, showed that the lifting of approximate zeros is always controlled by <u>some</u> "approximation function", once approximation of formal roots is at all possible. See [PP] and [KP], Kap. II.) Thus it seems interesting to stick to the one-dimensional case, and try to elucidate a bit the nature of the constants c and d of (0.).

As for both c and d (i.e., their best possible choices), they are trivially invariants of the ideal $f.R[X]$; $X = (X_1, \ldots, X_s)$. But since (0.) clearly holds for $f \in K[X]^r$ (where K is the quotient field of R) it seems clumsy not to invoke the geometry over K. In so doing, we deliberately sacrifice all invariance properties of d (e.g., multiply f by a non-unit in R) and will from now on restrict attention to the constant c which does remain an invariant of the ideal $f.K[X]$. But since an automorphism of $K[X]$ may switch points of R^s to points with non-integral components, c is certainly not an invariant of the algebraic geometry over K. The peculiar role which $R \subset K$ plays in (0.) is not germane to algebraic geometry.

When investigating c we may as well suppose K to be complete with repect to $| \ |$, because by (0.) we can a fortiori approximate zeros in the completion arbitrarily well by K-rational points.

K being complete, power series suggest themselves (see, e.g., the use of Puiseux Expansion in (3.6)below). If we want to let f consist of power series rather than polynomials, we have to insist, for (0.) to make sense, that they be convergent on all of R^s. That is to say, we are naturally led to replace $K[X]$ with the Banach algebra $K\langle X\rangle$ of "strictly convergent" power series, and use non-archimedean analysis as initiated by Tate[T].

At this point there is no reason to exclude non-archimedean fields which are not discretely (but still real-) valued -e.g , the case K algebraically closed will be interesting.

It turns out (see §1 below) that (0.) generalizes to the situation $f \in K\langle X\rangle^r$, and c can easily be seen to be invariant under automorphisms of $K\langle X\rangle$. Whether invariance holds for isomorphisms $K\langle X\rangle/(f) \cong K\langle Y\rangle/(g)$ as well, is one of the interesting problems to which we do not know the general answer yet.

Substituting R^s, in the statement of (0.), by smaller and smaller neighborhoods of a fixed $z \in R^s$ produces in the limit a constant $c(f,z)$ which can be shown to be an invariant of the local ring $K\{X\}/(f_z)$ of f at z. The question arises whether the old ("global") c is not greater than $\sup\{c(f,z) : z \in R^s\}$. We can prove this (and thereby solve the invariance problem for the global c) in some special situations; on the other hand, we do not know any counterexample.

In the last section of this paper we present a collection of examples (mostly plane curves) which contains some quite unexpected phenomena, but generally confirms the power series approach as very much appropriate.

ACKNOWLEDGEMENTS

In picking up the necessary non-archimedean analysis, I was helped by conversations with W. Bartenwefer, S. Bosch and R. Kiehl (on the phone). - Since most of what follows is already in [S] , thanks are due to M. Kneser for proposing the subject and taking constant interest in its development.

§1 THE THEOREM

If not specified otherwise, K will denote a field which is complete with respect to the non-trivial non-archimedean (multiplicative) valuation $| \ | : K \longrightarrow \mathbb{R}$. R is its valuation ring. $K\langle X\rangle$ resp. $K\{X\}$, denote the ring of strictly convergent (i.e., the coefficients converge to 0 in K), resp. convergent (in some neighborhood of the origin) power series in $X = (X_1, \ldots, X_s)$. The Banach norm on $K\langle X\rangle$ is again denoted by $| \ | : |\sum_{\nu \in \mathbb{N}^s} a_\nu \cdot X^\nu| = \max |a_\nu|$; so is the max-norm on K^s.

Let $f = (f_1, \ldots, f_r) \in K\langle X\rangle^r$; $\mathfrak{a}$ the ideal of $K\langle X\rangle$ generated by f ; and $A = K\langle X\rangle/\mathfrak{a}$ the "affinoid " algebra of f.

(1.1) Theorem : There exist real constants $c \geq 0$ and d, $0 < d \leq 1$, depending on f, such that for all $x \in R^s$, and all real α, $0 < \alpha < 1$, if $|fx| < \alpha^c . d$ then there is $y \in R^s$, $fy = 0$, such that $|x-y| < \alpha$.

Proof : Following Greenberg [G1], we let $f \in R\langle X\rangle$ for simplicity; assume without loss of generality that $\mathfrak{a}$ is prime, and induct on $\dim A$, the case $\dim A = -1$ being trivial. There are then two cases to consider.

<u>Case 1</u>. A analytically separable over K

This means A is absolutely reduced or, equivalently, $A \hat{\otimes}_K K^{1/p}$ is reduced, if $p = \mathrm{char}(K) > 0$. (See [BK] 4.2, 4.4)

For a multi-index $i = (i_1, \ldots, i_n)$, $1 \leq i_1 < \ldots < i_n \leq r$, $1 \leq n \leq s$, let D_i denote the ideal generated in $R\langle X\rangle$ by the $n \times n$ minors of the jacobian $(\partial f_{i_\nu} / \partial X_j)$, and F_i the conductor

$((f_{i_1}, \ldots, f_{i_n}) : (f_1, \ldots, f_r))$ in $R\langle X\rangle$. Then for a system $g = (g_1, \ldots, g_k)$, $g_j \in \Sigma D_i.F_i$, generating $\Sigma D_i.F_i.K\langle X\rangle$, the irreducible components of the locus of (f,g) have smaller dimension than **A** since we are in the separable case. Thus there are constants c', d', for (f,g), say $c' \geqq 1$ and $d' < 1$, and we may take a shortcut through Greenberg's original argument in this case by invoking R. Elkik's version of Hensel's lemma (see [E], p. 555-558, noting her remarks p.560 and 587 f.). This tells us that for all $0 < \alpha \leqq 1$, $x \in R^s$ such that $|fx| < \alpha < |gx|^2$, there is $y \in R^s$, $fy = 0$, satisfying $|x-y| < \alpha/|gx|$. So we may choose $c = 2c'$ and $d = d'^2$ for f.

<u>Case 2.</u> **A** analytically inseparable over K.

There is $g \in R^{1/p}\langle X\rangle$ which is not in $\mathfrak{a}K^{1/P}\langle X\rangle$, but $g^P \in f.R\langle X\rangle$. Pick a complete field $K^P \subset K' \subset K$ containing all the coefficients of g^P, such that K' is a K^P-Banach space of countable type. Let $\{b_n\}$ be a topological K^P-base of K'.
(In this situation even topological p-bases always exist : [K],1.4.)
Then $\{b_n\}$ is t-orthogonal for some $0 < t \leqq 1$ by the closed graph theorem ([R], 3.7), i.e for any $\{u_n\} \subset K^P$ which make $\Sigma u_n . b_n$ convergent one has $|\Sigma u_n b_n| \geqq t. \max |u_n b_n|$.

Therefore we can write : $g^P = \Sigma g_n^P . b_n$, where $g_n \in K\langle X\rangle$.
Then not all $g_n \in \mathfrak{a}$ because ideals in $K^{(1/P)}\langle X\rangle$ are closed, [GR1] I§4.
So by induction we get c', d' for the system (f, g_{n_o}), for some n_o, and clearly $c = p.c'$ and $d = t . \min(1, |b_{n_o}|). d'^P$ will do for f.

REMARKS.

1) From G. Pfister I learned the particularly nice idea to write g^P in a K^P-base; see [KP], p. 125.

2) In a Oberwolfach talk, recorded in [O], W.D. Geyer gave a proof of the inseparable case of (0.) which, in essence amounts to a differentiated version of the argument given in case 2 above. Suffice it to say that Geyer's proof can also be carried over to the analytic setting of (1.1) using the (absolute) module of continuous differentials $\Omega(K/K^P)$, which is constructed by an appropriate

generalization of [BK] § 2. (One has to be a little careful about working with the general Banach tensor products, in case $[K : K^p]$ is ∞.)

3) M. Kneser [Kn] gives an elementary proof of (0.) in characteristic 0, covering also the "global" version of the theorem as in [G2]. He inducts on a partial ordering of the systems of polynomials. Using Weierstraß Preparation for $K\langle X\rangle$([GR_1], I§2), thereby making it slightly less elementary, this proof can be modified to yield (1.1) in characteristic 0.

4)The following informations I owe to R. Transier:

a) Writing congruences mod.$\{x \in R : v(x) \geq \alpha\}$, and letting d be in the value group Γ of the (additive) valuation v, Greenberg's argument should yield (the additive form of) (0.) in characteristic 0, at least if $\Gamma \hookrightarrow \mathbb{R}^n$, i.e. v has finite rank.

b) A model -theoretic approach to (0.) and various strong approximation theorems is given in [BD].

(1.2) Corollary : Any f without zeros in R^s has the set $\{|fx| : x \in R^s\}$ bounded away from zero.

Write $c(f,R^s)$ for the minimal $c \geq 0$ such that there is d verifying the claim of (1.1) . So $c(f,R^s) = 0$ in (1.2).

(1.3) Corollary : If f has only simple zeros in R^s, then $c(f,R^s) \leq 1$.
Proof : Apply Elkik's lemma to f, using g is bounded away from 0. .

Similarly one gets :
(1.4) If the singular set of f on R^s is smooth, and (f,g)(where g is as in Elkik's lemma) gives it the reduced structure, then $c(f,R^s) \leq 2$.

In the introduction we anticipated :
(1.5) Proposition : $c(f, R^s)$ is an invariant of $\mathfrak{a}$, and does not change under automorphisms of $K\langle X\rangle$.
Proof: The first claim is clear since each series in $K\langle X\rangle$ has bounded denominators. For the second note that every automorphism is an isometry. [GR1] , I§1.3, and thus induces an automorphism of R^s. In fact, even d is invariant under automorphisms.

(1.6) Question : Is $c(f, R^s)$ invariant under different representations of **A** ?

At present we do not see how to attack this problem directly.

§2 THE POINTWISE CONSTANT

We add to the notation introduced so far : $c(f,U)$, for any $U \subset R^s$, is the minimal $c \geq 0$ such that the claim of (1.1) holds for some $0 < d \leq 1$, with R^s replaced by U everywhere. Note that $c(f,U)$ may be ∞. On the other hand, the pointwise constant
$c(f,z) = \inf\{ c(f,U) : z \in U \subset R^s \text{ open}\}$
is certainly finite for any $z \in R^s$ by (1.1). Moreover from the ultrametric property of $| \ |$ it follows that in the definition of $c(f,z)$ it suffices to inf over all small polydisks around z. Trivially one has $c(f,z) = c(f_z, 0)$ for the Taylor expansion f_z of f at z.
Note that $c(f,z)$ is defined whenever f is convergent near z.

(2.1) Proposition : $c(f,z)$ is an invariant of the local ring $K\{X\}/f_z . K\{X\}$.

Proof : That $c(f,z)$ is an invariant of $f.K\{X\}$, follows from (1.5) and the fact that $f(aX) \in K\langle X\rangle^r$ for suitable $a \in K^*$. Similarly, the coordinate series of a substitution automorphism φ of $K\{X\}$ may be supposed to lie in $R\{X\}$, which gives $c({}^{\varphi}f,U) = c(f,\varphi U)$ for small U,and hence invariance under φ ,because φ is locally invertible. Finally, the general case can be reduced to that of an automorphism by [GR1], II § 3.5, using (2.3) below.

(2.2) Corollary : If z is a simple point of f , then $c(f,z) = 1$.
Proof : Implicit functions !

For any $g \in K\langle X\rangle$ write $m(g)$ for the lowest degree of a monomial occuring in g; and $m(f) = \min\{ m(f_i) : 1 \leq i \leq r\}$. We already used:
(2.3) Proposition : $c(f,0) \geq m(f)$.
This follows easily from
(2.4) Lemma : There is a constant $\gamma \in \mathbb{R}$, depending on f, such that for all sufficiently small β , all $0 < \alpha < \beta\gamma \leq \gamma$, there is $x \in R^s$ satisfying $|x| < \alpha/\gamma$ and $|x-y| \geq \alpha$ for all $y \in R^s$, $fy = 0$, $|y| < \beta$.

Proof : Define the analytic tangent cone to f at 0: $\mathcal{C}$, to be the set of all $x \in K^s$ such that there exist sequences $\{y_i\} \subset R^s$, all $fy_i = 0$, and $\{a_i\} \subset K$ satisfying $\lim y_i = 0$ and $\lim a_i . y_i = x$.
This is a closed proper $(f \neq 0)$ subset of K^s which approximates the locus of f near 0. Hence we find the x's of the lemma in the cone over a small ("γ") open set not meeting $\mathcal{C}$.

(2.5) Corollary : For $z \in R^s$:

$fz \neq 0 \quad \Leftrightarrow \quad c(f,z) = 0 \quad \Leftrightarrow \quad 0 \leqq c(f,z) < 1$

z simple pt. of $f \quad \Leftrightarrow \quad c(f,z) = 1 \quad \Leftrightarrow \quad 1 \leqq c(f,z) < 2$

Here the proof of the last equivalence requires picking suitable coordinates, and a refinement of (2.4), which says that $c(f,0) \geqq m(g)$ for a subsystem g of f such that the tangent cone of the complement of g in f is strictly bigger than that of f.

Note : We do not know any singularity with $2 < c(f,z) < 3$.
No conjecture is made that the following statement is valid in general :
(2.6) Proposition : $c(f,R^s) = \sup \{c(f,z) : z \in R^s\}$, in each of the following cases :

(i) K is locally compact
(ii) f has only finitely many singular points in R^s

Proof : Only "$\leqq$" requires proof, and (i) is straightforward.
Letting g be as in Elkik's lemma, (f,g) is bounded away from 0, by (1.1), outside small disks around the finitely many singularities.(ii) now follows easily.
Using (2.5) we see that (2.6) holds also in the special situation of (1.4).
From (2.1) we get :

(2.7) Corollary : Under any of the assumptions in (2.6) ,question (1.6) has a positive answer.

§ 3 EXAMPLES AND A FORMULA

(3.1) If K is algebraically closed and $f \in K\langle X\rangle$ a hypersurface, then $c(f,0) = m(f)$.

Proof : Choose X_s transversal to f at 0 , and apply Weierstraß Preparation in $K\{X\}$ to find "$\leqq$". "$\geqq$" follows from (2.3).

(3.2) In the situation of (3.1), $c(f,R^s) \leqq \widetilde{m}(f)$, where $\widetilde{m}(f)$ is the highest degree of a monomial in f having the same $K\langle X\rangle$ -norm as f.

Proof : By [GR1], p. 407 , we can choose coordinates such that Weierstraß Preparation in $K\langle X\rangle$, applied to f , gives a polynomial of degree $\widetilde{m}(f)$.

Note that, usually, $\widetilde{m}(f) \neq \sup\{m(f_z) : z \in R^s\}$, so that (3.2) tends to be far from best possible. Also note that (3.2) implies $c(f,R^s) \leqq \deg f$, if f is a polynomial.

(3.3) Question : Is $c(f,R^s) \leqq \deg f$, for $f \in K[X]$, over any field K

(at least, if f is homogeneous) ?

(3.1) generalizes to systems of equations (giving the maximal $m(f_i)$ as c), provided that the tangent cones of the single equations are "wide enough apart" from each other. Unfortunately we do not have any smooth theorem to offer on this; but there should be equality between $c(f,0)$ and the maximal finite component of Hironaka's invariant ν^*([H],p.155), for a class of "nice transversal" complete intersections over algebraically closed fields. On the other hand, no general result can be expected.:

(3.4) Example : Over any complete K ; for all integers $k,\ell,m,n \geq 1$ such that $k<\ell$, $m<n$, $k \leqq m$, $nk > \ell m$, letting $f(X_1,X_2) = (X_2^k - X_1^\ell , X_2^m - X_1^n) \in K[X]^2$, we get $c(f,R^2) = c(f,0) = m.\ell/k$.

In particular, c takes on non-integral values in $\mathbb{Q}$, > 4 , regardless of the ground field. The computation of (3.4) is straightforward. In general, of course, non algebraically closed fields present a serious lack of rational points :

(3.5) Example : Let $m,n > 1$. Assume $a \neq 0$ is not an m-th power in K. Then for $f = X_2^m - a.X_1^{m.n}$ one finds
$c(f,R^2) = c(f,0) \geq m.n$ over K, while over $K(a^{1/m}, \mu_m)$
$c(f,R^2) = c(f,0) = m$.

For plane curves, this example is typical in that, after a finite field extension, one gets some kind of a definite constant. More precisely, for $f \in K\langle X_1,X_2\rangle$, one finds a finite extension $L \supset K$ such that, in $L\{X_1, X_2\}$, f decomposes into a product of (locally) convergent power series f_i, each of which is irreducible in $\bar{K}\{X_1, X_2\}$; for some algebraically closed complete field $\bar{K} \supset L$. The f_i are the absolutely analytically irreducible components of f, or the "branches" of f near the origin. Thus the problem of determining $c(f,0)$ for a plane curve f, naturally splits up into two fairly different parts :

Case 1

$f \in K\{X_1, X_2\}$ is absolutely analytically irreducible . Changing co-ordinates so that X_1 is a tangential parameter, and therefore X_2 transversal, and applying Weierstraß Preparation in $K\{X\}$, we get the Puiseux Expansion of f , provided that $p = \mathrm{char}(K)$ does not divide the multiplicity $m = m(f)$. (See, e.g., [P] ,[AM](3.5))

$$f(X_1, X_2) = \prod_{\zeta^m = 1} (X_2 - P(\zeta(a. X_1)^{1/m})) , \text{ some } a \in K^x,$$

$$P(T) = \sum_{k>m} b_k T^k \in K\{T\}, \quad \gcd\{k : b_k \neq 0\} = 1.$$

The factor a may be deleted by invariance of $c(f,0)$.
We let $k_0 = m$, $k_{i+1} = \min\{k > k_i : b_k \neq 0 \text{ and } m_i = \gcd\{k_0,\dots,k_i\} \nmid k\}$

Here $i = 1, \dots, g$, where $m_g = 1$.

These constants k_i, m_i are closely related to (but not the same as) the "characteristic pairs" of f at 0. They are easily computed from the Newton Polygon of f. The use of **these** particular constants is motivated by the proof of the following intriguing

(3.6) Formula : If $|\ |$ is discrete, then

$$c(f,0) = \sum_{i-1}^{g-1} (m_{i-1} - m_i) \frac{k_i}{m} + m_{g-1} \cdot \frac{k_g}{m} =: M$$

This formula produces non-integral values of c, if $g \geq 2$. And for $X_2^2 - X_1^3$, e.g, we find $c = 3$ (instead of 2 over algebraically closed fields).

Proof : First, we do "$\geq$". Choose a uniformizing parameter q for R and a fixed m-th root $q^{1/m}$. Then set $x = (x_1, x_2) = (q^n, \sum_{m<k<k_g} b_k q^{kn/m})$

where $n >> 0$.

Then $|fx| = \prod_{\zeta} | x_2 - P(\zeta q^{n/m}) | \leq (|q|^n)^M . d$, for some d, as one sees by partitioning $Z = \{\zeta : \zeta^m = 1\}$:

$Z = \bigcup_{i=1}^{g} Z_i$, $Z_i = \{\zeta : \zeta^{m_{i-1}} = 1, \zeta^{m_i} \neq 1\}$ $(i = 1, \dots, g-1)$,

$Z_g = \{\zeta : \zeta^{m_{g-1}} = 1\}$. Then $\# Z_i = m_{i-1} - m_i$ $(1 \leq i \leq g-1)$, and $\# Z_g = m_{g-1}$,

and $\operatorname{ord}_T(P(\zeta T) - P(T)) = k_i$, if $\zeta \in Z_i$, $\zeta \neq 1$.

Furthermore assume n is divisible by m/m_{g-1}, but not by m. (There are infinitely many such n.) Choose $t \in \mathbb{Z}$, $t \geq 0$, such that every unit in R which is congruent to 1 mod. q^{t+1} is an m-th power in R. (m is not divisible by the characteristic). Then let $y = (y_1, y_2)$ be a zero of f close to the origin, so that $y_2 = P(v)$ for some v, $v^m = y_1$. It suffices to show that, if $|x_1 - y_1| < |q|^{n+t}$, one has $P(\zeta v) \notin K$, for all ζ. So let $y_1 = u.q^n$, $u \in 1 + q^{t+1}.R$.

Then $\sum_{m<k<k_g} b_k(\zeta v)^k \in K(\zeta)$, while $\sum_{k\geq k_g} b_k(\zeta v)^k \notin K$.

Therefore, if $P(\zeta v) \in K$, then the latter sum is in $K(\zeta)$, which is impossible since $|P(\zeta v) - P(\zeta' v)| > 0$ for $\zeta \neq \zeta'$ (we had $n \gg 0$.)

For the converse inequality, $|\ |$ need not be discrete :

Suppose $|fx| < \alpha^M . d$ (d to be determined later ; $\alpha << 1$), where $x = (x_1, x_2)$ is close to the origin. Write u for a fixed $x_1^{1/m}$.

Then partitioning Z as before we find $1 \leq i \leq g$ such that

$|x_2 - P(\zeta_0 u)| < \alpha^{k_i/m} . d_i$, where $\zeta_0 \in Z_i$ with

$|x_2 - P(\zeta_0 u)| \leq |x_2 - P(\zeta u)|$ for all $\zeta \in Z_i$.

If now $|x_1| < \alpha$, take the origin as the zero y we seek.

So suppose $|x_1| \geq \alpha$. Then we will show that, in fact

(+) $|x_2 - P(\zeta_0 u)| < |x_2 - P(\zeta u)|$, for all $\zeta \in Z$, $\zeta \neq \zeta_0$.

This will allow us to apply Krasner's lemma to $P(\zeta_0 u)$ with respect to x_2 over K, so that we get $P(\zeta_0 u) \in K(x_2) = K$, and find $(x_1, P(\zeta_0 u))$ as the zero sought for.

To verify (+), note that for $\zeta \in Z_j$ with $j \neq i$, you even get
$|x_2 - P(\zeta_0 u)| < \alpha^{k\ell/m} . d_i \leq |P(\zeta u) - P(\zeta_0 u)|$, with $\ell = \min(i,j)$.

In perfect analogy, we can do away with the $\zeta \in Z_i$ such that $\zeta_0^{-1} . \zeta \in Z_i$. So suppose there is a counterexample $\zeta_1 \in Z_i$ to (+), such that $\zeta_0^{-1} . \zeta_1 \in Z_{i+j}$, j minimal.

Then $|x_2 - P(\zeta_1 u)| \geq |x_1|^{k_{i+j}/m} . d_i$. Using all bounds on $|x_2 - P(\zeta u)|$ for various ζ, which follow from the above relations, one ultimately gets $|fx| \geq \alpha^M . d$ -contradiction.

<u>Case 2</u> : $f = \prod_i f_i$, where the $f_i \in K\{X_1, X_2\}$ are absolutely analytically irreducible. While in Case 1 , we were able to produce a fairly satisfactory formula for c , here it seems not at all easy, in general, to predict how the $c(f_i, 0)$ "piece together" to yield $c(f,0)$.

Assuming $|\ |$ to be discrete and analyzing the fact that x in the first part of the proof of (3.6) approximates the tangent cone ($= X_1$ - axis) more rapidly than the origin one can show that

$c(f,0) = c(f_1,0) + \sum_{i\neq 1} m(f_i)$, if $c(f_1, 0) - m(f_1)$ is maximal, provided that the tangent cones of the f_i are all different from each other. This last condition can trivially be dropped, if the number of f_i's is just $m(f)$ (use (2.2) and (2.3)). But beyond that case, many weird things can happen, as is documented by the following examples which are easily checked ($p = \text{char } K \nmid m$; $n > m > 1$, $(m,n) = 1$):

(3.7) $c((X_2^m - X_1^n) \cdot X_2, 0) = m + 1.$

(Recall that $c(X_2^m - X_1^n, 0) = n.$)

(3.8) $c((X_2^m - X_1^n).(X_2 - X_1^\ell), 0) = n/\ell + 1$; if $\ell > 1$, and $n \geq \ell \cdot m$. (Note that ℓ does not appear in the c of either branch !)

In (3.8) as well as in the next example, $|\ |$ is assumed to be discrete.

(3.9) $c((X_2^m - X_1^n) \cdot (X_2^k - X_1^\ell), 0) = \ell + n,$

if $\ell > k > 1$; $(k,\ell) = 1$.

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