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Norbert Schappacher

Periods of Hecke Characters

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*A Rosita*

*qui a su débloquent la rédaction  
de ce travail  
janvier 1986*

Налево беру и направо,  
И даже, без чувства вины,  
Немного у жизни лукавой  
И все – у ночной тишины.

*Анна Ахматова*

## INTRODUCTION

In two papers – n<sup>o</sup> 12 and 14 of [He], published in 1918 and 1920 – E. Hecke introduced what he called “Größencharaktere” of algebraic number fields, with a view to extending the theory of L-functions and their applications in analytic number theory. In the early 1950’s the arithmetic and geometric significance of those of Hecke’s characters that take algebraic values began to appear in two different, if overlapping, lines of thought. (Both of these had been anticipated in special cases by Eisenstein exactly one hundred years earlier; but none of the mathematicians working on them in the fifties seems to have been aware of their precursor at the time.) First Weil, testing a conjecture of Hasse, investigated algebraic curves over  $\mathbb{Q}$  with the property that the number of  $F_p$  rational points on their reductions modulo  $p$  can be computed in terms of exponential sums. This led him to a study of “Jacobi sums as ‘Größencharaktere’”. Secondly Deuring, developing one aspect of Weil’s examples, proved that the (Hasse-Weil) L-function of an elliptic curve with complex multiplication is a (product of) Hecke L-function(s). This was then quickly generalized to higher dimensional  $CM$  abelian varieties by Shimura and Taniyama, with Weil providing clarification, for instance, on the Hecke characters employed in the theory.

Both approaches cover only very limited classes of algebraic Hecke characters. Jacobi sum characters were confined to cyclotomic (today: abelian) fields, and in general, not every algebraic Hecke character of such a field is given by Jacobi sums. – The product of several Hecke characters each one of which is attached to a  $CM$  abelian variety does no longer occur in the  $L$ -function of an abelian variety.

This last difficulty disappears in a theory of motives, as proposed by Grothendieck. There one associates with every (smooth projective) algebraic variety, in some sense, its “universal cohomology” which is an object of what Grothendieck termed a Tannakian category, and may therefore be viewed as a representation of some proalgebraic group. The product of Hecke characters then corresponds essentially to the tensor product of representations. Until the mid seventies such a category of motives existed only conjecturally; the morphisms were to be defined by the cohomology classes of algebraic correspondences, and conjectures on the existence of sufficiently many algebraic cycles had to be used to show that the construction actually yielded a Tannakian category – see [Sa]. Granting this, the semi-simplicity of the Galois action on  $l$ -adic cohomology and Tate’s conjectures, one could show that a motive defined over a number field is determined up to isomorphism by its (“Hasse-Weil”)  $L$ -function (defined using the étale cohomology of the motive). Consequently, two motives (which may be constructed from different varieties, but are) attached to the same Hecke character would have to be isomorphic, and in particular, would have the same periods (defined by “integrating” de Rham cohomology classes “against” Betti cohomology of the motive.)

This uniqueness principle is at the center of our work. We peruse a variety of consequences of it that can be proven, either because an analogous uniqueness principle is available in a slightly different framework – see next paragraph – or because of the special situation considered – this is the case in Chapter V. Applications include a refined version of the so called formula of Chowla and Selberg, deduced from the comparison of the motive of a basic Jacobi sum Hecke character of an imaginary quadratic field  $K$  to elliptic curves with complex multiplication by  $K$  – see Chapter III; refinements of Shimura’s monomial period relations; generalizations of the formula of Chowla and Selberg to arbitrary abelian number fields – Chapter IV; and the study of motives for the theta

series of Hecke characters of imaginary quadratic fields - Chapter V.

That we can actually prove theorems, not merely do an exegesis of conjectures, hinges on two insights by P. Deligne. First, he saw that one could actually construct a theory of motives by weakening the requirement on the morphisms; they no longer have to be algebraic but only “absolute Hodge” correspondences – see Chapter I, Section 2. (This idea may in fact go back to Grothendieck: see the first consequence of the Hodge conjecture discussed in §4 of [Gro].) Henceforth, when we speak of “motives”, we refer to this existing theory. Second, Deligne was able to show that, on an abelian variety over  $\bar{\mathbb{Q}}$ , every Hodge cycle is an absolute Hodge cycle. This consequence of the Hodge conjecture provides enough absolute Hodge cycles to prove the uniqueness principle for motives of algebraic Hecke characters, within the category of motives generated by abelian varieties – see Chapter I, Theorem 5.1.

In fact, for every algebraic Hecke character of a number field  $K$ , there exists a unique motive in the category of motives over  $K$  generated by abelian varieties with potential complex multiplication. Deligne has shown around 1980 that this category is equivalent to the representations of (a subgroup of) the Taniyama group, a group scheme which had been introduced by Langlands. This structure theorem also links the motivic interpretation of Hecke characters to that proposed by Serre in [S4] more than ten years earlier. It was also the starting point for G. Anderson’s comprehensive motivic theory of Gauss and Jacobi sums, and their relations to representations of the Taniyama group, a theory which he worked out between 1982 and 1984 – see [A1] and [A2]. In Anderson’s formalism, the basic observation that Fermat hypersurfaces provide motives for Jacobi sum Hecke characters of cyclotomic fields is extended to a class of characters of abelian number fields which is likely to include all sensible candidates of Hecke characters of “Jacobi sum type”. We make essential use of Anderson’s theory when dealing with Jacobi sum Hecke characters.

Thus, I really “take on the left and on the right” very substantial results obtained by others, and numerous little chats with many people have found their way into the “silent hours of the night” during which these pages were written.

\* \* \*

It was my intention, in writing up the paper, to also provide a viable introduction to the background theories. More precisely, the reader should get an idea of what they are like, without however being offered complete proofs. I hope there will be readers to whom my blend of explanations and quotes appeals, and is actually helpful.

CHAPTER 0 should be completely readable for anyone with some very basic knowledge of algebraic number theory. It covers the elementary (as opposed to geometric) theory of algebraic Hecke characters, including their interpretation via Serre’s groups  $S_m$ , and the definition and basic properties of Jacobi sum Hecke characters according to G. Anderson. (The Jacobi sum characters of imaginary quadratic fields are largely treated without reference to Anderson, by way of a fundamental example which is used in Chapter III.)

CHAPTER I falls into five parts.

I §1 presents the Shimura-Taniyama theory of complex multiplication of abelian varieties with a view to introducing motives for Hecke characters. The existence of the Hecke character attached to a  $CM$  abelian variety is derived using a transcendence result which implies - see [Henn] - that every semisimple abelian  $E$ -rational  $\lambda$ -adic representation is locally algebraic - cf. I, 1.4.

I §2 reviews the theory of motives for absolute Hodge cycles. We hope that our shortcut through this theory can serve as a reading guide for [DMOS], Chapter II, and also to the corresponding sections of [A2]. Deligne’s fundamental theorem on absolute Hodge cycles of abelian varieties is only quoted from [DMOS], Chapter I, because its proof would have led us too far away from the geometric

study of Hecke characters.

I §§3 - 5 cover the “naive” theory of motives for Hecke characters. In §4, a motive for every algebraic Hecke character is constructed “by hand”, out of Artin motives and  $CM$  abelian varieties. Its uniqueness up to isomorphism, in the category of motives generated by abelian varieties, is derived from Deligne’s theorem in §5.

I §6 treats the theory of the Taniyama group and its relation with the category of motives  $CM_{\mathbb{Q}}$ . While in the previous sections of chapter I the reader should be able to survive with a certain knowledge of algebraic geometry, this section is deliberately sketchy. In fact, we shall make very little use of it in later chapters – except through Anderson’s theory. Also, Milne is preparing a book on this subject which will also deal with Shimura varieties.

I §7 briefly reviews Anderson’s theory of motives for Jacobi sum Hecke characters, and also his ulterior motives. For all the details the reader is referred to his papers.

CHAPTER II is the technical heart of this work. The formalism of the periods of motives in general and motives for Hecke characters in particular, is unfolded here. This “arithmetic linear algebra” is carried out in great generality. I am afraid this does not exactly simplify the notation and understanding of this chapter. But I do hope that this treatment of periods – which, by the way, is essentially due to Deligne – will be useful for further investigations. This chapter also contains a brief review of Deligne’s rationality conjecture for special values of Hecke L-functions. This case of the conjecture is now a theorem by virtue of recent important results of Blasius [Bl] and Harder (unpublished). However, Blasius’s motivic treatment of the periods  $c^+$  is not included in my exposition because he himself applies it to questions similar to those discussed here – see his forthcoming paper [Bl’].

At the end of chapter II, after discussing the periods of Jacobi sum Hecke



characters starting from the example of Fermat hypersurfaces, we deduce some relations between values of the  $\Gamma$  function at rational numbers which were first conjectured and proved by Deligne.

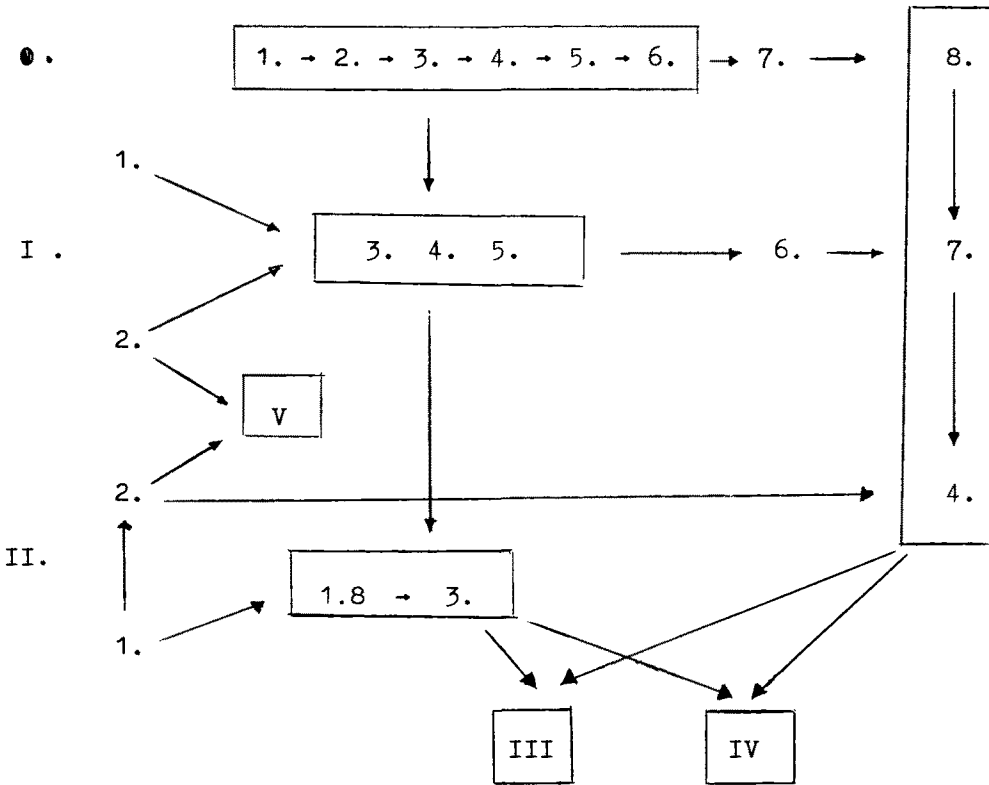
CHAPTER III is devoted to the so called formula of Chowla and Selberg. We prove a refined version of it, and show (in III §3) that it “generates” all period relations produced by Jacobi sum Hecke characters of imaginary quadratic fields. An interesting feature of the motivic treatment of this formula is that here, it is often convenient to deduce an identity of Hecke characters from an analytically accessible period relation – rather than going the other way around. – Cf. also II, 3.5.

CHAPTER IV treats Shimura’s relations between periods of *CM* abelian varieties and generalizations of the Chowla-Selberg formula to abelian fields. The most remarkable feature here is the serious discrepancy between the potential of the method and the scarcity of information about concrete situations to which the method applies. In the Chowla-Selberg case it is often possible to determine explicitly every single character whose periods contribute to the formula. But over an arbitrary abelian field such explicit identities are usually not available, and so – in spite of the inherent precision of the method – one is led to weaken the period relations in order to get sensible statements.

Compared to the preceding chapters, CHAPTER V is really written “in shorthand”. It starts by reviewing very briefly U. Jannsen’s recent construction of an honest regard (absolute Hodge cycle) motive for every newform  $f$  on  $\Gamma_1(N) \subset SL_2(\mathbf{Z})$  of weight  $\geq 2$ . Then we proceed to show that this motive “lies in”  $\mathcal{CM}_{\mathbf{Q}}$  if  $f$  has complex multiplication. This has to be done by hand, using Deligne’s conjecture for the critical values of these modular forms.

It is a pleasure to acknowledge the hospitality of the Max-Planck-Institut für Mathematik at Bonn, where I stayed from October 1983 through January 1985. About half of this work was written there, not little influenced by Harder's interest in these questions, and his willingness to let me organize his seminar in the winter 1984 - 85 on motives for absolute Hodge cycles. Most of the suggestions explicitly acknowledged in the text I obtained through my stay in Bonn. – For the excellent typing my hearty thanks go to K. Deutler at Bonn, C. Giesecking at Göttingen.

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Here as well as in the internal references throughout the text, roman numerals denote chapters - chapter zero being represented by 0 - , and an expression of the form n.m.l , n.m , or n. refers to the corresponding formula, theorem, paragraph or section within the given chapter.

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CHAPTER ZERO:

Algebraic Hecke Characters

In this chapter we review the elementary theory of algebraic Hecke characters and fix some basic notation.

1. Definition

Let  $K$  and  $E$  be two number fields, i.e., finite extensions of  $\mathbb{Q}$ . Let  $\mathfrak{f}$  be a non-zero integral ideal of  $K$ , and  $T = \sum n_{\sigma} \sigma \in \mathbb{Z} [\text{Hom}(K, \bar{E})]$  a  $\mathbb{Z}$ -linear combination of embeddings of  $K$  into a fixed algebraic closure  $\bar{E}$  of  $E$ .

Definition: [cf. [SGA 4 $\frac{1}{2}$ ], Sommes trig. § 5]: An algebraic Hecke character  $\chi$  of  $K$  with values in  $E$ , of infinity-type  $T$  and conductor dividing  $\mathfrak{f}$ , is a group homomorphism

$$\chi: I_{\mathfrak{f}} \rightarrow E^*$$

from the group  $I_{\mathfrak{f}}$  of ideals of  $K$  prime to  $\mathfrak{f}$  to the multiplicative group of  $E$ , such that, for any ideal  $(\alpha) \in I_{\mathfrak{f}}$  generated by an  $\alpha \in K^*$  with  $\alpha \equiv 1 \pmod{\mathfrak{f}}$ , and  $\alpha$  totally positive (i.e.,  $\alpha^{\rho} > 0$  for all real embeddings  $\rho: K \hookrightarrow \mathbb{R}$ , symbolically:  $\alpha \gg 0$ ), one has

$$\chi((\alpha)) = \alpha^T = \prod_{\sigma} (\alpha^{\sigma})^{n_{\sigma}}.$$

It is understood that, if  $\mathfrak{f} | \mathfrak{f}'$ , characters of conductor dividing  $\mathfrak{f}$  are identified with the corresponding characters of conductor dividing  $\mathfrak{f}'$  obtained by restricting to  $I_{\mathfrak{f}} \subseteq I_{\mathfrak{f}'}$ . The smallest  $\mathfrak{f}$  (in the sense of divisibility) such that  $\chi$  extends to a character of conductor dividing  $\mathfrak{f}$  is called the conductor of

$\chi$ , and denoted  $f_{\chi}$ . - Note that the subgroup of ideals  $(\alpha)$  with  $\alpha \gg 0$  and  $\alpha \equiv 1 \pmod{f}$  has finite index in  $I_f$ .

## 2. Algebraic homomorphisms

Recall from [SGA 4 $\frac{1}{2}$ ], Sommes trig. § 5, the various ways to view the infinity-type  $T$  of an algebraic Hecke character. In general, an algebraic homomorphism  $t: K^* \rightarrow E^*$  is a group homomorphism such that either one of the following equivalent conditions is satisfied.

(a) For any basis  $\{e_i \mid i = 1, \dots, n\}$  of  $K$  over  $\mathbb{Q}$ , there is a rational function  $f \in E(X_1, \dots, X_n)$  such that

$$t\left(\sum a_i e_i\right) = f(a_1, \dots, a_n),$$

for all  $(a_i) \in \mathbb{Q}^n$ .

(b)  $t$  is induced by a homomorphism of algebraic groups over  $E$

$$R_{K/\mathbb{Q}} \mathbb{G}_m \times_{\mathbb{Q}} E \rightarrow \mathbb{G}_m.$$

(c)  $t$  is induced by a homomorphism of algebraic groups over  $\mathbb{Q}$

$$R_{K/\mathbb{Q}} \mathbb{G}_m \rightarrow R_{E/\mathbb{Q}} \mathbb{G}_m.$$

(d) There is  $T = \sum n_{\sigma} \sigma \in \mathbb{Z}[\text{Hom}(K, \bar{E})]$  such that for all  $\alpha \in K^*$ ,

$$t(\alpha) = \alpha^T = \prod_{\sigma} (\alpha^{\sigma})^{n_{\sigma}}.$$

(e) Decompose  $K \otimes_{\mathbb{Q}} E = \prod_j F_j$  (finite product of fields). There are integers  $m_j$  such that

$$t = \prod_j N_{F_j/E}^{m_j}.$$



As explained in loc. cit., the equivalence of (a) through (c) follows from elementary facts about algebraic groups, and (d), (e) are reformulations of (b) using the identification of the character group of  $R_{K/\mathbb{Q}} \mathbb{G}_m$  over  $\bar{E}$  with  $\mathbb{Z}^{\text{Hom}(K, \bar{E})}$ . An analogous reformulation of (c) will be given in § 4. In the sequel we will often identify a type  $T$  like in (d) with the algebraic homomorphism  $t$  defined by it. Note that  $T$  gives rise to an algebraic homomorphism  $K^* \rightarrow E^*$  if and only if  $n_\sigma = n_{\tau\sigma}$ , for every  $\tau \in \text{Gal}(\bar{E}/E)$ . This is the case if  $T$  is the infinity-type of an algebraic Hecke character with values in  $E$ .

### 3. Infinity-types and algebraic Hecke characters

It is not true that, conversely, every algebraic homomorphism  $K^* \rightarrow E^*$  occurs as infinity-type of an algebraic Hecke character of  $K$  with values in  $E$ . The first obvious constraint is that such an infinity-type has to kill all totally-positive units  $\equiv 1 \pmod{\mathfrak{f}}$  in  $\mathfrak{o}_K$ . As these are of finite index in  $\mathfrak{o}_K^*$ , the proof of Dirichlet's unit theorem implies that there is an integer  $w$  such that, for any embedding  $\bar{E} \hookrightarrow \mathbb{C}$ , inducing an action of complex conjugation,  $\sigma \mapsto \bar{\sigma}$ , on  $\text{Hom}(K, \bar{E})$ , and for any  $\sigma \in \text{Hom}(K, \bar{E})$ , one has

$$(3.1) \quad n_\sigma + n_{\bar{\sigma}} = w.$$

$w$  is called the weight of  $T$  (or of  $\chi$ ).

Thus, for any complex conjugation of  $\bar{E}$ , we find

$$\chi \cdot \bar{\chi} = \mathbb{N}_{K/\mathbb{Q}}^w,$$

where  $\mathbb{N}_{K/\mathbb{Q}}(\mathfrak{a}) = \#(\mathfrak{o}_K/\mathfrak{a})$  for an integral ideal  $\mathfrak{a}$  of  $K$ . (In fact, this is true on a subgroup of finite index of  $I_{\mathfrak{f}}$ , and  $\mathbb{R}_+^*$  is torsion-free.) Therefore the values of an algebraic Hecke character are pure, in the sense that all embeddings into  $\mathbb{C}$  have the same absolute value. Similarly, they are what we shall call (for want of a better term) numbers of CM-type:

An algebraic number  $\alpha$  is of CM-type if there is a (necessarily unique) conjugate  $\alpha'$  of  $\alpha$  such that, for all embeddings

$$\tau: \mathbb{Q}(\alpha, \alpha') \rightarrow \mathbb{C},$$

one has  $\overline{\tau(\alpha)} = \tau(\alpha')$ .

To make more explicit the restriction on the existence of algebraic Hecke characters imposed by the homogeneity condition  $n_\sigma + n_{\bar{\sigma}} = \text{cst.}$ , let  $K'$  be the subfield of  $K$  consisting of all  $\alpha \in K$  that are of CM-type. So,  $K'$  is either totally real or a CM-field (i.e., a totally imaginary quadratic extension of a totally real field). Then in our infinity-type  $T = \sum n_\sigma \cdot \sigma$ ,  $n_\sigma$  depends only on  $\sigma|_{K'}$ , because  $n_\sigma + n_{\bar{\sigma}}$  is independent of the choice of complex conjugation. So one gets an element

$$T' = \sum_{(\sigma|_{K'})} n_{(\sigma|_{K'})} \cdot (\sigma|_{K'}) \in \mathbb{Z}[\text{Hom}(K', \bar{\mathbb{E}})].$$

The fact that  $n_\sigma + n_{\bar{\sigma}}$  is independent of  $\sigma$  means this:

- (a) if  $K'$  is totally real, then  $T' \in \mathbb{Z} \cdot \Sigma \tau$  (summed over all  $\tau: K' \hookrightarrow \bar{\mathbb{E}}$ );
- (b) if  $K'$  is a CM-field, then  $T'$  belongs to the subgroup of  $\mathbb{Z}[\text{Hom}(K', \bar{\mathbb{E}})]$  generated by the CM-types  $\{\sum n_\sigma \sigma \mid n_\sigma \in \{0, -1\}, n_\sigma + n_{\bar{\sigma}} = -1\}$ .

Since algebraic Hecke characters of finite order are precisely those whose infinity-type is trivial, we see that

- (a) if  $K'$  is totally real, then every algebraic Hecke character  $\chi$  of  $K$  is of the form  $\chi = \mu \cdot \mathbb{N}_{K/\mathbb{Q}}^{w/2}$ , where  $\mu$  is of finite order and  $w \in 2\mathbb{Z}$ .
- (b) if  $K'$  is a CM-field, we have  $\chi = \mu \cdot (\varphi \circ \mathbb{N}_{K/K'})$  for some algebraic Hecke character  $\varphi$  of  $K'$ , and  $\mu$  a character of finite order of  $K$ .

Consequently, the field of values of an algebraic Hecke character is either  $\mathbb{Q}$  or a CM-field.

Every algebraic homomorphism  $T = \sum n_\sigma \cdot \sigma: K^* \rightarrow \mathbb{E}^*$  which

satisfies the homogeneity condition  $n_{\sigma} + n_{\bar{\sigma}} = \text{cst.}$  as above is the infinity-type of some algebraic Hecke character  $\chi$  of  $K$  with values in a finite extension of  $E$ . The construction of  $\chi$  is straightforward: Choose an ideal  $f$  such that

$$\{\varepsilon \in \mathcal{O}_K^* \mid \varepsilon \gg 0 \text{ and } \varepsilon \equiv 1 \pmod{f}\}^T = 1,$$

and take any extension of

$$\chi \mid \{(\alpha) \mid \alpha \gg 0 \text{ and } \alpha \equiv 1 \pmod{f}\}$$

to all of  $I_f$  (cf., e.g., [Shim], Lemma 7.45). Since roots of values of  $T$  are extracted in this process one cannot do without extending  $E$ , in general. If we insist on keeping  $E$ , we may pass to  $T \cdot N_{L/K}$ , for a suitable finite (abelian) extension  $L$  of  $K$ . Fixing  $E$  and  $K$ , all one can assert in general is that  $m \cdot T$ , for a suitable  $m \in \mathbb{N}$ , will come from an algebraic Hecke character of  $K$  with values in  $E$ . (Cf. [DP], 8.2.) - We shall not be concerned with the problem of bounding the conductor of the character  $\chi$  which we have just shown to exist. For this, cf. [Schm], I.2.

#### 4. The Hodge decomposition

A homomorphism  $R_{K/\mathbb{Q}} \mathbb{G}_m \rightarrow R_{E/\mathbb{Q}} \mathbb{G}_m$  over  $\bar{E}$  (cf. §2, (c)) is a system of characters of  $R_{K/\mathbb{Q}} \mathbb{G}_m$  indexed by  $\text{Hom}(E, \bar{E})$ . This yields a description of infinity types whose relation with the title of this section will become apparent in the next chapter (see in particular I, 1.7, 4.2, 6.1.5; cf. [DP], 8.2).

Let  $t: K^* \rightarrow E^*$  be an algebraic homomorphism, and  $\tau \in \text{Hom}(E, \mathbb{C})$ . Then  $\tau \circ t: K^* \rightarrow (E^\tau)^*$  is again an algebraic homomorphism whose type will be written

$$T_\tau = \sum n(\sigma, \tau) \cdot \sigma,$$

where  $\sigma$  now ranges over all embeddings of  $K$  into the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ , or simply  $\sigma: K \rightarrow \mathbb{C}$ .

If  $T = \sum_{\eta:K \rightarrow \bar{E}} n_{\eta} \cdot \eta$  is the type of  $t$ , then  $n_{\eta} = n(\sigma, \tau)$  if and only if  $\sigma = \tilde{\tau} \circ \eta$ , for any extension  $\tilde{\tau}$  of  $\tau$  to  $\bar{E}$ .

We have  $n(\sigma, \tau) = n(\alpha\sigma, \alpha\tau)$  for all  $\alpha \in \text{Aut } \mathbb{C}$  because  $t: R_{K/\mathbb{Q}} \mathbb{E}_m \rightarrow R_{E/\mathbb{Q}} \mathbb{E}_m$  is defined over  $\mathbb{Q}$  (cf. I, 6.1.2). Furthermore, if  $t$  is the infinity-type of an algebraic Hecke character  $\chi$  of weight  $w$ , then

$$w = n(\sigma, \tau) + n(c\sigma, \tau) = n(\sigma, \tau) + n(\sigma, c\tau),$$

for any  $\sigma \in \text{Hom}(K, \mathbb{C})$ ,  $\tau \in \text{Hom}(E, \mathbb{C})$ , where  $c =$  complex conjugation.

## 5. Adèles

Algebraic Hecke characters may, of course, be read on the idèles  $K_{\mathbb{A}}^*$  of  $K$ : cf. [W],[1955c] where algebraic Hecke characters were introduced as characters of the idèle class group "of type  $(A_0)$ ".

First, given  $\chi$  as in § 1, there clearly exists a unique group homomorphism

$$\chi_{\mathbb{A}}: K_{\mathbb{A}}^* \rightarrow E^*$$

such that

- (a)  $\chi_{\mathbb{A}}^{-1}(1)$  is open in  $K_{\mathbb{A}}^*$ ;
- (b)  $\chi_{\mathbb{A}}|_{K^*} = T: K^* \rightarrow E^*$ ;
- (c)  $\chi_{\mathbb{A}}(\pi_{\mathfrak{p}}) = \chi(\mathfrak{p})$ , for all prime ideals  $\mathfrak{p}$  of  $K$  not dividing  $f$ .

Here,  $\pi_{\mathfrak{p}}$  denotes any idèle having a uniformizing parameter at  $\mathfrak{p}$ , and 1 at all other components.

Since  $\chi_{\mathbb{A}}$  takes values in  $E^*$ , it could not be an idèle class character, and its restrictions to individual completions of  $K$  are not very interesting. But this can be changed by conveniently "localizing over  $E$ ":

Being an algebraic homomorphism,  $T$  induces a continuous homomorphism  $K_{\mathbb{A}}^* \rightarrow E_{\mathbb{A}}^*$  - see, e.g., condition (c) of § 2. Given any place  $\lambda$  of  $E$ , denote by  $T_{\lambda}$  the composite with projection onto the  $\lambda$ -component of  $E_{\mathbb{A}}$ :

$$T_\lambda : K_{\mathbf{A}}^* \xrightarrow{T} E_{\mathbf{A}}^* \rightarrow E_\lambda^* ,$$

and write

$$\chi_\lambda = \chi_{\mathbf{A}} \cdot T_\lambda^{-1} : K_{\mathbf{A}}^* \rightarrow E_\lambda^* .$$

Then  $\chi_\lambda$  is an idèle class character, i.e., a continuous homomorphism  $K_{\mathbf{A}}^*/K^* \rightarrow E_\lambda^*$ .

If  $\lambda$  is a finite place,  $E_\lambda$  is a totally disconnected topological space, so  $\ker \chi_\lambda$  contains the connected component of 1 in  $K_{\mathbf{A}}^*$ . By class-field-theory,  $\chi_\lambda$  factorizes:  $\text{Gal}(K^{\text{ab}}/K) \rightarrow E_\lambda^*$  as the 1-dimensional  $\lambda$ -adic Galois representation with

$$\chi_\lambda(\text{Frob } \mathfrak{p}) = \chi(\mathfrak{p}) \in E^* \hookrightarrow E_\lambda^* ,$$

for any prime ideal  $\mathfrak{p}$  of  $K$  not dividing  $f \cdot N\lambda$ . Here,  $\text{Frob } \mathfrak{p}$  is a "geometric Frobenius" at  $\mathfrak{p}$ , i.e., we normalize the reciprocity map of class-field-theory to be the reciprocal of the Artin map. This is done to comply with [DP]. Note that the "eigenvalues" of  $\text{Frob } \mathfrak{p}$  with respect to this  $\lambda$ -adic representation are purely of absolute value  $(N\mathfrak{p})^{w/2}$ .

If  $\lambda$  is a complex place of  $E$ , we get two (possibly equal) continuous homomorphisms, or "quasi-characters of the idèle class group" in the sense of [Tt] or [W3], chap. VII:

$${}_1\chi_\lambda, {}_2\chi_\lambda : K_{\mathbf{A}}^*/K^* \rightarrow E_\lambda^* \xrightarrow{\sim} \mathbb{C}^* ,$$

according to the two continuous isomorphisms  $E_\lambda \cong \mathbb{C}$ .

If  $\lambda$  is a real place, there is just one such character  $\chi_\lambda : K_{\mathbf{A}}^*/K^* \rightarrow \mathbb{R}^* \hookrightarrow \mathbb{C}^*$ .

In another language, we get for each infinite place  $\lambda$ , one or two automorphic forms on  $\text{GL}(1, K_{\mathbf{A}})$ .

## 6. L-functions

To every complex embedding  $\tau : E \rightarrow \mathbb{C}$  is attached the "Größen-

charakter" (in Hecke's sense)  $\tau \circ \chi$ . If  $\tau$  induces the infinite place  $\lambda$  of  $E$ , then  $\tau \circ \chi$  corresponds to (one of) the idèle class character(s)  $(j)\chi_\lambda$ . Consider the Hecke L-function

$$L(\chi^\tau, s) = \sum_{\substack{(\mathfrak{a}, f_\chi)=1 \\ \mathfrak{a} \subset \mathfrak{o}_K}} (\tau \circ \chi)(\mathfrak{a}) \cdot \mathbf{N} \mathfrak{a}^{-s} = \prod_{\mathfrak{p} \nmid f_\chi} (1 - (\tau \circ \chi)(\mathfrak{p}) \cdot \mathbf{N} \mathfrak{p}^{-s})^{-1}$$

(for  $\operatorname{Re}(s) > \frac{w}{2} + 1$ ).

We write formally

$$L^*(\chi, s) = (L(\chi^\tau, s))_{\tau: E \rightarrow \mathbb{C}},$$

so that  $L^*(\chi, s)$  is an array of L-functions taking values in  $\mathbb{C}^{\operatorname{Hom}(E, \mathbb{C})} = E \otimes_{\mathbb{Q}} \mathbb{C}$ .

Recall the general form of the functional equation of the  $L(\chi^\tau, s)$  - cf. [He], p. 272 f; [Tt] or [W3], VII-7. Put

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right); \quad \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s) \Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s} \Gamma(s).$$

For a real place  $v$  of  $K$  (whose existence implies, in the notation of § 4, that all  $n(\sigma, \tau)$  are equal to  $\frac{w}{2}$ ), put

$$L_v(\chi^\tau, s) = \Gamma_{\mathbb{R}}\left(s + \varepsilon - \frac{w}{2}\right),$$

where  $\varepsilon = 0$  or  $1$ , such that the  $v$ -component  $\chi_v^\tau: K_v^* \rightarrow \mathbb{C}^*$  of  $\chi_\lambda$  satisfies  $\chi_v^\tau(-1) = (-1)^{\varepsilon + w/2}$ . For a complex place  $v$  of  $K$ , corresponding to the pair  $\sigma, \bar{\sigma}: K \rightarrow \mathbb{C}$  of complex embeddings of  $K$ , put

$$L_v(\chi^\tau, s) = \Gamma_{\mathbb{C}}(s - \inf(n(\sigma, \tau), n(\bar{\sigma}, \tau))).$$

Then, setting

$$\Lambda(\chi^\tau, s) = \prod_{v|\infty} L_v(\chi^\tau, s) \cdot L(\chi^\tau, s),$$

we get a meromorphic continuation to the whole complex plane

with functional equation of the type

$$\Lambda(\chi^\tau, s) = \varepsilon(\chi^\tau, s) \cdot \Lambda((\chi^\tau)^{-1}, 1 - s),$$

where  $\varepsilon(\chi^\tau, s) = W(\chi^\tau) \cdot \{|d_K| \cdot \text{Inf}_\chi\}^{1/2-s}$ , for some constant  $W(\chi^\tau)$  of absolute value 1, and  $d_K$  the discriminant of  $K$  (over  $\mathbb{Q}$ ). - As  $\overline{\chi^\tau} = \mathbf{N}^w \cdot (\chi^\tau)^{-1}$ , this functional equation may be rewritten as one relating  $L(\chi^\tau, s)$  to  $L(\overline{\chi^\tau}, w+1-s)$ .

## 7. Serre's group

In [S&], chap. II, Serre has given an interpretation of algebraic Hecke characters which generalizes the definitions (b) or (c) of algebraic homomorphisms recalled in § 2.

7.1 Put:

$$U_{\mathbf{A}, f} = \left\{ (x_v) \in K_{\mathbf{A}}^* \left| \begin{array}{l} x_v > 0 \text{ if } v \text{ is real} \\ x_v \equiv 1 \pmod{f_v} \text{ if } v|f \\ x_v \in \mathfrak{o}_v^* \text{ if } v \text{ is finite} \end{array} \right. \right\}.$$

Then  $\chi_{\mathbf{A}}$  factorizes through  $K_{\mathbf{A}}^*/U_{\mathbf{A}, f}$ , and we have the diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & K^*/U_f & \rightarrow & K_{\mathbf{A}}^*/U_{\mathbf{A}, f} & \rightarrow & C_f \rightarrow 1 \\ & & \searrow T_{\mathbf{A}} & & \swarrow \chi_{\mathbf{A}} & & \\ & & & & & & E^* \end{array}$$

where  $U_f = U_{\mathbf{A}, f} \cap K^*$ , and  $C_f$  is the ray class group of  $K$  mod  $f$ . Recall that  $C_f$  is a finite abelian group. Now,  $K^*/U_f$  is the group of  $\mathbb{Q}$ -rational points of the  $\mathbb{Q}$ -torus

$$Z_{K, f} = (R_{K/\mathbb{Q}} \mathbb{G}_m) / \overline{\Gamma_f},$$

where  $\overline{\Gamma_f}$  is the Zariski-closure of a suitable arithmetic subgroup  $\Gamma_f$  of  $R_{K/\mathbb{Q}} \mathbb{G}_m$ . Serre shows how to construct a  $\mathbb{Q}$ -algebraic group  $S_{K, f}$  of multiplicative type (i.e.,  $S_{K, f}$

is the product of a torus by a finite abelian group) which is an extension of  $C_f$  by  $Z_{K,f}$  such that  $S_{K,f}(\mathbb{Q}) = K^*/U_{\mathbb{A},f}$ . In fact, define  $S_{K,f}$  via its character group:

$$X(S_{K,f}) = \text{Hom}(S_{K,f} \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{G}_m/\overline{\mathbb{Q}}) =$$

$$= \left\{ (\eta, \xi) \left| \begin{array}{l} \eta: K^*/U_{\mathbb{A},f} \rightarrow \overline{\mathbb{Q}}^* \text{ homomorphism} \\ \xi \in \text{Hom}(Z_{K,f} \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{G}_m/\overline{\mathbb{Q}}) \\ \text{and } \eta = \xi \text{ on } K^*/U_f \end{array} \right. \right\}$$

(as  $\text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ -module).

7.2 An algebraic Hecke character of  $K$  with values in  $E$  can then be viewed as a representation defined over  $E$  of algebraic groups

$$S_K = \varprojlim_f S_{K,f} \rightarrow \text{GL}(1),$$

or equivalently, as a homomorphism of algebraic groups defined over  $\mathbb{Q}$ :

$$S_K \rightarrow R_{E/\mathbb{Q}} \mathbb{G}_m.$$

7.3  $S_K$  sits in the exact sequence of  $\mathbb{Q}$ -algebraic groups (obtained as projective limit over  $f$ ):

$$(7.3.1) \quad 1 \rightarrow Z_K \rightarrow S_K \rightarrow \text{Gal}(K^{\text{ab}}/K) \rightarrow 0,$$

where  $Z_K$  can be described as follows.

7.3.2 Given an algebraic Hecke character of  $K$  with values in  $E$ , as a representation  $S_K \xrightarrow{\chi} R_{E/\mathbb{Q}} \mathbb{G}_m$ , its infinity-type is obtained simply by restricting to  $Z_K: Z_K \xrightarrow{T} R_{E/\mathbb{Q}} \mathbb{G}_m$  (cf. §4) and  $Z_K$  is the largest quotient of  $R_{K/\mathbb{Q}} \mathbb{G}_m$  through which all infinity-types of algebraic Hecke characters of  $K$  factorize.

Let  $\overline{\mathbb{Q}}$  be the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ , and consider temporarily all fields  $K$  as subfields of  $\overline{\mathbb{Q}}$ . Write  $\tau: E \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$ . Then  $\tau \circ T$  can be interpreted as a character  $T_\tau: Z_K \rightarrow \mathbb{G}_m$  over  $\overline{\mathbb{Q}}$ . Denoting complex conjugation on  $\overline{\mathbb{Q}}$  by  $c$ , and by



$G_{CM} \subset \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  the subgroup fixing all algebraic numbers of CM-type, it follows that the character group of  $Z_K$  is given by:

$$X(Z_K) = \left\{ \lambda \in X(R_{K|\mathbb{Q}} \mathbb{G}_m) \left| \begin{array}{l} \lambda^s = \lambda, \text{ for all } s \in G_{CM} \\ \lambda(c\sigma) + \lambda(\sigma) \text{ indep. of } \sigma \end{array} \right. \right\},$$

where we identify  $X(R_{K|\mathbb{Q}} \mathbb{G}_m) = \mathbb{Z} [\text{Hom}(K, \overline{\mathbb{Q}})]$ , and define  $\lambda^s(\sigma) = \lambda(s^{-1} \circ \sigma)$ , for  $s \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ ,  $\sigma \in \text{Hom}(K, \overline{\mathbb{Q}})$ .

7.3.3 Consequently, if  $K' \subset K$  is the field of numbers of CM-type in  $K$ , then  $Z_K = Z_{K'}$ , and

$$Z_{K'} = \begin{cases} Z_{\mathbb{Q}} = \mathbb{G}_m / \mathbb{Q}, & \text{if } K' \text{ is totally real} \\ R_{K'|\mathbb{Q}} \mathbb{G}_m / \ker(N_{K'/K_0} : R_{K'|\mathbb{Q}} \mathbb{G}_m \rightarrow R_{K_0|\mathbb{Q}} \mathbb{G}_m), & \text{if } K' \text{ is a CM-field with } K_0' \text{ as} \\ & \text{maximal totally real subfield.} \end{cases}$$

In particular, for  $K_1 \subset K_2$ , the norm maps

$$N_{K_2/K_1} : R_{K_2|\mathbb{Q}} \mathbb{G}_m \rightarrow R_{K_1|\mathbb{Q}} \mathbb{G}_m$$

factor through  $Z_{K_2} \rightarrow Z_{K_1}$ , allowing us to define

$$Z = \varprojlim_K Z_K = \varprojlim_{K \text{ CM-field}} Z_K.$$

The infinity-types of all algebraic Hecke characters can be regarded as characters of  $Z$  (identifying  $T_\tau$  on  $R_{K_1|\mathbb{Q}} \mathbb{G}_m$  with  $T_\tau \circ N_{K_2/K_1}$ ):

$$X(Z) = \left\{ f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z} \left| \begin{array}{l} f \text{ locally constant} \\ f^s = f, \text{ for all } s \in G_{CM} \\ f(c\sigma) + f(\sigma) \text{ indep. of } \\ \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \end{array} \right. \right\}$$

7.3.4 Thus, some invariants of  $\chi$  can be viewed as homomorphisms of (pro-) algebraic groups. E.g.,

$$X(Z) \rightarrow \mathbb{Z} = X(\mathbb{G}_m)$$

$$f \rightarrow f(1) + f(c) (=w)$$

$$f \rightarrow f(1) (=n(\sigma, \tau))$$

gives rise to:

$$\tilde{w}: \mathbb{G}_m \rightarrow Z(/Q)$$

$$\tilde{\mu}: \mathbb{G}_m \rightarrow Z(/E)$$

7.4 The sequence 7.3.1 admits a natural section over the finite idèles  $\mathbb{Q}_{\mathbb{A}}^* f$  whose construction is reminiscent of the way in which we passed from  $\chi_{\mathbb{A}}$  to  $\chi_{\lambda}$ , for a finite place  $\lambda$  of  $E$ , in § 5. As  $S_K(Q) = \varprojlim_{\mathfrak{f}} K_{\mathbb{A}}^*/U_{\mathbb{A}, \mathfrak{f}}$ , there is a natural continuous map

$$f: K_{\mathbb{A}}^* \rightarrow S_K(Q) \hookrightarrow S_K(\mathbb{Q}_{\mathbb{A}} f) .$$

On the other hand,  $Z_K(\mathbb{Q}_{\mathbb{A}} f)$  is also a quotient of  $K_{\mathbb{A}}^*$ , whence a continuous map

$$g: K_{\mathbb{A}}^* \rightarrow Z_K(\mathbb{Q}_{\mathbb{A}} f) \rightarrow S_K(\mathbb{Q}_{\mathbb{A}} f) .$$

$f$  and  $g$  obviously agree on  $K^*$ , and as  $S_K(\mathbb{Q}_{\mathbb{A}} f)$  is a totally disconnected topological space, the quotient  $f/g$  factors through a continuous homomorphism

$$\varepsilon: \text{Gal}(K^{ab}/K) \rightarrow S_K(\mathbb{Q}_{\mathbb{A}} f)$$

which is the section sought.

Given an algebraic Hecke character as a homomorphism of  $\mathbb{Q}$ -algebraic groups

$$\chi: S_K \rightarrow R_{E/Q} \mathbb{G}_m ,$$

we can recover  $\chi_{\mathbb{A}}$  as the map induced by  $\chi$  on the  $\mathbb{Q}$ -rational points of  $S_K, R_{E/Q} \mathbb{G}_m$ . As for  $\chi_{\lambda}$ , for  $\lambda$  a finite place of  $E$ , it is the  $\lambda$ -component of

$$\text{Gal}(K^{ab}/K) \xrightarrow{\varepsilon} S_K(\mathbb{Q}_{\mathbb{A}} f) \xrightarrow{\chi/\mathbb{Q}_{\mathbb{A}} f} R_{E/Q} \mathbb{G}_m(\mathbb{Q}_{\mathbb{A}} f) = E_{\mathbb{A}}^* f .$$

It is obvious how to mimick the construction of  $fg^{-1}$  at the infinite place of  $\mathbb{Q}$ . The result will no longer factor through  $\text{Gal}(K^{ab}/K)$ , but gives the characters  $\chi_{\lambda}$ , for  $\lambda|\infty$ , introduced in § 5.

## 8. Jacobi sum Hecke characters

### 8.0.1 History

Although special cases are already present in Eisenstein [Ei 3] the notion of Gauss or Jacobi sums viewed as Hecke characters really starts with André Weil: [W II], 1952d, for the cyclotomic case; [W III], 1974d, over abelian number fields. Cf. also the beautiful [W III], 1974c. Several authors have then extended the class of characters amenable to Weil's method - see [Kb],[KL],[Li] -, and proved results about special values of their L-functions (to wit, special cases of Lichtenbaum's "Γ-hypothesis"): [Br],[BL],[Li].

On the other hand, a thoroughly geometric study of Weil's Jacobi sum Hecke characters - with a view to majorize exponential sums - was done by Deligne in [SGA 4<sub>2</sub><sup>1</sup>], Sommes trig. - The "motivic" picture of Jacobi sum characters over cyclotomic fields is implicitly discussed in [DMOS], I § 7. Recently, G. Anderson took up the subject introducing, in [A2], a very smooth and efficient formalism as well as a geometric interpretation for a class of Jacobi sum Hecke characters which includes all Hecke characters of abelian fields that have ever been proposed as candidates of Jacobi sum Hecke characters. More precisely, I checked that Anderson's class coincides with the one defined by Kubert,[Kb]. The geometric interpretation makes it seem very unlikely that new reasonable candidates for Jacobi sum Hecke characters of abelian fields can be proposed. Anderson's work (in fact, essentially already the earlier [A1]) definitely links up the "Γ-hypothesis" with Deligne's rationality conjecture of [DP]. This has actually been the starting point of the present work - see [GS'], announcement made after Cor. 1.2.

8.0.2 In this section we shall, as of 8.2, introduce Anderson's class of Jacobi sum Hecke characters and briefly discuss, in 8.4, the corresponding notion of Stickelberger ideal (of an abelian number field) - which is easily seen to coincide with Sinnott's ,[Sin]. - Our account of [A2] will continue in I § 7, where we describe Anderson's motives

for Jacobi sum Hecke characters - also touching upon his 'ulterior motives' -, and will be concluded in II § 4, with the calculation of their periods in terms of values of the  $\Gamma$ -function at rational numbers. - Proofs will often be replaced by a reference.

To make things more concrete we begin, in 8.1, with a family of examples of Jacobi sum Hecke characters (all included already in [W III], 1974d) which will play a prominent role in chapter III.

### 8.1 The basic Jacobi sum character of an imaginary quadratic number field.

Let  $K = \mathbb{Q}(\sqrt{-D})$  be the imaginary quadratic number field of discriminant  $-D < -8$ . (The exceptional cases  $D = 3, 4, 8$  will be treated in 8.3.2.) Pick an embedding

$$K \hookrightarrow L = \mathbb{Q}(\mu_D) = \mathbb{Q}(e^{2\pi i/D}) \subset \mathbb{C}.$$

$K$  is the unique quadratic subfield of  $L$ . For a prime  $\mathfrak{P}$  of  $L$  not dividing  $D$  write  $L(\mathfrak{P}) = \mathbb{Z}[\mu_D]/\mathfrak{P}$  the residue field and  $\chi_{D,\mathfrak{P}}$  the  $D$ -th power residue symbol modulo  $\mathfrak{P}$ : for  $x \in L(\mathfrak{P})$ ,

$$\chi_{D,\mathfrak{P}}(x) \in \mu_D \cup \{0\} \quad \text{and} \quad \chi_{D,\mathfrak{P}}(x) \pmod{\mathfrak{P}} = x^{\frac{\text{NF}-1}{D}}.$$

Finally, put  $\mathfrak{e}(z) = \exp(2\pi i z)$ , and denote by  $\text{tr}$  the trace map from  $L(\mathfrak{P})$  down to its prime field  $\mathbb{F}_{\mathfrak{P}}$ . The basic Jacobi sum character of  $K$  is given by its values on prime ideals  $\mathfrak{p} \nmid D$  of  $K$  as follows.

$$(8.1.1) \quad J_D(\mathfrak{p}) = \prod_{\mathfrak{P}|\mathfrak{p}} \left( - \sum_{x \in L(\mathfrak{P})} \chi_{D,\mathfrak{P}}(x) e((\text{tr } x)/\mathfrak{p}) \right),$$

where  $\mathfrak{P}$  runs over the primes of  $L$  dividing  $\mathfrak{p}$ . Extending multiplicatively to the group  $I_{(D)}$  of all ideals prime to  $D$  in  $K$ , this gives a homomorphism

$$I_{(D)} \rightarrow K^*$$

as is easily seen from the behaviour of Gauss sums under conjugation: see [W III], 1974c, § 1. But it is by no means obvious, a priori, that  $J_D$  is a Hecke character, i.e., that it "admits a conductor"  $f$ , as in § 1. Suppose we knew this. Then Stickelberger's theorem would give us the infinity type of  $J_D$ , as follows. Identify as usual

$$\begin{aligned} (\mathbb{Z}/D\mathbb{Z})^* &\xrightarrow{\sim} \text{Gal}(L/\mathbb{Q}) \\ \alpha &\longrightarrow \sigma_\alpha: \zeta \rightarrow \zeta^\alpha \quad (\zeta \in \mu_D) \end{aligned}$$

Write the Dirichlet character corresponding to  $K$  as

$$(8.1.2) \quad \begin{aligned} \varepsilon: (\mathbb{Z}/D\mathbb{Z})^* &\longrightarrow \{\pm 1\} \\ \varepsilon(\alpha) &= \begin{cases} 1 & \text{if } \sigma_\alpha|_K = 1 \\ -1 & \text{if } \sigma_\alpha|_K = c \text{ (complex conjugation)} \end{cases} \end{aligned}$$

Lift  $\varepsilon$  back to  $\mathbb{Z}$  when convenient, also extending it to numbers not prime to  $D$  by 0. Thus  $\varepsilon(p) = \left(\frac{-D}{p}\right)$  (Legendre's symbol), for all rational primes  $p$ . Define, for  $\alpha$  running over  $(\mathbb{Z}/D\mathbb{Z})^*$

$$(8.1.3) \quad n_1 = \sum_{\varepsilon(\alpha) = -1} \left\langle \frac{\alpha}{D} \right\rangle; \quad n_c = \sum_{\varepsilon(\alpha) = +1} \left\langle \frac{\alpha}{D} \right\rangle,$$

where

$$(8.1.4) \quad \langle \cdot \rangle: \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q} \text{ is the representative in } [0, 1) \text{ of a class mod } \mathbb{Z}.$$

Then an easy calculation, starting, e.g., from [W III], 1974c, § 15, shows that

$$J_D(\mathfrak{p}) \cdot \rho_K = \mathfrak{p}^{(n_1 + n_c)}$$

Now, the trick of Gauss as a young man, and the analytic class number formula of Dirichlet give the two equations (remember that  $D \neq 3, 4$ ):

$$n_1 + n_c = \varphi(D)/2$$

$$n_1 - n_c = h_D \quad ,$$

where  $\varphi(D) = \#(\mathbb{Z}/D\mathbb{Z})^*$  is Euler's  $\varphi$ -function, and  $h_D$  is the class number of  $K$ . Therefore the infinity type of  $J_D$  would have to be

$$(8.1.5) \quad T_D = \frac{1}{2} \left[ \left( \frac{\varphi(D)}{2} + h_D \right) \cdot 1 + \left( \frac{\varphi(D)}{2} - h_D \right) \cdot c \right] .$$

This does give an algebraic homomorphism of  $K^*$  into itself because, by genus theory,  $\frac{\varphi(D)}{2} \equiv h_D \pmod{2}$  - this is why we had to exclude  $D = 8$  also!

It is proved in [W III], 1974d, that  $J_D$  is actually a Hecke character of  $K$ , with defining ideal  $\mathfrak{f}$  dividing a power of  $D$ . Alternatively, this follows from Anderson's interpretation: see I § 7.

## 8.2 Anderson's formalism

The latent reference for this subsection is [A2], § 2.

8.2.1 Let  $\mathbb{B}$  be the free abelian group on  $\mathbb{Q}/\mathbb{Z} \setminus \{0\}$ . For  $\underline{a} = \sum n_a [a] \in \mathbb{B}$ , let  $m(\underline{a})$  be the order of the subgroup of  $\mathbb{Q}/\mathbb{Z}$  generated by  $\{a \in \mathbb{Q}/\mathbb{Z} \mid n_a \neq 0\}$ . Extend the function 8.1.4 to  $\mathbb{B}$  by the rule

$$\langle \underline{a} \rangle = \sum n_a \langle a \rangle .$$

Let  $\overline{\mathbb{Q}}$  be the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ , and let  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  act on  $\mathbb{B}$  via its action on roots of 1: Writing  $\Psi: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \hat{\mathbb{Z}}^*$  the cyclotomic character defined by  $\zeta^s = \zeta^{\Psi(s)}$ , for all  $s \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and  $\zeta \in \overline{\mathbb{Q}}^*$  a root of 1, we set

$$\underline{a}^s = \left( \sum n_a [a] \right)^s = \sum n_a [\Psi(s)a] .$$

Given a number field  $K \subset \overline{\mathbb{Q}}$ , write  $\mathbb{B}_K = \mathbb{B}^{G(\overline{\mathbb{Q}}/K)}$  the subgroup of elements invariant under  $\text{Gal}(\overline{\mathbb{Q}}/K)$ . Given  $K$

and  $\underline{a} \in \mathbb{B}_K$ , define

$$\theta_K(\underline{a}) : G(\overline{\mathbb{Q}}/\mathbb{Q})/G(\overline{\mathbb{Q}}/K) \rightarrow \mathbb{Q}$$

$$\sigma \mapsto \langle \sigma^{-1} \underline{a} \rangle .$$

In the application,  $K$  will be abelian over  $\mathbb{Q}$  and  $\theta_K$  will be read on  $\text{Gal}(K/\mathbb{Q})$ .

Also let  $\mathbb{B}^0 = \{ \underline{a} \in \mathbb{B} \mid \sum n_a a = 0 \text{ in } \mathbb{Q}/\mathbb{Z} \}$ , and

$$\mathbb{B}_K^0 = \mathbb{B}^0 \cap \mathbb{B}_K .$$

8.2.2 Let  $p$  be a rational prime and let  $\mathbb{B}_{(p)}$  be the subgroup of  $\mathbb{B}$  generated by elements of the form

$$\sum_{j=1}^f [p^j a] ,$$

where  $f$  is a positive integer and  $0 \neq a \in \mathbb{Q}/\mathbb{Z}$  is such that  $(p^f - 1)a = 0$ .

We assume that, for every rational prime  $p$ , an extension of the  $p$ -adic absolute value  $|\cdot|_p$  to  $\overline{\mathbb{Q}}$  has been chosen. So, in any number field  $L(\subset \overline{\mathbb{Q}})$ , there is a privileged prime divisor  $\mathfrak{P}$  of  $p$ .

There is a unique homomorphism

$$g_p : \mathbb{B}_{(p)} \rightarrow \overline{\mathbb{Q}}^*$$

such that, for all integral powers  $q = p^f \geq p$ , and all  $0 \neq a \in \mathbb{Q}/\mathbb{Z}$  with  $(q - 1)a = 0$ , one has

$$g_p \left( \sum_{j=1}^f [p^j a] \right) = - \sum_{\zeta^{q-1}=1} \zeta^{-\langle a \rangle (q-1)} \cdot e \left( \frac{t(q, \zeta)}{p} \right)$$

with  $t(q, \zeta) \in \mathbb{Z}$  and  $t(q, \zeta) \equiv \sum_{j=1}^f \zeta^{p^j} \pmod{\mathfrak{P}}$ , for  $\mathfrak{P}$  the chosen prime over  $p$  in the field  $\mathbb{Q}(\mu_{q-1})$ . This is Anderson's version of the theorem of Hasse and Davenport. - Note the - in the exponent of  $\zeta$  !

8.2.3 Let  $K \subset \overline{\mathbb{Q}}$  be a number field which is abelian over  $\mathbb{Q}$ . Let  $\underline{a} \in \mathbb{B}_K$  and  $\mathfrak{p}$  a prime ideal of  $K$  with  $\mathfrak{p} \nmid m(\underline{a})$ . Call  $p$  the rational prime below  $\mathfrak{p}$ , and write  $D(K, \mathfrak{p}) \subset \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  the open subset of all  $s$  such that  $\mathfrak{p}^s$  is the privileged prime above  $p$  in  $K$ . Thus  $D(K, \mathfrak{p})$  consists of full left cosets of the decomposition group  $D(p)$  of  $\mathbb{Z}/p\mathbb{Z}$ , as well as of  $G(\overline{\mathbb{Q}}/K)$ .  
Put

$$g_K(\underline{a}, \mathfrak{p}) = g_p \left( \sum_{\sigma \in D(K, \mathfrak{p})/G(\overline{\mathbb{Q}}/K)} \sigma^{-1} \underline{a} \right),$$

where  $g_p$  was defined in 8.2.2: To see that  $\sum \sigma^{-1} \underline{a}$  actually lies in  $\mathbb{B}_{(p)}$ , note that

$$\mathbb{B}_{(p)} = \mathbb{B}^{D(p)} \cap \{ \underline{a} \in \mathbb{B} \mid p \nmid m(\underline{a}) \}.$$

8.2.4 A Jacobi sum Hecke character (according to Anderson) is a character of the form  $J_K(\underline{a})$ , where:

- $K$  is an abelian number field,  $K \subset \overline{\mathbb{Q}}$ ,
- $\underline{a} \in \mathbb{B}_K^0$ ,

and  $J_K(\underline{a})$  is given on prime ideals  $\mathfrak{p}$  of  $K$  not dividing  $m(\underline{a})$  by the rule

- $J_K(\underline{a})(\mathfrak{p}) = g_K(\underline{a}, \mathfrak{p})$ .

The fact that  $J_K(\underline{a})$  is actually a Hecke character of  $K$ , with defining ideal dividing a power of  $m(\underline{a})$ , hinges on the condition  $\underline{a} \in \mathbb{B}_K^0$ , and can either be dug out of [Kb], or derived from Anderson's geometric interpretation: see I § 7.

8.2.5 Elementary properties of Gauss sums imply that  $J_K(\underline{a})$  is galois equivariant:

$$[J_K(\underline{a})(\mathfrak{a})]^s = J_K(\underline{a})(\mathfrak{a}^s) = J_K(\underline{a}^s)(\mathfrak{a}),$$



for all  $s \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and any ideal  $\underline{a}$  of  $K$  prime to  $\mathfrak{m}(\underline{a})$ . In particular,  $J_K(\underline{a})$  takes values in  $K^*$ .

8.2.6 It is plain from the construction that, for  $L/K$  a finite extension and  $\underline{a} \in \mathbb{B}_K^0 \subset \mathbb{B}_L^0$ , one has

$$J_L(\underline{a}) = J_K(\underline{a}) \circ \mathbb{N}_{L/K} \quad .$$

8.2.7 Stickelberger's theorem implies that the infinity type of  $J_K(\underline{a})$  is  $\theta_K(\underline{a})$  - defined in 8.2.1 - , which takes values in  $\mathbf{Z}$  if  $\underline{a} \in \mathbb{B}^0$ .

### 8.3 Example 8.1 revisited

8.3.1 Let us first write our basic characters of 8.1 in Anderson's notations. So let  $K = \mathbb{Q}(\sqrt{-D})$  be of discriminant  $-D < -8$ . Put

$$\underline{a}_D = \sum_{\substack{j=1 \\ \varepsilon(j)=-1}}^D \left[ \frac{j}{D} \right] \quad .$$

We find (8.1.3) that

$$n_1 = \langle \underline{a}_D \rangle \quad \text{and} \quad n_c = \langle c\underline{a}_D \rangle \quad .$$

Therefore, by the remark following 8.1.5, one has

$$\underline{a}_D \in \mathbb{B}^0 \quad .$$

Since  $\underline{a}_D$  clearly belongs to  $\mathbb{B}_K$ , the character  $J_K(\underline{a}_D)$  is well defined in Anderson's setup, and it is an easy exercise to check that

$$J_D = J_K(\underline{a}_D) \quad .$$

8.3.2 We shall now define, in Anderson's notation, a basic Jacobi sum character for each of the imaginary quadratic fields not treated in 8.1 and 8.3.1, i.e., for  $D = 3, 4, 8$ .

In all three cases the class number  $h_D$  is 1, and (in analogy with 8.1.5 for  $D = 3, 4$ ) we shall define  $J_D$  so that its infinity type  $T_D$  is  $1.1 + 0.c = 1$ . All we have to do is give an element  $\underline{a}_D \in \mathbb{B}_{\mathbb{Q}(\sqrt{-D})}^0$ , for  $D = 3, 4, 8$ , such that

$$\theta_{\mathbb{Q}(\sqrt{-D})}(\underline{a}_D)(\sigma) = \begin{cases} 1 & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma = c \end{cases} .$$

For  $D = 3, 4$ , we have tried to make a "classical" choice of  $\underline{a}_D$  - see I. 7.5.

We propose as basic characters,  $J_D = J_K(\underline{a}_D)$  with

$$\begin{aligned} \underline{a}_3 &= 2\left[\frac{2}{3}\right] - \left[\frac{1}{3}\right] \\ \underline{a}_4 &= \left[\frac{1}{2}\right] + \left[\frac{3}{4}\right] - \left[\frac{1}{4}\right] \\ \underline{a}_8 &= -\left[\frac{1}{2}\right] + \left[\frac{5}{8}\right] + \left[\frac{7}{8}\right] \quad . \end{aligned}$$

#### 8.4 The Stickelberger ideal

8.4.1 Definition. Let  $K$  be an abelian number field. The Stickelberger ideal of  $K$  is the ideal of the group ring  $\mathbb{Z}[\text{Gal}(K/\mathbb{Q})]$  consisting precisely of the infinity types of all Jacobi sum Hecke characters of  $K$ . It is denoted  $\text{St}_K$  .

It is not hard to check that our Stickelberger ideal  $\text{St}_K$  coincides with the one defined by Sinnott in [Sin] . The main property of  $\text{St}_K$  which we shall have occasion to use is the following

8.4.2 Proposition. Let  $A_K \subset \mathbb{Z}[\text{Gal}(K/\mathbb{Q})]$  be the set of infinity types of all algebraic Hecke characters of  $K$  . Then  $\text{St}_K$  is a subgroup of finite index in  $A_K$  .

See [Sin], Theorem 2.1.

In general, it is very hard to describe  $St_K$  inside  $A_K$ , and even to give an explicit formula for the index  $[A_K:St_K]$ .

An interesting case in which both can be done takes us back to our initial example of 8.1, resp. to 8.3.2.

8.4.3 Lemma Let  $K = \mathbb{Q}(\sqrt{-D})$  be any imaginary quadratic number field,  $-D$  its discriminant. Then  $St_K$  consists precisely of the types

$$k \cdot T_D + j \cdot (1 + c)$$

with  $k, j \in \mathbb{Z}$ . The index  $[A_K:St_K] = h_D$  .

(Recall that  $T_D$  was defined in 8.1.5, resp. 8.3.2.)

Proof. First observe that the given types are actually contained in  $St_K$ . This is true by construction for  $T_D$ , and  $1 + c$  is the infinity type of the norm  $\mathbb{N}$ , i.e. of (say)  $J_K(\underline{a}_3 + c\underline{a}_3)$ , where  $\underline{a}_3$  is as in 8.3.2 (but  $K = \mathbb{Q}(\sqrt{-D})$  - cf. 8.2.6).

Secondly, the index of the set of types described is  $h_D$ . In fact, (8.1.5)

$$T_D - \frac{1}{2} \left( \frac{\varphi(D)}{2} - h_D \right) \cdot (1 + c) = h_D \cdot 1 ,$$

unless  $D = 8$  - in which case  $A_K = St_K$  according to 8.3.2.

On the other hand, it follows from Theorem 2.1 combined with Theorem 5.3 of [Sin] that  $[A_K:St_K] = h_D$ , in our case. - But, to be sure, our quadratic fields do not really merit this quote: In fact, suppose  $\underline{b} \in \mathbb{B}_K^0$ , and  $K = \mathbb{Q}(\sqrt{-D})$  with  $D > 8$ . Put  $m = m(\underline{b})$  and decompose  $\underline{b} = \sum_{1 < d | m} \underline{b}_d$  with

$$\underline{b}_d = \sum_{\substack{i=1 \\ (i,m)=\frac{m}{d}}}^m n_i \left[ \frac{i}{m} \right] .$$

Then  $\underline{b}_d \in \mathbb{B}_{\mathbb{Q}(\mu_d)}$ , and since the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  respects this decomposition of  $\underline{b}$  we find that

$$\underline{b}_d \in \begin{cases} \mathbb{B}_{\mathbb{Q}} & \text{if } K \not\subset \mathbb{Q}(\mu_d) \\ \mathbb{B}_K & \text{if } K \subset \mathbb{Q}(\mu_d) \end{cases} .$$

In the first case, if  $d \neq 2$ , it follows that  $\underline{b}_d$  is a multiple of  $\prod_{j=1}^d \left[ \frac{j}{d} \right]$ , and therefore in  $\mathbb{B}_{\mathbb{Q}}^0$ . So it contributes to  $J_K(\underline{b})$  a Hecke character of  $\mathbb{Q}$ , i.e., a multiple of  $1 + c$  to the infinity type. We are therefore reduced to elements  $\underline{b}$  of the form

$$\underline{b} = n_m \left[ \frac{1}{2} \right] + \prod_{D|d|m} \underline{b}_D .$$

Now  $\underline{b}_d \in \mathbb{B}_K$  implies that

$$\langle \underline{b}_d \rangle = r \cdot \prod_{\substack{j=1 \\ (j,d)=1 \\ \varepsilon(j)=-1}}^d \frac{j}{d} + s \cdot \prod_{\substack{j=1 \\ (j,d)=1 \\ \varepsilon(j)=+1}}^d \frac{j}{d} = r \cdot n_1(d) + s \cdot n_c(d),$$

with  $n_1(D) = n_1$  and  $n_c(D) = n_c$ , as in 8.1.3.

Now it is easy to check that

$$n_1(d) - n_c(d) = (n_1 - n_c) \prod_{p|d} (1 - \varepsilon(p)) = h_D \cdot \prod_{p|d} (1 - \varepsilon(p)).$$

This means that, if  $n \cdot 1$  is the infinity type of a Jacobi sum Hecke character of  $K$ , then  $h_D | n$ .

q.e.d.

CHAPTER ONE:

Motives for Algebraic Hecke Characters

This chapter contains an exposition of the less elementary and more geometric parts of the theory of algebraic Hecke characters. None of the results is original, but all the main theorems are fairly recent, so this is almost the first time that they are explicitly put together with a view to providing a "motivic" theory of Hecke characters. Compare however [A2] and [B1]. More precisely, we indicate a proof of Conjecture 8.1 of [DP] in the setting of certain motives for absolute Hodge cycles. We start out with the key example of the theory:

1. Abelian varieties with complex multiplication

1.1 Let  $K$  and  $E$  be two number fields (of finite degree over  $\mathbb{Q}$ ), and let  $A$  be an abelian variety defined over  $K$  such that  $2 \dim A = [E : \mathbb{Q}]$ . Denote by  $\text{End}_K A$  the ring of endomorphisms of  $A$  that are defined over  $K$ , and assume there is an embedding of  $\mathbb{Q}$ -algebras

$$E \hookrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \text{End}_K A,$$

which will be fixed throughout. Then we say that  $A/K$  has complex multiplication by  $E$ . For any prime power  $\ell^n$  in  $\mathbb{Z}$ , denote by  $A[\ell^n]$  the kernel of multiplication by  $\ell^n$  on  $A$ , and define as usual

$$T_\ell(A) = \varprojlim_n A[\ell^n](\bar{K}), \quad V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Here  $\bar{K}$  is some fixed algebraic closure of  $K$ . There is a natural faithful action of  $\text{End } A$  on  $T_\ell(A)$ , and therefore of  $E$  on  $V_\ell(A)$ . As  $K$  is of characteristic 0,  $T_\ell(A)$  is a free  $\mathbb{Z}_\ell$  module of rank  $2 \dim A$ , and  $V_\ell(A)$  is a free  $E \otimes \mathbb{Q}_\ell$  module of rank 1. The action of  $\text{End}_K A$  commutes with the natural action of  $\text{Gal}(\bar{K}/K)$  on  $T_\ell(A)$  and  $V_\ell(A)$ . So the Galois representation on  $V_\ell(A)$  is  $E \otimes \mathbb{Q}_\ell$ -linear, and splits up as a sum of 1-dimensional  $\lambda$ -adic representations, for the places  $\lambda$  of  $E$  dividing  $\ell$ ,

$$\chi_\lambda : \text{Gal}(\bar{K}/K) \rightarrow E_\lambda^* .$$

1.2 The formation of  $T_\ell$  and  $V_\ell$  is, of course, not restricted to abelian varieties with complex multiplication by a field  $E$  as above. And the part of the "Weil-conjectures" proved by Weil himself implies that the system of Galois representations  $T_\ell(A)$ , for  $\ell$  varying over all rational primes, is a strictly compatible system of ( $\mathbb{Q}$ -) rational representations. Recall what this means: For a finite place  $\mathfrak{p}$  of  $K$ , choose a decomposition group  $D_\mathfrak{p} \subseteq \text{Gal}(\bar{K}/K)$  and an element  $\text{Frob } \mathfrak{p} \in D_\mathfrak{p}$  whose class modulo the inertia subgroup  $I_\mathfrak{p} \subseteq D_\mathfrak{p}$  corresponds to the automorphism  $x \mapsto x^{1/\mathbb{N}\mathfrak{p}}$  of the residue field extension defined by  $D_\mathfrak{p}$ . Then  $\text{Frob } \mathfrak{p}$  is well defined modulo  $I_\mathfrak{p}$ , and up to conjugation in  $\text{Gal}(\bar{K}/K)$ , and is called a geometric Frobenius element at  $\mathfrak{p}$ . With these notations the strict compatibility of the  $T_\ell(A)$  means that there is a finite set  $S$  of places of  $K$  - to wit, the places where  $A$  has bad reduction - such that for all rational primes  $\ell, \ell'$  and any  $\mathfrak{p}$  with  $\mathfrak{p} \notin S$  and  $\mathfrak{p} \nmid \ell\ell'$ ,  $T_\ell(A)$  and  $T_{\ell'}(A)$  are unramified at  $\mathfrak{p}$ , and the "characteristic polynomials" (which are then independent of the choice of  $\text{Frob } \mathfrak{p}$ )

$$\det(1 - \text{Frob } \mathfrak{p} \cdot X | T_\ell(A)) \quad \text{and} \quad \det(1 - \text{Frob } \mathfrak{p} \cdot X | T_{\ell'}(A))$$

have coefficients in  $\mathbb{Q}$  and are equal. - Cf. [ST]. Weil's theorems also tell us that all the eigenvalues of  $\text{Frob } \mathfrak{p}$  on  $T_\ell(A)$  are algebraic numbers purely of absolute value  $(\sqrt{\mathbb{N}\mathfrak{p}})^{-1}$ .

1.3 In the case of complex multiplication, the system  $(\chi_\lambda)$  of  $\lambda$ -adic representations,  $\lambda$  varying over all finite places of  $E$ , is itself a strictly compatible system of  $E$ -rational Galois representations. That is to say, for every finite place  $\mathfrak{p}$  of  $K$  not in the set of bad reduction  $S$ , there is a number  $\chi(\mathfrak{p}) \in E^*$  such that for any finite place  $\lambda$  of  $E$  with  $\mathfrak{p} \nmid \mathbb{N}\lambda$ ,  $\chi(\mathfrak{p})$  maps to  $\chi_\lambda(\text{Frob } \mathfrak{p})$  under  $E \hookrightarrow E_\lambda$ . To prove this, one has to study the reduction mod  $\mathfrak{p}$  of the abelian variety  $A$ : The Galois action of  $\text{Frob } \mathfrak{p}$  reduces to the

geometric Frobenius endomorphism on the reduction  $\tilde{A}_p$  over the residue class field of  $K$  at  $p$ . This endomorphism lies in the center of the algebra  $\mathbb{Q} \otimes_{\mathbb{Z}} \text{End}(\tilde{A}_p)$ , and therefore lifts back to an element  $\chi(p) \in E \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \text{End}_K A$ .  $\chi(p)$  is unique because reduction of endomorphisms is injective. It is the number sought. Cf. [LCM]. 2 §3.

It follows from the theory of Shimura and Taniyama - see [ShT], cf. [LCM], Chap. 4 - that  $\chi$  extends multiplicatively to an algebraic Hecke character, i.e., " $\chi$  admits a conductor". But we prefer to deduce this from a much more general result which will be used later on:

1.4 Proposition: Let  $\chi_\lambda : \text{Gal}(\bar{K}/K) \rightarrow E_\lambda^*$ , for all finite places  $\lambda$  of  $E$ , be a strictly compatible system of E-rational  $\lambda$ -adic representations of  $K$ . Then there is an algebraic Hecke character  $\chi$  of  $K$  with values in  $E$  such that, for every finite place  $\lambda$  of  $E$ ,  $\chi_\lambda$  is the  $\lambda$ -adic representation attached to  $\chi$  (defined in Chapter 0, §5).

This proposition is just a variant of the main theorem of [Henn], which in turn is a corollary of a result in transcendence theory. In fact, Henniart proves that any abelian semisimple E-rational  $\lambda$ -adic Galois representation of  $K$  is locally algebraic. This means that there is a homomorphism of group-schemes over  $E_\lambda$ ,

$$T_\lambda : Z/E_\lambda \rightarrow \mathbb{G}_m/E_\lambda \quad (\text{where } Z = R_{K/\mathbb{Q}} \mathbb{G}_m)$$

such that the restriction of  $T_\lambda/E_\lambda : Z(E_\lambda) \rightarrow E_\lambda^*$  to the subgroup

$$\prod_{\mathfrak{v}|\mathbb{N}\lambda} K_{\mathfrak{v}}^* = Z(\mathbb{Q}_\ell) \subseteq Z(E_\lambda)$$

coincides with the reciprocal of the composite map

$$\prod_{\mathfrak{v}|\mathbb{N}\lambda} K_{\mathfrak{v}}^* \rightarrow K_A^* \xrightarrow{\text{Frob}} \text{Gal}(K^{\text{ab}}/K) \xrightarrow{\text{repres.}} E_\lambda^*$$

on a suitable neighbourhood of 1. - Note that this condition

is the analogue, for a finite place  $\lambda$ , of the existence of a defining ideal for the representation: see [S $\ell$ ], III - 2. This is the reason why proposition 1.4 follows from Henniart's theorem.

1.5 Let us come back to the abelian variety  $A$  over  $K$  with complex multiplication by  $E$ . Let  $\chi$  be the algebraic Hecke character of  $K$  with values in  $E$  giving the  $\lambda$ -adic representations  $\chi_\lambda$  of  $A$ , i.e., giving the action of  $\text{Gal}(\bar{K}/K)$  on the torsion points of  $A$ .

The Tate-conjecture proved by Faltings - cf. [Sch2], in particular 4.2 - implies that, for every  $\ell$ , the  $\mathbb{Q}_\ell$ -subalgebra of  $\text{End}_{\mathbb{Q}_\ell} V_\ell(A)$  generated by the action of  $\text{Gal}(\bar{K}/K)$  is the commutant of  $\text{End}_K A \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ . Since  $E \otimes \mathbb{Q}_\ell$  is its own commutant in  $\text{End}_{\mathbb{Q}_\ell} V_\ell(A)$  it follows that  $E = \mathbb{Q}(\chi)$  - the field generated over  $\mathbb{Q}$  by the values of  $\chi$  - if and only if  $E = \mathbb{Q} \otimes_{\mathbb{Z}} \text{End}_K A$ , i.e., if and only if  $A$  is simple over  $K$ . In particular,  $E$  is a CM-field in that case (it cannot be totally real, as  $\chi\bar{\chi} = \mathbb{N}^{-1}$ ) - but this proof is of course a little heavy handed for this elementary fact.

1.6 The character  $\chi$  has weight  $-1$ , and, moreover, its infinity-type  $T$  is what we call a CM-type of  $K$  (see 0 §3). To determine  $T : K^* \rightarrow E^*$ , extend it to a map from ideals of  $K$  to ideals of  $E$  so that

$$\chi(\mathfrak{p}) \cdot \mathcal{O}_E = T(\mathfrak{p}),$$

for almost all prime ideals  $\mathfrak{p}$  of  $K$ . The prime ideal decomposition of  $\chi(\mathfrak{p}) \cdot \mathcal{O}_E$  can be determined from the fact that

$$\chi(\mathfrak{p}) \in E \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \text{End}_K A \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \text{End}(\tilde{A}_{\mathfrak{p}})$$

reduces to the geometric Frobenius on  $\tilde{A}_{\mathfrak{p}}$ , by letting  $\text{End } A$  and  $\text{End } \tilde{A}_{\mathfrak{p}}$  act on the tangent spaces  $\text{Lie } A$  and  $\text{Lie } \tilde{A}_{\mathfrak{p}}$  - see [Gi]. Viewing  $\text{Lie } A$  as a  $K \otimes_{\mathbb{Q}} E$  module - cf. [ST], §7 - the final result can be stated like this;



$$T(x) = \det_E(x \otimes 1 \mid \text{Lie } A)^{-1} \in E^*.$$

Recall in passing that the algebraic homomorphism

$$\begin{aligned} E^* &\rightarrow K^* \\ y &\mapsto \det_K(1 \otimes y \mid \text{Lie } A)^{-1} \end{aligned}$$

(or rather, its reciprocal) is often called the CM-type of  $A$ , and  $T$  (or rather,  $-T$ ) is called its "reflex-type" (on  $K$ ).

1.7 An interesting way of rephrasing this description of the infinity-type of  $\chi$  is provided by the Hodge decomposition of the first singular homology of  $A$ . For  $\sigma : K \rightarrow \mathbb{C}$ , write

$$H_1^\sigma(A, \mathbb{C}) = H_1((A \times_{K, \sigma} \mathbb{C})(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H_\sigma^{-1, 0} \oplus H_\sigma^{0, -1}.$$

Then

$$H_\sigma^{0, -1} = \text{Lie}(A \times_{K, \sigma} \mathbb{C}) = (\text{Lie } A) \otimes_{K, \sigma} \mathbb{C}.$$

But  $\text{Lie } A$  is also an  $E$  module. For any  $\tau : E \rightarrow \mathbb{C}$ , define  $n(\sigma, \tau)$  to be  $-1$  or  $0$  according as the action of  $E$  on  $\text{Lie } A$  agrees with the action of  $E$  via  $E \xrightarrow{\tau} \mathbb{C}$  on the subspace  $H_\sigma^{-1, 0}$ , or not (in which case  $n(\sigma, \bar{\tau})$  will be  $-1$ ). These integers  $n(\sigma, \tau)$  describe the Hodge decompositions of the  $H_1^\sigma(A)$ , for all  $\sigma$ , as follows: Since  $E \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \text{End}_K A$ , every  $H_1^\sigma(A, \mathbb{Q})$  has the structure of an  $E$  vector space (of dimension 1). The direct factor  $H_1^\sigma(A, \mathbb{Q}) \otimes_{E, \tau} \mathbb{C}$  of  $H_1^\sigma(A, \mathbb{C})$  lies in  $H_\sigma^{n(\sigma, \tau), (-1-n(\sigma, \tau))}$ .

On the other hand, the identity  $T(x) = \det_E(x \otimes 1 \mid \text{Lie } A)^{-1}$  means that the  $n(\sigma, \tau)$ 's are precisely the integers attached to  $\chi$  in Chapter 0, §4. Later on in this chapter, we shall systematically generalize this kind of correspondence between infinity-types and Hodge structures on an  $E$ -vector space of dimension one.

1.8 Sticking to our preference for the geometric Frobenius

over the arithmetic one, and taking the  $V_\ell(A)$  (rather than their duals  $H_{\text{ét}}^1(A/\bar{K}, \mathbb{Q}_\ell)$ ) as the  $\ell$ -adic representations characterizing  $A$ , we are led, by the motivic formalism explained in 6.5.9 below, to define the (false!) "Hasse-Weil" L-function of  $A$  over  $K$ , for  $\text{Re}(s) > \frac{1}{2}$ , by:

$$L(A/K, s) = \prod_p \det(1 - \text{Frob}_p \cdot \mathbb{N}_p^{-s} | V_\ell(A)_{I_p})^{-1},$$

where  $I_p$  is an inertia-subgroup at  $p$ ;  $p$  runs over all finite places of  $K$ , and it is understood that the determinant is calculated using some prime number  $\ell$  such that  $p \nmid \ell$ .  
- Cf. [ST].

Since  $V_\ell(A) = \bigoplus_{\lambda | \ell} \chi_\lambda$ , we see immediately that

$$L(A/K, s) = \prod_{\tau: E \rightarrow \mathbb{C}} L(\chi^\tau, s),$$

in the notation of 0 §6.

**1.8.1 Warning:** Our function  $L(A/K, s)$  is not the usual Hasse-Weil L-function of  $A$  over  $K$  as employed for instance in the standard formulation of the conjectures of Birch and Swinnerton-Dyer:

$$\begin{aligned} L_{\text{usual}}(A/K, s) &= \prod_p \det(1 - \text{Frob}_p \cdot \mathbb{N}_p^{-s} | H_{\text{ét}}^1(A/\bar{K}, \mathbb{Q}_\ell)_{I_p})^{-1} \\ &= \prod_p \det(1 - \text{Frob}_{\text{arith}}^p \cdot \mathbb{N}_p^{-s} | V_\ell(A)_{I_p})^{-1}, \end{aligned}$$

with  $V_\ell(A)_{I_p}$  denoting coinvariants under inertia. Under the functional equation (which for general  $A$  is only conjectured),  $L(A/K, s)$  corresponds to  $L_{\text{usual}}(A/K, 1-s)$ .

The L-function  $L(A/K, s)$  is attached to  $A/K$  without reference to the fact that  $A$  has complex multiplication by  $E$ . In the presence of complex multiplication it is, however, more adequate to consider the array of L-functions

$$L^*(A/K, s) = L^*(\chi, s) = (L(\chi^\tau, s))_{\tau: E \rightarrow \mathbb{C}}$$

taking values in  $E \otimes_{\mathbb{Q}} \mathbb{C}$ .

In order to find "geometric" objects over  $K$  whose L-functions include all L-functions of algebraic Hecke characters of  $K$ , we have to pass from abelian varieties (with complex multiplication) to motives.

## 2. Motives for absolute Hodge cycles

The lecture notes [DMOS] contain the first detailed exposition of a theory of motives over fields of characteristic zero which does not depend on unproven conjectures. They will be our constant frame of reference when we are dealing with motives. The other main source for the kind of questions treated here is of course Deligne's article [DP] which, however, insists on the general formalism, not attaching any specific meaning to the word motive, and using a hierarchy of conjectures when needed. In this section, we shall quickly review the main concepts and results from the general theory of motives as constructed in [DMOS], II § 6.- For a somewhat different setup of largely the same theory, see [A2].

### 2.1 ABSOLUTE HODGE CYCLES (Reference: [DMOS], I § 1, § 2)

Let  $K$  be a field which can be embedded into  $\mathbb{C}$ , and  $X$  a smooth projective algebraic variety over  $K$ . To every place of  $\mathbb{Q}$ , we can attach a cohomology theory of varieties  $X$  over  $K$ :

At infinity, take the algebraic de Rham cohomology

$$H_{\text{DR}}^*(X) = H_{\text{DR}}^*(X/K) = H^*(X_{\text{Zar}}, \Omega_{X/K}^*)$$

For all  $n$ ,  $H_{\text{DR}}^n(X)$  is a  $K$ -vector space equipped with a descending filtration  $F^*$ , the Hodge filtration.

All the finite primes  $\ell$  of  $\mathbb{Q}$  can be treated simultaneously: denote by  $\mathbb{O}_{\mathbb{A}f}$  the ring of finite adèles of  $\mathbb{Q}$ , and put

$$H_{\mathbf{A}f}^*(X) = H_{\mathbf{A}f}^*(X/K) = \left\{ \varinjlim_m H^*((X \times_K \bar{K})_{\text{ét}}, \mathbf{Z}/m\mathbf{Z}) \right\} \otimes_{\mathbf{Z}}^{\wedge} \mathbb{Q}_{\mathbf{A}f},$$

where  $\bar{K}$  is an algebraic closure of  $K$ . The  $H_{\mathbf{A}f}^n(X/K)$  are  $\mathbb{Q}_{\mathbf{A}f}$ -modules with a natural action of  $\text{Gal}(\bar{K}/K)$ . The  $\mathbb{Q}_{\ell}$ -component of  $H_{\mathbf{A}f}^*(X)$  will be written  $H_{\ell}^*(X)$ .

For any embedding  $\sigma: K \rightarrow \mathbb{C}$ , denote by  $\sigma X$  the extension of scalars  $X \times_{K, \sigma} \mathbb{C}$ , and by

$$H_{\sigma}^*(X) = H^*(\sigma X(\mathbb{C}), \mathbb{Q})$$

the rational singular cohomology (resp.,  $H_{\sigma}^{\mathbb{C}}(X)$  the rational singular homology) of  $\sigma X(\mathbb{C})$ . The  $H_{\sigma}^*(X)$  are rational Hodge structures, i.e.,  $\mathbb{Q}$ -vector spaces together with a decomposition of the complexifications

$$H_{\sigma}^n(X) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

such that  $H^{p,q}$  and  $H^{q,p}$  are interchanged by complex conjugation. Whenever  $K$  is given as a subfield of  $\mathbb{C}$  (e.g.,  $K = \mathbb{Q}, \mathbb{R}$ ),  $H_{\sigma}^*$ , for  $\sigma$  the inclusion  $K \subset \mathbb{C}$ , will be written  $H_B^*$ , the letter  $B$  standing for "Betti". (See also 6.0.)

In these cohomology theories, define the Tate twist as follows (we write  $\mu_m(\bar{K}) = \{\zeta \in \bar{K}^* \mid \zeta^m = 1\}$ ).

$$\mathbb{Q}_{\text{DR}}(1) = K$$

$$\mathbb{Q}_{\text{DR}}(-1) = K$$

$$\mathbb{Q}_{\text{DR}}(1) = F^{-1} \supset F^0 = 0$$

$$\mathbb{Q}_{\text{DR}}(-1) = F^1 \supset F^2 = 0$$

$$\mathbb{Q}_{\mathbf{A}f}(1) = \varinjlim_m \mu_m(\bar{K}) \otimes_{\mathbf{Z}} \mathbb{Q}_{\mathbf{A}f}$$

$$\mathbb{Q}_{\mathbf{A}f}(-1) = \text{Hom}(\mathbb{Q}_{\mathbf{A}f}(1), \mathbb{Q}_{\mathbf{A}f})$$

$$\mathbb{Q}_B(1) = 2\pi i \mathbb{Q} \subset \mathbb{C}$$

$$\mathbb{Q}_B(-1) = \frac{1}{2\pi i} \cdot \mathbb{Q}$$

$$\mathbb{Q}_B(1) \otimes_{\mathbb{Q}} \mathbb{C} = H^{-1, -1}$$

$$\mathbb{Q}_B(-1) \otimes \mathbb{C} = H^{1, 1}$$

the involution  $F_{\infty}$  (see II, 1.6.1) acts on  $\mathbb{Q}_B(1)$  as  $-1$

the involution  $F_{\infty}$  acts on  $\mathbb{Q}_B(-1)$  as  $-1$

For  $m \in \mathbf{Z}, m > 0$ ,

For  $m \in \mathbf{Z}, m < 0$ ,

$$\mathbb{Q}..(m) = \mathbb{Q}..(1)^{\otimes m}$$

$$\mathbb{Q}..(m) = \mathbb{Q}..(-1)^{\otimes -m}$$

For all  $m \in \mathbb{Z}$ ,

$$H_{\mathbb{Q}}^*(X)(m) = H_{\mathbb{Q}}^*(X) \otimes \mathbb{Q}..(m).$$

For every embedding  $\sigma: \bar{K} \hookrightarrow \mathbb{C}$  of the fixed algebraic closure  $\bar{K}$  of  $K$  into  $\mathbb{C}$ , there is the total comparison isomorphism

$$(2.1.1) \quad H_{\mathbb{Q}}^*(X)(m) \otimes_{\mathbb{Q}} (\mathbb{C} \times \mathbb{A}^f) \xrightarrow{\cong} H_{\text{DR}}^*(\sigma X) \times H_{\mathbb{A}^f}^*(\sigma X)(m),$$

the filtration on  $H_{\text{DR}}^*(\sigma X) = H_{\text{DR}}^*(X) \otimes_{K, \sigma} \mathbb{C}$  being given by

$$\bigoplus_{p' \geq p} H^{p', q'} \xrightarrow{\cong} F^p.$$

Note that  $\sigma$  induces an isomorphism

$$H_{\mathbb{A}^f}^*(X \times_K \bar{K}/K) \cong H_{\mathbb{A}^f}^*(\sigma X/\mathbb{C}),$$

and  $H_{\mathbb{A}^f}^*(X \times_K \bar{K}/\bar{K})$  is just  $H_{\mathbb{A}^f}^*(X/K)$ , with the action of  $\text{Gal}(\bar{K}/K)$  forgotten.

Abbreviate  $H_{\text{DR}}^*(X)(m) \times H_{\mathbb{A}^f}^*(X)(m)$  to  $H_{\mathbb{A}}^*(X)(m)$ . For  $p \in \mathbb{Z}$ ,  $p > 0$ , an element  $t \in H_{\mathbb{A}}^{2p}(X/K)(p)$  is called a Hodge cycle (of codimension  $p$ ) over  $K$  relative to  $\sigma: \bar{K} \rightarrow \mathbb{C}$ , if

$$(i) \quad t \in H_{\sigma}^{2p}(X)(p) \subset H_{\sigma}^{2p}(X)(p) \otimes (\mathbb{C} \times \mathbb{Q}_{\mathbb{A}^f}) \quad (\text{by } 2.1.1),$$

(ii) the  $H_{\text{DR}}$ -component  $t_{\text{DR}}$  of  $t$  lies in

$$F^0 H_{\text{DR}}^{2p}(X)(p) = F^p H_{\text{DR}}^p(X).$$

The algebraic condition (ii) is clearly equivalent to the analytic one:

$$(ii)' \quad t_{\text{DR}} \in H^{0,0} \subset H_{\sigma}^{2p}(X)(p) \otimes \mathbb{C} \quad (\text{by } 2.1.1_{\text{DR}}).$$

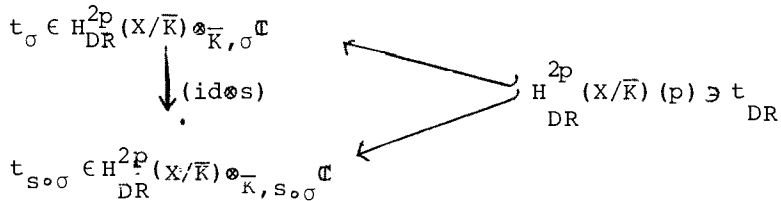
Condition (i) means that the components of  $t \in H_{\mathbf{A}}^{2p}(X)(p)$  all correspond to a single element in  $H_{\sigma}^{2p}(X)(p)$ , under the various comparison isomorphisms: between Betti and de Rham, Betti and étale cohomologies.

An absolute Hodge cycle on  $X$  over  $K$  (of codimension  $p$ ) is an element  $t \in H_{\mathbf{A}}^{2p}(X/K)(p)$  which is a Hodge cycle relative to all  $\sigma: \bar{K} \rightarrow \mathbb{C}$ . The  $\mathbb{Q}$ -vector space of all these absolute Hodge cycles is denoted by  $C_{\text{AH}}^p(X/K)$ , or  $C_{\text{AH}}^p(X)$  if the reference to  $K$  is clear or irrelevant (e.g.,  $K = \bar{K}$  - see below). Clearly,  $\dim_{\mathbb{Q}} C_{\text{AH}}^p(X/K) < \infty$ . The definition of  $C_{\text{AH}}^p$  we have given does not easily betray its virtues. - Looked at "from the side of Betti cohomology",  $C_{\text{AH}}^p(X/K)$  is isomorphic to the  $\mathbb{Q}$ -vector space of arrays  $(t_{\sigma} | \sigma: \bar{K} \rightarrow \mathbb{C})$ , where

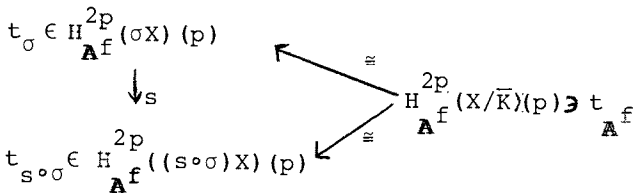
(i)  $t_{\sigma} \in H_{\sigma}^{2p}(X)(p) \cap H^{0,0}$ ,

such that, fixing any  $\sigma: \bar{K} \rightarrow \mathbb{C}$ , we have for all  $s \in \text{Aut } \mathbb{C}$ :

(ii)  $t_{\sigma}$  and  $t_{s \circ \sigma}$  correspond to the same element  $t_{\text{DR}} \in H_{\text{DR}}^{2p}(X/\bar{K})$  under the Betti-de Rham comparison isomorphism



(iii)  $t_{\sigma}$  and  $t_{s \circ \sigma}$  correspond to the same element  $t_{\mathbf{A}^f} \in H_{\mathbf{A}^f}^{2p}(X)(p)$  under the Betti-étale comparison isomorphism:



(iv)  $t_{\sigma} = t_{\sigma \circ g}$ , for all  $g \in \text{Gal}(\bar{K}/K)$ .

Note that (iv) makes sense because  $X$  is defined over  $K$ , and

so  $H_{\sigma}^1(X) = H_{\sigma \circ g}^1(X)$ . Given the compatibilities (ii) and (iii), the Galois-action may also be read on  $H_{DR}$  or  $H_{\mathbf{A}^f}$ , and (iv) may be replaced by either

$$(iv)' \quad t_{DR} \in H_{DR}^{2p}(X/K),$$

- which is what we used in our first definition of absolute Hodge cycles - , or by

$$(iv)'' \quad t_{\mathbf{A}^f} \in [H_{\mathbf{A}^f}^{2p}(X)(p)]^{\text{Gal}(\bar{K}/K)}.$$

The following proposition sums up the fundamental rationality properties of absolute Hodge cycles.

2.1.2 Proposition: a) If  $L \supset K$  is still embeddable into  $\mathbb{C}$ , then the natural map

$$C_{AH}^p(X/\bar{K}) \rightarrow C_{AH}^p(X/\mathbb{L})$$

is an isomorphism.

b)  $\text{Gal}(\bar{K}/K)$  acts on  $C_{AH}^p(X/K)$  through a finite quotient.

To prove a), one has to invoke the theory of the Gauss-Manin connection. As for b), we have already seen that  $\text{Gal}(\bar{K}/K)$  stabilizes  $C_{AH}^p(X/\bar{K})$ . This action being continuous and  $\mathbb{Q}$ -linear on a finite dimensional  $\mathbb{Q}$ -vector space it factors through a finite quotient.

The crucial result justifying in a way the theory we are about to develop is Deligne's

2.1.3 Theorem: If  $K$  is algebraically closed, and  $X$  is an abelian variety over  $K$ , then every cycle  $t \in H_{\mathbf{A}^f}^{2p}(X)(p)$  which is a Hodge cycle relative to one embedding  $\sigma: K \rightarrow \mathbb{C}$  is an absolute Hodge cycle.

If  $K$  is not algebraically closed the conclusion will hold for cycles  $t$  whose  $H_{DR}$ - or  $H_{\mathbf{A}^f}$ -component is fixed by  $\text{Gal}(\bar{K}/K)$ .

The proof starts with the "exceptional" Hodge cycles on abelian varieties with some complex multiplications, studied in [Weil; 1977 c] and used more generally in [Gr 2]. They are shown to be absolute Hodge cycles by a deformation argument very much reminiscent of Gross' paper. From there, Deligne goes on to CM-abelian varieties first, and passes to the general case by another deformation argument. --See [DMOS], chap. I.

2.1.4 Remark. Every algebraic cycle, i.e., every element of  $H_{\mathbb{A}}^{2p}(X)(p)$  coming from an algebraic subvariety of  $X$  of codimension  $p$ , via the cycle maps in de Rham and étale cohomology, is an absolute Hodge cycle. The Hodge conjecture states that any cycle which is a Hodge cycle relative to one  $\sigma$  is algebraic. In this sense Deligne's result proves part of the Hodge conjecture for abelian varieties.

## 2.2 MOTIVES (Reference: [DMOS], II § 6)

Let  $K$  as before be a field embeddable into  $\mathbb{C}$ . The construction of the category  $M_K$  of motives over  $K$ , via absolute Hodge cycles, proceeds roughly as follows.

Step 1. Let  $CV_K$  be the category with objects written  $h(X)$ , for  $X$  varying over smooth projective algebraic varieties defined over  $K$ , and morphisms the  $\mathbb{Q}$ -vector spaces defined by  $\text{Hom}(h(X), h(Y)) = C_{\text{AH}}^n(X \times Y)$ , if  $X$  is connected of dimension  $n$ , and by additivity, via  $h(X \sqcup Y) = h(X) \oplus h(Y)$ , in general.

To understand this definition of morphisms, note that

$$C_{\text{AH}}^{n+p}(X \times Y) \hookrightarrow H_{\mathbb{A}}^{2(n+p)}(X \times Y)(n+p) = \bigoplus_{\mathbb{F}} \text{Hom}_{\mathbb{F}}(H^r(X), H^{r+2p}(Y)(p))$$

(by Künneth and duality), so that  $\text{Hom}(h(X), h(Y))$  really gives a family of maps between the graded Betti, resp. de Rham, resp. étale cohomology of  $X$  and  $Y$ .

Taking the cycle of the graph of a  $K$ -morphism  $X \rightarrow Y$  yields a contravariant functor  $V_K \rightarrow CV_K$ , where  $V_K$  is the usual category of smooth projective  $K$ -varieties.



It is essential to consider  $CV_K$  as a tensor category (cf. [DMOS], II § 1), the tensor product being given by

$$h(X) \otimes h(Y) = h(X \times Y),$$

with obvious associativity and commutativity constraints, and  $h(\text{pt})$  as identity object.

Step 2. Let  $\dot{M}_K^+$  be the pseudoabelian (or: "Karoubian") envelope of  $CV_K$ . This means we formally adjoint objects to  $CV_K$  to insure that every idempotent in  $\text{End}(h(X))$ , for any  $X$ , arises from a splitting  $h(X) = M' \oplus M''$  in  $\dot{M}_K^+$ . The objects of  $\dot{M}_K^+$  can be represented explicitly as pairs  $(M, p)$ , with  $M$  in  $CV_K$  and  $p \in \text{End}(M)$ ,  $p^2 = \text{id}_M$ . The morphisms are given by

$$\text{Hom}((M, p), (N, q)) = \frac{\{f: M \rightarrow N \mid f \circ p = q \circ f\}}{\{f \mid f \circ p = 0 = q \circ f\}}.$$

For every  $X$ , there is a standard decomposition, in  $\text{End}(h(X))$ , of  $\text{id}_{h(X)}$  into a sum of pairwise orthogonal idempotents

$$\text{id}_{h(X)} = p^0 + p^1 + p^2 + \dots$$

(actually a finite sum): take  $p^r$  to be the projection

$$\oplus H^*(X) \rightarrow H^r(X),$$

in all cohomology theories. In terms of absolute Hodge cycles, look at the Künneth components of the diagonal  $\Delta \subset X \times X$ :

$$\begin{aligned} H^{2n}(X \times X)(n) &= \bigoplus_{i=0}^{2n} H^{2n-i}(X) \otimes H^i(X) \\ \text{cl}(\Delta) &= \sum_{i=0}^{2n} \pi^i. \end{aligned}$$

For all  $i$ , one has  $\pi^i \in C_{\text{AH}}^n(X \times X)$ .

So, for every  $X \in V_K$  and  $0 \leq r \leq 2 \dim X$ , there is an object  $h^r(X) \in \dot{M}_K^+$  which singles out the  $r$ -th cohomology groups of  $X$ . Whence a grading on the objects of  $\dot{M}_K^+$ .

The tensor structure on  $\dot{M}_K^+$  is defined by

$$(M, p) \otimes (N, q) = (M \otimes N, p \otimes q).$$

It respects the grading in the sense that the "Künneth formula" holds:

$$(X \otimes Y)^r = \bigoplus_{s+t=r} X^s \otimes Y^t,$$

i.e., one has to check that the Künneth components are absolute Hodge cycles.

Step 3. We now introduce the Tate twist into  $\dot{M}_K^+$ . The motive  $h^2(\mathbb{P}^1) \in \dot{M}_K^+$  has precisely the cohomology groups denoted by  $\mathbb{Q}(-1)$  in 2.1. Let  $\dot{M}_K$  be the category obtained from  $\dot{M}_K^+$  by inverting the fully faithful functor  $M \mapsto M \otimes h^2(\mathbb{P}^1)$  on  $\dot{M}_K^+$ . In down-to-earth terms, this means that the objects of  $\dot{M}_K$  can be represented as pairs

$$(M, m), \text{ with } M \in \dot{M}_K^+ \text{ and } m \in \mathbb{Z}.$$

For morphisms we have

$$\text{Hom}((M, m), (N, n)) = \text{Hom}_{\dot{M}_K^+}(M \otimes (h^2(\mathbb{P}^1))^{\otimes k-m}, N \otimes (h^2(\mathbb{P}^1))^{\otimes k-n}),$$

for any  $k \geq m, n$ .

This definition is independent of  $k$ , and thus allows to define the composition of morphisms by choosing  $k$  sufficiently large.

We write  $(M, m)$  as  $M(m)$  or  $M \otimes \mathbb{Q}(m)$ .  $\dot{M}_K^+$  is a full subcategory of  $\dot{M}_K$  via  $M \mapsto M(0)$ . The tensor structure on  $\dot{M}_K$  is given by

$$M(m) \otimes N(n) = (M \otimes N)(m+n).$$

The grading on  $\dot{M}_K^+$  extends to  $\dot{M}_K$  by

$$M(m)^r = M^{r-2m}.$$

Step 4.  $\dot{M}_K$  is almost the category of motives we want. Its only technical (but important) shortcoming is the sign convention relating the grading of the objects to the tensor structure. The point is that a good category of motives should be equivalent to the category of representations of a group scheme - i.e., should be a tannakian category (see 2.3 below!). In such a category, the rank of a representation (i.e., the trace of its identity-morphism - see [DMOS], II § 1.7) is simply the dimension of the underlying space, i.e., a positive integer. But in  $\dot{M}_K$ , the rank of  $h(X)$  turns out to be the Euler-Poincaré characteristic which may of course be negative. - To put it another way, the problem is that the cup product which yields the identification of  $h(X \times Y)$  with  $h(X) \otimes h(Y)$  is not commutative.

This problem can be overcome by tampering with the commutativity constraint of the tensor structure on  $\dot{M}_K$ :

$$\dot{\Psi}: M \otimes N \xrightarrow{\sim} N \otimes M, \quad \dot{\Psi} = \oplus \dot{\Psi}^{p,q} \quad \text{where}$$

$$\dot{\Psi}^{p,q}: M^p \otimes N^q \xrightarrow{\sim} N^q \otimes M^p.$$

The corrected constraint is defined by:

$$\Psi: M \otimes N \xrightarrow{\sim} N \otimes M, \quad \Psi = \oplus \Psi^{p,q} \quad \text{where}$$

$$\Psi^{p,q} = (-1)^{pq} \dot{\Psi}^{p,q}.$$

$\dot{M}_K$  with the commutativity constraint  $\dot{\Psi}$  replaced by  $\Psi$ , is denoted by  $M_K$ , and called the category of motives over  $K$  (constructed with absolute Hodge cycles).  $M_K$  is a tannakian category, in the sense to be explained below.

2.2.1. One shows that  $M_K$  is a semisimple tannakian category (i.e., every exact sequence in  $M_K$  splits), see [DMOS], II 6.5, and that  $\text{End } M$ , for every object  $M$  of  $M_K$  is a semisimple  $\mathbb{Q}$ -algebra.

2.2.2 For practical purposes, it is often sufficient to identify a motive  $M$  in  $M_K$  with the string of its realizations in the different cohomology theories:

$$H_{\sigma}(M), H_{\text{DR}}(M), H_{\ell}(M) \quad (\sigma: K \rightarrow \mathbb{C}; \ell \text{ a rational prime}).$$

These realizations are formally defined by extending the cohomology functors  $H_{\bullet}: V_K \rightarrow A_{\bullet}$  to  $M_K$ , where  $A_{\bullet}$  is the corresponding target category:  $A_{\mathbb{B}} =$  rational Hodge structures;  $A_{\text{DR}} =$  filtered finite-dimensional  $K$ -vector spaces;  $A_{\ell} =$  finite dimensional  $\mathbb{Q}_{\ell}$ -vector spaces with  $\text{Gal}(\bar{K}/K)$ -action. For each of the cohomology theories, the extension of  $H_{\bullet}$  is possible because the categories  $A_{\bullet}$  are Karoubian, have a tensor structure with Künneth formula, and the Tate twist is defined (see 2.1).

Then,  $\text{Hom}(M, N)$ , for  $M, N \in M_K$ , consists precisely of the systems of maps

$$(f_{\mathbb{A}}) = (f_{\text{DR}}, f_{\ell} \mid \text{all } \ell)$$

such that

$$f_{\text{DR}}: H_{\text{DR}}(M) \rightarrow H_{\text{DR}}(N)$$

is a  $K$ -linear map preserving the Hodge filtrations, and for every prime number  $\ell$ ,

$$f_{\ell}: H_{\ell}(M) \rightarrow H_{\ell}(N)$$

is a  $\mathbb{Q}$ -linear map with  $f_{\ell}^{\sigma} = f_{\ell}$ , for all  $\sigma \in \text{Gal}(\bar{K}/K)$ , and such that, for any embedding  $\sigma: K \rightarrow \mathbb{C}$ , there exists a  $\mathbb{Q}$ -linear map

$$f_{\sigma}: H_{\sigma}(M) \rightarrow H_{\sigma}(N)$$

such that  $f_{\sigma} \otimes (\mathbb{C} \times \mathbb{Q}_{\mathbb{A}} f)$  corresponds to  $f_{\mathbb{A}}$  under the comparison isomorphism (2.1.1)

$$H_{\sigma}(M) \otimes (\mathbb{C} \times \mathbb{Q}_{\mathbb{A}} f) \xrightarrow{\sim} H_{\mathbb{A}}(\sigma M).$$

(Here we have used the fact that the functor  $\times_{K, \sigma} \mathbb{C}$ , or in general  $\times_K K': V_K \rightarrow V_{K'}$ , extends to a functor  $M_K \rightarrow M_{K'}$ ). Note that  $f_{\sigma}$  has to respect the Hodge decomposition.

$f$  is an isomorphism, if at least one of  $f_{\sigma}, f_{\text{DR}}, f_{\ell}$  is.

2.2.3 Remark: For any field extension  $K' \supset K$ , the functor extension of scalars,  $\times_K K' : M_K \rightarrow M_{K'}$ , carries over from varieties to motives,  $\times_K K' : M_K \rightarrow M_{K'}$ . If  $K' \supset K$  is finite, so does the functor restriction of scalars  $R_{K'/K} : V_{K'} \rightarrow V_K$ , defined by

$$\begin{array}{ccc}
 X & & X \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 \text{Spec } K' & & \text{Spec } K
 \end{array}
 \begin{array}{l}
 \nearrow \\
 \nwarrow
 \end{array}
 \text{Spec } K'$$

(or classically, according to [We 4], 1.3.) Note, however, that  $R_{K'/K} : M_{K'} \rightarrow M_K$  is not  $\otimes$ -compatible.

### 2.3 TANNAKIAN PHILOSOPHY (Reference: [DMOS], II §§ 1-5; [Sa])

2.3.1 Let  $k$  be a field, and  $G$  an affine group scheme over  $k$ , i.e., a representable group valued functor on  $k$ -algebras, or again, the inverse limit of affine  $k$ -algebraic groups (=affine group schemes of finite type over  $k$ ). - Cf. [Wa]. The category  $\text{Rep}_k(G)$  of finite dimensional representations of  $G$  over  $k$  (=algebraic morphisms  $G \rightarrow \text{GL}(V)$ , with a finite dimensional  $k$ -vector space  $V$ ) has the following properties:

- $\text{Rep}_k(G)$  is a  $k$ -linear abelian category;
- $\text{Rep}_k(G)$  is a  $\otimes$ -category - cf. [DMOS], II § 1 - with commutativity and associativity constraint, unit object whose algebra of endomorphisms is  $k$ ,  $\ast$ -compatible Hom-objects, and duals such that each object is isomorphic to its double dual;
- there is a  $k$ -linear,  $\otimes$ -compatible functor

$$\omega : \text{Rep}_k(G) \rightarrow \text{Vec}_k ;$$

$\omega$  is faithful, additive and exact.

Namely, take for  $\omega$  the functor forgetting the  $G$ -action on  $V$ .

A (neutralized) tannakian category (over  $k$ ) is a pair  $(C, \omega)$  consisting of a  $\otimes$ -category  $C$  satisfying the first two properties listed for  $\text{Rep}_k(G)$  above, and a  $\otimes$ -functor  $\omega: C \rightarrow \text{Vec}_k$  verifying the third set of conditions.

"Tannakian philosophy" exploits the fundamental theorem to the effect that there is a 1-1 correspondence between affine  $k$ -group schemes  $G$  and neutralized tannakian categories over  $k$ :

$$\begin{array}{ccc} G & \rightarrow & (\text{Rep}_k(G), \omega_{\text{forget}}) \\ \text{Aut}^{\otimes \omega} & \leftarrow & (C, \omega), \end{array}$$

where  $\text{Aut}^{\otimes \omega}$  is the group valued functor on  $k$ -algebras  $R$  such that  $(\text{Aut}^{\otimes \omega})(R)$  consists of all  $R$ -linear,  $\otimes$ -compatible automorphisms of the functor  $X \mapsto \omega(X) \otimes_k R$  on  $C$ . - Thus, it is shown that  $\text{Aut}^{\otimes \omega}$  can be represented by an affine group scheme  $G$  over  $k$ , and that  $\omega$  defines an equivalence of  $\otimes$ -categories  $C \rightarrow \text{Rep}_k(G)$ .

2.3.2 Here is a portion of the dictionary between affine group schemes and tannakian categories which results from the fundamental correspondence between them. - Cf. [Sa], p. 156 f; [DMOS], pp. 138-144; and [A2], 3.4 - 3.6.

Suppose  $(C, \omega)$  and  $(C', \omega')$  are neutralized tannakian categories over  $k$  with corresponding affine  $k$ -group schemes  $G$  and  $G'$ . Any additive  $\otimes$ -compatible functor  $F: C' \rightarrow C$  such that  $\omega' = \omega \circ F$  induces a  $k$ -morphism  $F^\#: G \rightarrow G'$ .

- (a) Suppose  $k$  is of characteristic 0. Then  $C$  is semisimple (i.e., every exact sequence in  $C$  splits) if and only if  $G$ , i.e., its connected component  $G^\circ$ , is (pro-)reductive.
- (b) Suppose the equivalent conditions of (a) are verified. Then  $F^\#$  is faithfully flat if and only if  $F$  is fully faithful.
- (c)  $F^\#$  is a closed immersion if and only if every object of  $C$

is isomorphic to a subquotient of  $F(X)$ , for some object  $X$  of  $\mathcal{C}'$ .

- (d) The objects  $\{X_i \mid i \in I\}$  of  $\mathcal{C}$  generate the tannakian category  $\mathcal{C}$  (i.e., every object of  $\mathcal{C}$  is isomorphic to an object obtained from the  $\{X_i\}$  by a finite number of operations of the following kind: tensor product, dual, direct sum, subquotient) if and only if, for every  $k$ -algebra  $R$ , the obvious map

$$G(R) \rightarrow \prod_{i \in I} \text{Aut}_R(\omega(X_i) \otimes_k R)$$

is injective.

2.3.3 As an example of a tannakian category specified by generators, consider this definition of the Mumford-Tate group of an abelian variety (cf. [DMOS], pp. 39-47 and p. 63 f; see also 6.0 below):

Let  $K = \bar{K}$  be an algebraically closed field, and  $\sigma: K \rightarrow \mathbb{C}$ . Let  $A$  be an abelian variety defined over  $K$ , and denote by  $\langle A \rangle$  the smallest full tannakian subcategory of  $M_K$  containing  $h^1(A)$  and  $\mathbb{Q}(1)$ . As  $\otimes$ -functor on  $\langle A \rangle$  we take the restriction to  $\langle A \rangle$  of  $H_\sigma: M_K \rightarrow \text{Vec}_\mathbb{Q}$ . Then  $(\langle A \rangle, H_\sigma)$  corresponds to an affine group scheme  $\text{MT}(A)$  over  $\mathbb{Q}$ , called the Mumford-Tate group of  $A$ .

From 2.3.2(d) we see that

$$\text{MT}(A) \hookrightarrow \text{GL}(H_\sigma^1(A)) \times \mathbb{G}_m.$$

But we know more: Deligne's fundamental theorem 2.1.3 implies that

$$H_\sigma: \langle A \rangle \rightarrow \text{Hod}_\mathbb{Q}$$

is a fully faithful functor to the category of rational Hodge structures - see 2.4.3 below. Thus, writing

$$V = H_\sigma^1(A) \quad (\text{as a } \mathbb{Q}\text{-Hodge structure}),$$

$$\mathbb{T}^{a,b,m} = V^{\otimes a} \otimes V^{\vee \otimes b} \otimes \mathbb{Q}(m),$$

for  $a, b, m \in \mathbb{Z}$ ;  $a, b \geq 0$ , and calling Hodge cycles in  $T^{a,b,m}$  those elements of  $T^{a,b,m} \otimes_{\mathbb{Q}} \mathbb{C}$  that are pure of type  $(0,0)$  in the Hodge decomposition of  $T^{a,b,m} \otimes_{\mathbb{Q}} \mathbb{C}$ , we find that  $MT(A)$  is the  $\mathbb{Q}$ -algebraic subgroup of  $GL(V) \times \mathbb{G}_m$  fixing all Hodge cycles in all spaces  $T^{a,b,m}$ .

The third description of  $MT(A)$  is this: define

$\mu: \mathbb{G}_m \rightarrow GL(V) \times \mathbb{G}_m$  over  $\mathbb{C}$  by  $\mu(z) = (\mu_1(z), z)$  and  $\mu_1(z)(v) = (z \cdot v^{1,0}) + v^{0,1}$  ( $z \in \mathbb{C}^*$ ,  $v = v^{1,0} + v^{0,1} \in H_{\sigma}^1(A) \otimes \mathbb{C}$ .) Then  $MT(A)$  is the smallest  $\mathbb{Q}$ -algebraic subgroup  $U$  of  $GL(V) \times \mathbb{G}_m$  such that  $U(\mathbb{C}) \supset \mu(\mathbb{C}^*)$ .

$MT(A)$  is reductive. This follows alternatively from the existence of a polarization on  $A$  - which is fixed by  $MT(A)$  because it defines a Hodge cycle in  $T^{2,0,1}$ , and the existence of which forces  $MT(A)$  to have a compact real form -, or from the semisimplicity of  $\langle A \rangle$ , by 2.3.2(a). In fact the whole category of motives  $M_K$  is semisimple (i.e., every exact sequence in  $M_K$  splits), by [DMOS], II. 6.5, and  $\langle A \rangle$  is a full tannakian subcategory of  $M_K$ .

Finally,  $MT(A)$  is a torus if and only if all simple factors (up to isogeny)  $A_i$  of  $A$  admit complex multiplication by a CM-field  $E_i$  with  $[E_i:\mathbb{Q}] = 2 \dim A_i$ .

2.3.4 We have already quoted that, for any field  $K$  admitting an embedding  $\sigma: K \rightarrow \mathbb{C}$ , the category  $M_K$  of motives (for absolute Hodge cycles) over  $K$  is a semisimple tannakian category, equipped with the  $\otimes$ -functor  $H_{\sigma}: M_K \rightarrow \text{Vec}_{\mathbb{Q}}$ . The corresponding affine group scheme over  $\mathbb{Q}$  is denoted  $G(\sigma)$ . It is proreductive, according to 2.3.2(a).  $G(\sigma)$  as a whole looks prohibitively big and uncontrollable. In order to make it appear less outlandish, it is called the motivic Galois group. This terminology takes its clue from the classical son of  $G(\sigma)$  to be discussed in the next paragraph.

## 2.4 SPECIAL MOTIVES (Reference: [DMOS], II § 6)

We shall need later on a few subcategories of  $M_K$ .



2.4.1 Artin motives:  $M_K^0$

Let  $CV_K^0$  be the subcategory of  $CV_K$  (2.2, step 1) formed by the  $h(X)$  with  $X$  a variety over  $K$  of dimension zero. For such an  $X$ , the  $\bar{K}$ -rational points  $X(\bar{K})$  are just a finite set with a  $\text{Gal}(\bar{K}/K)$ -action, so consider the finite dimensional rational representation  $\mathbb{Q}^{X(\bar{K})}$  of  $\text{Gal}(\bar{K}/K)$ , where we may view  $\text{Gal}(\bar{K}/K)$  as a constant group scheme over  $K$ . In  $CV_K^0$ , one has

$$\begin{aligned} \text{Hom}(h(X), h(Y)) &= C_{\text{AH}}^0(X \times Y) \\ &= \left( \mathbb{Q}^{X(\bar{K}) \times Y(\bar{K})} \right)^{\text{Gal}(\bar{K}/K)} \\ &= \text{Hom}_{\text{Gal}(\bar{K}/K)}(\mathbb{Q}^{X(\bar{K})}, \mathbb{Q}^{Y(\bar{K})}). \end{aligned}$$

Whence a fully faithful functor  $CV_K^0 \rightarrow \text{Rep}_{\mathbb{Q}}(\text{Gal}(\bar{K}/K))$  into the tannakian category of finite-dimensional rational representations of  $\text{Gal}(\bar{K}/K)$ . Let  $M_K^0$  be the smallest tannakian subcategory of  $M_K$  containing  $CV_K^0$ . Thus there is an equivalence of  $\otimes$ -categories between this category  $M_K^0$  of (Emil) Artin motives and  $\text{Rep}_{\mathbb{Q}}(\text{Gal}(\bar{K}/K))$ .

For future reference, let us list the realizations of an Artin motive  $M \in M_K^0$ . We think of  $M$  as a representation of  $\text{Gal}(\bar{K}/K)$ , and denote by  $M_S$  the underlying finite-dimensional  $\mathbb{Q}$ -vector space.

For all  $\sigma: K \hookrightarrow \mathbb{C}$ ,  $H_{\sigma}(M) = M_S$

$$H_{\sigma}(M) \otimes \mathbb{C} = H^{0,0}$$

Hence for every prime number  $\ell$ ,

$$H_{\ell}(M) = M \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell} \quad (\text{as } \text{Gal}(\bar{K}/K)\text{-module}).$$

To determine the de Rham realization write  $M$  as  $\text{Spec } A$ , with  $A = \prod K_i$ , where the  $K_i \supset K$  are finitely many number fields. Now,

$$\begin{aligned}
H_{\text{DR}}(\text{Spec } K_i) &= K_i \quad (\text{as } K\text{-vector space}) \\
&= (K_i \otimes_K \bar{K})^{\text{Gal}(\bar{K}/K)} \\
&= (\bar{K}^{\text{Hom}_K(K_i, \bar{K})})^{\text{Gal}(\bar{K}/K)} \\
&= (\mathbb{Q}^{\text{Hom}_K(K_i, \bar{K})})_{\otimes_{\mathbb{Q}} \bar{K}}^{\text{Gal}(\bar{K}/K)}.
\end{aligned}$$

Therefore,

$$H_{\text{DR}}(M) = (M \otimes_{\mathbb{Q}} \bar{K})^{\text{Gal}(\bar{K}/K)}.$$

#### 2.4.2 Abelian varieties: $M_K^{\text{av}}$

Let  $M_K^{\text{av}}$  be the tannakian subcategory of  $M_K$  generated by motives of abelian varieties and Artin motives over  $K$ . Since all of  $h(X)$  is given by the exterior algebra of  $h^1(X)$ , for an abelian variety  $X$ ,  $M_K^{\text{av}}$  is already generated by the  $h^1(X)$  and  $M_K^0$ .

2.4.3 Theorem. If  $K$  is algebraically closed, and  $\sigma: K \rightarrow \mathbb{C}$  is any embedding, then the  $\otimes$ -functor

$$H_{\sigma}: M_K^{\text{av}} \rightarrow \text{Hod}_{\mathbb{Q}}$$

into the tannakian category of rational Hodge structures is fully faithful.

This is an easy reformulation of Deligne's theorem 2.1.3 above.

Let us give a proof that would work for any category  $\mathcal{C}$  of motives generated by varieties of which one could prove that, over algebraically closed fields, every Hodge cycle on them was absolutely Hodge: We have to show that any  $\mathbb{Q}$ -linear map

$f_{\sigma}: H_{\sigma}(M) \rightarrow H_{\sigma}(N)$ , for  $M, N \in \mathcal{C}$ , which over  $\mathbb{C}$  respects the Hodge decompositions comes from an "absolute Hodge cycle on

$M \times N$ ". By the comparison isomorphisms  $f_{\sigma}$  induces a  $\mathbb{Q}_{\mathbb{A}^f}$ -linear map  $f_{\mathbb{A}^f}: H_{\mathbb{A}^f}(M/K) \rightarrow H_{\mathbb{A}^f}(N/K)$ , and a  $\mathbb{C}$ -linear map

$f_{\text{DR}, \mathbb{C}}: H_{\text{DR}}(M/\mathbb{C}) \rightarrow H_{\text{DR}}(N/\mathbb{C})$  respecting the Hodge filtrations. There is a field  $L \supset K$ , say  $L = \bar{L}$ , of finite transcendence degree over

$K$ , and an extension  $\tilde{\sigma}$  of  $\sigma$  from  $K$  to  $L$  such that  $f_{DR, \mathbb{C}}$  is already defined over  $L^{\tilde{\sigma}}$ . Then  $(f_{DR, L}, f_A, f)$  is a Hodge cycle on  $M \times N$  over  $L$  relative to  $\tilde{\sigma}$ . By assumption on  $\mathbb{C}$ , as  $L$  is algebraically closed it is an absolute Hodge cycle. Now Proposition 2.1.2(a) shows that it can already be defined over  $\bar{K} = K$ .

Certain classes of algebraic varieties are known to have motives isomorphic in  $M_K$  to objects of  $M_K^{av}$ . E.g., curves (via their jacobians), but also K3-surfaces and Fermat varieties. We shall recall these results as we need them.

Since our main concern is with algebraic Hecke characters we are eventually going to concentrate on the subcategory of  $M_K^{av}$  generated by abelian varieties with (potential) complex multiplication (and Artin motives). First, however, we have to explain how motives can be related to algebraic Hecke characters.

### 3. Motives of rank 1

3.0 The notion of complex multiplication for abelian varieties generalizes to motives in the following way. Let  $K$  be a field embeddable into  $\mathbb{C}$  and  $E$  a number field of finite degree over  $\mathbb{Q}$ . The category  $M_K(E)$  of motives over  $K$  with coefficients in  $E$  has objects the pairs  $(M, \theta)$ , with  $M$  a motive over  $K$  (i.e., an object of  $M_K$ ), and  $\theta: E \rightarrow \text{End}(M)$  an embedding of  $\mathbb{Q}$ -algebras. The morphisms in  $M_K(E)$  are the obvious ones, respecting the  $E$ -structures.  $M_K(E)$  is a  $\otimes$ -category via

$$(M, \theta) \otimes_E (M', \theta') = (N, \iota)$$

where  $N$  is the direct factor of  $M \otimes M'$  on which

$$\begin{array}{ccc} E \hookrightarrow \text{End}(M) & \searrow & \text{End}(M \otimes M') \\ E \hookrightarrow \text{End}(M') & \nearrow & \end{array}$$

agree to define  $\iota$ . For an alternative description of  $M_K(E)$ ,

see [DP], 2.1, "langage B".

The E-structure  $\theta: E \rightarrow \text{End}(M)$  defines E-module structures on all the realizations of a motive  $(M, \theta)$  in  $M_K(E)$ . Thus, for  $\sigma: K \rightarrow \mathbb{C}$ ,  $H_\sigma(M)$  is an E-rational Hodge structure, i.e., an E-vector space with a decomposition of  $E \otimes \mathbb{C}$ -modules

$$H_\sigma(M) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p,q} H^{p,q}$$

such that complex conjugation interchanges  $H^{p,q}$  and  $H^{q,p}$ . (Cf. 6.0/1) Forgetting this Hodge decomposition of  $H_\sigma(M)$ , the pair  $(M_K(E), H_\sigma)$  is a neutralized tannakian category over E, and the corresponding E-group scheme is  $G(\sigma) \times_{\mathbb{Q}} E$ , the motivic Galois group considered over E. (This is most easily seen in "langage B" quoted above.) In other words, there is an equivalence of categories induced by  $H_\sigma$ ,

$$M_K(E) \rightarrow \text{Rep}_E(G(\sigma)/E).$$

The rank of  $(M, \theta)$  in  $M_K(E)$  is defined to be the trace of identity in the corresponding representation:

$$\text{rk}(M, \theta) = \text{rk}_E M = \dim_E H_\sigma(M).$$

(This is, of course, independent of  $\sigma$ .)

The de Rham realization of a motive  $(M, \theta)$  - or, as we shall simply write, M - in  $M_K(E)$  is a filtered  $E \otimes K$ -module, free of rank  $\text{rk}_E M$ .

For all prime numbers  $\ell$ ,  $H_\ell(M)$  is a free  $E \otimes \mathbb{Q}_\ell$ -module of rank  $\text{rk}_E M$ , with an  $E \otimes \mathbb{Q}_\ell$ -linear action of  $\text{Gal}(\bar{K}/K)$ . Since  $E \otimes \mathbb{Q}_\ell = \prod_{\lambda|\ell} E_\lambda$ ,  $\lambda$  ranging over the places of E dividing  $\ell$ ,

there is a decomposition of  $\text{Gal}(\bar{K}/K)$ -representations

$$H_\ell(M) = \bigoplus_{\lambda|\ell} H_\lambda(M),$$

the  $H_\lambda(M) = H_\ell(M) \otimes_{E \otimes \mathbb{Q}_\ell} E_\lambda$  being the  $\lambda$ -adic Galois representations of M.

The realizations of  $M \otimes_E M'$  are simply the  $E$ -linear tensor products of the realizations of  $M, M' \in M_K(E)$ .

The functors on  $M_K$ : extension and restriction of the base field  $K$  clearly induce functors on  $M_K(E)$  - cf. 2.2.2. In addition, if  $E \subset E'$  is a (necessarily finite) extension, there are functors

$$\otimes_E^{E'} : M_K(E) \rightarrow M_K(E')$$

$$\downarrow_E : M_K(E') \rightarrow M_K(E)$$

of extension and restriction of the field of coefficients.

$\downarrow_E$  is simply forgetting the  $E'$ -action, except for that of  $E$ .  
 $\otimes_E^{E'}$  sends  $M$  into  $M \otimes_E^{E'}$ , where  $E' \in M_K$  is the first component of the unit object  $(E', \theta')$  of  $M_K(E')$ .

Assume now that  $K$  (as well as  $E$ ) is a number field of finite degree over  $\mathbb{Q}$ .

3.1 Proposition (cf. [DP], 8.1 (iii)) Let  $M \in M_K(E)$  with  $\text{rk}_E M = 1$ . If the system of the  $H_\lambda(M)$ , for all finite places  $\lambda$  of  $E$ , is a strictly compatible system of  $E$ -rational  $\lambda$ -adic Galois-representations  $\chi_\lambda$  over  $K$ , then there is an algebraic Hecke character  $\chi$  of  $K$  with values in  $E$  such that for all  $\lambda$  and almost all primes  $\mathfrak{p}$  of  $K$ ,  $\chi(\mathfrak{p}) = \chi_\lambda(\text{Frob } \mathfrak{p})$ .

This is just an application of Proposition 1.4 above.

3.2 Remark: Absolute Hodge cycles do not lend themselves to reduction mod  $\mathfrak{p}$  - or at least, we do not know how to prove that they do. This is why it is not even known, for a general motive  $M$  in  $M_K$ , that the  $H_\lambda(M)$ , for all rational primes  $\lambda$ , form a compatible system of rational  $\lambda$ -adic representations. This is true for  $M \in CV_K$  by the "Weil conjectures", but it cannot be shown to carry over to all motives constructed in Step 2 of the construction of  $M_K$ . - On the positive side however, it will be shown in § 6 that every rank 1 motive in  $M_K^{\text{av}}(E)$  has strictly compatible  $\lambda$ -adic representations, and therefore defines a Hecke character. So, Proposition 3.1. will be shown to be far from vacuous.

We have now motivated the basic notion in the "geometric" theory of algebraic Hecke characters:

3.3. Definition. Let  $\chi$  be an algebraic Hecke character of  $K$  with values in  $E$ . A motive  $M \in M_K(E)$  is said to be a motive for  $\chi$ , if  $\text{rk}_E M = 1$  and for all finite places  $\lambda$  of  $E$ , and all prime ideals  $\mathfrak{p}$  of  $K$  with  $\mathfrak{p} \nmid f_{\chi} \cdot N\lambda$ , the  $\lambda$ -adic representation  $H_{\lambda}(M)$  of  $\text{Gal}(\bar{K}/K)$  is unramified at  $\mathfrak{p}$  and a geometric Frobenius element  $\text{Frob } \mathfrak{p} \in \text{Gal}(\bar{K}/K)$  acts on  $H_{\lambda}(M)$  via multiplication by  $\chi(\mathfrak{p})$ .

In other words,  $M$  is a motive for  $\chi$ , if  $H_{\lambda}(M) = \chi_{\lambda}$ , in the notation of 0 § 5.

The typical example of a motive for an algebraic Hecke character is an abelian variety with complex multiplication - see § 1.

#### 4. A standard motive for a Hecke character

Let  $K$  be embeddable into  $\mathbb{C}$ . Let  $CM_K$  be the Tannakian subcategory of  $M_K$  (equivalently: of  $M_K^{\text{av}}$ ) generated by the Artin motives over  $K$  and by the motives  $h^1(A)$ , where  $A$  is an abelian variety over  $K$  which, over  $\bar{K}$ , has complex multiplication (in the sense that  $\text{End}_{\bar{K}} A$  contains a number field  $E$  of degree  $[E:\mathbb{Q}] = 2 \dim A$ .) Given a number field  $E$  (of finite degree over  $\mathbb{Q}$ ), we can consider the category  $CM_K(E)$  of motives  $M$  in  $CM_K$  that are equipped with an  $E$ -action,  $E \rightarrow \text{End}(M)$ .

4.1 Theorem. Suppose  $K$  is a number field. For any algebraic Hecke character  $\chi$  of  $K$  with values in  $E$ , there exists a motive  $M(\chi) \in CM_K(E)$  which is a motive for  $\chi$ , in the sense of 3.3.

#### Elementary and direct proof of 4.1

4.1.0 If  $\chi$  is of the form  $\mu \cdot N^{w/2}$ , for a character of finite order  $\mu$  on  $\text{Gal}(\bar{K}/K)$  with values in  $E^*$ , then we can write down a motive for  $\chi$  in  $CM_K(E)$  like this:

$$M(\chi) = [\mu] \otimes_E E\left(\frac{-w}{2}\right),$$

where  $[\mu] \in M_K^0(E)$  is the rank 1 Artin motive for  $\mu$  with coefficients in  $E$  (i.e.,  $s \in \text{Gal}(\bar{K}/K)$  acts on  $E$  via multiplication by  $\mu(s) \in E^*$ ), and  $E\left(\frac{-w}{2}\right) = \mathbb{Q}\left(\frac{-w}{2}\right) \otimes_{\mathbb{Q}} E$ ,  $E$  being here the unit object in  $M_K(E)$ .

Thus calling  $K'$  the field of all numbers of CM-type in  $K$ , let us henceforth assume, without loss of generality, that  $K'$  and  $E$  are CM-fields - cf.  $\mathbb{O}$  § 3.

4.1.1 We now treat the case that the infinity-type of  $\chi$  is of the form  $T = \phi' \circ N_{K/K}$ , with a CM-type  $\phi'$  of  $K'$ , i.e.,  $\chi$  is of weight  $-1$  and the invariants  $n(\sigma, \tau)$ , for  $\sigma \in \text{Hom}(K, \mathbb{C})$ ,  $\tau \in \text{Hom}(E, \mathbb{C})$ , introduced in  $\mathbb{O}$  § 4 are all either  $-1$  or  $0$ . Exchanging the rôles of  $K$  and  $E$  these same  $n(\sigma, \tau)$  also define a CM-type  $\phi$  of the field

$$E_0 = \mathbb{Q}(K'^{\phi'}) = \mathbb{Q}(K^T) \subset E,$$

called the reflex type of  $\phi'$ . By a theorem of Casselman, [ShiL], Theorem 6, there is an abelian variety  $A$  defined over  $K$  with complex multiplication by the ring of integers of  $E$ , of CM-type  $(E, \phi \circ N_{E/E_0})$ , such that  $h_1(A)$  is a motive for  $\chi$ . - In fact, if one tries to be very neat,  $A$  may be constructed as being a direct summand in  $h_1(R_{L|K}B)$ , for a suitable (abelian) extension  $L$  of  $K$  such that  $\chi \circ N_{L|K}$  takes values in  $E_0^*$ , and  $B$ , provided by Casselman, an abelian variety over  $L$  of CM-type  $(E_0, \phi)$  with character  $\chi \circ N_{L|K}$  - cf. [GS], théorème 4.1.

4.1.2 All infinity-types of algebraic Hecke characters of  $K$  are  $\mathbb{Z}$ -linear combinations of the CM-types discussed in 4.1.1. - This is an easy exercise, observing the homogeneity condition  $n(\sigma, \tau) + n(c\sigma, \tau) = w$ , cf.  $\mathbb{O}$  §§ 3, 4.

4.1.3 We can now start on the general case of 4.1 (always under the assumption that  $K'$  is a CM-field). Write the infinity-type  $T$  of  $\chi$  as

$$T = \prod_i T_i^{n_i} ,$$

with  $n_i \in \mathbf{Z}$  and  $T_i$  CM-types like in 4.1.1. There is a finite extension field  $E' \supseteq E$  such that, for all  $i$ , there exists an algebraic Hecke character  $\chi_i$  of  $K$  with values in  $E'$  of infinity-type  $T_i$  - see 0 § 3. Let  $A_i$  be an abelian variety attached to  $\chi_i$  as in 4.1.1. Put

$$M' = (\otimes_{E'} h_1(A_i)^{\otimes_{E'} n_i}) \otimes_{E'} [\mu]$$

where  $\mu = \chi(\prod \chi_i^{-n_i})$  is of finite order, and  $[\mu]$  is the Artin motive for  $\mu$  in  $M_K^0(E')$ . Then  $M'$  is a motive for  $\chi$ , if we consider  $\chi$  to take values in  $E'$ , rather than  $E$ . Thus it remains to "descend the coefficients".

4.1.4 There is a finite (abelian) extension  $L$  of  $K$  such that all characters  $\chi_i$  used in 4.1.3 take their values in  $E^*$  when composed with  $N_{L|K}$ . Therefore, taking, for simplicity,  $L$  such that every  $\chi_i \circ N_{L|K}$  takes its values in the corresponding reflex-field  $E_{\circ, i}$  (see 4.1.1), we see that  $M' \times_K L$  is of the form  $M_L \otimes_{E'} E'$ , with  $M_L$  a direct factor of the motive:

$$\left\{ \otimes_{E'} [(h_1(B_i)^{\otimes_{E_{\circ, i}} n_i}) \otimes_{E_{\circ, i}} E] \right\} \otimes_{E'} [\mu \circ N_{L|K}]_{E'} ,$$

the  $B_i$  being as  $B$  in the last sentence of 4.1.1. So,  $M_L$  is a motive for  $\chi \circ N_{L|K}$  in  $CM_L(E)$ . In other words, there is a projector (an absolute Hodge cycle)  $\pi \in \text{End}_{M_L}(M' \times_K L)$  carving out the  $E$ -structure  $M_L$ . We have to show that  $\pi$  is already defined over  $K$ . Since it is an absolute Hodge cycle it is enough to show that its  $\mathbf{A}^f$ -component is invariant under  $\text{Gal}(\bar{K}/K)$ . But  $M'$  is a motive for the character  $\chi$  which takes values in  $E$ . So  $\pi$  cannot possibly be affected by the action of  $\text{Gal}(\bar{K}/K)$  on  $H_{\mathbf{A}^f}(M')$ .

4.2 Remark. In view of 1.7 above, the motive  $M(\chi)$  for  $\chi$  which was just constructed has its Hodge structure determined by the infinity-type of  $\chi$ . Explicitly, for all  $\sigma: K \rightarrow \mathbb{C}, \tau: E \rightarrow \mathbb{C}$ , one finds

$$H_{\sigma}(M(\chi)) \otimes_{E, \tau} \mathbb{C} \subset H^{n(\sigma, \tau), w - n(\sigma, \tau)} .$$



(See 0 § 4 for the notation.)

The proof of theorem 4.1 which we have presented is "elementary and direct" in that it starts immediately from the geometry of the varieties that generate  $CM_K$ , and does not use Deligne's theorem 2.1.3 about Hodge cycles on abelian varieties. It does use Casselman's theorem, i.e., the Shimura-Taniyama reciprocity law for abelian varieties with complex multiplication.

Using 2.1.3 it is possible to gain much more insight into the structure of  $CM_K$ , reproving theorem 4.1 (via the interpretation of algebraic characters given in 0 § 7) and generalizing the Shimura-Taniyama reciprocity law. Specifically, what one has to do is to identify  $CM_{\mathbb{Q}}$  with  $Rep_{\mathbb{Q}}(\mathcal{T})$ , for the Taniyama group  $\mathcal{T}$ . We shall sketch this in § 6 below, thereby obtaining additional information about all rank 1 motives constructed from abelian varieties.

### 5. Unicity of $M(\chi)$

Let  $K$  and  $E$  be number fields of finite degree over  $\mathbb{Q}$ , and consider the category of motives  $M_K^{av}(E)$ . These are the motives in  $M_K^{av}$  - see 2.4.2 - with an  $E$ -action. Since  $CM_K$  is a subcategory of  $M_K^{av}$ , there always exists - by theorem 4.1 - a motive  $M(\chi)$  in  $M_K^{av}(E)$  for a given algebraic Hecke character  $\chi$  of  $K$  with values in  $E$ .

5.1 Theorem. Up to isomorphism, there is only one motive  $M(\chi)$  in  $M_K^{av}(E)$  for a given algebraic Hecke character  $\chi$  of  $K$  with values in  $E$ .

Proof. Let  $M$  in  $M_K^{av}(E)$  be any motive for  $\chi$ . For any  $\sigma: K \hookrightarrow \mathbb{C}$ ,  $H_{\sigma}(M)$  is a Hodge structure of rank 1 over  $E$ . In particular, it is indecomposable and therefore pure of some weight  $w$  (cf. 6.0 below). The relations

$$H_{\sigma}(M) \otimes_{E, \tau} \mathbb{C} \subset H^{n(\sigma, \tau), w - n(\sigma, \tau)}$$

define invariants  $n(\sigma, \tau)$  for all  $\tau: E \rightarrow \mathbb{C}$ . They actually satisfy  $n(\alpha\sigma, \alpha\tau) = n(\sigma, \tau)$ , for all  $\alpha \in \text{Aut } \mathbb{C}$  (because  $E$  operates on  $M$  through absolute Hodge cycles.) So by 0 §§ 3 and 4, there is some algebraic Hecke character  $\Psi$  of some number field  $L \supset K$  with values in  $E$  having the  $n(\sigma, \tau)$ 's as its invariants. Let  $M(\Psi) \in \text{CM}_L(E)$  be the motive for  $\Psi$  constructed in 4.1. By remark 4.2 the Hodge structure  $H_{\tilde{\sigma}}(M(\Psi))$ , for every embedding  $\tilde{\sigma}: L \rightarrow \mathbb{C}$  extending  $\sigma: K \hookrightarrow \mathbb{C}$ , is  $E$ -compatibly isomorphic to  $H_{\sigma}(M)$ . By theorem 2.4.3,  $M$  and  $M(\Psi)$  are isomorphic in  $M_K^{\text{av}}(E)$ . In view of 2.1.2(b), they are isomorphic over some finite extension  $L'$  of  $L$ . Recalling that  $M$  was a motive for  $\chi$ , and  $M(\Psi)$  for  $\Psi$ , we find that  $\chi \circ N_{L'/K} = \Psi \circ N_{L'/L}$ . Hence the  $n(\sigma, \tau)$  which by construction describe the infinity-type of  $\Psi$  are also the invariants attached to the character  $\chi$ . Thus we have shown that, if  $M$  is an arbitrary motive for  $\chi$  in  $M_K^{\text{av}}(E)$ , its Hodge realizations  $H_{\sigma}(M)$  are those determined by (the infinity-type of)  $\chi$ , as in 4.2. This establishes an isomorphism (an absolute Hodge cycle) between  $M$  and our standard motive  $M(\chi)$  over  $\bar{K}$ . But  $H_{\mathbb{A}^f}(M)$  and  $H_{\mathbb{A}^f}(M(\chi))$  are isomorphic  $\text{Gal}(K/\bar{K})$ -representations by definition. So the isomorphism is defined over  $K$ .

q.e.d.

5.2 So far we have verified all but part (iii) of conjecture 8.1 in [DP] for the category  $M_K^{\text{av}}$ . (In fact, our description in 4.2 of the Hodge decomposition of  $M(\chi)$  is equivalent to the characterization of the Hodge filtration given by Deligne in [DP], 8.1 (iv).) As to [DP], 8.1 (iii), it will be shown in § 6 that every motive  $M$  in  $\text{CM}_K(E)$  has strictly compatible  $E$ -rational  $\lambda$ -adic representations  $H_{\lambda}(M)$ . By proposition 3.1, this will settle [DP], 8.1 (iii) for the category  $\text{CM}_K$ . But in fact, it will automatically take care also of the rank 1 motives in  $M_K^{\text{av}}$ :

5.3 Remark Every motive of rank 1 in  $M_K^{\text{av}}(E)$  is isomorphic to a motive of  $\text{CM}_K(E)$ .

Proof. Let  $M$  be in  $M_K^{\text{av}}(E)$ ; let  $\sigma: K \hookrightarrow \mathbb{C}$ , and assume that  $\dim_E H_{\sigma}(M) = 1$ . Then  $H_{\sigma}(M)$  is an  $E$ -rational Hodge

structure of the kind described in 4.2 and the proof of 5.1. It occurs as  $H_{\tilde{\sigma}}(N)$ , for some  $N$  in  $CM_L(E)$ , for some number field  $L$  with  $K \subset L \subset \bar{K}$  and  $\tilde{\sigma}$  extending  $\sigma$  to  $L$ . By 2.4.3,  $M \times_{K, \bar{K}}$  is isomorphic to a motive in  $CM_{\bar{K}}(E)$ . This being true over some finite extension  $L'$  of  $K$  (and  $L$ ), we see that  $M$  is isomorphic to a motive in  $CM_K(E)$ . (E.g.,  $M$  occurs as a direct factor in  $R_{L'/K}N$ .)

## 6. Representations of the Taniyama group

The affine group scheme over  $\mathbb{Q}$  corresponding, by tannakian philosophy, to the neutralized category of motives  $(CM_{\mathbb{Q}}, H_B)$  - see § 4 above for the notation - is (isomorphic to) the Taniyama group introduced by Langlands in [Lg], 5. This fact was first proved by Deligne: see [DMOS], IV. Using a formalism of Tate's completed by an argument of Deligne - see [LCM], chap. 7 - , the proof can be given much more explicitly. This second proof is certainly part of the folklore on this subject - I myself am indebted to G. Anderson for explaining it to me - and J.S. Milne is preparing a book which will contain it in detail. In this section we shall give an extremely sketchy account of how this proof proceeds, and then apply the theorem to settle the only question left open in 5.2. The results of this section will not be substantially used in the sequel. They are however, essential for G. Anderson's formalism (section 7), and they complete the picture we are drawing of motives for Hecke characters. - The first two subsections: 6.0 and 6.1, are more detailed than the rest because they give a more thorough basis to things that have been used before: rational Hodge structures and the Serre group. The definition of CM Hodge structures in 6.1 was suggested by R. Pink.

### 6.0 Rational Hodge structures

6.0.0 A rational Hodge structure of weight  $w$  is a finite dimensional  $\mathbb{Q}$ -vector space  $V$  equipped with a decomposition

$$V \otimes \mathbb{C} = \bigoplus_{\substack{p+q=w \\ p,q \in \mathbb{Z}}} V^{pq}$$

such that  $(1 \otimes c) V^{pq} = V^{qp}$ , for  $c =$  complex conjugation.

A rational Hodge structure is a finite direct sum of rational Hodge structures of fixed weights. A homomorphism of rational Hodge structures  $V_1, V_2$  is a  $\mathbb{Q}$ -linear map  $f: V_1 \rightarrow V_2$  such that, for all  $p, q \in \mathbb{Z}$ , one has

$$(f \otimes 1_{\mathbb{C}}) V_1^{pq} \subset V_2^{pq} .$$

6.0.1 Reformulated in a tannakian way, the extra structure on the  $\mathbb{Q}$ -vector space  $V$  amounts to a representation  $h: \mathbb{S} \rightarrow GL(V)$  defined over  $\mathbb{R}$ , where  $\mathbb{S} = \mathbb{R}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$  - see [DeH II], 2.1. The translation is given by the rule

$$h(z) \Big|_{V^{pq}} = \text{multiplication by } z^p \bar{z}^q ,$$

for  $z \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$ .

The inclusion  $\mathbb{R}^* \hookrightarrow \mathbb{C}^*$  gives rise to a canonical map  $\overset{\circ}{w}: \mathbb{G}_m \rightarrow \mathbb{S}$  over  $\mathbb{R}$ . Given a rational Hodge structure  $V$ , we set  $w = h \circ \overset{\circ}{w}: \mathbb{G}_m \rightarrow GL(V)$ . If  $V$  is of weight  $n$ , then  $w(\lambda)$  acts as multiplication by  $\lambda^n$  on  $V$ ; this justifies the letter  $w$ , and implies that, among the "real Hodge structures"  $h: \mathbb{S} \rightarrow GL(V)/\mathbb{R}$ , the rational Hodge structures are precisely those for which  $w$  is defined over  $\mathbb{Q}$ . Over  $\mathbb{C}$ , denote by  $\mu: \mathbb{G}_m \rightarrow GL(V)/\mathbb{C}$  the complex cocharacter given by the rule

$$\mu(z) \Big|_{V^{pq}} = \text{multiplication by } z^p ,$$

for  $z \in \mathbb{C}^* = \mathbb{G}_m(\mathbb{C})$ . Its (imagewise) complex conjugate

$$\bar{\mu}: z \mapsto (v^{pq} \mapsto \bar{z}^q \cdot v^{pq})$$

is algebraic (not over  $\mathbb{C}$  but) over  $\mathbb{R}$ , and  $(\mu \bar{\mu})$  takes values in  $GL(V \otimes \mathbb{R})$  on  $\mathbb{C}^*$ . Thus it defines an algebraic homomorphism  $\mathbb{S} \rightarrow GL(V)$  over  $\mathbb{R}$ , which is none other than  $h$ . Either  $h$  or  $\mu$  suffice to characterize the Hodge structure on a given  $\mathbb{Q}$ -vector space  $V$ . This is convenient, for instance, in defining the

tensor product of rational Hodge structures via the tensor product of real  $\mathbb{S}$ -representations. - Rational Hodge structures form a  $\mathbb{Q}$ -linear  $\otimes$ -category.

6.0.2 Define the Mumford-Tate group of a rational Hodge structure  $V$  by generalizing 2.3.3:  $MT(V)$  is the  $\mathbb{Q}$ -algebraic subgroup of  $GL(V) \times \mathbb{E}_m$  that fixes all elements of pure type  $(0,0)$  in all spaces of the form

$$T^{a,b,m} = V^{\otimes a} \otimes V^{\otimes b} \otimes \mathbb{Q}(m),$$

where  $a, b, m \in \mathbb{Z}$ ;  $a, b \geq 0$ ;  $T$  being viewed as tensor product of the natural representations of  $GL(V)$  on  $V$  and  $V^{\vee}$ , and the representation: multiplication by  $\lambda^{-1}$  of  $\mathbb{E}_m$  on  $\mathbb{Q}(1)$ . Equivalently ([DMOS], I. 3.4.),  $MT(V)$  is the smallest  $\mathbb{Q}$ -algebraic subgroup of  $GL(V) \times \mathbb{E}_m$  such that  $MT(V)(\mathbb{C})$  contains the image of

$$\begin{aligned} \mathbb{C}^* &\rightarrow GL(V \otimes \mathbb{C}) \times \mathbb{C}^* \\ z &\rightarrow (\mu(z), z) . \end{aligned}$$

It is often convenient to identify  $MT(V)$  with its first projection. Thus  $MT(V)$  becomes the smallest  $\mathbb{Q}$ -algebraic subgroup of  $GL(V)$  the  $\mathbb{C}$ -rational points of which contain the image of  $\mu$ . The first description then runs:

$$MT(V)(\mathbb{Q}) = \left\{ \gamma \in GL(V) \left\{ \begin{array}{l} \text{for all } a, b, m, \text{ and all} \\ t \in V^{\otimes a} \otimes V^{\otimes b} \cap (V^{\otimes a} \otimes V^{\otimes b})^{m,m}, \\ \text{there is } \lambda \in \mathbb{Q}^* \text{ such that } \gamma(t) = \lambda^{m_t} t \end{array} \right. \right\}$$

6.0.3 As  $MT(V)(\mathbb{C})$  receives the cocharacter  $\mu$ , one also has

$$\begin{aligned} h: \mathbb{S} &\rightarrow MT(V) \quad \text{over } \mathbb{R} , \\ \text{and } w: \mathbb{E}_m &\rightarrow MT(V) \quad \text{over } \mathbb{Q} . \end{aligned}$$

$\mu$  and  $w$  have more or less surfaced already in chapter 0, 7.3.4, and we are now going to reconsider the Serre group  $Z$  in the context of rational Hodge structures of CM type.

## 6.1 CM Hodge Structures (cf. also [DMOS], III. 1)

6.1.0 Definition: Let  $V$  be a rational Hodge structure,  $MT(V)$  its Mumford-Tate group (6.0.2) and  $w: \mathbb{G}_m \rightarrow MT(V)/\mathbb{Q}$  the associated cocharacter (6.0.1/3).  $V$  is called a CM Hodge structure if  $MT(V)$  is a torus and  $(MT(V)/w(\mathbb{G}_m))(\mathbb{R})$  is compact.

CM Hodge structures form a  $\otimes$ -subcategory of all rational Hodge structures. We are going to show that the Mumford-Tate group of a CM Hodge structure is a quotient of the Serre group  $Z$ , introduced in 0, 7.3.3. Recall that  $Z = \varprojlim Z_K$ , where  $K$  runs over number fields and  $Z_K$  is the quotient of  $R_{K/\mathbb{Q}} \mathbb{G}_m$  by a sufficiently small arithmetic subgroup. Observe that the maps  $\tilde{w}$  and  $\tilde{\mu}$  defined in 0, 7.3.4 actually give rise to cocharacters of  $R_{K/\mathbb{Q}} \mathbb{G}_m$  (defined over  $\mathbb{R}$ , and  $\mathbb{C}$ , resp.), for all number fields  $K$  (embedded into  $\overline{\mathbb{Q}}$ , as explained in 0, 7.3.2).

6.1.1 Lemma Let  $V$  be a rational Hodge structure such that  $MT(V)$  is a torus.

(i) For any sufficiently large number field  $K \hookrightarrow \overline{\mathbb{Q}}$ , there exists a unique homomorphism

$$\nu: R_{K/\mathbb{Q}} \mathbb{G}_m \rightarrow MT(V)$$

of  $\mathbb{Q}$ -algebraic groups rendering the following diagram commutative.

$$\begin{array}{ccc} R_{K/\mathbb{Q}} \mathbb{G}_m \times \mathbb{C} & \xrightarrow{\nu \times \mathbb{C}} & MT(V) \times \mathbb{C} \\ \tilde{\mu} \swarrow & & \searrow \mu \\ & \mathbb{G}_m / \mathbb{C} & \end{array}$$

(ii) For  $K \subset L$ , the maps  $\nu$  are compatible via  $N_{L/K}$ .

(iii)  $\nu$  is faithfully flat.

Proof. (i) Translate into a statement on character groups; the requirement that  $\nu$  be defined over  $\mathbb{Q}$  then forces

(for  $K$  normal over  $\mathbb{Q}$  splitting  $MT(V)$ ):

$$\nu^*(f) = \sum_{\sigma \in G(K/\mathbb{Q})} \mu^*(f^{\sigma^{-1}}), \quad \sigma \in X(R_{K/\mathbb{Q}}\mathbb{E}_m),$$

for  $f \in X(MT(V))$ .

(ii) follows from the uniqueness of  $\nu$ .

(iii) expresses the fact that  $\mu$  "generates" the  $\mathbb{Q}$ -algebraic group  $MT(V)$ .

6.1.2 Remark Care has to be taken with the galois action(s) on  $Z$ : cf. [Lg], p. 219/20; [DMOS], III (1.3), (1.8); [DMOS], IV, (B)... Since we just used a galois invariance in the proof of (i), let us make explicit our setup: we define a left action of  $G(\overline{\mathbb{Q}}/\mathbb{Q})$  on

$$R_{K/\mathbb{Q}}\mathbb{E}_m(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^*)^{\text{Hom}(K, \overline{\mathbb{Q}})} \quad \text{by the rule}$$

$$((z_\sigma)_{\sigma \in \text{Hom}(K, \overline{\mathbb{Q}})})^s = (z_{s^{-1} \circ \sigma})_\sigma, \quad \text{for } s \in G(\overline{\mathbb{Q}}/\mathbb{Q})$$

This transports to a left action on characters

$$f: R_{K/\mathbb{Q}}\mathbb{E}_m \rightarrow \mathbb{E}_m/\overline{\mathbb{Q}} \quad \text{via } : f^s((z_\sigma)) = f((z_\sigma)^{s^{-1}}).$$

Identifying  $f$  with  $\sum_\sigma n_\sigma \sigma \in \mathbb{Z}[\text{Hom}(K, \overline{\mathbb{Q}})]$ , one finds

$$(\sum_\sigma n_\sigma \sigma)^s = \sum_\sigma n_{s^{-1} \circ \sigma} \sigma.$$

This is the action we have used on characters of  $R_{K/\mathbb{Q}}\mathbb{E}_m$  (cf. 0, §§ 2 und 4). Passing to the quotient  $Z_K$  and to the limit  $Z$ , this yields, for a character  $f: G(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z}$  of  $Z$  (as in 0. 7.3.4):  $f^s(t) = f(s^{-1}t)$ , for  $s, t \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . (Note that  $f^{(st)} = (f^t)^s$ .)

There is, of course, another left action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on  $X(Z)$ , namely by right translation:

$$(s \cdot f)(t) = f(ts).$$

In the preceding setup, it is induced by the rule

$(z_\sigma)_\sigma^S = (z_{\sigma_S})_\sigma$  on  $R_{K/\mathbb{Q}}\mathbb{G}_m(\overline{\mathbb{Q}})$  - where  $K$  has now to be assumed normal over  $\mathbb{Q}$  (and always embedded into  $\overline{\mathbb{Q}}$ ). - This second action will become relevant as of 6.2.0.

6.1.3 Proposition [R. Pink] Let  $V$  be a rational Hodge structure such that  $MT(V)$  is a torus. Choose  $K$  so that  $\nu$  exists as in 6.1.1. Then the following are equivalent.

- (i)  $V$  is a CM Hodge structure.
- (ii)  $\ker \nu$  contains an arithmetic subgroup.
- (iii) The subgroup  $X(MT(V)) \xrightarrow{\nu^*} X(R_{K/\mathbb{Q}}\mathbb{G}_m)$  is contained in  $X(Z_K)$ .

Proof. (i)  $\Rightarrow$  (ii). Assume (without loss of generality) that  $K$  is totally imaginary, say  $[K:\mathbb{Q}] = 2r$ , and consider the diagram

$$\begin{array}{ccc}
 \begin{array}{c} \mathbb{S}^r \\ \circlearrowleft \nu \\ \downarrow N_{\mathbb{C}/\mathbb{R}} \\ (\mathbb{G}_m/\mathbb{R})^r \end{array} & = & R_{K/\mathbb{Q}}\mathbb{G}_m \times \mathbb{R} \xrightarrow{\nu \times \mathbb{R}} MT(V) \times \mathbb{R} \\
 & \swarrow \tilde{\nu} & \nearrow w \times \mathbb{R} \\
 & \mathbb{G}_m/\mathbb{R} &
 \end{array}$$

whose commutativity is easily checked using 1.6.1 (i).

According to Dirichlet, the units in  $R_{K/\mathbb{Q}}\mathbb{G}_m$  map to  $(\mathbb{G}_m/\mathbb{R})^r$  with a finite kernel. As  $N_{\mathbb{C}/\mathbb{R}} \circ \tilde{\nu}(x) = x^2$ , any sufficiently small arithmetic subgroup  $U$  of  $R_{K/\mathbb{Q}}\mathbb{G}_m$  is contained in  $\tilde{\nu}^{-1}(\mathbb{G}_m/\mathbb{R})^r$ , and since  $(MT(V)/w(\mathbb{G}_m))(\mathbb{R})$  is compact we can achieve that  $\nu(U)(\mathbb{R}) \subset w(\mathbb{G}_m/\mathbb{R})$ . Now assume furthermore that  $U \subset \ker N_{K/\mathbb{Q}}$ , the norm-1-subgroup of  $R_{K/\mathbb{Q}}\mathbb{G}_m$ . Then  $U(\mathbb{R}) \cap \tilde{\nu}^{-1}(\mathbb{G}_m/\mathbb{R})$  is obviously finite. Thus, in view of the commutative triangle above, we conclude that  $U \subset \ker \nu$ , provided again that  $U$  is sufficiently small.

(ii) and (iii) just express the two possible definitions of  $Z_K$  - as quotient of  $R_{K/\mathbb{Q}}\mathbb{G}_m$ , or via its character group.



Finally, assuming (iii), we find (cf. 0, 7.3.2)

$$X(MT(V)/w(\mathbb{E}_m)) \subset \{ \lambda \in X(R_{K/\mathbb{Q}} \mathbb{E}_m) \mid \lambda(\sigma) = -\lambda(c\sigma) \text{ for all } \sigma: K \hookrightarrow \overline{\mathbb{Q}} \}$$

So,  $c$  acts as  $-1$  on  $X(MT(V)/w(\mathbb{E}_m))$ , and  $(MT(V)/w(\mathbb{E}_m))(\mathbb{R})$  is a quotient of

$$\ker(N_{\mathbb{E}/\mathbb{R}}: \mathbb{S} \rightarrow \mathbb{E}_m(\mathbb{R})^r),$$

that is, a quotient of a product of  $S^1$ 's, and therefore compact.

q.e.d.

6.1.4 Corollary The  $\otimes$ -category of all CM Hodge structures, together with the functor which, to a CM Hodge structure, associates its underlying  $\mathbb{Q}$ -vector space is a neutralized tannakian category over  $\mathbb{Q}$ , with the Serre group  $Z$  (see 0, 7.3.3) as corresponding affine group scheme.

It is clear how the mapping  $\nu$  of 6.1.1 defines a representation of  $Z$  if the conditions of 6.1.3 are satisfied. So the proof of 6.1.4 is obvious, but let us illustrate the corollary in our principal

6.1.5 Example. Let  $\chi$  be an algebraic Hecke character of  $K$  with values in  $E$ , and let  $M(\chi) \in CM_K(E)$  be the standard motive for  $\chi$  constructed in § 4. We know - remark 4.2 - that the Hodge structures  $H_\sigma(M(\chi))$ , for  $\sigma: K \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$  are given by

$$H_\sigma(M(\chi)) \otimes_{E, \tau} \mathbb{C} \subset H^{n(\sigma, \tau)}, \quad w = n(\sigma, \tau),$$

the  $n(\sigma, \tau)$  describing the infinity type of  $\chi$  as in 0, § 4. Every  $H_\sigma(M(\chi))$  is a CM Hodge structure: in fact, the elements of  $E$  define endomorphisms of the motive  $M(\chi)$ , and therefore in particular elements of type  $(0,0)$  in  $T^{1,1} = H_\sigma(M(\chi))^\vee \otimes H_\sigma(M(\chi))$ , so that

$$\mathrm{MT}(H_{\sigma}(M(\chi))) \subset \mathrm{GL}_{\mathbb{E}}(H_{\sigma}(M(\chi)))/\mathbb{Q} = \mathrm{R}_{\mathbb{E}/\mathbb{Q}}\mathbb{G}_{\mathbb{m}};$$

on the other hand, we shall show that, for all  $\sigma$ , the corresponding map  $v_{\sigma}$  factors through  $Z_L$ , for a suitable number field  $L$ . More precisely, remember that, when working with the Serre group  $Z$ , we consider all number fields as embedded into  $\overline{\mathbb{Q}}$  (cf. **0**, 7.3.2). Pick a finite Galois extension  $L$  of  $\mathbb{Q}$ ,  $L \subset \overline{\mathbb{Q}}$ , which contains  $K$  and  $\mathbb{E}$ . Then

$$v_{\sigma}: \mathrm{R}_{L/\mathbb{Q}}\mathbb{G}_{\mathbb{m}} \rightarrow \mathrm{MT}(H_{\sigma}(M(\chi))) \subset \mathrm{R}_{\mathbb{E}/\mathbb{Q}}\mathbb{G}_{\mathbb{m}}$$

exists, and is given by the formula in the proof of 6.1.1 above. Now,  $\mu^*(\sum_{\tau} n_{\tau} \cdot \tau) = \sum_{\tau} n(\sigma, \tau) n_{\tau}$ , and thus one easily finds that  $v_{\sigma}$ , as a homomorphism over  $\mathbb{Q}$ :  $\mathrm{R}_{L/\mathbb{Q}}\mathbb{G}_{\mathbb{m}} \rightarrow \mathrm{R}_{\mathbb{E}/\mathbb{Q}}\mathbb{G}_{\mathbb{m}}$ , is given precisely by the array of numbers  $\{n(s\sigma, \tau) \mid s \in G(L/\mathbb{Q}), \tau \in \mathrm{Hom}(\mathbb{E}, \overline{\mathbb{Q}})\}$ ; cf. **0**, § 4. As the  $n(\sigma', \tau)$ 's come from the infinity-type of a Hecke character, we know - **0**, 7.3.2 - that  $v_{\sigma}$  factorizes through  $Z_L$ , and therefore defines a representation  $\tilde{v}_{\sigma}: Z \rightarrow \mathrm{R}_{\mathbb{E}/\mathbb{Q}}\mathbb{G}_{\mathbb{m}}/\mathbb{Q}$  - which then corresponds to  $H_{\sigma}(M(\chi))$  via 6.1.4. If  $\sigma$  is just the fixed embedding of  $K$  into  $\overline{\mathbb{Q}}$ , this representation  $v_{\sigma}$  is nothing but the infinity type of  $\chi$ , viewed as a representation of  $Z$  as in **0**, 7.3.2.

6.1.6 Corollary Let  $\overline{\mathbb{Q}}$  be the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ , and  $\sigma$  some embedding  $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ . The pair  $(\mathrm{CM}_{\overline{\mathbb{Q}}}, H_{\sigma})$  - see § 4 above - is a neutralized tannakian category over  $\mathbb{Q}$  with  $Z$  as corresponding group scheme.

Proof. We show that  $H_{\sigma}$  establishes an equivalence of categories between the motives  $\mathrm{CM}_{\overline{\mathbb{Q}}}$  and all CM Hodge structures. By theorem 2.4.3 we are reduced to showing that

- (a) if  $M$  is a motive in  $\mathrm{CM}_{\overline{\mathbb{Q}}}$ , then  $H_{\sigma}(M)$  is a CM Hodge structure;
- (b) every CM Hodge structure arises in this way.

Since there are no non trivial Artin motives over  $\overline{\mathbb{Q}}$ ,

for (a) it suffices to show that  $H_1^{\sigma}(A)$  is a CM Hodge structure if  $A/\overline{\mathbb{Q}}$  is an abelian variety with complex multiplication as in 1.1 - but this is a special case of 6.1.5. As to (b), any representation of  $Z$  breaks up - over a suitable number field  $E$  - into a direct sum of characters  $\lambda: Z \rightarrow R_{E/\overline{\mathbb{Q}}}^{\mathbb{C}_m}$ . Any such  $\lambda$  is the infinity type of some Hecke character  $\chi$  with values in  $E$ , defined over a suitable number field  $K \subset \overline{\mathbb{Q}}$ . We have seen in 6.1.5 how these infinity types arise from the Hodge structures  $H_{\sigma}(M(\chi)) = H_{\sigma}(M(\chi) \times_K \overline{\mathbb{Q}})$ .

6.1.7 Remark It follows (for example, from 6.1.4 and the definition of  $CM_{\overline{\mathbb{Q}}}$  ...) that every CM Hodge structure is polarizable. - In fact, it is more customary - cf. [Lg], p. 215 f - to define a CM Hodge structure as a polarizable rational Hodge structure whose Mumford-Tate group is abelian.

## 6.2 Taniyama extensions

We now start setting up the formalism for determining the group scheme corresponding to  $(CM_{\overline{\mathbb{Q}}}, H_B)$ . Proofs are essentially omitted.

6.2.0 Definition. A Taniyama extension is an exact sequence of affine group schemes over  $\mathbb{Q}$

$$1 \rightarrow Z \xrightarrow{i} T \xrightarrow{\varphi} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow 1,$$

where  $Z$  is the Serre group, together with a homomorphism of topological groups  $\varepsilon: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow T(\mathbf{A}^f)$  such that  $\varphi_{\mathbf{A}^f} \circ \varepsilon = \text{id}$ .

Implicit in this definition is the requirement that the action of  $T$  on  $Z$  by conjugation - which, as  $Z$  is abelian, factors through  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  - be the second galois action described in 6.1.2.

Any Taniyama extension may be written as inverse limit of sequences (with finite adelic splittings  $\varepsilon_E$ )

$$(6.2.1) \quad 1 \rightarrow Z_E \rightarrow T_E \rightarrow \text{Gal}(E^{\text{ab}}/\mathbb{Q}) \rightarrow 1$$

over finite normal extensions  $E$  of  $\mathbb{Q}$  contained in  $\bar{\mathbb{Q}}$ .

Given 6.2.1, choose any set theoretic splitting

$$a_E: \text{Gal}(E^{\text{ab}}/\mathbb{Q}) \rightarrow T_E(E)$$

and define, for  $s \in \text{Gal}(E^{\text{ab}}/\mathbb{Q})$ ,

$$(6.2.2) \quad c_E(s) = \varepsilon_E(s)^{-1} a_E(s) \pmod{Z_E(E)},$$

a class in  $Z_E(E_{\mathbf{A}^f})/Z_E(E)$  which is independent of the choice of  $a_E$ . Next, for  $\lambda$  any character  $Z_E \rightarrow \mathbb{C}_m^*$  defined over  $\bar{\mathbb{Q}}$  and  $s \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ , define the "cocycle":

$$(6.2.3) \quad c_E(s, \lambda) = \lambda(c_E(s)) \in E_{\mathbf{A}^f}^*/E^*.$$

These finite idèle classes have the following properties, valid for all  $s, t \in \text{Gal}(E^{\text{ab}}/\mathbb{Q})$ ;  $\lambda, \lambda' \in \text{Hom}_{\bar{\mathbb{Q}}} (Z_E, \mathbb{C}_m^*)$ ;  $\lambda^s$  being defined as in 6.1.2.

6.2.4 Lemma (i)  $c_E(st, \lambda) = c_E(s, \lambda^t) c_E(t, \lambda)$ .

(ii)  $c_E(t, \lambda)^s = c_E(t, \lambda^s)$ .

(iii) If  $F \supset E$ , then  $c_F(s, \lambda \circ N_{F/E}) = c_E(s, \lambda)$ .

(iv) If  $c = \text{complex conjugation}$ , then:  $c_E(c, \lambda) = 1$  if and only if  $\varepsilon(c) \in T(\mathbb{Q})$ .

(v)  $c_E(s, \lambda) c_E(s, \lambda') = c_E(s, \lambda \cdot \lambda')$ .

(In (iii), use that  $E_{\mathbf{A}^f}^*/E^* \hookrightarrow F_{\mathbf{A}^f}^*/F^*$ , by Hilbert 90.)

6.2.5 Proposition. Two Taniyama extensions  $T$  and  $T'$  are isomorphic (as exact sequences of affine group schemes over  $\mathbb{Q}$  with finite adelic splittings), if and only if the corresponding cocycles  $c_E, c'_E$  are equal, for all  $E$ .

### 6.3 The group scheme for $(CM_{\mathbb{Q}}, H_B)$ .

6.3.0 Let  $\mathcal{U}$  be the affine group scheme over  $\mathbb{Q}$  which corresponds, by tannakian philosophy, to the neutralized category of motives  $(CM_{\mathbb{Q}}, H_B)$ , defined in § 4 above. Then  $\mathcal{U}$  is naturally endowed with the structure of a Taniyama extension

$$1 \rightarrow Z \xrightarrow{i} \mathcal{U} \xrightarrow{\varphi} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow 1.$$

In fact,  $\varphi$  is given (via 2.3.2, (b)) by the fact that the Artin motives (2.4.1) are contained in  $CM_{\mathbb{Q}}$  (note that  $\mathcal{U}$  is, in fact, proreductive: cf. 2.3.4); and  $i$  corresponds via 2.3.2 (c) to the functor  $CM_{\mathbb{Q}} \rightarrow CM_{\overline{\mathbb{Q}}}$ ,  $M \mapsto M \times_{\mathbb{Q}} \overline{\mathbb{Q}}$  (the condition of 2.3.2, (c) is satisfied because every object  $M$  of  $CM_{\overline{\mathbb{Q}}}$  is defined over some number field  $K$ , and  $R_{K/\mathbb{Q}}^M \in CM_{\mathbb{Q}}$  contains  $M$  as a direct factor if viewed over  $\overline{\mathbb{Q}}$ ), where we make use of 6.1.6. The exactness is then straightforward; the fact that the galois action on  $Z$  comes out right is slightly more subtle (cf. [DMOS], IV, B), or at least confusing ... - Finally, the finite adelic splitting  $\varepsilon$  required for a Taniyama extension, comes from the fact that the étale realization  $H_{\mathbb{A}^f}(M)$  of a motive  $M$  of  $CM_{\mathbb{Q}}$  carries an action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , and is, on the other hand, isomorphic to  $H_B(M) \otimes_{\mathbb{A}^f}$ .

6.3.1 We shall now write down, generalizing Tate (cf. [LCM], 7 § 3; [Bl], § 4), a "cocycle"  $g_E(s, \lambda)$ , for every CM field  $E$  normal over  $\mathbb{Q}$ , which can be easily shown to be the cocycle corresponding to  $\mathcal{U}$  in the setup of 6.2. Let  $M \in CM_{\overline{\mathbb{Q}}}(\mathbb{Q})$  be of rank 1. For any  $s \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , the conjugate motive  $M^s$  is defined in  $CM_{\overline{\mathbb{Q}}}$  and carries the action of  $E$  transported by  $s$ , with respect to which it is also of rank 1 over  $E$ . Thus, fixing identifications

$$\begin{array}{ccc} E & \xrightarrow[\sim]{\theta} & H_B(M) \\ E & \xrightarrow[\sim]{\xi} & H_B(M^s) \end{array}$$

(where  $H_B$  denotes the realization  $H_\sigma$ , for  $\sigma$  the identical embedding  $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ ), there is an element  $a \in E_{\mathbb{A}^f}^*$  such that the following diagram commutes.

$$\begin{array}{ccc}
 E_{\mathbb{A}^f} & \xrightarrow{\theta \otimes \mathbb{Q}_{\mathbb{A}^f}} & H_{\mathbb{A}^f}^f(M) \\
 \downarrow \cdot a & & \downarrow s \\
 E_{\mathbb{A}^f} & \xrightarrow{\zeta \otimes \mathbb{Q}_{\mathbb{A}^f}} & H_{\mathbb{A}^f}^f(M^S)
 \end{array}$$

6.3.2 Theorem Up to multiplication by an element of  $E^*$ ,  $a$  depends only on  $s$  and on the representation  $\lambda: \mathbb{Z} \rightarrow R_{E/\mathbb{Q}} \mathbb{G}_m/\mathbb{Q}$  corresponding to the CM Hodge structure  $H_B(M)$ .

The proof is easy from what we already know - but it does, of course, use the absolute Hodge cycle theorem. -  
 Rewriting  $\lambda: \mathbb{Z} \rightarrow R_{E/\mathbb{Q}} \mathbb{G}_m/\mathbb{Q}$  as  $\lambda: \mathbb{Z}_E \rightarrow \mathbb{G}_m$ , defined over  $E$ , we shall write the class  $a \cdot E^*$  as  $g_E(s, \lambda)$ , thereby defining the cocycle characterizing  $\mathfrak{U}$ .

6.3.3 Lemma (i)  $g_E(s, \lambda)$  has the properties analogous to (i), (ii), (iii) and (v) of 6.2.4.

(ii)  $g_E(c, \lambda) = 1$ , for all  $\lambda$ .

(iii)  $g_E(s, \lambda)^{1+c} = \psi(s)^{-w} \cdot E^*$ ,

where  $\psi(s) \in \hat{\mathbb{Z}}^*$  is such that  $\zeta^s = \zeta^{\psi(s)}$ , for every root of unity  $\zeta \in \bar{\mathbb{Q}}^*$ , and  $w$  is the weight of the Hodge structure  $H_B(M)$ .

In the case where  $\lambda$  is a CM type (i.e.,  $M = H_1(A)$ , for some CM abelian variety  $A/\bar{\mathbb{Q}}$ ) and  $\lambda^S = \lambda$ , the class  $g_E(s, \lambda)$  is given by the Shimura-Taniyama reciprocity law. This class field theoretic description of  $g_E(s, \lambda)$  will be generalized in 6.4, and the fact that it does describe  $g_E$  will be equivalent to the isomorphism between  $\mathfrak{U}$  and the Taniyama group ...

#### 6.4 The Taniyama group

We proceed to define Tate's second cocycle  $f_E(s, \lambda)$ - generalizing it the same way we generalized  $g_E$  in 6.3: cf. [LCM], 7 §§ 1,2, and [B1], § 4.

##### 6.4.0 First, generalize Tate's "half transfer":

Given a CM field  $E$  normal over  $\mathbb{Q}$ , choose a system of representatives (remember that  $E \subset \overline{\mathbb{Q}}$ )

$$v: \text{Hom}(E, \overline{\mathbb{Q}}) \rightarrow \text{Gal}(E^{\text{ab}}/\mathbb{Q})$$

such that  $v(\tau)|_E = \tau$  and  $v(c\tau) = cv(\tau)$ , for  $c =$  complex conjugation. Then, for  $s \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and  $\lambda: Z_E \rightarrow \mathbb{C}_m^*$  over  $E$  (or over  $\overline{\mathbb{Q}}$ ), write  $\lambda$  as

$$\lambda = \sum_{\tau: E \hookrightarrow \overline{\mathbb{Q}}} n_\tau \tau \quad ,$$

and set

$$V_E(s, \lambda) = \prod_{\tau: E \hookrightarrow \overline{\mathbb{Q}}} (v(s\tau)^{-1} \cdot (s|_{E^{\text{ab}}}) \cdot v(\tau))^{-n_\tau} \quad ,$$

an expression well defined in  $\text{Gal}(E^{\text{ab}}/\mathbb{Q})$ .

6.4.1 Notations In the situation of 6.4.0, denote by  $r_E: E_{\mathbf{A}_f}^*/E^* \rightarrow \text{Gal}(E^{\text{ab}}/E)$  the reciprocal of the classical Artin map, i.e.,  $r_E$  sends a uniformizer  $\pi$  to a geometric Frobenius at  $\pi$ . - Recall the cyclotomic character  $\Psi$  defined in 6.3.3 (iii), and note that one has

$$r_{\mathbb{Q}}(\Psi(s)) = s|_{\mathbb{Q}^{\text{ab}}} \quad ,$$

for all  $s \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . - Finally, for  $\lambda$  as before, write as usual  $w = n_\tau + n_{c\tau}$  (any  $\tau$ ) the weight of  $\lambda$  (or: of the corresponding Hodge structure).

6.4.2 Proposition/Definition. With the preceding notations,  
there exists, for any  $s \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , a unique class  
 $f_E(s, \lambda) \in E_{\mathbb{A}^f}^*/E^*$ , such that

$$(i) \quad r_E(f_E(s, \lambda)) = v_E(s, \lambda),$$

and (ii)  $f_E(s, \lambda)^{1+c} = \psi(s)^{-w} \cdot E^*$ .

For the proof, cf. [LCM], 7, 2.2.

6.4.3 One can show that  $f_E(s, \lambda)$  gives the cocycle attached to the Taniyama group  $\mathcal{T}$  defined by Langlands in [Lg], § 5 - except possibly for a certain number of renormalizations, of the sort carried out in [DMOS], III. We have not taken the time to check the details, and for us  $\mathcal{T}$  will be the Taniyama extension characterized by  $f_E(s, \lambda)$  (whose existence is not proved here.) We do however call this  $\mathcal{T}$  the Taniyama group.

## 6.5 The Main Theorem, Consequences

6.5.1 Theorem. The cocycles  $f_E(s, \lambda)$  and  $g_E(s, \lambda)$  are  
equal: the Taniyama extension  $\mathcal{U}$  corresponding to  $(CM_{\mathbb{Q}}, H_B)$   
is isomorphic to the Taniyama group  $\mathcal{T}$ .

The reader may obtain a complete proof of  $f_E = g_E$  from [LCM], chap. 7 : the one crucial property of  $g$  (therefore of "e") not demonstrated in Lang - theorem 4.2 of [LCM], 7 - has simply been built into our motivic construction of  $g_E(s, \lambda)$ ! In translating Lang's setup into our notations one has to identify a CM abelian variety  $A/\overline{\mathbb{Q}}$  with the motive  $H_1(A)$  - cf. 1.1 above. Thus instead of the CM types  $\phi$  (of weight + 1) in Lang, we consider representations  $\lambda$  with  $n_\tau = -1$  or 0, of weight -1.

6.5.2 We shall use the following notation for the Taniyama group  $\mathcal{T}$  :



$$\begin{array}{ccccccc}
 1 & \rightarrow & \mathbb{Z} & \xrightarrow{j} & \tau & \xrightarrow{\phi} & \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow 1 \\
 & & & & \downarrow & & \\
 & & & & \tau(\mathbb{A}^f) & \leftarrow & 
 \end{array}$$

We write  ${}_K\tau = \phi^{-1}(\text{Gal}(\overline{\mathbb{Q}}/K))$ , for any number field  $K \subset \overline{\mathbb{Q}}$ . And if  $E$  is a finite normal extension of  $\mathbb{Q}$ , also contained in  $\overline{\mathbb{Q}}$ , such that  $K \subset E^{\text{ab}}$ , then we write  ${}_K\tau_E$  for the image of  ${}_K\tau$  in the quotient  $\tau_E$  (6.2.1) of  $\tau$ . One might call

$$\begin{array}{ccccccc}
 1 & \rightarrow & \mathbb{Z} & \rightarrow & {}_K\tau & \rightarrow & \text{Gal}(\mathbb{Q}/K) \rightarrow 1 \\
 & & & & \downarrow & & \\
 & & & & {}_K\tau(\mathbb{A}^f) & \leftarrow & \alpha|_G(\overline{\mathbb{Q}}/K)
 \end{array}$$

a Taniyama extension over  $K$ . It is the inverse limit of

$$\begin{array}{ccccccc}
 (6.5.3) & 1 & \rightarrow & \mathbb{Z}_E & \rightarrow & {}_K\tau_E & \rightarrow \text{Gal}(E^{\text{ab}}/K) \rightarrow 1 \\
 & & & & & \downarrow & \\
 & & & & & {}_K\tau_E(\mathbb{A}^f) & \leftarrow \alpha_E
 \end{array}$$

Note that  ${}_K\tau_E$  is abelian, if  $K \supset E$ .

6.5.4 Scholion. Let  $K \subset \overline{\mathbb{Q}}$  be a number field, and write  $H_B$  for the fibre functor  $H_\sigma$  on  $CM_K$ , with  $\sigma =$  the inclusion  $K \hookrightarrow \overline{\mathbb{Q}}$ . Then  ${}_K\tau$  is the affine group scheme corresponding to  $(CM_K, H_B)$ .

Cf. [DMOS], p. 265.

6.5.5 Scholion. Let  $K \subset \overline{\mathbb{Q}}$  be a number field which is galois over  $\mathbb{Q}$ . Then the structure  ${}_K\tau_K$  (i.e., 6.5.3 with  $E = K$ ) is isomorphic to Serre's group  $S_K$ , i.e. to the sequence  $\mathbf{0}$ , 7.3.1, equipped with the section  $\varepsilon$  of  $\mathbf{0}$ , 7.4. - Equivalently, there is an isomorphism

$${}_K\mathcal{U}^{\text{ab}} \cong S_K$$

compatible with the finite adelic splittings  $\alpha$  and  $\varepsilon$ .

Two proofs of 6.5.5 are possible: First, a direct proof using only the cocycle  $f_E$  characterizing  $\mathcal{U}$  - cf. [Lg], p. 224; second, using the fact that  $\mathcal{U} \cong \mathcal{U}$ , one can exploit the existence of  $M(\chi)$  in  $CM_K$ , for any Hecke character  $\chi$  of  $K$ , to identify  $\text{Hom}_{\mathbb{Q}}({}_K\mathcal{U}, \mathbb{G}_m)$  with  $\text{Hom}_{\mathbb{Q}}(S_K, \mathbb{G}_m)$  - cf. [DMOS], IV, (D) and (E).

6.5.6 The first proof of 6.5.5 would provide us with a new method to construct  $M(\chi)$  (via 6.5.1): Viewed as a representation  ${}_K\mathcal{U} \rightarrow R_{E/\mathbb{Q}}\mathbb{G}_m$  over  $\mathbb{Q}$ , the motive  $M(\chi)$  is simply the Hecke character  $\chi$  of  $K$  with values in  $E$ , interpreted as in  $\mathbb{Q}$ , 7.2,

$$S_K \xrightarrow{\chi} R_{E/\mathbb{Q}}\mathbb{G}_m \quad \text{over } \mathbb{Q},$$

and pulled back to  ${}_K\mathcal{U}$ , via the canonical map  ${}_K\mathcal{U} \rightarrow {}_K\mathcal{U}^{\text{ab}}$ .

6.5.7 Corollary. Let  $K$  and  $E$  be number fields, and  $K \subset \overline{\mathbb{Q}}$ . Let  $M$  be a motive in  $CM_K(E)$ . Then the system of  $\lambda$ -adic representations, for  $\lambda$  running over the finite places of  $E$ ,

$$H_\lambda(M) = H_\ell(M) \otimes_E \otimes_{\mathbb{Q}_\ell} E_\lambda \quad (\text{where } \lambda | \ell)$$

is a strictly compatible system of  $E$ -rational Galois representations.

Recall that the statement of the corollary means that there is a finite set  $\Sigma$  of places of  $K$  such that for any prime ideal  $\mathfrak{p}$  of  $K$  not contained in  $\Sigma$ , and any place  $\lambda$  of  $E$  with  $\mathfrak{p} \nmid N\lambda$ , the Galois representation  $H_\lambda(M)$  is unramified at  $\mathfrak{p}$  (so that the action of a geometric Frobenius element  $\text{Frob } \mathfrak{p}$  at  $\mathfrak{p}$  on  $H_\lambda(M)$  is well defined), and the "characteristic polynomial"

$$\det_{E_\lambda} (1 - \text{Frob } \mathfrak{p} \cdot X \mid H_\lambda(M)) \in E_\lambda[X]$$

actually has coefficients in  $E \hookrightarrow E_\lambda$  which are independent of the place  $\lambda$ .

One proof of 6.5.7 (via 6.5.1) uses the fact - due to Langlands, [Lg], p. 226/227; cf. [DMOS], III, 3.17 - that there is a natural homomorphism  $W_\mathbb{Q} \rightarrow \mathcal{T}(\mathbb{C})$ , where  $W_\mathbb{Q}$  is the global Weil group of  $\mathbb{Q}$ . See [DMOS], IV, remarques 2,3 (p. 262). - As Greg Anderson has pointed out (see [A2], 5.7), 6.5.7 as well as a few other important corollaries of 6.5.1 can also be obtained using R. Brauer's induction lemma:

6.5.8 Lemma The Grothendieck group of  $\text{Rep}_\mathbb{C}(\mathcal{T} \times \mathbb{C})$  is generated by the representations of the form

$$\text{Ind}_{K/\mathbb{Q}}(\chi),$$

for number fields  $K$  and characters  $\chi: \mathcal{T} \rightarrow \mathbb{C}_m^\times/\mathbb{C}^\times$ .

In fact,  $\mathcal{T} \times \mathbb{C}$  is the inverse limit of  $\mathbb{C}$ -algebraic groups whose connected components are tori. And the Grothendieck group of  $\text{Rep}_\mathbb{C}(G)$ , for  $G$  a  $\mathbb{C}$ -algebraic group such that  $G^0$  is a torus, is generated by the representations induced from 1 dimensional characters of subgroups of finite index.

Now, to deduce 6.5.7 from 6.5.8, assume (without loss of generality, applying  $R_{K/\mathbb{Q}}$ ) that  $K = \mathbb{Q}$  in the claim of 6.5.8, and note simply that

$$\det_E (1 - \text{Frob } \mathfrak{p} \cdot X \mid \text{Ind}_{K/\mathbb{Q}}(\chi)) = \prod_{\mathfrak{p} \mid \mathfrak{p}} (1 - \chi(\mathfrak{p}) \cdot X) \in E[X],$$

at good places  $\mathfrak{p}$ .

6.5.9 The corollary 6.5.7 allows to unconditionally define the L-function of a motive  $M$  in  $CM_K(E)$  (or, equivalently,  $M$  in  $CM_{\mathbb{Q}}(E)$ ): For  $s \in \mathbb{C}$  with  $\text{Re}(s) \gg 0$ , put (cf. 1.8 above):

$$L^*(M/K, s) = \prod_{\mathfrak{p}} \det_E (1 - \text{Frob}_{\mathfrak{p}} \cdot \mathbb{N}\mathfrak{p}^{-s} \mid H_{\lambda}(M)^{\mathbb{I}_{\mathfrak{p}}})^{-1},$$

where  $\mathfrak{p}$  runs over all finite primes of  $K$ , and the determinant is calculated using any  $\lambda$  such that  $\mathfrak{p} \nmid \mathbb{N}\lambda$ . It is well known that the product converges for  $\text{Re}(s)$  sufficiently big, defining an element

$$L^*(M/K, s) \in E \otimes \mathbb{C} \cong \mathbb{C}^{\text{Hom}(E, \mathbb{C})}.$$

(With our definition of strict compatibility, we have no control, a priori, about the Euler factors of the primes  $\mathfrak{p}$  in the bad set  $\Sigma$ . This problem disappears however, in the light of 6.5.8!)

If  $E = \mathbb{Q}$  (i.e.,  $M$  is considered without  $E$  action), then we write simply  $L(M/K, s)$  for the ("Hasse-Weil") L-function of the motive  $M$ .

Recall that, as functions on  $\mathbb{C}$  (a priori on  $\{\text{Re}(s) \gg 0\}$ ), the following L-functions coincide:

$$L^*(M/K, s) = L^*(R_{K/\mathbb{Q}}M/\mathbb{Q}, s).$$

(This generalizes the identity recalled before 6.5.9.)

In terms of L-functions, 6.5.8 reads:

6.5.10 Corollary. For any motive  $M$  in  $CM_K$ , there exist number fields  $L_1, \dots, L_n$  and, for every  $i = 1, \dots, n$ , an algebraic Hecke character  $\chi_i$  of  $L_i$  with values in a suitable field  $E \subset \mathbb{C}$ , and an integer  $m_i$ , such that

$$L(M/K, s) = \prod_{i=1}^n L(\chi_i, s)^{m_i}.$$

Here,  $L(\chi_1, s)$  is the L-function  $L(\chi_1^\tau, s)$  of  $\mathbf{0}$ , § 6, with  $\tau: E \hookrightarrow \mathbb{C}$ .

6.5.11 Amazingly enough, the same line of thought also gives an alternative proof of the unicity theorem 5.1! For this, we refer to [A2], 5.7.5.

## 6.6 Motives of rank 1 arising from abelian varieties

From 3.1, 5.3 and 6.5.7, we can now deduce that conjecture [DP], 8.1, (iii) is also true - like all the rest of conjecture [DP], 8.1 - in the category  $\mathcal{M}_K^{\text{av}}$  of motives over the number field  $K$  which can be constructed from the cohomology of abelian varieties (with or without complex multiplication). Joined with 5.1, this gives the

6.6.1 Theorem Every motive  $M$  in  $\mathcal{M}_K^{\text{av}}(E)$ , for a number field  $E$ , of rank 1 over  $E$ , is isomorphic in  $\mathcal{M}_K^{\text{av}}$  to a motive  $M(\chi)$  - see 4.1 - , for some algebraic Hecke character  $\chi$  of  $K$  with values in  $E$ .

## 7. Anderson's motives for Jacobi sum characters

This section continues  $\mathbf{0}$  § 8.

### 7.1 The basic example (Reference: [DMOS], I § 7)

7.1.1 For integers  $m \geq 0$ ,  $n > 1$ , let  $X_m^n \xrightarrow{i} \mathbb{P}^{n-1}$  be the Fermat hypersurface of dimension  $n - 2$  and degree  $m$ , given in projective coordinates by the equation

$$x_1^m + \dots + x_n^m = 0.$$

The twisted primitive cohomology motive

$$h_{\text{prim}}(X_m^n)(-1) = [h(X_m^n)/i^*h(\mathbb{P}^{n-1})] \otimes \mathbb{Q}(-1),$$

a priori an object of  $\mathcal{M}_{\mathbb{Q}}$ , decomposes over  $\mathbb{Q}(\mu_m)$  under

the action of the group

$$G_m^n = \left( \bigoplus_{i=1}^n \mu_m \right) / (\text{diagonal}) \subset \text{Aut} \left( X_m^n / \mathbb{Q}(\mu_m) \right).$$

Specifically, write the characters of  $G_m^n$  as

$$\underline{a} = \sum_{i=1}^n [a_i] \in \bigoplus_{i=1}^n \frac{1}{m} \mathbb{Z}/\mathbb{Z}, \quad \sum_{i=1}^n a_i = 0, \quad \text{all } a_i \neq 0,$$

$$\text{with } \underline{a}((\zeta_1, \dots, \zeta_n) \pmod{\text{diag.}}) = \prod_{i=1}^n \zeta_i^{a_i m}.$$

Define the eigenmotive  $h_{\text{prim}}(X_m^n)_{\underline{a}}$  as the image of  $[h_{\text{prim}}(X_m^n) \otimes \mathbb{Q}(\mu_m)] \times \mathbb{Q}(\mu_m)$  under the projector

$$P_{\underline{a}} = \frac{1}{\#G_m^n} \sum_{g \in G_m^n} C(g) \otimes \underline{a}(g)^{-1},$$

where  $C(g)$  is  $g$ , viewed as endomorphism (= absolute Hodge correspondence) of  $h(X_m^n)$ . Here,  $\mathbb{Q}(\mu_m)$  in the tensor product is (the first component of) the unit object of  $M_{\mathbb{Q}}(\mathbb{Q}(\mu_m))$  - cf. 3.0 above. One shows that  $h_{\text{prim}}(X_m^n)_{\underline{a}}$  is an object of  $M_{\mathbb{Q}}(\mathbb{Q}(\mu_m))(\mathbb{Q}(\mu_m))$  of rank 1, and that its L-function is given, in terms of the Jacobi sum Hecke characters of **0**, 8.2, by the relation [see 6.5.9 for the notation] :

$$L^*(h_{\text{prim}}(X_m^n)_{\underline{a}}(-1)/\mathbb{Q}(\mu_m), s) = (L(J_{\mathbb{Q}(\mu_m)}(c_{\underline{a}})^{\tau}, s))_{\tau \in \text{Hom}(\mathbb{Q}(\mu_m), \mathbb{C})}$$

7.1.2 As Jacobi sum Hecke characters are galois equivariant - see **0**, 8.2.5 - all the components of this array of L-functions are actually equal (as meromorphic functions on  $\mathbb{C}$ ). On the other hand, the sum of projectors

$$\bigoplus_{\sigma \in \text{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})} P_{\sigma \underline{a}}$$

(the galois action being that <sup>of</sup> **0**.8.2.1) is an absolute Hodge correspondence of  $h(X_m^n) \wedge \mathbb{Q}(\mu_m)$  defined over  $\mathbb{Q}$ . Thus, there is a motive  $M(\underline{a})$  in  $M_{\mathbb{Q}}$ , with coefficients in  $\mathbb{Q}$  (the action of  $G_m^n$  is only defined over  $\mathbb{Q}(\mu_m)$ ) of rank  $[\mathbb{Q}(\mu_m) : \mathbb{Q}]$  such that

$$M(\underline{a})(1) \times \mathbb{Q}(\mu_m) \cong h_{\text{prim}}(X_m^n)_{\mathbb{C}\underline{a}}$$

in  $M_{\mathbb{Q}}(\mu_m)$  ; consequently

$$L(M(\underline{a})/\mathbb{Q}, s) = L(J_{\mathbb{Q}}(\mu_m)(\underline{a}), s)$$

(an identity of functions on  $\mathbb{C}$ .)

7.1.3 The motive  $h(X_m^n)$ , and therefore also  $M(\underline{a})$ , is (isomorphic to) a motive in  $M_{\mathbb{Q}}^{\text{av}}$ , and in fact, by the same token, in  $CM_{\mathbb{Q}}$  - cf. 5.3.

This is shown by Shioda induction: see [A2], § 9 for a detailed proof of how to express  $h(X_m^n)$  in terms of the Fermat curves  $X_m^2, X_m^3$  and of  $\mathbb{P}^1$ . As the Jacobian of Fermat curves are well known to admit complex multiplication (over  $\mathbb{Q}(\mu_m)$ ) this directly proves the stronger assertion.

7.1.4 Thus we have indicated how to construct, for  $\underline{a}$  as above, a motive  $M(\underline{a})$  in  $M_{\mathbb{Q}}^{\text{av}}$ , and in fact a representation of the Taniyama group, whose L-function is the Hecke L-function of  $J_{\mathbb{Q}}(\mu_m)(\underline{a})$ . Note that, in view of 6.5.7, this already proves that  $J_{\mathbb{Q}}(\mu_m)(\underline{a})$  is a Hecke character, and more precisely, a Hecke character unramified outside  $m$  - because this is true of the  $\ell$ -adic representations of  $X_m^n$ .

Anderson, in [A1], and especially in [A2], has generalized this construction of  $M(\underline{a})$  to all Jacobi sum Hecke characters, in the sense of **0**, 8.2.4.

7.2 Anderson's first theorem (Reference: [A1] or [A2])

Let  $K$  be an abelian number field and  $\underline{a} \in \mathbb{B}_K^0$  (for the notation, see **0**, 8.2). There is a motive  $M_K(\underline{a})$  in  $CM_{\mathbb{Q}}$  of rank  $[K:\mathbb{Q}]$  such that

$$L(M_K(\underline{a}), s) = L(J_K(\underline{a}), s).$$

Upon extension of scalars,  $M_K(\underline{a}) \times K$  acquires the structure of a motive of rank 1 in  $CM_K(K)$ , which is a motive for  $J_K(\underline{a})$ , in the sense of 3.3.

The crucial point about this theorem is that  $M_K(\underline{a})$  is constructed from the cohomology of Fermat hypersurfaces  $X_m^n$ . This will make it possible to calculate the periods of  $M_K(\underline{a})$  in terms of values of the  $\Gamma$ -function at rational numbers: see II § 4! In [A1],  $M_K(\underline{a})$  is explicitly constructed as sitting in twisted Fermat hypersurfaces; in [A2], the theorem is no longer stated the way we just did but rather embedded in a much more general formalism which we shall now sketch very roughly.

### 7.3 Anderson's ulterior motives (Reference: [A2] )

#### 7.3.1 An arithmetic Hodge structure $W$ of weight $w \in \mathbb{Z}$ is

• a finite dimensional  $\mathbb{Q}$  vector space  $W_B$ , with a decomposition

$$W_B \otimes \mathbb{C} = \bigoplus_{\substack{a, b \in \mathbb{Q} \\ a+b=w}} W^{a,b}$$

such that  $(1 \otimes c)W^{a,b} = W^{b,a}$ , for  $c =$  complex conjugation;

• a  $\mathbb{Q}$ -linear subspace  $W_{DR}$  of  $W_B \otimes \mathbb{C}$  of the same dimension as  $W_B$  such that

(i) for all  $a \in \mathbb{Q}/\mathbb{Z}$ , writing

$$F^a(W_B \otimes \mathbb{C}) = \bigoplus_{\substack{a' \in \mathbb{Q} \\ a' \geq a}} W^{a', w-a'} \quad \text{and} \quad F^a W_{DR} = W_{DR} \cap F^a(W_B \otimes \mathbb{C}),$$

one has

$$(F^a W_{DR}) \otimes \mathbb{C} = F^a(W_B \otimes \mathbb{C});$$

(ii) there is a  $\mathbb{Q}$ -linear involution  $F_\infty: W_B \rightarrow W_B$  making the following triangle commute.



$$\begin{array}{ccc}
 W_B \otimes \mathbb{C} & \xrightarrow{F_\infty \otimes c} & W_B \otimes \mathbb{C} \\
 & \searrow & \swarrow \\
 & W_{DR} &
 \end{array}$$

An arithmetic Hodge structure is a finite direct sum of arithmetic Hodge structures of fixed weights.

It is obvious how the notion of arithmetic Hodge structure is a generalization of the "part at infinity" of a motive over  $\mathbb{Q}$  - cf. [DP], 1.4, for the last requirement, (ii). We shall write  $\omega_\infty(M)$  for the arithmetic Hodge structure given by a motive  $M$  in  $M_\mathbb{Q}$ . The fractional exponents permitted in the decomposition of  $W_B \otimes \mathbb{C}$  will be needed to accommodate individual Gauss sums ...

A morphism of arithmetic Hodge structures is a  $\mathbb{Q}$ -linear map of the  $W_B$ 's which respects the  $W^{a,b}$ 's and the  $W_{DR}$ 's. Just like Hodge structures, arithmetic Hodge structures form a tannakian category over  $\mathbb{Q}$ , say  $AH$ , neutralized by the functor  $W \mapsto W_B$  into  $\mathbb{Q}$  vector spaces.

7.3.2 Write  $2\pi i \hat{\mathbb{Z}}$  for the Pontrjagin dual of  $\mathbb{Q}/\mathbb{Z}$ , and consider the pairing

$$\begin{array}{ccc}
 2\pi i \hat{\mathbb{Z}} \times \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathbb{C}^* \\
 (2\pi i n, a) & \longmapsto & \langle 2\pi i n, a \rangle = \exp(2\pi i \langle na \rangle),
 \end{array}$$

where the function  $\langle \cdot \rangle$  on  $\mathbb{Q}/\mathbb{Z}$  was defined in **0**, 8.1.4. Given  $0 \neq a \in \mathbb{Q}/\mathbb{Z}$ , define  $\gamma(a): 2\pi i \hat{\mathbb{Z}} \rightarrow \mathbb{C}^*$  by

$$\gamma(a)(2\pi i n) = \langle 2\pi i n, a \rangle \cdot \Gamma(\langle -a \rangle).$$

For each integer  $m \geq 1$ , we define the arithmetic Hodge structure of weight 1,  $E_m$  by:

$$\bullet \quad (E_m)_B = \left\{ e: 2\pi i \hat{\mathbb{Z}} \rightarrow \mathbb{Q} \mid \begin{array}{l} e \text{ factors through } 2\pi i (\hat{\mathbb{Z}}/m\hat{\mathbb{Z}}) \\ \text{and } \sum_{j \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}} e(2\pi i m j) = 0 \end{array} \right\}$$

- $E_m^{\langle a \rangle, \langle -a \rangle} = \mathbb{C} \cdot \gamma(a)$ , if  $0 \neq a \in \frac{1}{m} \mathbb{Z}/\mathbb{Z}$
- $(E_m)_{\text{DR}} = \sum_{0 \neq a \in \frac{1}{m} \mathbb{Z}/\mathbb{Z}} \mathbb{Q} \cdot \gamma(a)$ .

7.3.3 Call  $\tilde{CM}$  the smallest tannakian subcategory of  $AH$  containing  $\omega_\infty(M)$ , for all objects  $M$  of  $CM_\mathbb{Q}$ , and  $E_m$ , for all  $m \geq 1$ . Write  $\tilde{\mathcal{T}}$  for the affine group scheme over  $\mathbb{Q}$  which corresponds to  $(\tilde{CM}, W \mapsto W_B)$ . The  $\mathbb{Q}$  vector space  $(E_m)_B$ , viewed as a representation of  $\tilde{\mathcal{T}}$ , is denoted  $\mathbb{I}E_m$  by Anderson, and he defines

$$\mathbb{I}E = \varinjlim \mathbb{I}E_m,$$

using the inclusions of arithmetic Hodge structures  $E_m \rightarrow E_n$ , for  $m|n$ .

By tannakian philosophy, the functor  $\omega_\infty: CM_\mathbb{Q} \rightarrow \tilde{CM}$  corresponds to a morphism

$$\tilde{\phi}: \tilde{\mathcal{T}} \rightarrow \mathcal{T}$$

with  $\mathcal{T}$  the Taniyama group (6.5).

Furthermore, there is an arrow

$$\tilde{j}: 2\pi i \hat{\mathbb{Z}} \rightarrow \tilde{\mathcal{T}}$$

which arises as follows.

7.3.4 Let  $V$  be a  $\mathbb{Q}$  vector space with an admissible  $\mathbb{Q}$ -linear action of  $2\pi i \hat{\mathbb{Z}}$ . Then  $V$  can be decomposed into eigenspaces over  $\mathbb{C}$ :

$$V \otimes \mathbb{C} = \bigoplus_{a \in \mathbb{Q}/\mathbb{Z}} V(a),$$

and one finds, for all  $s \in \text{Aut } \mathbb{C}$ , that

$$(1\otimes s) V(a) = V(\Psi(s|\overline{\mathbb{Q}})a)$$

with  $\Psi$  the cyclotomic character: see  $\mathbb{Q}$ , 8.2.1; or 6.4.1. Conversely, every decomposition respecting the galois action on  $\mathbb{Q}/\mathbb{Z}$  comes from an (admissible) representation  $2\pi i \hat{\mathbb{Z}} \rightarrow GL(V)$ .

Now, given a representation  $W$  of  $\tilde{\mathcal{G}}$ , and  $a \in \mathbb{Q}/\mathbb{Z}$ , put

$$W(a) = \bigoplus_{p, q \in \mathbb{Z}} W^{p+\langle a \rangle, q-\langle a \rangle}$$

The decomposition  $W \otimes \mathbb{C} = \bigoplus W(a)$  is compatible with the galois action on  $\mathbb{Q}/\mathbb{Z}$  - look at  $E_m!$  - , and therefore comes from a representation  $2\pi i \hat{\mathbb{Z}} \rightarrow GL(W)$ . This action of  $2\pi i \hat{\mathbb{Z}}$  depends naturally on  $W$  and thus defines the desired morphism  $\tilde{j}: 2\pi i \hat{\mathbb{Z}} \rightarrow \tilde{\mathcal{G}}$ . Since motives have honest regard Hodge structures: with integral exponents, it is plain that the image of  $\tilde{j}$  is contained in the kernel of  $\tilde{\phi}$ . - And more is true:

#### 7.4 Anderson's second theorem ([A2], Theorem 8)

##### 7.4.1 The sequence

$$1 \rightarrow 2\pi i \hat{\mathbb{Z}} \xrightarrow{\tilde{j}} \tilde{\mathcal{G}} \xrightarrow{\tilde{\phi}} \mathcal{G} \rightarrow 1$$

is exact.

7.4.2 Recall from 7.1.1 the motive  $h_{\text{prim}}(X_m^n)(-1)$ . According to 7.1.3, it may be viewed as giving a representation of the Taniyama group  $\mathcal{G}$ , and hence, via  $\tilde{\phi}$ , a representation of  $\tilde{\mathcal{G}}$ .

There is an isomorphism of  $\tilde{\mathcal{G}}$  representations

$$h_{\text{prim}}(X_m^n)(-1) \cong (\mathbb{E}_m^{\otimes n})^{2\pi i \hat{\mathbb{Z}}}$$

The superscript denotes, of course, the subspace of elements invariant under  $\tilde{j}(2\pi i \hat{\mathbb{Z}})$ .

7.4.3 Use the embedding  $\tilde{j}$ , like in 7.3.4, to decompose

$$\mathbb{E} \otimes \mathbb{C} = \bigoplus_{a \in \mathbb{Q}/\mathbb{Z}} \mathbb{E}(a).$$

Note that  $\dim_{\mathbb{C}} \mathbb{E}(a) = 1$  or  $0$  according as  $a \neq 0$  or  $a = 0$  in  $\mathbb{Q}/\mathbb{Z}$ . Recall from 6.5.2 our notations for the Taniyama group.

For all  $\tilde{t} \in \tilde{\mathcal{T}}(\mathbb{C})$ , calling  $s \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  the image  $\phi \circ \tilde{\phi}(\tilde{t})$ , and for all  $0 \neq a \in \mathbb{Q}/\mathbb{Z}$ , one has

$$\tilde{t} \mathbb{E}(a) = \mathbb{E}(\psi(s)^{-1}a).$$

7.4.4 Last but not least Anderson proves that there is a substitute for the Galois action on the eigenspaces  $\mathbb{E}(a)$ , which shows their relation to Gauss sums. In stating it he makes use of a fixed chosen embedding of  $\mathbb{Q}_{\ell}$  into  $\mathbb{C}$ , for every  $\ell$ . In fact, recall (8.2.2) that in the treatment of Jacobi sum Hecke characters we also fixed, at least, an extension of the absolute value  $|\cdot|_{\ell}$  from  $\mathbb{Q}$  to  $\bar{\mathbb{Q}}$ , for every  $\ell$ .

Let  $p$  and  $\ell$  be two distinct rational primes, and write  $\text{Frob } p \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  for a geometric Frobenius element (well determined up to  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}^{\text{ab}})$  and up to the inertia group of the chosen extension of  $|\cdot|_p$ .) Call  $\alpha_{\ell}$  the  $\ell$ -component of the splitting  $\alpha$  of  $\phi_{\mathbb{A}^f}$  - see 6.5.2.

There exists  $\tilde{F}(p, \ell) \in \tilde{\mathcal{T}}(\mathbb{C})$  satisfying

- $\alpha_{\ell}(\text{Frob } p) = \tilde{\phi}(\tilde{F}(p, \ell))$
- for all positive integers  $f$  and all  $0 \neq a \in \mathbb{Q}/\mathbb{Z}$  such that  $(p^f - 1)a = 0$  in  $\mathbb{Q}/\mathbb{Z}$ , one has

$$\text{tr}_{\mathbb{C}}(\tilde{F}(p, \ell)^f | \mathbb{E}(a)) = g_p \left( \sum_{i=1}^f [p^i a] \right),$$

with  $g_p$  as in 8.2.2.

7.4.5 In the setup of Anderson's second theorem, the motives  $M_K(\underline{a})$  of 7.2 are obtained like this: First, it is enough to consider elements  $\underline{a} = \sum_{0 \neq a \in \mathbb{Q}/\mathbb{Z}} n_a [a] \in \mathbb{B}_K^0$  with  $n_a \geq 0$  for all  $a$ . For such  $\underline{a}$ , put

$$\mathbb{E}(\underline{a}) = \bigoplus_a \mathbb{E}(a)^{\otimes n_a}.$$

$M_K(\underline{a})$  then appears as a  $\mathbb{Q}$ -rational representation of  $\tilde{\mathcal{U}}$  such that

$$M_K(\underline{a}) \otimes \mathbb{C} = \bigoplus_{\sigma \in \text{Gal}(K/\mathbb{Q})} \mathbb{E}(\sigma \underline{a}).$$

As  $\underline{a} \in \mathbb{B}_K^0$ , the action of  $2\pi i \hat{\mathbb{Z}}$  on  $M_K(\underline{a})$  is trivial which, by 7.4.1, makes it a representation of  $\mathcal{U}$ , i.e., a motive in  $CM_{\mathbb{Q}}$ .

For all details of Anderson's construction we refer to [A2]. Our discussion of this work will be taken up again in II § 4 where we give an account of his period calculations.

## 7.5 Elliptic curves

Let us mention in passing the geometric reasons that have motivated our choices of the basic characters of the exceptional imaginary quadratic fields: 0, 8.3.2.

It is easily checked that  $J_3$  is the Hecke character of the elliptic curve

$$u^3 + v^3 = 1, \text{ or of } y(1-y) = x^3,$$

the latter one being  $\mathbb{Q}$ -isogenous to the first one. In fact,  $u^3 + v^3 = 1$  is also the strong Weil curve for  $\Gamma_0(27)$ . These coincidences seemed to give some geometric privilege to this choice of  $J_3$ .

On a more historical basis,  $J_4$  was chosen because it is the Hecke character of the famous elliptic curve  $y^2 = x^3 - 4x$

which corresponds to the elliptic integral measuring the arc length of the lemniscate - cf. [HS], §1.

As to  $\mathbb{Q}(\sqrt{-2})$ , any Jacobi-sum Hecke character of infinity type 1 corresponds to a  $\mathbb{Q}$ -curve, in the sense of [Gr 1], §11. But we do not know of any such curve that has attracted individual interest.

## CHAPTER TWO:

### The Periods of Algebraic Hecke Characters

Although we introduce in this chapter the basic notion of our work, the constructions we have to present are quite formal. More precisely, we review what may be called the "arithmetic linear algebra" needed for our purposes:

- In § 1, we define the periods of a motive  $M$  in  $\mathcal{M}_K(E)$  -  $K$  and  $E$  number fields -, and the periods  $c^\pm(M)$ , for  $M$  in  $\mathcal{M}_\mathbb{Q}(E)$ , introduced by Deligne [DP] to formulate his rationality conjecture for critical values of L-functions of motives. The periods of a Hecke Character  $\chi$  of  $K$  with values in  $E$  are simply those of any motive  $M$  attached to  $\chi$  in the sense of I.3.3, or the  $c^\pm(R_{K/\mathbb{Q}}M)$ .
- Deligne's rationality conjecture (and its proof in the case of Hecke characters) is recalled (resp. quoted) in § 2.
- § 3 is devoted to the study of the behaviour of our periods "under twisting". Very similar calculations have also been done by Blasius.
- § 4 continues and closes our discussion of Anderson's motives for Jacobi sum Hecke characters by recalling their periods. They will be needed in chapters III and IV.

#### 1. The periods of a motive

1.0 Let  $K$  and  $E$  be finite extensions of  $\mathbb{Q}$ , and let  $M$  be a motive defined over  $K$  with coefficients in  $E$ , of rank  $r$  over  $E$ . Thus, in the notation of I.3.0,  $M$  is an object of  $\mathcal{M}_K(E)$ . But the linear algebra which we are about to present would work in any sensible theory of motives, not just the particular one using absolute Hodge cycles with which we work here. The constructions of this section are all (essentially) present in various sections of [DP].

### 1.1 Definition of $p(M)$

The component at infinity of the comparison isomorphism I,2.1.1 yields an isomorphism of free  $K \otimes E \otimes \mathbb{C}$  modules of rank  $r$ ,

$$(1.1.1) \quad I : \bigoplus_{\sigma} H_{\sigma}(M) \otimes \mathbb{C} \xrightarrow{\sim} H_{DR}(M) \otimes \mathbb{C} ,$$

where an unmarked  $\otimes$  is always over  $\mathbb{Q}$  and  $\sigma$  runs through all distinct embeddings of  $K$  into  $\mathbb{C}$ ; so the  $K$ -linear structure on the left hand side is obtained from identifying  $K \otimes \mathbb{C}$  with  $\mathbb{C}^{\text{Hom}(K, \mathbb{C})}$ .

The  $K \otimes E \otimes \mathbb{C}$  modules compared in 1.1.1 are extensions of scalars up to  $\mathbb{C}$  of

- the  $E^{\text{Hom}(K, \mathbb{C})}$  module  $\bigoplus_{\sigma} H_{\sigma}(M)$ , on the left;
- the  $K \otimes E$  module  $H_{DR}(M)$ , on the right.

Choose bases  $\gamma_1, \dots, \gamma_r$  - where  $\gamma_i = (\gamma_{i\sigma})_{\sigma}$  with  $H_{\sigma}(M) = \bigoplus_{i=1}^r E \cdot \gamma_{i\sigma}$ , for all  $\sigma$  -, resp.  $\omega_1, \dots, \omega_r$ , of these sub-structures, and define

$$p(M) \in (K \otimes E \otimes \mathbb{C})^* = (\mathbb{C}^*)^{\text{Hom}(K, \mathbb{C})} \times \text{Hom}(E, \mathbb{C})$$

to be the determinant of the matrix representing the isomorphism 1.1.1 relative to these bases.

Changing the bases multiplies  $p(M)$  by an element of  $(E^*)^{\text{Hom}(K, \mathbb{C})}$ , resp. of  $(K \otimes E)^*$ . Thus,  $p(M)$  will be regarded modulo these operations, defining a class

$$p(M) \in (K \otimes E \otimes \mathbb{C})^* / (E^*)^{\text{Hom}(K, \mathbb{C})} \cdot (K \otimes E)^* .$$

1.1.2 Remark. Let  $\det_E M$  be the maximal  $E$ -linear exterior power of  $M$ , i.e., the direct factor of  $M^{\otimes_E r}$  in  $\mathcal{M}_K(E)$  whose  $\sigma$ -realization (for any  $\sigma : K \hookrightarrow \mathbb{C}$ ) is  $\bigwedge_E^r H_{\sigma}(M)$ . Then  $\det_E M$  is a motive of rank 1 over  $E$ , and it is clear from the definition of  $p$  that



$$(1.1.3) \quad p(\det_E M) = p(M) .$$

## 1.2 Components of $p(M)$

Although  $p(M)$  is really a class  $\text{mod}(E^*)^{\text{Hom}(K, \mathbb{C})} (K \otimes E)^*$ , we continue to think of our period, via representatives, as an array of complex numbers, in  $(\mathbb{C}^*)^{\text{Hom}(K, \mathbb{C})} \times \text{Hom}(E, \mathbb{C})$ , and we write

$$(1.2.1) \quad p(M) = (p(M; \sigma, \tau))_{\sigma, \tau} ,$$

with  $p(M; \sigma, \tau) \in \mathbb{C}^*$ , for all  $\sigma \in \text{Hom}(K, \mathbb{C})$ ,  $\tau \in \text{Hom}(E, \mathbb{C})$ .

Similarly, we occasionally write, for  $\sigma : K \hookrightarrow \mathbb{C}$ ,

$$p(M; \sigma) = (p(M; \sigma, \tau))_{\tau} \in (E \otimes \mathbb{C})^* ,$$

and for  $\tau : E \hookrightarrow \mathbb{C}$ ,

$$p(M; ; \tau) = (p(M; \sigma, \tau))_{\sigma} \in (K \otimes \mathbb{C})^* .$$

(If  $K=E$ , the two roles of this field still have to be neatly separated!)

All these components are actually determinants (with respect to certain bases) of comparison isomorphisms derived from 1.1.1. For example, decompose our basis elements  $w_i$  as

$$(1.2.2) \quad w_i = (w_{i\sigma})_{\sigma} \in H_{\text{DR}}(M) \otimes \mathbb{C} = \bigoplus_{\sigma} H_{\text{DR}}(M) \otimes_{K, \sigma} \mathbb{C} .$$

By its very construction, 1.1.1 is the direct sum of isomorphisms

$$(1.2.3) \quad I_{\sigma} : H_{\sigma}(M) \otimes \mathbb{C} \xrightarrow{\sim} H_{\text{DR}} \otimes_{K, \sigma} \mathbb{C} .$$

So,  $p(M; \sigma)$  is the determinant of  $I_{\sigma}$  with respect to the bases  $\{v_{i\sigma}\}_i$  and  $\{w_{i\sigma}\}_i$ . - Note that  $\{w_{i\sigma}\}_i$  is a  $K^{\sigma} \otimes E$  basis of  $H_{\text{DR}}(M) \otimes_{K, \sigma} K^{\sigma}$ .

## 1.3 Field of coefficients

1.3.1 If  $E' \supset E$  is a (finite) extension, and  $M' = M \otimes_E E'$  - see I, 3.0 -, then  $H_{\sigma}(M') = H_{\sigma}(M) \otimes_E E'$ , for all  $\sigma$ , and

$H_{DR}(M') = H_{DR}(M) \otimes_E E'$  . Hence,  $p(M')$  is simply the image of  $p(M)$  under the natural map

$$(K \otimes E \otimes \mathbb{C})^* \hookrightarrow (K \otimes E' \otimes \mathbb{C})^* .$$

Observe that, if we know that  $M'$  in  $\mathcal{M}_K(E')$  is of the form  $M \otimes_E E'$  , for some  $M$  in  $\mathcal{M}_K(E)$  , then  $p(M)$  can be recuperated from  $p(M')$  because, assuming  $E'/E$  to be galois with group  $G$  and letting  $G$  act trivially on  $K$  , one has

$$[(K \otimes E' \otimes \mathbb{C})^* / (E'^*)^{\text{Hom}(K, \mathbb{C})} (K \otimes E')^*]^G = (K \otimes E \otimes \mathbb{C})^* / (E^*)^{\text{Hom}(K, \mathbb{C})} (K \otimes E)^*$$

In fact, use the exact sequence

$$1 \rightarrow E'^* \rightarrow (E'^*)^{\text{Hom}(K, \mathbb{C})} (K \otimes E)^* \rightarrow \frac{(E'^*)^{\text{Hom}(K, \mathbb{C})} (K \otimes E')^*}{(E'^*)^{\text{Hom}(K, \mathbb{C})} \cap (K \otimes E')^*} \rightarrow 1 ,$$

as well as "Hilbert 90" for  $E'^*$  and  $(K \otimes E')^*$  to conclude that

$$H^1(G, (E'^*)^{\text{Hom}(K, \mathbb{C})} (K \otimes E')^*) = 0 .$$

1.3.2 Suppose now that  $M'$  in  $\mathcal{M}_K(E')$  is given, and that  $M = M' |_E$  , in the notation of I,3.0. Using a basis  $\{e'_i\}$  of  $E'$  over  $E$  to obtain bases  $\{\gamma_i\}$  and  $\{\omega_i\}$  for  $M$  from those chosen for  $M'$  , one finds that

$$p(M) = N_{E'/E}(p(M')) .$$

In terms of components, this means that

$$p(M; \tau) = \prod_{\tau' |_E = \tau} p(M'; \tau') ,$$

for  $\tau \in \text{Hom}(E, \mathbb{C})$  and  $\tau' \in \text{Hom}(E', \mathbb{C})$  restricting to  $\tau$  .

#### 1.4 Field of definition

1.4.1 If  $K' \supset K$  is a finite extension and  $M$  is defined over  $K$  , then  $p(M \times_K K')$  is clearly the image in  $(K' \otimes E \otimes \mathbb{C})^* / (E^*)^{\text{Hom}(K', \mathbb{C})} (K' \otimes E)^*$  of  $p(M)$  via the natural map

$$K \otimes E \otimes \mathbb{C} \hookrightarrow K' \otimes E \otimes \mathbb{C} .$$

In practice it is often convenient to extend the base field in order to have eigendifferentials for the action of  $E$  defined over  $K'$  - cf. for example, 1.5.2 below. However, unlike 1.3.1, extending  $K$  does in general throw away information about  $p(M)$  : If  $K'/K$  is galois with group  $G$  acting trivially on  $E$ , then

$$H^1(G, (E^*)^{\text{Hom}(K'\mathbb{C})} \cdot (K' \otimes E)^*) \cong \text{Hom}(G, E^*) .$$

Put another way, different  $K'/K$ -forms of  $M'$  will in general have different periods  $p(M)$  .

1.4.2 Suppose now that  $M'$  in  $\mathcal{M}_{K'}(E)$  is given, and define  $M = R_{K'/K} M'$ , in  $\mathcal{M}_K(E)$  . Then  $H_{\text{DR}}(M)$  is  $H_{\text{DR}}(M')$ , but considered as  $K \otimes E$  module. Therefore, for every  $\sigma : K \hookrightarrow \mathbb{C}$ ,

$$H_{\text{DR}}(M) \otimes_{K, \sigma} \mathbb{C} = \bigoplus_{\sigma' | K = \sigma} H_{\text{DR}}(M') \otimes_{K', \sigma'} \mathbb{C} ,$$

where  $\sigma'$  varies over the embeddings of  $K'$  that restrict to  $\sigma$  on  $K$  . Thus, if  $w_i^! = (w_{i\sigma'}^!,)_{\sigma'}$ , - like in 1.2.2 - make up a basis of  $H_{\text{DR}}(M')$  over  $K' \otimes E$ , then  $\{w_{i\sigma'}^! | i=1, \dots, r'; \sigma' | K = \sigma\}$  is a basis of  $H_{\text{DR}}(M) \otimes_{K, \sigma} \mathbb{C}$  over  $E \otimes K \otimes_{K, \sigma} \mathbb{C}$  . Here,  $r'$  is the rank of  $M'$  over  $E$  . And since

$$H_{\sigma}(M) = \bigoplus_{\sigma' | K = \sigma} H_{\sigma'}(M') ,$$

it follows that

$$(1.4.3) \quad p(M; \sigma) = \left( \prod_{\sigma' | K = \sigma} p(M'; \sigma') \right) D_{\sigma} ,$$

where  $D_{\sigma} \in (E \otimes \mathbb{C})^*$  will now be computed. This factor comes in because the  $\{w_{i\sigma'}^!\}$  are not necessarily a basis of  $H_{\text{DR}}(M) \otimes_{K, \sigma} K^{\sigma}$  .

First, given the basis  $\{w_i^! | i=1, \dots, r'\}$  of  $H_{\text{DR}}(M')$  over  $K' \otimes E$ , and choosing a basis  $\{\alpha_s\}$  of  $K'$  over  $K$ , take

$\{\alpha_s \cdot \omega_i^1\}_{i,s}$  as basis of  $H_{DR}(M)$  over  $K \otimes E$ . For every  $\sigma : K \hookrightarrow \mathbb{C}$ , the factor  $D_\sigma$  is then the determinant of  $\text{id}_{H_{DR}(M)} \otimes_{K,\sigma} \mathbb{C}$ , relative to the bases  $\{\omega_{i,\sigma'}^1\}_{i,\sigma'}|_{K=\sigma}$ , on the left, and  $\{(\alpha_s \omega_i^1)_\sigma\}_{i,s}$ , on the right. Thus,

$$(1.4.4) \quad D_\sigma = \det((\alpha_s^{\sigma'})_{s,\sigma'}|_{K=\sigma})^{-r'} \in \mathbb{C}^* \hookrightarrow (E \otimes \mathbb{C})^* .$$

Second, note that the undeterminacy of the determinants

$$(1.4.5) \quad \delta(K'/K, \sigma) = \det((\alpha_s^{\sigma'})_{s,\sigma'}|_{K=\sigma})$$

is just what is allowed for in the definition of  $p(M)$  :

On the one hand, changing the basis  $\{\alpha_s\}$  of  $K'/K$  multiplies  $\delta(K'/K, \sigma)$  by  $k^\sigma$ , for some element  $k \in K^*$ , so that the array

$$(1.4.6) \quad \delta(K'/K) = (\delta(K'/K, \sigma))_\sigma \in (K \otimes \mathbb{C})^* \hookrightarrow (K \otimes E \otimes \mathbb{C})^*$$

gets multiplied by  $k \in K^* \hookrightarrow (K \otimes E)^*$ . On the other hand, 1.4.5 gives  $\delta(K'/K, \sigma)$  only up to a sign since no ordering was imposed on  $\{\sigma'|_K = \sigma\}$ , and in general, there does not seem to be a reasonable way to fix these signs simultaneously in  $\sigma$ . We are saved by the fact that  $p(M)$  is only well defined up to factors in  $(E^*)^{\text{Hom}(K, \mathbb{C})}$  !

It is plain that  $\delta(K'/K)^2 \in K^* \hookrightarrow (K \otimes E \otimes \mathbb{C})^*$ . Thus, calling  $\epsilon(M') \in \{0, 1\}$  the rank of  $M'$  over  $E$  taken modulo 2 we have shown:

$$(1.4.7) \quad p(R_{K'/K}^{M'}) = N_{K'/K}(p(M')) \delta(K'/K)^{\epsilon(M')} .$$

1.4.8 Remark. Recall the following characterizations of  $\delta(K'/K)$ , in the case that  $K'$  is normal over  $K$ . For  $\sigma' : K' \hookrightarrow \mathbb{C}$  call  $\sigma$  the restriction of  $\sigma'$  to  $K$ . Then  $K^\sigma(\delta(K'/K, \sigma)) \subset K'^{\sigma'}$  is the (at most quadratic) extension of  $K^\sigma$  such that  $\text{Gal}(K'^{\sigma'}/K^\sigma(\delta(K'/K, \sigma)))$  is the kernel of the sign character

$$\begin{aligned} \text{Gal}(K'^{\sigma'}/K^\sigma) &\longrightarrow \{\pm 1\} \\ s &\longmapsto \left( \begin{array}{l} \text{sign of the permutation } t \mapsto st \\ \text{of the set } \text{Gal}(K'^{\sigma'}/K^\sigma) \end{array} \right) \end{aligned}$$

This and the condition that  $\delta(K'/K, \sigma)^2 \in (K^\sigma)^*$  characterize the numbers defined by 1.4.5.

Furthermore, by the classical theory of the discriminant, there is an ideal  $b$  of  $K^\sigma$  such that

$$\delta(K'/K, \sigma)^2 \cdot \sigma_{K^\sigma} = \mathcal{D}_{K', \sigma' / K^\sigma} \cdot b^2,$$

where  $\sigma_{K^\sigma}$  is the ring of integers of  $K^\sigma$  and  $\mathcal{D}_{K', \sigma' / K^\sigma}$  is the relative discriminant ideal of  $K', \sigma'$  over  $K^\sigma$ .

## 1.5 Examples

1.5.0 Let  $n \in \mathbb{Z}$  and consider the  $n$ -th Tate motive  $\mathbb{Q}(n)$  in  $\mathcal{M}_{\mathbb{Q}}$  - cf. I, 2.1 and I, 2.2, Step 3. The comparison isomorphism

$$\mathbb{Q}_{\mathbb{B}}(n) \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{Q}_{\text{DR}}(n) \otimes \mathbb{C}$$

is simply the identity on  $\mathbb{C}$ . Rational bases are, say,  $(2\pi i)^n$  on the left, and 1 on the right. Therefore

$$p(\mathbb{Q}(n)) = (2\pi i)^n \in \mathbb{C}^*.$$

1.5.1 Let  $A$  be an abelian variety with complex multiplication by  $E$  (necessarily a CM field) defined over  $K$ , as in I, 1.1. Assume that the galois closure of  $E$  over  $\mathbb{Q}$  can be embedded into  $K$ . Let  $\sigma \in \text{Hom}(K, \mathbb{C})$  and  $\tau \in \text{Hom}(E, \mathbb{C})$ , and recall the Hodge exponents  $n(\sigma, \tau)$  of  $H_1(A)$  defined in I, 1.7. Then there is a (holomorphic or antiholomorphic) 1-form

$$0 \neq \omega_{\sigma, \tau} \in H_{\text{DR}}^1(A^\sigma / K^\sigma) \cap H^{-n(\sigma, \tau), 1+n(\sigma, \tau)}$$

such that, for all  $e \in E \hookrightarrow \text{End}_{K^\sigma}(A^\sigma)$ , one has

$$e^*(\omega_{\sigma, \tau}) = e^\tau \cdot \omega_{\sigma, \tau}.$$

(Note that, by assumption on  $K$  and  $E$ ,  $e^\tau \in K^\sigma$ .)

Choosing any nonzero rational cycle  $\gamma_\sigma$ , so that  $H_1^\sigma(A) = E \cdot \gamma_\sigma$ , we find that

$$(1.5.2) \quad p(H_1(A); \sigma, \tau) = \int_{\gamma_\sigma} \omega_{\sigma, \tau},$$

up to the usual indeterminacy.

By the definition of an abelian variety,  $A$  admits a polarization, i.e., a correspondence

$$\psi : H^1(A) \times H^1(A) \rightarrow \mathbb{Q}(-1),$$

and its Rosati involution necessarily induces complex conjugation  $c$  on  $E$ . So,  $\psi$  gives an  $E$  semilinear isomorphism

$$(1.5.3) \quad H^1(A) \simeq H_1(A)(-1).$$

(We have used the fact that  $H_1(A) = H^1(A)^\vee$ .) Hence,

$$(1.5.4) \quad \begin{aligned} p(H_1(A); \sigma, \tau) &= 2\pi i \cdot p(H_1(A)(-1); \sigma, \tau) \\ &= 2\pi i \cdot p(H^1(A); \sigma, \tau c) \\ &= \frac{2\pi i}{p(H_1(A); \sigma, \tau c)} \end{aligned}$$

This relation generalizes Legendre's period relation from elliptic curves to abelian varieties - here in the case of complex multiplication. It allows to express all periods of  $A$  in terms of  $2\pi i$  and periods of holomorphic 1-forms on  $A$ .

There is also the following relation, which is valid under quite general circumstances: see 1.6.6 below.

$$(1.5.5) \quad p(H_1(A); c\sigma, \tau) = \overline{p(H_1(A); \sigma, c\tau)}.$$

Since  $E$  is a CM field, the right hand side may also be written as the complex conjugate of  $p(H_1(A); \sigma, \tau c)$ .

## 1.6 Definition of $c^\pm(M)$

1.6.0 It follows from 1.4.7, 1.5.4, and 1.5.5 that, for any abelian variety  $A$  as in 1.5.2, with real periods  $p(H_1(A); \sigma, \tau)$ , the period  $p(R_{K/\mathbb{Q}}H_1(A))$  is essentially  $2\pi i$ . We shall

now recall Deligne's device to separate holomorphic from anti-holomorphic periods over  $\mathbb{Q}$ . We generalize it very slightly by working over a totally real field  $K$ .

1.6.1 So, let  $M$  be, as before, a motive with coefficients in  $E$  defined over  $K$ ; but assume that  $K$  is a totally real number field. Then, for every  $\sigma : K \hookrightarrow \mathbb{R} \subset \mathbb{C}$ , the realization  $H_{\sigma}(M)$  carries the involution

$$F_{\infty} : H_{\sigma}(M) \xrightarrow{\circlearrowright}$$

induced by complex conjugation on  $H_{A^f}(M)$ , or directly by

$$1 \times c : M \times_{K, \sigma} \mathbb{C} \rightarrow M \times_{K, c\sigma} \mathbb{C} = M \times_{K, \sigma} \mathbb{C}.$$

Clearly,  $F_{\infty} \otimes 1_{\mathbb{C}}(H_{\sigma}^{pq}) = H_{\sigma}^{qp}$ .

Write the  $+$  (resp.  $-$ ) eigenspace of  $F_{\infty}$  as

$$H^{\pm}(M) = \{ \gamma \in H_{\sigma}(M) \mid F_{\infty} \gamma = \pm \gamma \}.$$

Both are  $E$  submodules of  $H_{\sigma}(M)$ : being defined over  $K$ , the action of  $E$  on  $M$  commutes with  $F_{\infty}$ . Then

$$\dim_{E \otimes \mathbb{C}}((H_{\sigma}^+(M) \otimes \mathbb{C}) \cap \bigoplus_{p+q} H_{\sigma}^{pq}) = \dim_{E \otimes \mathbb{C}}(\bigoplus_{p>q} H_{\sigma}^{pq}).$$

In order to include the  $H_{\sigma}^{pp}$ 's we impose the

1.6.2 Assumption: There is  $\pi \in \{+, -\}$  such that, for all  $\sigma : K \hookrightarrow \mathbb{C}$ , the involution  $F_{\infty} \otimes 1_{\mathbb{C}}$  acts as multiplication by  $\pi 1$  on all spaces  $H_{\sigma}^{pp}$  that occur in the Hodge decomposition of  $H_{\sigma}(M)$ .

This hypothesis will be made whenever we speak of the periods  $c^{\pm}(M)$ , to be defined presently.

1.6.3 The comparison isomorphism  $I_{\sigma}$  (1.2.3) transforms  $F_{\infty} \otimes c$  on  $H_{\sigma}(M) \otimes \mathbb{C}$  into  $1 \otimes c$  on  $H_{DR}(M) \otimes_{K, \sigma} \mathbb{C}$ . (See [DP], 1.4, for the proof; cf. I, 7.3.1 above.) Thus  $H_{\sigma}^+(M) \otimes \mathbb{R}$  is real with respect to the real structure

$I_{\sigma}^{-1}H_{\text{DR}}(M \times_{K,\sigma} \mathbb{R})$ , and  $H_{\sigma}^{-}(M)$  is purely imaginary.

Recall the comparison of the Hodge filtration on  $H_{\text{DR}}(M)$  with the Hodge decomposition of  $H_{\sigma}(M)$  :

$$F^p H_{\text{DR}}(M) \otimes_{K,\sigma} \mathbb{C} = I_{\sigma} \left( \bigoplus_{p' \geq p} H_{\sigma}^{p',q'} \right) .$$

With  $\pi$  as in 1.6.2, define the  $K \otimes E$  linear subspace  $F^{\pi}H_{\text{DR}}(M)$  of  $H_{\text{DR}}(M)$  by

$$F^{\pi}H_{\text{DR}}(M) \otimes \mathbb{C} = I \left( \bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \bigoplus_{p \geq q} H_{\sigma}^{p,q} \right) ,$$

and put

$$F^{-\pi}H_{\text{DR}}(M) \otimes \mathbb{C} = I \left( \bigoplus_{\sigma} \bigoplus_{p > q} H_{\sigma}^{p,q} \right) .$$

Note that, if  $H_{\sigma}(M)$  is homogeneous of even weight  $w = 2p$ , then  $F^{\pi}H_{\text{DR}}(M) = F^p H_{\text{DR}}(M)$  and  $F^{-\pi}H_{\text{DR}}(M) = F^{p+1}H_{\text{DR}}(M)$ . If  $w$  is odd, then 1.6.2 is vacuous, and  $F^{+}H_{\text{DR}}(M) = F^{-}H_{\text{DR}}(M) = F^{(w+1)/2}H_{\text{DR}}(M)$ .

Then a count of dimensions shows that the isomorphism  $I$  of 1.1.1 induces isomorphisms of free  $K \otimes E \otimes \mathbb{C}$  modules

$$(1.6.4) \quad I^{\pm} : \bigoplus_{\sigma} H_{\sigma}^{\pm}(M) \otimes \mathbb{C} \xrightarrow{\sim} H_{\text{DR}}^{\pm}(M) \otimes \mathbb{C} ,$$

where we have put

$$H_{\text{DR}}^{\pm}(M) = H_{\text{DR}}(M) / F^{\mp}H_{\text{DR}}(M) .$$

1.6.5 In analogy to  $p(M)$  above, we define (under the assumption that  $K$  is totally real, and that 1.6.2 holds):

$$c^{\pm}(M) \in (K \otimes E \otimes \mathbb{C})^* / (E^*)^{\text{Hom}(K,\mathbb{C})} . (K \otimes E)^*$$

to be the determinant of  $I^{\pm}$ , computed with respect to  $E$  bases of the  $H_{\sigma}^{\pm}(M)$ 's on the left, and a  $K \otimes E$  basis of  $H_{\text{DR}}^{\pm}(M)$ , on the right.



As with  $p(M)$ , write the coordinates:

$$c^\pm(M) = (c^\pm(M; \sigma, \tau))_{\sigma, \tau} \in (\mathbb{C}^*)^{\text{Hom}(K, \mathbb{C})} \times \text{Hom}(E, \mathbb{C}) .$$

Note that, if  $K = \mathbb{Q}$ , one has simply

$$c^\pm(M) \in (E \otimes \mathbb{C})^* / E^* .$$

1.6.6 Remark. In the general situation of 1.1, with no assumption on  $K$  or  $E$ , we have

$$F_\infty : H_\sigma(M) \rightarrow H_{\mathbb{C}\sigma}(M) ,$$

for all  $\sigma : K \hookrightarrow \mathbb{C}$ . So, one can choose  $E$  bases  $\{\gamma_{i\sigma}\}$  of  $H_\sigma(M)$  such that  $F_\infty(\gamma_{i\sigma}) = \gamma_{i\sigma}(c_\sigma)$ , for all  $\sigma$ . Since  $1 \otimes c$  coincides with  $F_\infty$  on  $H_{\text{DR}}(M)$  - via  $I$  - one sees that

$$p(M; c_\sigma, c\tau) = \overline{p(M; \sigma, \tau)} ,$$

up to the usual indeterminacy .

## 1.7 c and p

Consider the following

1.7.0 Situation. Let  $K$  and  $E$  be totally imaginary number fields, and  $M$  a motive with coefficients in  $E$  defined over  $K$ , of rank  $r$  over  $E$ . Let  $K_0$  be a totally real subfield of  $K$  (e.g.,  $K_0 = \mathbb{Q}$ ), and put  $M_0 = R_{K/K_0}(M)$ . Assume (for simplicity) that  $M$  is homogeneous of weight  $w$ . Suppose that, for all  $\sigma \in \text{Hom}(K, \mathbb{C})$  and  $\tau \in \text{Hom}(E, \mathbb{C})$ , the subspace

$$H_\sigma(M) \otimes_{E, \tau} \mathbb{C} \subset H_\sigma(M) \otimes \mathbb{C}$$

is of pure Hodge type  $(n(\sigma, \tau), w - n(\sigma, \tau))$ , for some  $n(\sigma, \tau) \in \mathbb{Z}$ . (If  $r = 1$ , this is automatic and has been used before; for instance, in the proof of I, 5.1.) Note that, for all  $s \in \text{Aut } \mathbb{C}$ , one has  $n(s\sigma, s\tau) = n(\sigma, \tau)$ . Finally, assume that  $H_\sigma^{\frac{w}{2}, \frac{w}{2}} = 0$ , for all  $\sigma : K \hookrightarrow \mathbb{C}$ .

Under these circumstances we shall now compute the periods  $c^\pm(M_0)$  in terms of  $p(M)$ , using basically the same method as

in 1.4.2 above. (Cf. [DP], 8.16.)

1.7.1 Let  $\sigma_0 \in \text{Hom}(K_0, \mathbb{C})$ , and start by choosing an  $E$  basis of  $H_{\sigma_0}^+(M_0) = [K_0 \oplus H_{\sigma_0}(M)]^+ : \sigma|_{K_0} = \sigma_0$

set  $\{\sigma : K \hookrightarrow \mathbb{C} \mid \sigma|_{K_0} = \sigma_0\}$  modulo the action of complex conjugation  $c$ . For each  $\bar{\sigma} = \{\sigma, c\sigma\} \in S(\sigma_0)$ , choose a basis  $\{\gamma_{i\sigma}\}_{i=1, \dots, r}$  of  $H_{\sigma}(M)$  over  $E$ , and take

$$\{\gamma_{i\sigma} + F_{\infty}(\gamma_{i\sigma}) \mid i=1, \dots, r; \{\sigma, c\sigma\} \in S(\sigma_0)\}$$

as  $E$  basis of  $H_{\sigma_0}^+(M_0)$ . - Note that  $F_{\infty}(\gamma_{i\sigma}) \in H_{c\sigma}(M)$  - see remark 1.6.6 -, and that our construction does not depend on the choice of the representatives  $\sigma \in \bar{\sigma}$ .

1.7.2 There is a unique direct factor  $(K \otimes E)^+$  of  $K \otimes E$  such that

$$(K \otimes E)^+ \otimes \mathbb{C} = \mathbb{C}^{\{(\sigma, \tau) \mid n(\sigma, \tau) < \frac{w}{2}\}} \subset K \otimes E \otimes \mathbb{C},$$

with  $n(\sigma, \tau)$  as in 1.7.0; and the quotient  $H_{\text{DR}}^+(M_0)$  of  $H_{\text{DR}}(M_0)$  is isomorphic (as  $K_0 \otimes E$  module) to the direct factor

$$H_{\text{DR}}(M) \otimes_{K \otimes E} (K \otimes E)^+ \text{ of } H_{\text{DR}}(M).$$

Starting from a basis  $\{\omega_i\}$  of  $H_{\text{DR}}(M)$  over  $K \otimes E$ , with components

$$\omega_i = (\omega_{i, \sigma, \tau})_{\sigma, \tau} \in \bigoplus_{\sigma, \tau} H_{\text{DR}}(M) \otimes_{K \otimes E, \sigma \otimes \tau} \mathbb{C} = H_{\text{DR}}(M) \otimes \mathbb{C},$$

consider the  $E \otimes K_0 \otimes_{K_0, \sigma_0} \mathbb{C}$  basis of  $H_{\text{DR}}^+(M_0) \otimes_{K_0, \sigma_0} \mathbb{C}$

$$\{\omega_{i, \bar{\sigma}} \mid i=1, \dots, r; \bar{\sigma} \in S(\sigma_0)\},$$

where  $\omega_{i, \bar{\sigma}, \tau} = \omega_{i, \sigma, \tau}$ , if  $\sigma \in \bar{\sigma}$  and  $n(\sigma, \tau) < \frac{w}{2}$ .

Then we find

$$(1.7.3) \quad c^+(M_0; \sigma_0, \tau) = \left( \prod_{\sigma|_{K_0} = \sigma_0} p(M; \sigma, \tau) \right) \cdot D^+(\sigma_0, \tau) \quad ,$$

$$n(\sigma, \tau) < \frac{w}{2}$$

where  $D^+ \in (K_0 \otimes E \otimes \mathbb{C})^*$  - well determined up to

$(E^*)^{\text{Hom}(K_0, \mathbb{C})} \cdot (K_0 \otimes E)^*$  - is the determinant of the identity on  $H_{\text{DR}}^+(M_0) \otimes \mathbb{C}$ , computed with respect to the basis  $\{w_{i, \bar{\sigma}}\}_{i, \sigma}$ , on the left, and some  $K_0 \otimes E$  basis of  $H_{\text{DR}}^+(M_0)$ , on the right. To compute  $D^+$  note first that  $(K \otimes E)^+$  is, in fact, a free  $K_0 \otimes E$  module, because  $K_0$  is totally real. Pick a basis of it:

$$\{e_j \mid j=1, \dots, \frac{[K : K_0]}{2}\} .$$

If  $\{w_i \mid i=1, \dots, r\}$  is the  $K \otimes E$  basis of  $H_{\text{DR}}(M)$  used before, with  $w_i$  projecting to  $w_i^+$  in  $H_{\text{DR}}^+(M_0)$ , then  $\{e_j w_i^+\}_{j, i}$  is a  $K_0 \otimes E$  basis of  $H_{\text{DR}}^+(M_0)$ . Thus, writing  $\delta^+ \in (K_0 \otimes E \otimes \mathbb{C})^*$  - well determined, as usual, up to

$(E^*)^{\text{Hom}(K_0, \mathbb{C})} \cdot (K_0 \otimes \mathbb{C})^*$  - the array with components

$$(1.7.4) \quad \delta^+(\sigma_0, \tau) = \det \left( e_j^\sigma \mid \begin{array}{l} j=1, \dots, [K : K_0]/2; \\ \sigma|_{K_0} = \sigma_0, \quad n(\sigma, \tau) < \frac{w}{2} \end{array} \right) ,$$

we see that

$$(1.7.5) \quad D^+ = (\delta^+)^{\varepsilon(M)} \quad ,$$

where  $\varepsilon(M) = r \pmod{2}$ .

1.7.6 Like in 1.4.8, let us also give an abstract characterization of  $\delta^+$  - cf. [HS], 4.5.

Start with one fixed  $\tau_0 \in \text{Hom}(E, \mathbb{C})$ . For each  $\sigma_0 : K_0 \hookrightarrow \mathbb{C}$  independently, choose  $\delta^+(\sigma_0, \tau_0) \in \mathbb{C}^*$  such that the group

$$\left\{ s \in \text{Gal}(\overline{\mathbb{Q}}/K^{\sigma_0}) \mid \begin{array}{l} \text{for all } \sigma : K \hookrightarrow \mathbb{C} \text{ with } \sigma|_{K_0} = \sigma_0 : \\ n(s\sigma, \tau_0) < \frac{w}{2} \Leftrightarrow n(\sigma, \tau_0) < \frac{w}{2} \end{array} \right\}$$

acts on  $\delta^+(\sigma_0, \tau_0)$  via the sign character

$$s \longmapsto \left( \begin{array}{l} \text{sign of the permutation of the set} \\ \{ \sigma|_{K_0} = \sigma_0 \mid n(\sigma, \tau_0) < \frac{w}{2} \} \text{ induced by } s \end{array} \right) .$$

It remains to define  $\delta^+(\sigma_0, \rho\tau_0)$ , for all  $\sigma_0 \in \text{Hom}(K_0, \mathbb{C})$  and  $\rho \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . We put

$$\delta^+(\sigma_0, \rho\tau_0) = \epsilon(\rho; \sigma_0) \cdot \delta^+(\rho^{-1}\sigma_0, \tau_0)^\rho ,$$

where the signs  $\epsilon(\rho; \sigma_0)$  are defined as follows. For each  $\sigma_0 : K_0 \hookrightarrow \mathbb{C}$ , choose an ordering of the set of infinite places of  $K$  lying above the place of  $K_0$  induced by  $\sigma_0$ . Note that, for any  $\tau$ , the set  $\{ \sigma|_{K_0} = \sigma_0 \mid n(\sigma, \tau) < \frac{w}{2} \}$  is in bijection with the places of  $K$  above  $\sigma_0$ . Then  $\epsilon(\rho; \sigma_0)$  is the sign of that permutation of the places above  $\sigma_0$  which transforms the chosen ordering into the image under  $\rho$  of the chosen ordering on the set of places above  $\rho^{-1}\sigma_0$ . - The choices made are compensated by the indeterminacy of  $\delta^+$  up to  $(E^*)^{\text{Hom}(K_0, \mathbb{C})}$ .

1.7.7 Let us now derive the analogue of 1.7.3 for  $c^-$ . In the notation of 1.7.1,

$$\{ \gamma_{i\sigma} - F_\infty(\gamma_{i\sigma}) \mid i=1, \dots, r; \{ \sigma, c\sigma \} \in S(\sigma_0) \}$$

is a basis of  $H_{\sigma_0}^-(M_0)$  over  $E$ . Note that here we have to take a particular choice of representatives  $\{ \sigma \}$  of  $S(\sigma_0)$ . One way to do this - which we adopt - is again to fix one embedding  $\tau_0 : E \hookrightarrow \mathbb{C}$ , and to use the set

$$\{ \sigma \mid \sigma|_{K_0} = \sigma_0 \text{ and } n(\sigma, \tau_0) < \frac{w}{2} \} .$$

Define accordingly,

$$\tilde{\omega}_{i, \bar{\sigma}} \in H_{\text{DR}}^-(M_0) \otimes_{K_0, \sigma_0} \mathbb{C} = H_{\text{DR}}^+(M_0) \otimes_{K_0, \sigma_0} \mathbb{C}$$

by their components:

$$\tilde{\omega}_{i, \bar{\sigma}, \tau} = \begin{cases} \omega_{i, \sigma, \tau} & , \text{ if } \sigma \in \bar{\sigma} \text{ and } n(\sigma, \tau) < \frac{W}{2} > n(\sigma, \tau_0) \\ -\omega_{i, \sigma, \tau} & , \text{ if } \sigma \in \bar{\sigma} \text{ and } n(\sigma, \tau) < \frac{W}{2} < n(\sigma, \tau_0) . \end{cases}$$

This gives

$$(1.7.8) \quad c^-(M_0; \sigma_0, \tau) = \left( \prod_{\substack{\sigma |_{K_0} = \sigma_0 \\ n(\sigma, \tau) < \frac{W}{2}}} p(M; \sigma, \tau) \right) \cdot D^-(\sigma_0, \tau) ,$$

where  $D^-$  is defined like  $D^+$  in 1.7.3, with  $\{\omega_{i, \bar{\sigma}}\}$  replaced by  $\{\tilde{\omega}_{i, \bar{\sigma}}\}$ . In particular, like in 1.7.5,

$$(1.7.9) \quad D^- = (\delta^-)^{\epsilon(M)} ,$$

where the quotient  $\delta^+/\delta^-$  is given - up to the usual indeterminacy - by the rule

$$(1.7.10) \quad \frac{\delta^+(\sigma_0, \tau)}{\delta^-(\sigma_0, \tau)} = (-1)^{\#\{\sigma |_{K_0} = \sigma_0 | n(\sigma, \tau) < \frac{W}{2} < n(\sigma, \tau_0)\}} ,$$

for all  $\sigma_0 \in \text{Hom}(K_0, \mathbb{C})$ ,  $\tau \in \text{Hom}(E, \mathbb{C})$ , and  $\tau_0$  as fixed above.

1.7.11 Corrigendum. Formula 1.7.10 emends our foolish negligence at the end of the proof of [GS], 9.3, and again in [GS'], 3.3. There we asserted, for  $K_0 = \mathbb{Q}$ ,  $K$  quadratic, and  $r=1$ , that  $c^+ = c^-$ . In "proving" Deligne's conjecture in that case, we compensated this mistake by overlooking the fact that the complex conjugate of  $2\pi i$  is  $-2\pi i$ , in the application of [DP], 5.18. The same false replacement of  $c^-$  for  $c^+$  still slipped into [HS], formula 11 - cf. instead, 3.1. below -, where it was finally caught by Blasius.

1.7.12 Lemma

(i)  $\delta^\pm$  depend only on  $K_0, K, E$ , and the family of "CM-  
types" of  $K$  ( $\{\sigma \in \text{Hom}(K, \mathbb{C}) | n(\sigma, \tau) < \frac{W}{2}\})_\tau : E \hookrightarrow \mathbb{C}$  .

$$(ii) \quad (\delta^\pm)^2 \in (E^*)^{\text{Hom}(K_0, \mathbb{C})} (K_0 \otimes E)^* .$$

(iii) Let  $K'/K$  be an extension of degree  $n$ , and denote by  $\delta'^\pm \in (K_0 \otimes E \otimes \mathbb{C})^*$  the  $\delta^+$ -factor relative to  $K'/K_0$ , and the exponents  $n(\sigma', \tau) = n(\sigma|_K, \tau)$ . Then, for all  $\sigma_0 \in \text{Hom}(K_0, \mathbb{C})$  and  $\tau \in \text{Hom}(E, \mathbb{C})$ :

$$\frac{\delta'^\pm(\sigma_0, \tau)}{\delta^\pm(\sigma_0, \tau)^n} = \prod_{\substack{\sigma|_{K_0} = \sigma_0 \\ n(\sigma, \tau) < \frac{w}{2}}} \delta(K'/K, \sigma) ,$$

where  $\delta(K'/K)$  has been defined in 1.4.5/6.

(iv) If  $K_0 = \mathbb{Q}$ , and  $K$  a CM field with maximal real subfield  $F$ , then - up to a factor in  $(E \otimes 1)^*$  - ,

$$\delta^+ = \sqrt{\text{disc}(F)} ,$$

where we take one and the same root of the absolute discriminant of  $F$ , for all embeddings of  $E$  into  $\mathbb{C}$ .

Parts (i) and (ii) are plain. In part (iii) note that the various indeterminacies actually do work out: If, for  $k \in K^*$ ,  $\delta(K'/K, \sigma)$  is replaced by  $k^\sigma \delta(K'/K, \sigma)$ , then we obtain in the formula the factor

$$\prod_{\substack{\sigma|_{K_0} = \sigma_0 \\ n(\sigma, \tau) < \frac{w}{2}}} k^\sigma \in (K_0^{\sigma_0} \cdot E^\tau)^* \subset \mathbb{C}^* .$$

The proof of (iii) is straightforward. (iv) is also easy to prove once one observes that  $\text{Hom}(F, \mathbb{C})$  naturally identifies itself with  $S(\text{id}_{\mathbb{Q}})$ . - Cf. [DP], 8.17; and 1.4.8 above.

### 1.8 Application to Hecke characters

We now resume the discussion of the "unique" motive  $M(\chi)$  in  $\mathcal{H}_K^{\text{av}}(E)$  which we have attached in chapter I (I,4, I,5; also

I, 6.5.6) to a given algebraic Hecke character  $\chi$  of  $K$  with values in  $E$ . Write its periods as

$$p(\chi) = p(M(\chi)) \in (K \otimes E \otimes \mathbb{C})^* / (E^*)^{\text{Hom}(K, \mathbb{C})} (K \otimes E)^*,$$

with components  $p(\chi; \sigma, \tau)$ . And if  $R_{K/\mathbb{Q}} M(\chi)$  satisfies hypothesis 1.6.2, it makes sense to write

$$c^\pm(\chi) = c^\pm(R_{K/\mathbb{Q}} M(\chi)) \in (E \otimes \mathbb{C})^* / E^*,$$

with components  $c^\pm(\chi; \tau)$ .

Recall that, by definition (I, 3.3),  $M(\chi)$  is of rank 1 over  $E$ , and that its Hodge decomposition is given by the invariants  $n(\sigma, \tau)$  attached to  $\chi$  in  $\mathbb{0}, 4$ , the weight of the Hodge structure being the weight  $w$  of  $\chi$ : see I, 6.1.5. It follows that, if none of the  $n(\sigma, \tau)$ 's equals  $\frac{w}{2}$ , then

$$(1.8.1) \quad c^\pm(\chi) = \delta^\pm(\chi) \left( \prod_{n(\sigma, \tau) < \frac{w}{2}} p(\chi; \sigma, \tau) \right)_\tau.$$

In fact, this is just a reformulation of 1.7.3, resp. 1.7.8, with  $\delta^\pm(\chi) \in (E \otimes \mathbb{C})^* / E^*$  equal to the factor given by 1.7.4, resp. 1.7.10, relative to the data  $\mathbb{Q}, K, E$ , and the  $n(\sigma, \tau)$ 's of  $\chi$  - see 1.7.12 (i).

1.8.2 In a nutshell, the observation which is basic to our work is that, by theorem I, 5.1, all these periods do not depend on the particular geometric construction of a motive  $M(\chi)$  in  $\mathcal{M}_K^{\text{av}}(E)$  attached to  $\chi$ .

As a first illustration of this principle we shall now give a list of six basic properties of the periods  $p(\chi)$  all of which follow from two different ways of writing the  $M(\chi)$  in question. These isomorphisms of motives are all easily checked on the  $\lambda$ -adic representations, i.e. precisely, by verifying that the motives on both sides of what we shall write as an equality are motives for one and the same character. In each case it is indicated, how the period relation follows from the corresponding isomorphism of motives. - The reader will notice

the analogy of our list to the table in [DB], § 2.

1.8.3 Let  $\chi$  and  $\chi'$  be two algebraic Hecke characters of  $K$  with values in  $E$ . Then

$$M(\chi \cdot \chi') = M(\chi) \otimes_E M(\chi') ; \quad p(\chi \cdot \chi') = p(\chi) \cdot p(\chi') .$$

Since the motives are of rank 1 over  $E$ , it is clear that the isomorphism on the left implies the simple period relation on the right.

1.8.4 Let  $E$  be  $\mathbb{Q}$ , and denote by  $\mathbb{N}$  the absolute norm of ideals of  $K$ . Then, for all  $n \in \mathbb{Z}$ ,

$$M(\mathbb{N}^n) = \mathbb{Q}(-n) \times_{\mathbb{Q}} K ; \quad p(\mathbb{N}^n) = (2\pi i)^{-n} .$$

This follows from 1.5.0 and 1.4.1. Note that, if  $K$  is also  $\mathbb{Q}$ , then  $(2\pi i)^{-n} = c^{(-)^n}(\mathbb{N}^n)$  - see I, 2.1 for the action of  $F_{\infty}$  on the Tate motive.

1.8.5 Remark. If  $\chi$  is the Hecke character of  $H_1(A)$ , for  $A$  an abelian variety with complex multiplication, like in I, 1, then  $\chi \cdot \bar{\chi} = \mathbb{N}^{-1}$ . Thus 1.8.3 and 1.8.4 reprove "Legendre's period relation", 1.5.4.

1.8.6 Let  $E'/E$  be a finite extension,  $\chi$  an algebraic Hecke character of  $K$  with values in  $E$ . Then, with  $i$  the inclusion  $E^* \hookrightarrow E'^*$  (also viewed as homomorphism of algebraic groups),

$$M(i \circ \chi) = M(\chi) \otimes_E E' ; \quad p(i \circ \chi) = i(p(\chi)) .$$

This is just an application of 1.3.1.

1.8.7 Let again  $E'/E$  be a finite extension, but let a Hecke character  $\chi'$  of  $K$  with values in  $E'$  be given. Then, denoting  $N_{E'/E}$  the norm homomorphism  $E'^* \rightarrow E^*$ ,

$$M(N_{E'/E} \circ \chi') = \det_E(M(\chi) | E) ; \quad p(N_{E'/E} \circ \chi') = N_{E'/E}(p(\chi)) .$$



Recalling that  $\det_E$  was defined in 1.1.2 (and the restriction of coefficients  $|_E$  in I, 3.0), the period relation is implied by the isomorphism, because of 1.1.3 and 1.3.2.

1.8.8 Let  $K'/K$  be a finite extension, and denote by  $N_{K'/K}$  the relative norm, on ideals of  $K'$ . Then, for  $\chi$  an algebraic Hecke character of  $K$ , and  $j : K^* \hookrightarrow K'^*$  the inclusion,

$$M(\chi \circ N_{K'/K}) = M(\chi) \times_K K' ; \quad p(\chi \circ N_{K'/K}) = j(p(\chi)) ,$$

as follows from 1.4.1.

1.8.9 Given again a finite extension  $K'/K$ , but a Hecke character  $\chi'$  of  $K'$  with values in  $E$ , write now  $j$  for the inclusion of ideals of  $K$  into ideals of  $K'$ , and  $N_{K'/K}$  for the norm  $K'^* \rightarrow K^*$ . Let  $\epsilon_{K'/K}$  be the finite order character of  $K$  which, via Artin reciprocity, corresponds to the character

$$\epsilon_{K'/K} : \text{Gal}(\bar{K}/K) \rightarrow \{\pm 1\}$$

$$s \longmapsto \left( \begin{array}{l} \text{sign of the permutation of the set} \\ G(\bar{K}/K)/G(\bar{K}/K') \text{ given by } s \end{array} \right)$$

Then the following isomorphism of motives is an easy generalization of Prop. 3.2 of [Mar], p. 35 f.

$$M(\epsilon_{K'/K} \cdot (\chi' \circ j)) = \det_E(R_{K'/K} M(\chi')) ; \quad p(\chi' \circ j) = N_{K'/K}(p(\chi')) .$$

The period relation follows from 1.1.3, 1.4.2, 1.8.3, and the fact that  $p(\epsilon_{K'/K}) = \delta(K'/K)$  which will be proved in 3.2 below.

1.8.10 It is usually clear how these formulas for the periods  $p$  can be used to derive relations between periods  $c^\pm$ , using 1.7, resp. 1.8.1. However, care must be taken not to apply 1.8.3 inside 1.8.1, unless both factors in question satisfy the conditions of 1.7.0 with the same system of "CM-types"  $\{\sigma \mid n(\sigma, \tau) < \frac{w}{2}\}_{\tau} \in \text{Hom}(E, \mathbb{C})$ .

See section 3 for the most common illustrations of this problem.

A relation deduced, with no such difficulty, by combining 1.7.12 (i), 1.7.12 (iii), 1.8.3 and 1.8.8, is the formula which plays a crucial role in [HS]: If  $K, K'$  and  $\chi$  are like in 1.8.8, and  $n = [K':K]$ , then

$$(1.8.11) \quad \frac{c^+(\chi \circ N_{K'/K})}{c^+(\chi^n)} = \left( \prod_{n(\sigma, \tau) < \frac{w}{2}} \delta(K'/K, \sigma) \right)_\tau \cdot \delta^+(\chi)^{n-1} .$$

## 2. Periods and L-values

As usual, let  $K$  and  $E$  be number fields, and  $\chi$  a Hecke character of  $K$  with values in  $E$ .

2.0 Let  $\tau \in \text{Hom}(E, \mathbb{C})$ . An integer  $s$  is called critical for (the L-function of)  $\chi^\tau$ , if the  $\Gamma$ -factors on both sides of the functional equation of  $L(\chi^\tau, \cdot)$  do not have

a pole at  $s$ . It is an easy exercise to work out what this means, using the formulas in  $\mathfrak{O}$ , § 6 - cf. [DP], 1.3, 3, 8.15 - :

2.0.1 If  $\{\sigma, \sigma\}$ , for  $\sigma \in \text{Hom}(K, \mathbb{C})$ , induces a complex place of  $K$ , then  $n(\sigma, \tau)$  has to be different from  $\frac{w}{2}$ , for a critical  $s$  to exist. Thus critical integers (for Hecke characters) can only occur, if  $K$  is totally real or totally imaginary, and in the latter case, one has a disjoint union

$$\text{Hom}(K, \mathbb{C}) \times \text{Hom}(E, \mathbb{C}) = \{(\sigma, \tau) \mid n(\sigma, \tau) < \frac{w}{2}\} \cup \{(\sigma, \tau) \mid n(\sigma, \tau) > \frac{w}{2}\} .$$

2.0.2 If  $K$  is totally real, the character  $\mu \mathbb{N}^n$  - with  $\mu$  of finite order and  $n \in \mathbb{Z}$  - admits critical integers  $s$  if and only if, for all infinite places  $v$  of  $K$ , the constants  $\epsilon_v$  defined for  $\mu_v^\tau$  in  $\mathfrak{O}$ , § 6 are equal to, say,  $\epsilon \in \{0, 1\}$ . Then the set of all critical  $s$  for  $\mu \mathbb{N}^n$  is

$$\{s > n \mid s \equiv n + \epsilon \pmod{2}\} \cup \{s \leq n \mid s \not\equiv n + \epsilon \pmod{2}\} .$$

2.0.3 If  $K$  is totally imaginary, critical  $s$  exist for  $\chi^\tau$ , if and only if, for all  $\sigma$ ,

$$n(\sigma, \tau) < \frac{w}{2} \quad \text{or} \quad n(\sigma, \tau) > \frac{w}{2} .$$

If this is so, then the set of critical integers for  $\chi$  - independently of  $\tau$  - is the interval

$$\left( \sup_{n(\sigma, \tau) < \frac{W}{2}} n(\sigma, \tau), \quad \inf_{n(\sigma, \tau) > \frac{W}{2}} n(\sigma, \tau) \right).$$

In all cases, we can therefore say "critical for  $\chi$ ", independently of  $\tau$ . - Recall the notation  $L^*(\chi, s)$  from 0, §6; and  $c^+(\chi)$  from 1.8.

2.1 Theorem [Siegel, Blasius, Harder]. If  $s$  is critical for  $\chi$ , then

$$\frac{L^*(\chi, s)}{c^+(\chi \cdot \mathbb{N}^{-s})} \in E \otimes 1 \hookrightarrow E \otimes \mathbb{C}.$$

In other words, Deligne's conjecture [DP], 2.8 is true "for all algebraic Hecke characters".

We have discussed the history as well as the important steps of the proof of this deep theorem in [HS], see especially §5. Let us just recall here that Siegel's contribution concerns the case when  $K$  is totally real; Blasius [B1] proves it for  $K$  a CM field, refining Shimura's groundbreaking proof of this case up to factors in  $\overline{\mathbb{Q}}^*$  - see [ShiA], [DP], 8.21. Finally, Harder's methods produce relations between L-values which allow one to carry over Blasius' result from CM fields to arbitrary totally imaginary fields by virtue of the analogous relations for the  $c^+$ -periods (cf. 3.2.7 below). The considerable technicalities of his argument have not been written up yet in the general case. Cf. However [Ha].

2.2 Remark. One of the key constructions in [B1] is the construction, for  $M = R_{K/\mathbb{Q}} M(\chi)$ , of a motive that plays a role analogous to  $\det_E M$  in 1.1.3, with  $p$  replaced by  $c^+$ . One can use this construction to derive all the formulae relative to  $c^\pm$  which we have presented.

### 3. Twisting

3.0 We continue to consider an algebraic Hecke character  $\chi$  of the number field  $K$  with values in the number field  $E$ . Assume that  $K$  is totally imaginary, and that  $\chi$  admits some critical integer  $s$  - see 2.0.1.

3.1 Write  $N$  for the absolute norm (of ideals of  $K$ ). Then

$$(3.1.1) \quad c^{\pm}(\chi N^{-1}) = (2\pi i)^{\frac{[K:\mathbb{Q}]}{2}} c^{\mp}(\chi).$$

[This rectifies formula (11) of [HS] - cf. 1.7.11.]

3.1.1 is a special case of [DP], 5.1.8, which follows from the characterizing properties of  $\mathbb{Q}(1)$  - see I, 2.1. In trying to derive 3.1.1 from 1.8.1 (for  $\chi N^{-1}$ ), 1.8.3, and 1.8.4, the subtlety is that  $c^{\pm}$  becomes  $c^{\mp}$  because, for  $\gamma_{\sigma} \in H_{\sigma}(M(\chi))$  and  $\alpha \in \mathbb{Q}_{\mathbb{E}}(1) \otimes E$ ,

$$\gamma_{\sigma} \otimes_{\mathbb{E}} \alpha + F_{\infty}(\gamma_{\sigma} \otimes_{\mathbb{E}} \alpha) = (\gamma_{\sigma} - F_{\infty}(\gamma_{\sigma})) \otimes_{\mathbb{E}} \alpha;$$

cf. 1.7.7.

3.2 Next, let  $\mu$  be a Hecke character of  $K$  of finite order, with values in  $E$ . (In view of 1.3.1, there is no loss of generality, for our period calculations, in assuming that  $\mu$  and  $\chi$  both take values in the same field  $E$ . This will be assumed in those of the following formulae which involve both  $\mu$  and  $\chi$ .) Then 1.8.3 gives:

$$(3.2.0) \quad p(\mu\chi) = p(\mu)p(\chi).$$

We shall compute  $p(\mu)$  using the explicit description of the Artin motive  $M(\mu)$  given in I, 2.4.1.

3.2.1 Let  $F$  be the finite abelian extension of  $K$  corresponding to  $\mu$  by class field theory. Thus, reading  $\mu$  on  $\Gamma = \text{Gal}(\bar{K}/K)$  - for some fixed algebraic closure  $\bar{K}$  of  $K$  -

via geometric Frobenii,  $F$  is the fixed field of  $\ker(\mu)$ . There are two actions of  $\Gamma$  on  $F \otimes E$  inducing the natural action of  $\Gamma$  on  $F \subset \bar{K}$ :

- the trivial action of  $\Gamma$  on  $E$  :  $(f \otimes e)^\gamma = f^\gamma \otimes e$ ;
- the action  $(f \otimes e)^{[\gamma]} = f^\gamma \otimes \mu(\gamma)e$ .

Clearly  $y^{[\gamma]} = (1 \otimes \mu(\gamma)) \cdot y^\gamma$ , for all  $y \in F \otimes E$  and  $\gamma \in \Gamma$ . - We shall usually work with the first action. In particular, this is the action for which we have the

3.2.2 Lemma  $H^1(\Gamma/\ker(\mu), (F \otimes E)^*) = 0$ .

Proof [suggested by M. Lorenz]: Let  $\Delta = \Gamma/\ker(\mu)$ , and write  $E = \mathbb{Q}[X]/(P)$ , with  $P \in \mathbb{Q}[X]$  irreducible. Then  $F \otimes E = F[X]/(P)$ . Factor  $P = P_1 \dots P_s$  in  $F[X]$ . Acting on  $F[X]$  through the coefficients  $\Delta$  permutes the ideals  $(P_1), \dots, (P_s)$ , since it stabilizes  $(P)$ . So, writing the orbits one by one, we have

$$(P) = \prod_{i=1}^t \prod_{j=1}^{s_i} (P_{ij}),$$

and for each  $i$ ,  $\Delta$  permutes the  $(P_{ij})$  transitively.

Then

$$(F \otimes E)^* = \prod_{i=1}^t A_i^*,$$

as  $\Delta$ -module, where

$$A_i = \prod_{j=1}^{s_i} F[X]/(P_{ij}).$$

Since  $H^1(\Delta, \prod_i A_i^*) = \bigoplus_i H^1(\Delta, A_i^*)$  we are reduced to the case

of a transitive  $\Delta$ -action. In other words, if  $\Delta_{i1}$  is the subgroup of  $\Delta$  stabilizing  $F[X]/(P_{i1})$ , then

$$A_i^* = \text{Ind}_{\Delta_{i1}}^{\Delta} ((F[X]/(P_{i1}))^*).$$

By Shapiro's lemma and Hilbert 90,

$$H^1(\Delta, A_i^*) = H^1(\Delta_{i1}, (F[X]/P_{i1}))^* = 0 .$$

q.e.d.

Applying the lemma to  $[\mu] \in H^1(\Gamma/\ker(\mu), (F \otimes E)^*)$ , we find a unit  $\xi \in (F \otimes E)^*$  such that, for all  $\gamma \in \Gamma$ ,

$$(3.2.3) \quad \xi^\gamma = (1 \otimes \mu(\gamma)) \xi .$$

(In fact, the lemma implies that any  $0 \neq \xi \in F \otimes E$  satisfying 3.2.3 lies in  $(F \otimes E)^*$ :)  $\xi$  is well-determined up to a factor in  $(K \otimes E)^*$ . - In terms of our second action, we have

$$(\bar{K} \otimes E)^{[\Gamma]} = \xi^{-1} (K \otimes E) \subset \bar{K} \otimes E .$$

Now, the motive  $M(\mu)$  really "is"  $E$  viewed as the one dimensional  $E$  linear representation of  $\Gamma$  given by  $\mu$ . On the other hand,  $H_{DR}(M(\mu)) = (\bar{K} \otimes E)^{[\Gamma]}$  - see I, 2.4.1 -, so  $\xi$  is a  $K \otimes E$  basis of  $H_{DR}(M(\mu))$ . Therefore, for each  $\sigma : K \hookrightarrow \bar{\mathbb{Q}} \subset \mathbb{C}$ , the period  $p(\mu; \sigma)$  can be computed like this: take 1 as  $E$ -basis of  $H_\sigma(M(\mu)) = E$ ; for any extension  $\tilde{\sigma} : \bar{K} \xrightarrow{\sim} \bar{\mathbb{Q}}$  of  $\sigma$ , the inverse of  $\xi^{\tilde{\sigma}} = \xi^{\tilde{\sigma}} \otimes \text{id}_E \in (\bar{\mathbb{Q}} \otimes E)^*$  is a  $K^\sigma \otimes E$  basis of  $H_{DR}(M(\mu)) \otimes_{K, \sigma} K^\sigma \subset H_{DR}(M(\mu)) \otimes_{K, \sigma} \mathbb{C} = E \otimes \mathbb{C}$ ; we find

$$(3.2.4) \quad p(\mu; \sigma) = \xi^{\tilde{\sigma}} .$$

Let us analyze the indeterminacy:  $\xi$  was well determined up to  $(K \otimes E)^*$ ; on the other hand, if we pick  $s\tilde{\sigma}$  instead of  $\tilde{\sigma}$ , with  $s \in \text{Gal}(\bar{\mathbb{Q}}/K^\sigma)$ , we find,

$$\begin{aligned} \xi^{s\tilde{\sigma}} &= \xi^{\tilde{\sigma}\tilde{\sigma}^{-1}s\tilde{\sigma}} = [ (1 \otimes \mu(\tilde{\sigma}^{-1}s\tilde{\sigma})) \cdot \xi ]^{\tilde{\sigma}} \\ &= (1 \otimes \mu(\tilde{\sigma}^{-1}s\tilde{\sigma})) \cdot \xi^{\tilde{\sigma}} . \end{aligned}$$

Thus, the array  $(\xi^{\tilde{\sigma}})_\sigma : K \hookrightarrow \mathbb{C} \in (K \otimes E \otimes \mathbb{C})^*$  is well determined up to a factor in  $(E^*)^{\text{Hom}(K, \mathbb{C})} (K \otimes E)^*$ , and we have

$$(3.2.5) \quad p(\mu) = (\xi^{\tilde{\sigma}})_\sigma \in \text{Hom}(K, \mathbb{C}) .$$

This establishes in particular the formula left unproven in 1.8.9 above. - Let us restate 1.8.9 for finite order characters, using the well known role of the transfer map in class field theory:

3.2.6 Let  $K'/K$  be a finite extension, and  $\mu'$  a character of finite order of  $K'$  (always with values in  $E$ ). Denote by  $\text{Ver}_{K'}^K : \text{Gal}(\bar{K}/K) \rightarrow \text{Gal}(\bar{K}/K')^{\text{ab}}$  the transfer map. Then

$$p(\mu' \circ \text{Ver}_{K'}^K) = N_{K'/K}(p(\mu')) .$$

This formula implies the following invariance lemma a special case of which was needed as formula (12) in [HS] - cf. also 1.8.11.

3.2.7 Lemma Let  $K$  and  $\chi$  be as in 3.0. Let  $K'/K$  be a finite extension and  $\chi'$  a Hecke character of  $K'$  with values in  $E$  (like  $\chi$ ), such that, for all  $\tau : E \hookrightarrow \mathbb{C}$ ,

$$\{\sigma' |_{K'} \mid \sigma' \in \text{Hom}(K', \mathbb{C}); n'(\sigma', \tau) < \frac{w'}{2}\} = \{\sigma \in \text{Hom}(K, \mathbb{C}) \mid n(\sigma, \tau) < \frac{w}{2}\} .$$

Then, for any  $\mu'$  as in 3.2.6, one has

$$\frac{c^{\pm}(\mu', \chi')}{c^{\pm}((\mu' \circ \text{Ver}_{K'}^K), \chi)} = \frac{c^{\pm}(\chi')}{c^{\pm}(\chi)} .$$

This follows from 1.8.1, 1.7.12(i), 1.8.3, and 3.2.6. Note that, unlike 3.1, the use of 1.8.3 inside 1.8.1 is licit here because  $K$  and  $K'$  are totally imaginary. In fact, even if we had, say,  $\mu' = \mu_0 \circ N_{K'/K_0}$ , for some totally real field  $K_0$  and with  $F_{\infty}$  acting on  $H_{\sigma_0}(M(\mu_0))$  as  $-1$  (i.e.,  $\mu_0$  involves a nontrivial sign character), no such signs would be visible over  $K'$ , and  $F_{\infty} : H_{\sigma_0}(M(\mu_0)) \rightarrow H_{\sigma_0}(M(\mu'))$  simply identifies these two spaces.

3.2.8 We shall now develop an analogue of 3.2.6, with  $N_{K'/K}$  replaced by Tate's "half transfer" - cf. I, 6.4.0. Let  $\chi$

and  $K$  be as before - but assume that  $K$  is a CM field. (We can always reduce to this case by 3.2.6: see 0, § 3.) Let  $K_0 \subset K$  be a totally real subfield of  $K$ . (The important case will be  $K_0 = \mathbb{Q}$ .) Fix an embedding  $\sigma_0 : K_0 \hookrightarrow \mathbb{C}$ , and consider  $K$  as embedded into  $\bar{\mathbb{Q}} \subset \mathbb{C}$ , by using some fixed extension of  $\sigma_0$  (which will not show up in the notation). Choose a system of representatives,

$$v : \text{Hom}_{K_0, \sigma_0}(K, \bar{\mathbb{Q}}) \rightarrow \text{Gal}(\bar{\mathbb{Q}}/K_0^{\sigma_0})$$

in such a way that  $v(c\sigma) = cv(\sigma)$ , for  $c =$  complex conjugation. For each  $\tau \in \text{Hom}(E, \mathbb{C})$ , define the "half transfer" map attached to  $\chi$  and  $\tau$ , relative to  $K_0, \sigma_0$ ,

$$V(\cdot, \tau) : \text{Gal}(\bar{\mathbb{Q}}/K_0^{\sigma_0}) \rightarrow \text{Gal}(K^{\text{ab}}/K)$$

by the rule

$$V(s, \tau) = \prod_{\sigma|_{K_0} = \sigma_0} [v(s\sigma)^{-1} s v(\sigma)]^{-n(\sigma, \tau)} \pmod{\text{Gal}(\bar{\mathbb{Q}}/K^{\text{ab}})} .$$

$V$  is independent of the choice of  $v$ , and for all  $s, t \in \text{Gal}(\bar{\mathbb{Q}}/K_0^{\sigma_0})$  and  $\tau \in \text{Hom}(E, \mathbb{C})$ , one has the cocycle relation

$$(3.2.9) \quad V(st, \tau) = V(s, t\tau) V(t, \tau) .$$

Now, let  $\mu$  be any finite order character on  $K$  with values in  $E$ . Define

$$\mu_0 : \text{Gal}(\bar{\mathbb{Q}}/K_0^{\sigma_0}) \rightarrow (E^*)^{\text{Hom}(E, \bar{\mathbb{Q}})}$$

by the rule

$$(3.2.10) \quad \mu_0(s) = (\mu(V(s^{-1}, \tau)))_{\tau} .$$

Let  $K \subset F \subset \bar{\mathbb{Q}}$  be as in 3.2.1, and define a left action of  $\text{Gal}(\bar{\mathbb{Q}}/K_0^{\sigma_0})$  on  $\text{Maps}(\text{Hom}(E, \bar{\mathbb{Q}}), (F \otimes E)^*)$  by using the trivial action on  $E^*$  and the natural actions on  $F^*$  and  $\text{Hom}(E, \bar{\mathbb{Q}})$  - cf. 3.2.1. Then there exists a unit

$$\eta \in (F \otimes E)^* \text{Hom}(E, \bar{\mathbb{Q}}) \hookrightarrow (E \otimes \mathbb{C})^* \text{Hom}(E, \mathbb{C})$$



such that, for all  $s \in \text{Gal}(\bar{\mathbb{Q}}/K_0^{\sigma_0})$ ,

$$(3.2.11) \quad \eta^s = (1 \otimes \mu_0(s)) \cdot \eta.$$

In fact, we can put (see 3.2.3/4, with  $\bar{K} = \bar{\mathbb{Q}}$  and  $\tilde{\sigma} = v(\sigma)$ ):

$$(3.2.12) \quad \eta = \left( \prod_{\sigma|_{K_0^{\sigma_0}} = \sigma_0} p(\mu; \sigma)^{n(\sigma, \tau)} \right)_{\tau}.$$

$\eta$  is determined by 3.2.11 up to a factor in

$$(K_0^{\sigma_0} \otimes E)^* (G(\bar{\mathbb{Q}}/K_0^{\sigma_0}) \setminus \text{Hom}(E, \bar{\mathbb{Q}})).$$

It is convenient to write  $\eta$  as a matrix  $(\eta_{\tau, \tau'})$ , with indices  $\tau, \tau' \in \text{Hom}(E, \bar{\mathbb{Q}})$ , and entries

$$(3.2.13) \quad \eta_{\tau, \tau'} = \prod_{\sigma|_{K_0^{\sigma_0}} = \sigma_0} (\xi^{v(\sigma) \otimes \tau'})^{n(\sigma, \tau)} \in \bar{\mathbb{Q}}^*.$$

Then, for  $s \in \text{Gal}(\bar{\mathbb{Q}}/K_0^{\sigma_0})$ ,

$$(3.2.14) \quad (\eta_{\tau, \tau'})^s = \frac{1}{\mu(V(s, \tau))^{s\tau'}} \eta_{s\tau, s\tau'},$$

and, for  $\eta_{\tau} = (\eta_{\tau, \tau'})_{\tau'} \in (\bar{\mathbb{Q}} \otimes E)^*$ , with  $s$  acting via the first action introduced in 3.2.1,

$$(3.2.15) \quad (\eta_{\tau})^s = (1 \otimes \mu(V(s, \tau)))^{-1} \eta_{s\tau}.$$

What makes these formulas interesting is their connection with the periods  $c^{\pm}$ , and thereby, via 2.1, with L-values:

3.3.0 Example. Let  $A$  be an abelian variety with complex multiplication by  $E$  defined over  $K$  - cf. I § 1. Call  $\chi$  its Hecke character:  $M(\chi) = H_1(A)$ . Then by 1.8.1, 1.8.3 and 3.2.13 (with  $K_0 = \mathbb{Q}$ ), one finds for any finite order character  $\mu$  of  $K$  (with values in  $E$ ):

$$(3.3.1) \quad \frac{c^{\pm}(\mu \cdot \chi)}{c^{\pm}(\chi)} = (\eta_{\tau, \tau})_{\tau} \in (E \otimes \bar{\mathbb{Q}})^*.$$

(The justification for applying 1.8.3 inside 1.8.1 is the same as in 3.2.7:  $K$  is totally imaginary.) Thus, by 3.2.14 with

s fixing  $\tau$ , the  $\tau$ -component  $\eta_{\tau, \tau}$  of this quotient of periods generates the abelian extension of  $E^\tau$  corresponding to the character  $\mu^\tau(V(\cdot, \tau))$  of  $\text{Gal}(\overline{\mathbb{Q}}/E^\tau)$ . And if, by chance, both  $L^*(\mu\chi, 0)$  and  $L^*(\chi, 0)$  are in  $(E \otimes \mathbb{C})^*$ , then their quotient in  $(E \otimes \mathbb{C})^*$  has the same property. Thus, in particular, for all  $\tau : E \hookrightarrow \overline{\mathbb{Q}}$ ,

$$(3.3.2) \quad \frac{L(\mu^\tau \chi^\tau, 0)}{L(\chi^\tau, 0)} \in (E^\tau)^{\text{ab}}, \quad \text{and} \quad \left\{ \frac{L^*(\mu\chi, 0)}{L^*(\chi, 0)} \right\}^{\text{order}(\mu)} \in E^* \subset (E \otimes \mathbb{C})^* .$$

3.3.3 It is easy to generalize the statements of this example to arbitrary Hecke characters  $\chi$  of  $K$ . Assume for simplicity, that  $s = 0$  is critical for  $\chi$ . Define  $\tilde{V}$  to be the transfer defined by the system of invariants

$$\tilde{n}(\sigma, \tau) = \begin{cases} -1 & \text{if } n(\sigma, \tau) < 0 \\ 0 & \text{if } n(\sigma, \tau) \geq 0 \end{cases} .$$

Then 3.3.1 holds for  $\chi$ , with  $\eta_{\tau, \tau}$  replaced by  $\tilde{\eta}_{\tau, \tau}$  - defined relative to  $\tilde{\mu}_0(s) = (\mu(\tilde{V}(s^{-1}, \tau)))_\tau$  instead of  $\mu_0$ . So here, too, 3.3.2 follows.

3.4 Finally, let us lift our convention 3.0, and consider the case that  $K$  is totally real (embedded into  $\overline{\mathbb{Q}}$ ). Assume for simplicity that  $s = 0$  is critical for the character  $\chi = \mu \cdot \mathbb{N}^n$ . Then (2.0.2)  $F_\infty$  acts on  $H_B(R_{K/\mathbb{Q}}^M(\mu))$  as  $(-1)^n$  if  $n > 0$ , and as  $-(-1)^n$ , if  $n \leq 0$ . Thus, if  $n \leq 0$ , putting  $\pi 1 = (-1)^e$ , we obtain

$$(3.4.1) \quad \begin{cases} c^+(\mu \mathbb{N}^n) = 1 = c^\pi(\mathbb{N}^n) \\ c^-(\mu \mathbb{N}^n) = p(R_{K/\mathbb{Q}}^M(\chi)) = p(\epsilon_{K/\mathbb{Q}} \cdot (\mu \circ \text{Ver}_K^{\mathbb{Q}}) \cdot \mathbb{N}^n) . \end{cases}$$

In the case  $n > 0$  the signs get reversed, and we find, for  $m = -n > 0$ ,

$$(3.4.2) \quad \frac{L(\mu, m)}{(2\pi i)^m} = \frac{c^+(\mu \mathbb{N}^n)}{c^\pi(\mathbb{N}^n)} = p(\epsilon_{K/\mathbb{Q}} \cdot (\mu \circ \text{Ver}_K^{\mathbb{Q}})) .$$

Since the construction of  $\mathfrak{g}$ , such that 3.2.3/4 hold, clearly

works over all base fields, we get in particular that  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  acts on  $p(\epsilon_{K/\mathbb{Q}} \cdot (\mu \circ \text{Ver}_K^{\mathbb{Q}})) \in (E \otimes \bar{\mathbb{Q}})^*$  via the character  $\epsilon_{K/\mathbb{Q}} \cdot (\mu \circ \text{Ver}_K^{\mathbb{Q}})$ .

But for all Dirichlet characters of  $\mathbb{Q}$ , such elements are classically given by Gauss sums, and more precisely by their "root numbers"; see [DP], 6.4, 6.5. This most incredible coincidence does NOT repeat itself over algebraic number fields  $K$  different from  $\mathbb{Q}$ ! In fact, the components of  $p(\mu)$  generate the corresponding abelian extensions of  $K$ , and can therefore not all lie in  $\mathbb{Q}^{\text{ab}}$ . - The last sentence of [HS], § 4 is therefore INCORRECT - and should never have been put in there in the first place.

3.5 Let  $K$  and  $E$  be arbitrary number fields.

3.5.1 Proposition Let  $M$  and  $M'$  be two motives in  $\mathcal{M}_K^{\text{av}}(E)$ , of rank 1 over  $E$ , such that

- (i)  $M$  and  $M'$  become isomorphic over  $\bar{K}$ ;
- (ii)  $p(M) = p(M')$  in  $(K \otimes E \otimes \mathbb{C})^*/(E^*)^{\text{Hom}(K, \mathbb{C})} (K \otimes E)^*$ .

Then  $M \cong M'$  in  $\mathcal{M}_K^{\text{av}}(E)$ .

Proof. By I, 6.6.1, we have  $M = M(\chi), M' = M(\chi')$ , for certain characters  $\chi, \chi'$  of  $K$  with values in  $E$ ; and (i) implies that  $\chi' = \mu \chi$ , for some finite order character of  $K$  - cf. 0, 3 and I, 6.15. Hence, by 1.8.3,

$$1 = \frac{p(M')}{p(M)} = p(\mu).$$

By 3.2.3/4, this means that  $\mu = 1$ , and I, 5.1 finishes the proof. (In fact, a direct argument can be given, using (ii) once more.)

3.5.2 The proof of 3.5.1 shows that the  $\bar{K}/K$ -forms of rank-1-motives in  $\mathcal{M}_K^{\text{av}}(E)$  are parametrized by the periods  $p(\mu) \in (K \otimes E \otimes K^{\text{ab}})^*$ , for  $\mu$  running over the finite order characters of  $K$ .

#### 4. The periods of Jacobi sum Hecke characters

4.0 The gamma function, i.e., the meromorphic continuation of

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^s \frac{dx}{x} \quad (\operatorname{Re} s > 0)$$

satisfies the following functional equations - for  $s \in \mathbb{C}$ , and  $m \in \mathbb{Z}$ ,  $m \geq 1$ .

$$(4.0.0) \quad s \Gamma(s) = \Gamma(1+s)$$

$$(4.0.1) \quad \prod_{j=0}^{m-1} \Gamma\left(\frac{s+j}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-s} \Gamma(s)$$

$$(4.0.2) \quad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} .$$

The first one implies that  $\Gamma$  induces a well-defined map

$$(4.0.3) \quad \Gamma : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{C}^*/\mathbb{Q}^* ,$$

and a folklore conjecture in transcendence theory says that all relations satisfied by (4.0.3), composed with

$$\mathbb{C}^*/\mathbb{Q}^* \rightarrow \mathbb{C}^*/\overline{\mathbb{Q}}^* ,$$

follow from (4.0.1) and (4.0.2) - see [LD], ex. 4.

#### 4.1 The basic example

Let us resume the situation of I, 7.1.1, assuming  $n \geq 3$ . On the affine open part

$$Y_2^m + \dots + Y_n^m = -1 \quad (Y_i = \frac{X_i}{X_1})$$

of the Fermat hypersurface  $X_m^n$ , the  $n-2$  form

$$\tilde{a}_2 \dots Y_n^{\tilde{a}_n} \frac{dY_2}{Y_2} \wedge \dots \wedge \frac{dY_{n-1}}{Y_{n-1}}$$

is an eigenform for the character  $\underline{a} = \sum [a_j]$  of  $G_m^n$ , if  $\frac{\tilde{a}_j}{m} \pmod{\mathbb{Z}} = -a_j$ , and  $\tilde{a}_j \gg 0$ . Its period against a suitable  $n-2$  simplex is computed as

$$(4.1.0) \quad (2\pi i)^{-1} \prod_{j=1}^n (1 - e^{2\pi i \cdot a_j}) \Gamma(-\langle a_j \rangle);$$

see [DMOS], I, 7.12-7.14. This allows us to compute the periods of the motive  $M(\underline{a}) \times \mathbb{Q}(\mu_m)$  which, by construction, has the structure of a motive with coefficients in  $\mathbb{Q}(\mu_m)$ . But  $M(\underline{a})$  is constructed in such a way that  $M(\underline{a}) \otimes E$  is isomorphic, in  $\mathcal{M}_{\mathbb{Q}}(\mathbb{Q}(\mu_m))$ , to  $R_{\mathbb{Q}(\mu_m)}/\mathbb{Q}(M(\underline{a}) \times \mathbb{Q}(\mu_m))$ . Thus we find, using 1.3.1, 1.4.2 - 1.4.8, and, in the case where 0 is critical for  $J(\underline{a})$ , 1.7,  $\emptyset$ , 8.2.7:

$$(4.1.1) \quad p(M(\underline{a})) = \sqrt{d(\mathbb{Q}(\mu_m))} \cdot \prod_{\substack{k=1 \\ (k,m)=1}}^{m-1} \prod_{j=1}^n \Gamma(-a_j k)^{-1}$$

$$(4.1.2) \quad c^+(M(\underline{a})) = \sqrt{d^+(\mathbb{Q}(\mu_m))} \cdot \prod_{\langle k\underline{a} \rangle > \langle -k\underline{a} \rangle} \prod_{j=1}^n \Gamma(a_j k)^{-1},$$

where  $d(\mathbb{Q}(\mu_m))$  (resp.  $d^+(\mathbb{Q}(\mu_m))$ ) is the discriminant of  $\mathbb{Q}(\mu_m)$  (resp. of the maximal totally real subfield of  $\mathbb{Q}(\mu_m)$  - see 1.7.12 (iv)). These expressions are well-determined up to a factor in  $\mathbb{Q}^*$ , as they should be for a motive in  $\mathcal{M}_{\mathbb{Q}}(\mathbb{Q})$ . Also, we claim that, if  $s = 0$  is critical for  $J(\underline{a})$ , then

$$(4.1.3) \quad c^-(M(\underline{a})) = \sqrt{d^-(\mathbb{Q}(\mu_m))} \cdot \prod_{\langle k\underline{a} \rangle > \langle -k\underline{a} \rangle} \prod_{j=1}^n \Gamma(a_j k)^{-1},$$

with  $d(\mathbb{Q}(\mu_m)) = d^+(\mathbb{Q}(\mu_m)) \cdot d^-(\mathbb{Q}(\mu_m))$ . In fact, by the behaviour of the discriminant in towers, we have

$$d^-(\mathbb{Q}(\mu_m)) = d^+(\mathbb{Q}(\mu_m)) \cdot \left[ \prod_k e^{\frac{2\pi i - k}{m}} - e^{\frac{2\pi i k}{m}} \right]^2$$

the product being over any set of representatives  $k$  of  $(\mathbb{Z}/m\mathbb{Z})^* \pmod{\{\pm 1\}}$ . This shows that  $(-\sqrt{d^-(\mathbb{Q}(\mu_m))})_{\tau} \in (\mathbb{Q}(\mu_m) \otimes \mathbb{C})^*$  equals  $\delta^-$  - given by 1.7.10 - , up to a factor in  $\mathbb{Q}(\mu_m)^*$ .

Note that - as it ought to be: [DP], 1.7 -  $c^+(M(\underline{a}))$  and  $i^{\varphi(m)}/2 c^-(M(\underline{a}))$  are real, where  $\varphi$  is Euler's phi function.

#### 4.2 Periods of Anderson's motives

If  $K$  is any abelian number field and  $\underline{a} \in \mathbb{B}_K^0$  - see 0, 8.2 -, then the periods of Anderson's motive  $M_K(\underline{a})$  - see I, 7.2; I, 7.4.5 - for the Jacobi sum character  $J_K(\underline{a})$  can be computed by formulas which immediately generalize 4.1.1-4.1.3. In fact, note that periods are built into the notion of arithmetic Hodge structure - see I, 7.3.1/2. - We shall only give the final expression that Anderson obtains for the periods corresponding to the critical values of all Jacobi sum Hecke characters. It contains 4.1.2 as a special case, and 4.1.3 follows from it via 3.1.1. - The formula for  $p(M_K(\underline{a}) \times K)$  is stated in 4.4.2.

4.2.1 If  $K$  is totally real, then  $F_\infty$  acts trivially on  $H_B(M_K(\underline{a}))$ , for any  $\underline{a} \in \mathbb{B}_K^0$ . (Essentially, this is so because  $J_K(\underline{a})$  is "pulled down" from some totally imaginary extension of  $K$ .) Thus, by 2.0.2 above, the critical values of the character

$J_K(\underline{a}) = \mu \cdot N^{\sum n_a}$  - with  $\underline{a} = \sum_a n_a [a]$  and  $\mu$  of finite order - are just the elements of

$$\{s \in 2\mathbb{Z}+1 \mid s \leq \sum_a n_a\} \cup \{s \in 2\mathbb{Z} \mid s > \sum_a n_a\} .$$

We put

$$\text{Crit}_K(\underline{a}) = 2\mathbb{Z} \cap \left( \sum_a n_a, \infty \right) .$$

4.2.2 If  $K$  is totally imaginary, then 2.0.3 implies that the critical  $s$  for  $J_K(\underline{a})$  are precisely those in

$$\text{Crit}_K(\underline{a}) = \left\{ s \in \mathbb{Z} \left| \begin{array}{l} \langle t\underline{a} \rangle < s \leq \langle t c \underline{a} \rangle \text{ for all } t \in G(\overline{\mathbb{Q}}/\mathbb{Q}) \\ \text{with } \langle t\underline{a} \rangle \leq \langle t c \underline{a} \rangle \end{array} \right. \right\}$$

Here, as usual,  $c$  denotes complex conjugation; the galois

action is that defined in  $\mathbb{Q}$ , 8.2.1.

4.2.3 Notation.  $d(K)$ , resp.  $d^+(K)$ , denotes the discriminant of  $K$ , resp. of the maximal totally real subfield of  $K$ ; and  $d(K) = d^+(K) \cdot d^-(K)$ . For all  $\underline{a} \in \mathbb{B}$ ,  $\underline{a} = \sum_{\mathfrak{a}} n_{\mathfrak{a}}[\mathfrak{a}]$ , extend 4.0.3 by the rule

$$\Gamma(\underline{a}) = \prod_{\mathfrak{a}} \Gamma(\mathfrak{a})^{n_{\mathfrak{a}}} \in \mathbb{C}^*/\mathbb{Q}^* .$$

4.2.4 For all abelian number fields  $K$ , all  $\underline{a} \in \mathbb{B}_K^0$  and all  $s = n \in \text{Crit}_K(\underline{a})$ , one finds

$$(4.2.5) \quad c^+(M_K(\underline{a}))(n) = \pi^{n[K:\mathbb{Q}]/2} |d^{(-)n}(K)|^{1/2} \prod_{\substack{\sigma \in G(K/\mathbb{Q}) \\ \langle \sigma \underline{a} \rangle \geq \langle \sigma c \underline{a} \rangle}} \Gamma(\sigma \underline{a})^{-1} .$$

### 4.3 Lichtenbaum's "Γ-hypothesis"

As Anderson points out, the period calculation 4.2.5, joined with theorem 2.1 above, yields the following theorem on the critical L-values of Jacobi sum Hecke characters which contains the most general formulation of what Lichtenbaum had called his  $\Gamma$ -hypothesis - see [Li], [KL].

4.3.1 Theorem. For all abelian number fields  $K$ , all  $\underline{a} \in \mathbb{B}_K^0$ , and all  $s = n \in \text{Crit}_K(\underline{a})$ ,

$$\pi^{-n[K:\mathbb{Q}]} |d^{(-)n}(K)|^{1/2} \prod_{\langle \sigma \underline{a} \rangle \geq \langle \sigma c \underline{a} \rangle} \Gamma(\sigma \underline{a}) \cdot L(J_K(\underline{a}), n) \in \mathbb{Q} .$$

Note that, in deriving this statement from 2.1, one has to use, as in 4.1, that  $M_K(\underline{a}) \otimes K$  is isomorphic, in  $\mathcal{M}_{\mathbb{Q}}(K)$ , to  $R_{K/\mathbb{Q}}(M_K(\underline{a}) \times K)$ , where  $M_K(\underline{a}) \times K$  has a natural structure of a motive with coefficients in  $K$  that makes it into a motive for  $J_K(\underline{a})$ . Recall also that  $J_K(\underline{a})$  is galois equivariant - see  $\mathbb{Q}$ , 8.2.5 - , so that the L-functions of all of its conjugates coincide.

Finally, it should be pointed out that, in the case that  $K$  is totally real, every critical  $s$  for  $J_K(\underline{a})$  either lies in  $\text{Crit}_K(\underline{a})$  or is related to an element of  $\text{Crit}_K(-\underline{a})$ , by the functional equation.

#### 4.4 $\Gamma$ -relations

4.4.1 Theorem. Let  $K \subset \overline{\mathbb{Q}}$  be an abelian number field. If  $\underline{a}, \underline{b} \in \mathbb{B}_K^0$  satisfy  $J_K(\underline{a}) = J_K(\underline{b})$ , then

$$(\Gamma(\underline{\sigma a}))_{\sigma \in G(K/\mathbb{Q})} = (\Gamma(\underline{\sigma b}))_{\sigma \in G(K/\mathbb{Q})} ,$$

in  $(K \otimes \mathbb{C})^*/K^*$ .

Proof. By construction, the motive  $M_K(\underline{a}) \times K$  has a natural structure of a motive with coefficients in  $K$  with respect to which it is a motive for  $J_K(\underline{a})$ . For all  $\sigma$ , one has

$$(4.4.2) \quad p(M_K(\underline{a}) \times K; \text{id}_K, \sigma) = \Gamma(\underline{\sigma c a})^{-1} .$$

(The fact that complex conjugation creeps into this formula is clearly seen in our basic example: I, 7.1./2, and 4.1.0 above.) By I, 5.1, the theorem follows.

q.e.d.

4.4.3 Corollary ([A2], 8.6) If  $\underline{a}$  and  $\underline{b}$  are in  $\mathbb{B}_K^0$  such that  $\text{Crit}_K(\underline{a}) \neq \emptyset$ , and

$$L(J_K(\underline{a}), s) = L(J_K(\underline{b}), s) ,$$

as meromorphic functions on  $\mathbb{C}$ , then

$$\langle \underline{\sigma a} \rangle \geq \prod_{\sigma \in G} \Gamma(\underline{\sigma a}) = \prod_{\sigma \in G} \Gamma(\underline{\sigma b}) = \langle \underline{\sigma b} \rangle \geq \prod_{\sigma \in G} \Gamma(\underline{\sigma b}) ,$$

in  $\mathbb{C}^*/\mathbb{Q}^*$ .



Proof. The hypothesis implies: immediately, that  $\text{Crit}_K(\underline{a}) = \text{Crit}_K(\underline{b})$  ; and, modulo an exercise in analytic number theory, that  $J_K(\underline{b}) = J_K(\tau \underline{a})$  , for some  $\tau \in G(K/\mathbb{Q})$  . Then, the theorem yields what is claimed, in view of  $\textcircled{0}$ , 8.2.7.

Known variants of the theorem used to be encouraging companions to the  $\Gamma$ -hypothesis when this was still unproven. Its motivic proof is a nice illustration of our central theme: how to derive period relations from character identities. More precisely, it is a compatibility result inside one family of motives for a class of Hecke characters. In that sense it is the analogue, for Anderson's motives, of Shimura's monomial relations, as derived from the standard motives of Hecke characters in chapter IV below.

4.4.4 A different instance of our main theme occurs when  $J_K(\underline{a})$  is of finite order, and  $M_K(\underline{a})$  is compared to an Artin motive. This was already pointed out by Deligne in [DP], 8.9 - 8.13. Let us briefly rederive the results in our setting.

By  $\textcircled{0}$ , 8.2.7, the Jacobi sum Hecke character  $J_K(\underline{a})$  is of finite order if and only if  $\langle \sigma \underline{a} \rangle = 0$  , for all  $\sigma \in \text{Gal}(K/\mathbb{Q})$  . If this is so, then - by I, 5.1 -  $M_K(\underline{ca}) \times K$  is isomorphic, in  $\mathcal{M}_K(K)$  , to the Artin motive of  $J_K(\underline{ca}) = J_K(\underline{a})^{-1}$  , and we deduce from 4.4.2, for  $M_K(\underline{ca})$  , and 3.2.4, with  $\tilde{\sigma} = \tau = \text{id}$  , the following theorem which contains conjecture 8.13 of [DP], and, together with 4.4.6, is equivalent to theorem 7.18 in [DMOS], chap. I. It also implies, of course, 4.4.1 and 4.4.3 above.

4.4.5 Theorem. For all abelian number fields  $K$  , and all  $\underline{a} \in \mathbb{B}_K^0$  such that  $\langle \sigma \underline{a} \rangle = 0$  for each  $\sigma \in G(K/\mathbb{Q})$  , reading the finite order character  $J_K(\underline{a})$  on  $\text{Gal}(\overline{\mathbb{Q}}/K)^{\text{ab}}$  , one has

(i)  $\Gamma(\underline{a}) \in \overline{\mathbb{Q}}^*/\mathbb{Q}^*$

(ii)  $\Gamma(\underline{a})^s = J_K(\underline{a})(s) \cdot \Gamma(\underline{a})$  , for all  $s \in G(\overline{\mathbb{Q}}/K)$  .

Part (i) was first proved directly by Koblitz and Ogus in the appendix to [DP]. - It is shown in [Sch  $\Gamma$ ] that a good deal of

(ii) can be derived from 4.0.1, 4.0.2 using only classical results on the arithmetic of Gauss sums.

4.4.6 As in [DMOS], I, 7.18, the preceding theorem can be complemented to give the behaviour of  $\Gamma(\underline{a})$  under all of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ :

(iii) For all  $t \in G(\overline{\mathbb{Q}}/\mathbb{Q})$ , the quotient  $\frac{\Gamma(\underline{a})}{\Gamma(\underline{a})^t}$  lies in  $K^*$ , and for all  $\sigma \in \text{Gal}(K/\mathbb{Q})$ , one has

$$\left( \frac{\Gamma(\underline{a})}{\Gamma(\underline{a})^t} \right)^\sigma = \frac{\Gamma(\sigma \underline{a})}{\Gamma(\sigma \underline{a})^t} .$$

Proof. Writing  $\xi$  the "period" 3.2.3 of the Artin motive of  $J_K(\underline{a})$ , we have

$$\Gamma(\underline{a})^t = \xi^{t \otimes t} ,$$

for all  $t \in G(\overline{\mathbb{Q}}/\mathbb{Q})$ . This and the galois equivariance  $\text{O}$ , 8.2.5:

$$J_K(\underline{a})(s) = J_K(t\underline{a})(t^{-1}st) \quad (s \in G(\overline{\mathbb{Q}}/K))$$

easily imply that  $\Gamma(\underline{a})/\Gamma(\underline{a})^t \in K^*$ .

The last claim is proved by analysing the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on  $\text{End}_{\overline{\mathbb{Q}}}(M_K(\underline{a})) = K$ : just imitate the argument on [DMOS], p. 93; the details are left to the reader.

CHAPTER THREE:

Elliptic Integrals and the Gamma Function

The subject of this chapter is a natural continuation of II, 4.4: - we now compare Anderson's motives for Jacobi sum Hecke characters to elliptic curves with complex multiplication. This gives essentially a refinement of the so-called formula of Chowla and Selberg - which originally is due to M. Lerch.

1. A formula of Lerch

1.0 Let  $K \xrightarrow{1} \mathbb{C}$  be an embedded imaginary quadratic field,  $-D$  its discriminant, and  $J_D = J_K(\underline{a}_D)$  the basic Jacobi sum Hecke character of  $K$  defined in  $\text{\textcircled{0}}$ , 8.1 for  $D \neq 3, 4, 8$  and in  $\text{\textcircled{0}}$ , 8.3 for arbitrary  $D$ . The infinity type  $T_D$  of  $J_D$  ( $\text{\textcircled{0}}$ , 8.1.5) is written

$$T_D = n_1 \cdot 1 + n_c \cdot c \quad ,$$

and  $h_D$  denotes the class number of  $K$ .

Let  $\chi$  be any fixed Hecke character of  $K$  (with values in some CM field  $E \supset K$ ) whose infinity type is  $-1$ . Then there exists a character of finite order  $\mu$  of  $K$ , with values in  $E$ , such that

$$(1.1) \quad \mu \cdot \chi^{h_D} = J_D^{-1} \cdot \mathbb{N}^{n_c} \quad .$$

Let us now compute the periods  $c^+$  of motives attached to both sides of the equation. They have to be equal by I, 5.1. On the left hand side, use II, 3.3.3 for the present  $\mu$ , and II, 1.8.1/3 as well as II, 1.7.12(iv). On the right, use II, 4.2.5 observing  $\text{\textcircled{0}}$ , 8.1.3 and  $\text{\textcircled{0}}$ , 8.3.1, and remembering that  $(M_K(-\underline{a}_D) \times K) \otimes E$  is a motive for  $J_D^{-1}$  considered as Hecke character of  $K$  with values in  $E$ . This gives, for  $D \neq 3, 4, 8$ :

$$(1.2) \quad (\tilde{\eta}_{\tau, \tau} \cdot p(\chi; \tau |_{K, \tau})^{h_D})_{\tau} = (\pi^{-n_c} |d(-)^{n_c}(K)|^{1/2} \prod_{\substack{j=1 \\ \epsilon(j)=1}}^{D-1} \Gamma(\frac{j}{D}))_{\tau}$$

where  $\tau$  runs over the complex embeddings of  $E$ , and  $\epsilon$  is the

Dirichlet character corresponding to  $K \subset \mathbb{Q}(\mu_D)$  .

Let  $F$  be any finite abelian extension of  $K$  such that  $\chi$  takes values in  $K^*$  on ideals which are norms from  $F$  . (Note that  $F$  has to contain the Hilbert class field  $H$  of  $K$  .) Then there exists an elliptic curve  $A$  defined over  $F$  such that  $H_1(A)$  is a motive for the Hecke character  $\psi = (\chi \circ N_{F/K})$  , considered as character of  $F$  with values in  $K$  . This is a special case of Casselman's theorem, i. e., theorem 6 in [Shi L] - cf. I, 4.1.1. Note that, in terms of  $A$  , the inclusion  $K \hookrightarrow F$  is given by the action of  $\text{End } A$  on the tangent space of  $A$  at the origin. - Cf. also [GS] § 4 .

From II, 1.8.6 and II, 1.8.8, we find that, for all  $\tau : E \hookrightarrow \mathbb{C}$  which restrict to  $\tau_0 : K \rightarrow \mathbb{C}$  ,

$$(1.2.1) \quad p(\chi; \tau_0, \tau)^{h_D \cdot [F:H]} = \prod_{\sigma|_K = \tau_0} p(\psi; \sigma, \tau_0) ,$$

up to a factor in  $E^*$  .

1.2.2 By II, 1.5.1 and II, 1.6.6,  $p(\psi; \sigma, \tau_0)$  is, independently of  $\tau_0$  and  $\sigma$  with  $\sigma|_K = \tau_0$  , equal (up to the usual indeterminacy) to  $\Omega_\sigma$  , a fundamental period of the elliptic curve  $A^\sigma/F^\sigma$  . In other words, the complex lattice  $\Lambda_\sigma$  corresponding to the pair  $(A^\sigma(\mathbb{C}), \omega^\sigma)$  , for a holomorphic 1-form  $\omega$  on  $A/F$  whose class is an  $F \otimes K$  basis of  $H_{DR}^1(A)$  , satisfies

$$\Lambda_\sigma \cdot \mathbb{Q} = \Omega_\sigma \cdot K \subset \mathbb{C} .$$

We now make the following

1.3 Assumption.  $F$  may be chosen to be the Hilbert class field  $H$  of  $K$  . In other words,  $\chi$  takes values in  $K^*$  on all principal ideals on which it is defined.

1.3.1. Remark. Characters  $\chi$  of type  $-1$  which satisfy 1.3 exist for all imaginary quadratic fields  $K$  - their construction is straightforward. The field of values  $E$  is then of degree  $h_D$  over  $K$  - see [Ro], cf. [Sch O], E . - It can have subfields which are galois over  $K$  only insofar as the few roots of unity in  $K^*$  afford Kummer extensions corresponding to elements in the class

group of  $K$  - cf. [Gr 1], § 15 for the case where  $h_D$  is odd (and  $\psi$  equivariant under complex conjugation).

1.3.2. Since  $J_D$  is galois equivariant, and therefore, in particular, takes values in  $K$ , and since  $h_D$  kills the class group of  $K$ , 1.1 and 1.3 imply that  $\mu$  takes values in  $K$ . Thus, if  $D \neq 3,4$ , then  $\mu$  is at most quadratic. In this case, the factor  $\tilde{\eta}_{\tau, \tau}$  simply becomes, independently of  $\tau$ , any non-zero element  $\Delta_\mu$  of  $K^{ab} \subset \bar{\mathbb{Q}}$  such that

$$\Delta_\mu^s = \mu(s)\Delta_\mu, \text{ for all } s \in \text{Gal}(\bar{\mathbb{Q}}/K).$$

By 1.2.1, 1.2.2, and 1.3.2, formula 1.2 becomes an identity of vectors with identical components:

If  $D \neq 3,4,8$ , then, up to a factor in  $K^*$ ,

$$(1.4) \quad \Delta_\mu \cdot \prod_{\sigma \in G(H/K)} \Omega_\sigma \underset{K^*}{\sim} \left(\frac{\sqrt{D}}{\pi}\right)^{\frac{1}{2}(\frac{\varphi(D)}{2} - h_D)} \prod_{j=1}^{D-1} \Gamma(\frac{j}{D}) \epsilon(j) = 1$$

1.4.1 Before discussing 1.4 let us write down the corresponding relations for  $D = 3,4,8$ . In these cases we simply take  $\chi = J_D^{-1}$  in 1.1: all three class numbers are 1. The corresponding elliptic curves  $A_D/K$  were briefly discussed in I, 7.5. They are actually defined over  $\mathbb{Q}$  and we have isomorphisms of motives in  $\mathcal{M}_{\mathbb{Q}}(\mathbb{Q})$ :

$$H^1(A_D) \cong M_K(\underline{a}_D),$$

because  $A_D$  is constructed such that  $L(H^1(A)/\mathbb{Q}, s) = L(J_D, s)$ .

Thus, writing  $\Omega_D = \int_{A_D(\mathbb{R})} \omega$  the real period of a nonzero differential of the first kind  $\omega$  on  $A_D/\mathbb{Q}$ , we find the following identities of classes in  $\mathbb{C}^*/\mathbb{Q}^*$ :

$$\Omega_3 \underset{\mathbb{Q}^*}{\sim} \Gamma(\underline{ca}_3) = \frac{\Gamma(\frac{1}{3})^2}{\Gamma(\frac{2}{3})}$$

$$\Omega_4 \underset{\mathbb{Q}^*}{\sim} \Gamma(\underline{ca}_4) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$$

$$\begin{aligned}
 (\text{by II, 4.0.2}) &= \frac{\Gamma(\frac{1}{4})^2}{\sqrt{2\pi}} \quad . \\
 \Omega_8 \underset{\mathbb{Q}}{\sim}^* \Gamma(\underline{ca}_8) &= \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{\sqrt{\pi}} \quad .
 \end{aligned}$$

To be sure, in the first two cases, it seems much more natural to go the other way: the formulas for  $\Omega_3$  and  $\Omega_4$  are classically well-known (cf. § 2 below), and they imply, by II, 3.5, that  $A_D \cong M_K(\underline{a}_D)$  over  $K$ , and in fact over  $\mathbb{Q}$ . This then shows what was claimed in I, 7.5: that, for  $D = 3, 4$ , the elliptic curves  $A_D$  described are such that  $H^1(A_D)$  is a motive for  $J_D$ .

1.4.2 Multiplying 1.4 with its complex conjugate yields a relation up to a rational number. In order to put it into a classical shape we apply II, 4.0.2 to the product on the right once, and use the following relation which is proved by arguments of the kind well-known in the context of the analytic class number formula for real quadratic fields:

$$(1.4.3) \quad \prod_{\substack{j=1 \\ \epsilon(j)=1}}^D \sin(\pi \frac{j}{D}) \underset{\mathbb{Q}}{\sim}^* \sqrt{D}^{h_D} .$$

This version holds for all  $D > 0$  such that  $-D$  is the discriminant of a quadratic field. In fact, the more natural right hand side,  $\sqrt{D}^{\varphi(D)/2}$  was replaced by  $\sqrt{D}^{h_D}$  in order to make it come out right for  $D = 8$ .

Thus, writing  $2m$  the number of units of  $K$ , we get the following relation, which can be checked for the exceptional cases  $D = 3, 4, 8$  directly from 1.4.1:

$$(1.4.4) \quad \Delta_\mu \bar{\Delta}_\mu \prod_{\sigma \in \overline{G(H/K)}} \left[ \frac{\sqrt{-D}}{\pi} \Omega_\sigma \bar{\Omega}_\sigma \right] \underset{\mathbb{Q}}{\sim}^* \left( \prod_{\mathbb{Z}/D\mathbb{Z}} \mathbb{Z} \right)^* \Gamma\left(\frac{j}{D}\right)^{\epsilon(j)m} .$$

Note that the complex conjugate  $\bar{\Delta}_\mu$  of  $\Delta_\mu$  is not intrinsically defined:  $K(\Delta_\mu)$  need not be a CM field. But, as  $\Delta_\mu^2 \in K^*$ ,  $\bar{\Delta}_\mu$  is well-determined up to a sign - which is inessential for 1.4.4.

1.4.5 Up to the interpretation of the factor  $\Delta_{\mu} \bar{\Delta}_{\mu}^{-1} \sqrt{D}^{h_D}$ , 1.4.4 is easily seen to be the exponential of a precise identity found by M. Lerch in 1897 (and rediscovered later by Chowla and Selberg), taken modulo  $\mathbb{Q}^*$  - see § 2 below. In this analytic identity, the factor  $\Delta_{\mu} \bar{\Delta}_{\mu}^{-1} \sqrt{D}^{h_D}$  appears as the 12-th root of  $\prod_{\sigma} \Delta(\Lambda_{\sigma})$ , where  $\Delta(\Lambda_{\sigma})$  is the discriminant of the lattice  $\Lambda_{\sigma}$  mentioned in 1.2.2. - See 1.5.6 below.

1.5.1 The left hand side of 1.4 - or of 1.4.4 - only depends on the field  $K$ . In fact, two elliptic curves over  $H$  coming from different characters  $\chi$  of  $K$  (like in 1.2) are twists of each other, by a finite order character of  $H$  of the form  $\mu \circ N_{H/K}$  - so we can use II § 3. Similarly, if any elliptic curve  $C/H$  with complex multiplication by  $K$  is given, it will be the twist of an  $A$  like in 1.2, by a finite order character of  $H$  - and again II § 3 tells us by which factor in  $H^{ab}$  to modify the left hand side of 1.4 in order to get the formula for the product of periods of the  $C^{\sigma}$ . - For the more general case where  $C$  is defined over some  $F \supset H$ , see § 3 below. There we shall also discuss<sup>a</sup> possible motivic interpretation of 1.4.4.

### 1.5.2 H/K-curves.

An elliptic curve  $A$  with complex multiplication by  $K$  defined over  $H$  is called an H/K-curve, if it is H-isogenous to all conjugates  $A^{\sigma}$ , with  $\sigma \in G(H/K)$ . If  $\psi$  is the Hecke character of  $H$  with values in  $K$  such that  $H_1(A) = M(\psi)$ , then  $A$  is an H/K-curve if and only if, for all  $\sigma \in G(H/K)$  and all ideals  $\mathfrak{a}$  of  $H$  on which  $\psi$  is defined, one has

$$\psi(\mathfrak{a}^{\sigma}) = \psi(\mathfrak{a}) .$$

If  $A$  is an H/K-curve, then

$$\psi^{h_D} = \psi \circ i \circ N_{H/K} ,$$

where  $i$  is the inclusion of ideals of  $K$  into ideals of  $H$ . The character  $\psi \circ i$  of  $K$  with values in  $K$  satisfies

$$(1.5.3) \quad \mu \cdot (\psi \circ i) = J_D^{-1} \cdot \mathbf{N}^{n_c} ,$$

for some character  $\mu$  of  $K$  of exponent two. Now we can imitate the arguments that have led us to 1.2, and from there to 1.4, obtaining 1.4 with the periods  $\Omega_\sigma$  of the  $A^\sigma$ . It is not always easy, however, to identify the character  $\mu$  for a given  $H/K$ -curve  $A$ .

#### 1.5.4 Standard $\mathbb{Q}$ -curves

Assume  $D > 3$  is odd, and recall from [Gr 1] § 11 the fundamental Hecke character of  $H$  attached to the field  $K = \mathbb{Q}(\sqrt{-D})$ : reading the Dirichlet character  $\epsilon$  of  $K$  as a character

$$\epsilon : \sigma_K / \sqrt{-D} \cdot \sigma_K \rightarrow \{\pm 1\} ,$$

every principal ideal of  $K$  prime to  $\sqrt{-D}$  admits an unique generator  $\alpha \in K^*$  with  $\epsilon(\alpha) = 1$ . Call  $\chi_D$  any extension to all ideals of  $K$  prime to  $D$ , so  $\chi_D$  is a Hecke character of  $K$  with values in some CM field  $E$  of degree  $h_D$  over  $K$ . Put  $\psi_D = \chi_D \circ N_{H/K}$  - this is the fundamental Hecke character of  $H$  with values in  $K$  we were alluding to above. We claim that

$$(1.5.5) \quad \chi_D^{h_D} = J_D \mathbf{N}^{-n_c} ,$$

with no twisting character  $\mu$ .

One way to prove (1.5.4) is by direct attack. - We leave this as an exercise to the reader, noting only that, if  $h_D$  is odd, one can get away without really looking at the definition of  $J_D$  - see [BL], lemma 3.4. - This direct proof of 1.5.4 verifies 1.4 with  $\Delta_\mu = 1$ , and  $\Omega_\sigma$  the period of  $A_D^\sigma$ , with  $A_D/H$  being the standard  $\mathbb{Q}$ -curve of  $K$ :  $H^1(A_D) = M(\psi_D)$ .

Alternatively, one can show that  $\mu = 1$  if one has an independent way of checking that 1.4 holds for the  $\Omega_\sigma$  of  $A_D$ , with  $\Delta_\mu = 1$  - cf. II, 3.5. Now, Gross has shown - see [Gr 1], 21.2.2 and [Gr 3], 5.6 - that the analytically proved relation 1.4.4 can be refined directly to yield exactly this: 1.4 for  $A_D$  with  $\Delta_\mu = 1$ . (In the case  $h_D$  odd, this elementary argument



of Gross was quite an important ingredient in the proof of the  $\Gamma$ -hypothesis for imaginary quadratic fields of odd class number: see [BL], a paper completed before the advent of Anderson's motives.)

1.5.6 For any  $D$ , let  $A/H$  and  $A'/H$  be elliptic curves with complex multiplication by  $K$ , with characters  $\psi, \psi'$ , respectively, so that

$$H_1(A) = M(\psi) \quad \text{and} \quad H_1(A') = M(\psi') .$$

Put  $v = \psi'/\psi$ . Assume we know  $\Delta_\mu$  in 1.4 applied to the periods  $\Omega_\sigma$  of  $A^\sigma$ . Then, for  $\Omega'_\sigma$  corresponding to  $A'^\sigma$ , one has

$$(1.5.7) \quad \prod_{\sigma} \Omega'_\sigma \underset{K^*}{\sim} \Delta(v \cdot i) \cdot \prod_{\sigma} \Omega_\sigma ,$$

where  $i$  is as in 1.5.3. This then determines the factor  $\Delta'_\mu$  which has to be used in 1.4 for the  $\Omega'_\sigma$ . - Cf. [BG], 10.5, where the analysis of the factors is finer than our motivic methods permit. There, Gross refers back to [Gr 4] § 4; cf. the analogous passages in [GS], §§ 4, 9. - The formula recalled after [GS], 10.1, implies that the two expressions  $\Delta_\mu \bar{\Delta}_\mu^{-1} \sqrt{-D}^{h_D}$  and  $\prod_{\sigma} \Delta(\Lambda_\sigma)$  - see 1.4.5 - do transform the same way under changement of the curve  $A/H$ . So, it is enough to check they are equal for one such  $A$ . This is what we have indicated, in 1.5.4, for  $D$  odd. - For curves that are no longer defined over  $H$ , see 3.2 below.

## 2. An historical aside

I am indebted to R. Szezech for pointing out to me that an analytic formula which implies 1.4.4 occurs as identity no. 163 in E. Landau's paper [La]. Thanks to Landau's bibliographical scrutiny, this paper contains references to what probably is a fairly complete history of this formula, prior to 1903.

Special cases, including  $\mathbb{Q}(\sqrt{-4})$  - the lemniscatic case - and  $\mathbb{Q}(\sqrt{-3})$  were known early in the 19<sup>th</sup> century, the main reference being Legendre's book [Le] - e.g., 1<sup>ère</sup> partie, n<sup>o</sup> 146, 147; pp. 209 f. By the middle of that century, the lemniscatic formula

of 1.4.1 above could be used, without further comment, by Eisenstein: [Ei 1], p. 186. - It is this part of the history that Chowla and Selberg were aware of when writing their papers: the announcement [CS] (see § 4 for our formula) and the final version [SC] (§§ 8, 12). - Cf. [WW], 22.8.

In the analytic proofs of 1.4.4, the  $\Gamma$ -values, or rather their logarithms, usually enter through the evaluation of

$$L'(\epsilon, 1) = - \sum_{n=1}^{\infty} \epsilon(n) \frac{\log n}{n} ,$$

via Kummer's series for  $\log \Gamma(x)$  - i.e., the identity derived in [Ku]. This part seems to have been done first by A. Berger - see [Be], p. 29/30 - as early as 1883. When Lerch rediscovered this argument in 1897 - [Ler], p. 302 f - Kronecker, using his "first" limit formula - cf. [We], VIII § 6 - , had already expressed  $L'(\epsilon, 1)$  in terms of various constants and (logarithms of) special values of theta series which correspond to the  $\Omega_{\sigma}$ 's in our notation - see [Kr], art. XVI, formula 7. Putting both parts together, Lerch deduces our identity (more precisely, its logarithm) as formula 26 of [Ler], p. 303.

Weil points out - [We 2], IX §§ 2,4 - that Lerch could have used his determination, in 1894, of the derivative at  $s=0$  of the Hurwitz zeta function, in order to relate  $L'(\epsilon, 1)$  to values of  $\log \Gamma$ . But using Kummer's series for this seems to have been closer to the taste of the day: in fact, J. de Séguier, a Jesuite professor of mathematics at the University of Angers, rediscovered this part of the proof in 1899 - see [dS 2] § 10 - although, as Landau does not fail to point out ([La], p. 177), he loses a factor of  $\frac{\pi}{2}$  along the way. de Séguier should have been especially well prepared to put together both parts of the proof because he had published, in 1894, a whole book - [dS 1] - on Kronecker's series of mémoires [Kr]. But our identity does not seem to have caught his interest.

Landau, in [La], gives new proofs of both parts of our identity. The part done first by Berger allows him to illustrate the use of divergent Dirichlet series - the theory of which he develops in the 3rd section of this paper -, see [La], 177-179.

I do not know of any occurrence of our identity between [La] and [CS], i.e., between 1903 and 1949.

Weil seems to have been the first to envisage a geometric proof of our identity - see [W III], 1976 b (and 1976 b\*). He did not succeed in general; but in a slightly later manuscript - [W III], 1977 c - he provided what Gross could then develop into an essential tool for his geometric proof of 1.4 up to factors in  $\overline{\mathbb{Q}}^*$  - see [Gr 2]. This proof tracks down periods along a family of abelian varieties which, at one point, contains (a factor of) the Jacobian of a Fermat curve - whence the  $\Gamma$ -values -, at another a power of the elliptic curve  $A$  - whence  $\Omega \cong \mathbb{Q}^*$  any  $\Omega_\sigma$ . Gross' deformation argument, in turn, provided a key step in Deligne's proof of the fundamental theorem I, 2.1.3 on absolute Hodge cycles on abelian varieties - see [DMOS], I, 4.8, 4.11 - : it enabled him to show that certain "exceptional cycles", presented in [W III], 1977 c as would-be obstructions to the Hodge conjecture, were at least absolutely Hodge. Since it is basically this theorem I, 2.1.3 - along with Anderson's motives - on which our deduction of 1.4 rests, the story of the geometric proofs has come almost full circle.

An amazing kind of a geometric revindication of the log in front of 1.4.4 which naturally comes out of the analytic proofs is provided by Falting's theory of the modular height of abelian varieties - cf. [DF], 1.5. Should the original identity really be viewed as an identity of the (logarithmic) heights of two ... ?

The attempt, in [Mor], to generalize the identity along analytic lines has, so far, not been linked to the geometric vein. The same can be said of the analogues for real quadratic fields in [Den]. If these theories have a geometric meaning it can be expected to be fairly different from the one we encountered with 1.4.

### 3. Twists and multiples

3.0 Either one of the following two properties characterize the imaginary quadratic fields among all CM fields.

3.0.1 The set of all CM types of  $K$  forms a  $\mathbb{Z}$  basis of the group of infinity types of all Hecke characters of  $K$  ;

3.0.2 Each element of  $\mathbb{Z}[\text{Hom}(K, \mathbb{C})]$  is the infinity type of a Hecke character of  $K$  .

Taking everything modulo  $\sum \sigma$  , the infinity type of the norm  $\mathbf{N}$  on  $K$  , i.e.,  $1+c$  if  $K$  is imaginary quadratic, these conditions amount to:

3.0.3 The group of all infinity types of Hecke characters of  $K$  , taken modulo  $\sum \sigma$  , is a free  $\mathbb{Z}$  module of dimension 1.

We know, from  $\mathbb{O}$ , 8.4.3, that the subgroup  $\text{St}_K$  of infinity types of Jacobi sum Hecke characters of  $K$  , taken modulo its element  $1+c$  , is precisely  $h \cdot (\mathbb{Z}[\text{Hom}(K, \mathbb{C})] / \sum \sigma)$  , for  $h$  the class number of the imaginary quadratic field  $K$  . The refinement 1.4 of Lerch's period relation 1.4.4 was deduced by writing the generator  $T_D$  of  $\text{St}_K / \sum \sigma$  as  $h \cdot \alpha \in h \cdot (\mathbb{Z}[\text{Hom}(K, \mathbb{C})] / \sum \sigma)$  . Since we are working in a onedimensional  $\mathbb{Z}$  module, the following remark is plain.

3.1 Remark. If, in the arguments 1.1 - 1.4, the character  $J_D$  is replaced by any Jacobi sum Hecke character of  $K$  (and  $n_c$  ,  $h_{D, \mu}$  are changed accordingly), then the period relations between elliptic integrals of CM type and values in  $\Gamma(\mathbb{Q})$  that one finds are all powers of 1.4, up to twisting by the norm or by finite order characters - see II § 3 - , and up to  $\Gamma$ -relations - which should all follow from II, 4.0.

3.1.1 In [St], Stern, proving a conjecture of Legendre, shows that at most  $\frac{\varphi(D)}{2}$  of the values  $\{\Gamma(\frac{j}{D}) \mid 0 < j < D\}$  are independent with respect to the relations II, 4.0.1/2. Unlike Landau - [La], p. 179 - we do not see 1.4 (or 1.4.4) as a relation which "allows to reduce the number of independent values" in this set. Instead, 1.4 goes beyond II, 4.0 in that it relates two different kinds of transcendental constants: elliptic integrals and  $\Gamma$ -values. This is also the use which is made of 1.4 in transcendence theory: to transport transcendence results from elliptic integrals to certain combinations of  $\Gamma$ -values ...

A remark similar to 3.1 also applies if we look at 1.4 from the point of view of the elliptic curves:

3.2 Let  $F$  be a finite extension of  $H$ , and  $A'/F$  an elliptic curve with complex multiplication by  $K$ . Write  $\psi'$  the Hecke character of  $F$  with values in  $K$  such that  $H_1(A') = M(\psi')$ . As in 1.5.6, let us compare  $A'$  to a curve  $A/H$ ,  $H_1(A) = M(\psi)$ , for which 1.4, with all of its constants, is assumed to be known. Then  $\psi' = \nu \cdot (\psi \circ N_{F/H})$ , for some finite order character  $\nu$  of  $F$ , with values in  $K$ . It follows, as in 1.5.7, that (assuming  $F/K$  galois, for simplicity)

$$(3.2.1) \quad \prod_{\sigma \in G(F/K)} \Omega'_\sigma \underset{K^*}{\sim} \Delta(\nu \circ i) \cdot \left( \prod_{\sigma \in G(H/K)} \Omega_\sigma \right)^{[F:H]},$$

where  $i$  is the inclusion of ideals of  $K$  into ideals of  $F$ . Then 1.4 yields, for  $D \neq 3, 4, 8$ ,

$$(3.2.2) \quad \frac{\Delta_{\mathbb{U}}^{[F:H]}}{\Delta(\nu \circ i)} \prod_{\sigma \in G(F/K)} \Omega'_\sigma \underset{K^*}{\sim} \left\{ \left( \frac{-\sqrt{D}}{\pi} \right)^{\frac{1}{2} \left( \frac{\varphi(D)}{2} - h_D \right)} \prod_{\substack{j=1 \\ \epsilon(j)=1}}^{D-1} \Gamma\left(\frac{j}{D}\right) \right\}^{[F:H]}.$$

3.3 Gross once asked me for a direct motivic interpretation of Lerch's relation 1.4.4. Remember that we have obtained 1.4.4 by mindlessly multiplying 1.4 with its complex conjugate. - I propose the following identity of periods in  $\mathbb{C}^*/\mathbb{Q}^*$  as an answer to this question:

$$(3.3.1) \quad c^+([\mathbb{R}_{K/\mathbb{Q}}^{M(u \cdot \chi^{h_D})}] |_{\mathbb{Q}}) = c^+(M(\underline{a}_D)(-n_c) \otimes K |_{\mathbb{Q}}).$$

In fact, 1.4.4 is deduced from 3.3.1, using the arguments of 1.2 - 1.4 as above, by virtue of II, 1.8.2, an analogue of II, 1.3.2 for  $c^+$ , and II, 1.6.6.

Note that, in case  $h_D = 1$ , 3.3.1 corresponds to studying the Hasse-Weil L-function of the elliptic curve  $A$ , instead of separating its CM factors  $L(\chi, s)$  and  $L(\bar{\chi}, s)$ .



## CHAPTER FOUR:

### Abelian Integrals with Complex Multiplication

In III we have studied the relations between periods of Hecke characters of imaginary quadratic fields and values of the gamma function. One aim of this chapter is to generalize these results to Hecke characters of abelian CM fields - see § 2. In order to do so, however, we first have to analyze a phenomenon which occurs for all CM fields  $K$  of degree  $[K : \mathbb{Q}] > 2$ : the monomial period relations implied by  $\mathbb{Z}$  linear relations among CM types of  $K$ . These relations were discovered by Shimura - see [Shi P], [Shi O] - ; their motivic version (up to factors in  $\overline{\mathbb{Q}}^*$ ) is already present in [DP], 8.18 - 8.23; and their motivic proof (up to  $\overline{\mathbb{Q}}^*$ ) was explained in [DB].

#### 1. Shimura's monomial relations

1.0 Let  $K$  be an algebraic number field, and  $\chi_i$ , for  $i=1, \dots, r$ , a collection of algebraic Hecke characters of  $K$  all of which take values in one number field  $E$ . Assume there are integers  $n_i$  such that

$$(1.0.0) \quad \prod_{i=1}^r \chi_i^{n_i} = \mu,$$

for some character of finite order  $\mu$ . Then, by II, 1.8.3, we get the period relation

$$(1.0.1) \quad \prod_{i=1}^r p(\chi_i)^{n_i} = p(\mu) \quad \text{in} \quad (K \otimes E \otimes \mathbb{C})^* / (E^*)^{\text{Hom}(K, \mathbb{C})} \cdot (K \otimes E)^*.$$

Furthermore, we know how to compute  $p(\mu)$  from  $\mu$ : - see II, 3.2. In particular, for all  $\sigma \in \text{Hom}(K, \mathbb{C})$ ,  $\tau \in \text{Hom}(E, \mathbb{C})$ , the complex number  $p(\mu; \sigma, \tau)$  lies in the composite of the maximal abelian extension of  $K^\sigma$  with  $E^\tau$ ;

$$(1.0.2) \quad p(\mu; \sigma, \tau) \in [(K^\sigma)^{\text{ab}} \cdot E^\tau]^* \subset \mathbb{C}^*.$$

#### 1.1 Shimura's basic relations

1.1.0 Assume that, in the situation of 1.0, each  $\chi_i$  is of weight  $-1$ , and that, for all  $\sigma \in \text{Hom}(K, \mathbb{C})$ ,  $\tau \in \text{Hom}(E, \mathbb{C})$ , the Hodge exponents  $n_i(\sigma, \tau)$  of  $\chi_i$  - see 0, § 4 - are all either  $-1$  or  $0$ ,

for all  $i = 1, \dots, r$ . Then 1.0.0, with unspecified  $\mu$  of finite order, is equivalent to a  $\mathbb{Z}$  linear relation between CM types of  $K$ . (Given  $r$  CM types of  $K$ , one has to choose  $E$  such that characters  $\chi_i$  with values in  $E$  exist, corresponding to the types.) As mentioned before - III, 3.0 - nontrivial such relations exist if and only if  $K$  contains a CM field of degree at least 4.

1.1.1 In the situation of 1.1.0, let us assume, without loss of generality, that  $E$  is a CM field, and let us fix embeddings

$K \xrightarrow{1} \mathbb{C}$ ,  $E \xrightarrow{1} \mathbb{C}$  which allow us to consider  $K$  and  $E$  as subfields of  $\mathbb{C}$ .

There exist abelian varieties  $A_i$  with complex multiplication by  $E$  defined over  $K$  such that  $H_1(A_i) = M(\chi_i)$  - see I, 4.1.1. If  $n_i(\sigma, \tau) = -1$ , there exists a holomorphic differential form  $\omega_{\sigma, \tau}^{(i)}$  on  $A_i^\sigma$ , defined over  $K^\sigma \cdot E^\tau \subset \mathbb{C}$ , such that  $e^*(\omega_{\sigma, \tau}^{(i)}) = e^\tau \cdot \omega_{\sigma, \tau}^{(i)}$ , for all  $e \in E$ , and

$$(1.1.2) \quad p(\chi_i; \sigma, \tau) = \int_{\gamma_\sigma^{(i)}} \omega_{\sigma, \tau}^{(i)}$$

(up to the usual indeterminacy), for any  $E$  basis  $\gamma_\sigma^{(i)}$  of  $H_1^\sigma(A_i)$  - cf. II, 1.5.1.

Putting  $\sigma = 1$  in 1.1.2 shows that 1.0.1 implies Shimura's basic period relations, as stated, e. g., in theorems 1.2 and 1.3 of [Shi P]. Note, however, that the passage to antiholomorphic periods is normalized differently in Shimura's paper; he simply inverts the corresponding holomorphic period, whereas we are obliged to also multiply by  $2\pi i$  - see II, 1.5.4.

The same translation also establishes propositions 1.4, 1.5, and 1.6 of [Shi P]. - Cf. [DP], 8.18 and [DB] for motivic interpretation and proof of all these relations in  $\mathbb{C}^*/\mathbb{Q}^*$ .

Instead of explicitly stating these results up to  $\overline{\mathbb{Q}}^*$ , let us discuss finer relations provided by our formalism of the  $p(\chi_i)$ .



## 1.2 Shimura's refinement

1.2.0 As before, let  $K \xrightarrow{1} \mathbb{C}$ ,  $E \xrightarrow{1} \mathbb{C}$  be embedded algebraic number fields;  $E$  a CM field. Let  $\chi$  be an algebraic Hecke character of  $K$  with values in  $E$ , and  $n(\sigma, \tau)$  its Hodge exponents. Call  $K_0 \subset K$  the fixed field of

$$\{s \in G(\overline{\mathbb{Q}}/\mathbb{Q}) \mid n(s1, \tau) = n(1, \tau), \text{ for all } \tau \in \text{Hom}(E, \mathbb{C})\}.$$

Likewise, let  $E_0 \subset E$  be the fixed field of

$$\{s \in G(\overline{\mathbb{Q}}/\mathbb{Q}) \mid n(\sigma, s1) = n(\sigma, 1), \text{ for all } \sigma \in \text{Hom}(K, \mathbb{C})\}.$$

$K_0$  and  $E_0$  are each either  $\mathbb{Q}$  or a CM field. From their definition, it follows that  $n$  descends to a function

$$n_0 : \text{Hom}(K_0, \mathbb{C}) \times \text{Hom}(E_0, \mathbb{C}) \rightarrow \mathbb{Z}$$

such that

$$(*) \quad n(\sigma, \tau) = n_0(\sigma|_{K_0}, \tau|_{E_0}),$$

for all  $\sigma \in \text{Hom}(K, \mathbb{C})$ ,  $\tau \in \text{Hom}(E, \mathbb{C})$ .

The following constructions will only depend on the function  $n_0$ , or equivalently, on the algebraic homomorphism

$$t : R_{K_0/\mathbb{Q}} \mathbb{G}_m \rightarrow R_{E_0/\mathbb{Q}} \mathbb{G}_m$$

defined by the  $n_0(\sigma_0, \tau_0)$ 's - see  $\textcircled{0}$ , § 2(c), and  $\textcircled{0}$ , § 4. For all finite extensions  $K_0 \subset L$  and  $E_0 \xrightarrow{i} F$ , the extended algebraic homomorphism

$$i \circ t \circ N_{L/K_0} : R_{L/\mathbb{Q}} \mathbb{G}_m \rightarrow R_{F/\mathbb{Q}} \mathbb{G}_m$$

is given by the function  $n$  on  $\text{Hom}(L, \mathbb{C}) \times \text{Hom}(F, \mathbb{C})$  defined by equation (\*). Thus, for  $L = K$ ,  $F = E$ , we find the infinity type of the Hecke character  $\chi$ . Writing  $w$  its weight, we have, for  $c = \text{complex conjugation}$ , that

$$n_0(\sigma_0, \tau_0) + n_0(\sigma_0, c\tau_0) = w ,$$

independent of  $(\sigma_0, \tau_0)$  .

1.2.1 There exists a finite extension  $E_0 \xrightarrow{i} F$  ,  $F$  a CM field, and a Hecke character  $\chi_0$  of  $K_0$  with values in  $F$  and infinity type  $i \circ t$  . - Furthermore, there exists a finite abelian extension  $L$  of  $K_0$  such that  $\tilde{\chi} = \chi_0 \circ N_{L/K_0}$  takes values in  $E_0$  .

1.2.2 Taken modulo  $[(K_0^{\tilde{\sigma}})^{\text{ab}} \cdot E_0^{\tau_0}]^*$  , the period  $p(\tilde{\chi}; \tilde{\sigma}, \tau_0)$  - for  $\tilde{\sigma} \in \text{Hom}(L, \mathbb{C})$  ,  $\tau_0 \in \text{Hom}(E_0, \mathbb{C})$  - depends only on  $t, \sigma_0 := \tilde{\sigma}|_{K_0}$  , and  $\tau_0$  . It will therefore be written

$$p(t; \sigma_0, \tau_0) \in \mathbb{C}^* / [(K_0^{\sigma_0})^{\text{ab}} \cdot E_0^{\tau_0}]^* .$$

If  $\chi_i$  , for  $i=1, \dots, r$  , are as in 1.1.0, and such that

$$(1.2.3) \quad \mu \cdot \chi = \prod_{i=1}^r \chi_i ,$$

for some  $\mu$  of finite order, then

$$(1.2.4) \quad p(t; 1, \tau_0) \underset{\mathbb{Q}^*}{\sim} \prod_{i=1}^r p(\chi_i; 1, \tau) ,$$

for any extension  $\tau$  of  $\tau_0$  to the common field of values  $E$  of  $\chi$  and the  $\chi_i$  . - Again, all that matters here, are the infinity types of the  $\chi_i$  ...

Relation 1.2.4 almost establishes conjecture 1.7 in [Shi P]. The only difference is that Shimura wants the period which we denote by  $p(t; 1, \tau_0)$  to be well defined up to  $[(K_0^{\text{ab}} \cdot E_0^{+\tau_0})^*]$  , where  $E_0^+$  is the maximal totally real subfield of  $E_0$  . To achieve this, we simply repeat remark 8.22 of [DP] in our context:

First, note the naturality of the formation of  $p(t)$  , which is easily proved on the level of Hecke characters, by transport of structure - cf. [DP], 8.18.4:

1.2.5 If  $\alpha : K'_0 \xrightarrow{\cong} K_0$  and  $\beta : E_0 \xrightarrow{\cong} E'_0$  are isomorphisms of fields, then

$$p(\beta\alpha; \sigma_0\alpha, \tau_0\beta^{-1}) = p(t; \sigma_0, \tau_0) .$$

From this, II, 1.6.6, and the fact that complex conjugation  $c$  induces a well defined automorphism of  $K_0$  as well as  $E_0$ , we obtain the equations

$$(1.2.6) \quad \overline{p(t; \sigma_0, \tau_0)} = p(t; c\sigma_0, c\tau_0) = p(t; \sigma_0 c, \tau_0 c) \\ = p(ctc; \sigma_0, \tau_0) = p(t; \sigma_0, \tau_0) .$$

(Note that  $(K_0^\sigma)^{ab}$  is stable under  $c$ , even though  $c$  does not in general commute with other automorphisms of  $(K_0^\sigma)^{ab}$ !)

Thus, by Hilbert 90, the periods  $p(t; \sigma_0, \tau_0)$  are represented by real numbers; they are well determined up to a factor in  $[(K_0^\sigma)^{ab} E_0^{\tau_0}]^*$ .

1.2.7 Remark. D. Blasius has informed me that he has not only found the above results independently, by the motivic formalism; but that he has also managed to prove conjecture 1.7 of [Shi P] adapting Shimura's proof - as in [Shi P], section 5 - , thus improving upon his 1981 Princeton thesis (unpublished) in which a partial result was obtained.

Let us now go on to examine a few standard properties of the  $p(t; \sigma_0, \tau_0)$  - cf. [DP], 8.18 for the relations up to  $\overline{\mathbb{Q}}^*$ .

1.2.8 For finite extensions  $K_0 \subset K'_0$ ,  $E_0 \xrightarrow{i} E'_0$ ,  $E_0$  a CM field, put

$$p(i \circ t \circ N_{K'_0/K_0}; \sigma', \tau') = p(t; \sigma' |_{K_0}, \tau' |_{E_0}) ,$$

but consider the left hand side as being well determined up to  $[(K'_0)^{\sigma'}]^{ab} (E_0^{\tau'})^*$ . - Cf. II, 1.8.6/8.

1.2.9 If  $u$  and  $u'$  are two algebraic homomorphisms  $K^* \rightarrow E^*$ , both satisfying  $n^{(i)}(\sigma, \tau) + n^{(i)}(\sigma, c\tau) = cst$ , independent of  $(\sigma, \tau)$ , then - in the sense of 1.2.8 - ,

$$p(u; \sigma, \tau) \cdot p(u'; \sigma, \tau) = p(u \cdot u'; \sigma, \tau) .$$

1.2.10 From 1.2.9 and II, 1.8.4, we get:

$$p(t; \sigma_0, \tau_0) \cdot p(t; \sigma_0, c\tau_0) = (2\pi i)^w ,$$

where

$$w = n_0(\sigma_0, \tau_0) + n_0(\sigma_0, c\tau_0).$$

1.2.11 "Reflex Principles"

"Reflex principles" - like theorem 2.3 of [Shi P]; theorem 1.2 of [Shi O]; proposition 8.20 of [DP] - are formal consequences of the general formalism of the periods concerned - i.e., in our case, of formulas 1.2.5, 1.2.8 - 1.2.10. They typically serve to trace the periods through arguments in which the rôles of the fields  $K_0$  and  $E_0$  above are interchanged. See, for instance, the use that Deligne makes of [DP], 8.20; cf. [HS], p. 35.

Theorem 1.2 in [Shi O] has the advantage of being particularly simple and general. But it seems to require a formalism of periods that are essentially only determined up to a factor in  $\bar{\mathbb{Q}}^*$ . - So, we content ourselves with a refined version of [DP], 8.20.

Let  $t^* : R_{E_0/\mathbb{Q}} \mathbb{G}_m \rightarrow R_{K_0/\mathbb{Q}} \mathbb{G}_m$  be the algebraic homomorphism defined by the invariants  $n_0^*(\tau_0, \sigma_0) = n_0(\sigma_0, \tau_0)$ . Define  $p(t^*; \tau_0, \sigma_0)$  as in 1.2.1/2, via  $p(\tilde{\chi}^*; \tilde{\tau}, \sigma_0)$ , for a suitable Hecke character  $\tilde{\chi}^*$  of some finite abelian extension  $F^*$  of  $E_0$  with values in  $K_0$ .

Assume, as in II, 1.8.1, that none of the  $n_0(\sigma_0, \tau_0)$ 's equals

$\frac{1}{2} w = \frac{1}{2}(n_0(\sigma_0, \tau_0) + n_0(\sigma_0, c\tau_0))$ . Then we have the following identity of classes in  $\mathbb{C}^*/(M^{ab})^*$ , where  $M$  is the smallest Galois extension of  $\mathbb{Q}$  contained in  $\mathbb{C}$  which contains both  $K_0$  and  $E_0$ .

$$(1.2.12) \quad \prod_{\substack{\sigma_0 \\ n_0(\sigma_0, 1) < \frac{w}{2}}} p(t^*; 1, \sigma_0) = \prod_{\substack{\sigma_0 \\ n_0(\sigma_0, 1) < \frac{w}{2}}} p(t; \sigma_0, 1) .$$

The proof is straightforward - cf. [DP], 8.20. - Further refinements of 1.2.12 may be treated using the construction of Blasius mentioned in II, 2.2 ...

1.2.13 For examples of these period relations, the reader should consult section 2 of [Shi P] . - As will be indicated in § 2, the precise relations of the form (1.0.1) are liable, in principle, to yield much more information in concrete circumstances (in particular, if  $\mu$  can be computed) than the coarser but more flexible periods  $p(t; \sigma, \tau_0)$  with their increased indeterminacy.

## 2. Abelian integrals and the gamma function

2.0 In this section we consider a finite imaginary abelian extension  $K$  of  $\mathbb{Q}$ , with Galois group  $G$ . We fix a privileged embedding  $K \xrightarrow{1} \mathbb{C}$ .

2.1 Let  $\chi$  be any algebraic Hecke character of  $K$  (with values in some number field  $E$ ). Write its infinity type as

$$(2.1.0) \quad t = \sum_{\sigma \in G} n_{\sigma} \sigma \in \mathbb{Z}[G] .$$

By  $\mathbb{Q}$ , 8.4.2, there exists for  $\chi$ ,

- a smallest positive (or: nonzero of smallest absolute value) integer  $h$ , such that there is
- an element  $\underline{a} \in \mathbb{B}_K^0$ , - see  $\mathbb{Q}$ , 8.2.1 -
- and a character of finite order  $\mu$  of  $K$ ,

satisfying,

$$(2.1.1) \quad \mu \cdot \chi^h = J_K(\underline{a}) .$$

Thus,  $\mu \cdot \chi^h$  takes values in  $K$ , and via I, 5.1 we can translate 2.1.1 into the period relation

$$(2.1.2) \quad p(\mu \chi^h; 1) = (\Gamma(\sigma c \underline{a})^{-1})_{\sigma \in G} \in (K \otimes \mathbb{C})^*/K^* ,$$

using II, 4.2.3 and II, 4.4.2.

2.1.3 The left hand side of 2.1.2 is easily expressed in terms of  $p(\mu)$  and  $p(\chi)$ ; so any information one has about  $\mu$  can, in principle, serve to relate  $p(\chi)$  to values of the gamma function.

But already the very explicit case discussed in chapter III shows that  $\mu$  will usually not be easy to determine from  $\chi$ ,  $h$  and  $\underline{a}$ .

2.1.4 Therefore, let us now discard finite order characters in 2.1.1. Then  $\underline{a}$  is determined by  $\chi$  and  $h$  up to addition of an element  $\underline{b} \in \mathbb{B}_K^0$  such that  $J_K(\underline{b})$  is of finite order. In other words,  $\underline{a}$  is such that, for all  $\sigma \in G$ , one has

$$(2.1.5) \quad \frac{1}{h} \langle \sigma^{-1} \underline{a} \rangle = n_\sigma .$$

By II, 4.4.5, this determines  $(\Gamma(\sigma \underline{ca}))_{\sigma \in G}$  up to a factor in  $(K^{ab})^* \subset \bar{\mathbb{Q}}^* \subset \mathbb{C}^*$ .

Using 1.2.6 and the notation of 1.2.8 above, it follows for all infinity types  $t$  as in 2.1.0 and all  $\underline{a} \in \mathbb{B}_K^0$  with 2.1.5 that, for all  $\sigma \in G$ , we have the period relation in  $\mathbb{R}^*/(K^{ab} \cap \mathbb{R})^*$ ,

$$(2.1.6) \quad p(t; 1, \sigma)^h = \Gamma(\sigma \underline{ca})^{-1} .$$

2.1.7 Considering both sides of 2.1.6 as representing classes in  $\mathbb{C}^*/\bar{\mathbb{Q}}^*$  this relation verifies Gross' period conjecture - see [Gr 2] § 4 - for all motives in  $\mathcal{M}_{\bar{\mathbb{Q}}}^{av}(K)$  of rank 1 over  $K$ . In fact, all these motives are determined by their Hodge realization (say, at  $\text{id}_{\bar{\mathbb{Q}}} : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ ); but they all come from motives of the form  $M(\chi)$  in  $\mathcal{M}_K^{av}(E)$ , for some  $\chi$  as above - see I, 5.3; I, 6.1.4, and I, 6.1.6. - This remark covers (and therefore indicates possible refinements of) the first two examples discussed by Gross in [Gr 2] p. 206/7; as for example 3 (p. 207/8), a motivic version of it will be established in chapter V below.

2.1.8 Suppose  $\mathcal{C}$  is a class of (smooth projective)  $\bar{\mathbb{Q}}$ -algebraic varieties for each of which one can show that every Hodge cycle on it is an absolute Hodge cycle. For  $L \subset \bar{\mathbb{Q}}$ , let  $\mathcal{M}[\mathcal{C}]_L$  be the smallest Tannakian subcategory of  $\mathcal{M}_L$  which contains  $\mathcal{CM}_L$  as well as all motives of the form  $h(X_L)$ , where  $X_L$  is a variety over  $L$  which becomes isomorphic, over  $\bar{\mathbb{Q}}$ , to a variety in  $\mathcal{C}$ . Then Gross' period conjecture holds for the Hodge realizations of all motives in

$\mathcal{M}[\mathcal{C}]_{\mathbb{Q}}(K)$  of rank 1 over  $K$ . Moreover, the results of this section would then extend to all motives in  $\mathcal{M}[\mathcal{C}]_K(E)$  attached to Hecke characters  $\chi$  of  $K$ , in the sense of I, 3.3. - This follows from our proof of I, 5.1; from I, 5.3 and I § 6.

2.1.9 We saw in 2.1.5 that working modulo finite order characters in 2.1.1 allows to express  $\underline{a}$  in terms of  $\chi$  and  $h$ . As for  $h$ , let us simply remark that it may always be replaced with the index  $[A_K : \text{St}_K]$  of  $\mathfrak{O}$ , 8.4. We refer to [Sin] for a number of results on this index.

2.2 Looking at things the other way around, let now  $\underline{a} \in \mathbb{B}_K^0$  be given. Then there exist

- a positive integer  $r$
- algebraic Hecke characters  $\chi_1, \dots, \chi_r$  of  $K$  like in 1.1.0 above
- integers  $n_1, \dots, n_r$
- a character of finite order  $\mu$  of  $K$

such that

$$(2.2.1) \quad \mu \cdot \prod_{i=1}^r \chi_i^{n_i} = J_K(\underline{a}) .$$

Let  $E$  be a finite extension of  $K$  which is a common field of values for  $\mu$  and all the  $\chi_i$ . Write  $j$  the inclusion  $(K \otimes \mathbb{C})^* \hookrightarrow (E \otimes \mathbb{C})^*$ . Then 2.2.1 translates into the period relation

$$(2.2.2) \quad p(\mu; 1) \cdot \prod_{i=1}^r p(\chi_i; 1)^{n_i} = j((\Gamma(\sigma \underline{ca})^{-1})_{\sigma \in G}) .$$

By section 1 above, this relation "does not depend" on the particular choice of  $\chi_i$ 's - in the sense that all such relations arising from  $\underline{a}$  can be seen to be equivalent without recourse to the right hand sides. - Conversely, all  $\Gamma$ -relations arising from comparing the left hand sides of two instances of 2.2.2, for two different  $\underline{a}, \underline{a}' \in \mathbb{B}_K^0$ , via 1.0.1 already follow from II, 4.4.5, and are therefore, if taken modulo  $\overline{\mathbb{Q}}^*$ , compatible with Rohrlich's conjecture - see II, 4.0 - , by virtue of the remark we made following theorem

II, 4.4.5. Shimura's feelings about this, expressed at the end of [Shi P] § 4, are therefore proven to have been correct.

We leave it to the reader to write weaker versions of 2.2.2, neglecting finite order characters in 2.2.1.

### 2.3 Biquadratrix

Let  $K \hookrightarrow \mathbb{C}$  be an abelian imaginary field of degree four over  $\mathbb{Q}$ .

2.3.1 If  $K$  is cyclic, then there is a simple abelian variety  $A$  with complex multiplication by  $K$ , defined, say, over  $K^{ab}$ , and all such simple abelian surfaces are isogenous to some conjugate of  $A$ . 2.2.2 relates certain products of  $\Gamma$  values to periods of the  $A^\sigma$ 's. E. g., when  $K = \mathbb{Q}(u_5)$ , one finds precisely the well known expressions of the periods of the Jacobian of  $X^5 + Y^5 = 1$  in terms of  $\Gamma$  - cf. II, 4.1, or Rohrlich's appendix to [Gr 2].

2.3.2 If  $K$  is not cyclic, call  $K^+$  its real quadratic subfield, and  $F_1, F_2$  the two distinct imaginary quadratic subfields of  $K$ . Every abelian variety  $A$  with complex multiplication by  $K$  is isogenous to the product of two CM elliptic curves - cf. [Sch A] for the exceptional rôle that such  $K$  play among all CM fields. All Hecke characters  $\chi$  of  $K$  like in 1.1.0 can be written as

$$(2.3.3) \quad \chi = \mu(\varphi_i \circ N_{K/F_i}) ,$$

for some  $i \in \{1, 2\}$ , and some Hecke character  $\varphi_i$  of  $F_i$  of infinity type  $\sigma_i$ , for some embedding  $\sigma_i : F_i \hookrightarrow \mathbb{C}$ ; and  $\mu$  some character of finite order of  $K$ . The only nontrivial relations of the form 1.0.0 over  $K$  are consequences of the relations  $\varphi_i \bar{\varphi}_i = N^{-1}$  over  $F_i$ . Thus, the corresponding period relations reproduce Legendre's relation on elliptic curves with complex multiplication by  $F_i$  - cf. II, 1.8.5. Similarly, it is easy to check that 2.2.2 does not produce any new relations between the elliptic integrals with CM by  $F_1$  or  $F_2$ , and  $\Gamma$  values, beyond what follows from II § 4 and III, 1.4 - cf. III § 3.



## CHAPTER FIVE:

### Motives of CM Modular Forms

Let  $K$  be a CM field with maximal totally real subfield  $K_0$ . Given a Hecke character  $\chi$  of  $K$  with corresponding theta series  $f$  (a Hilbert modular new-form relative to  $K_0$ ), there should be a motive  $M(f)$  for  $f$  whose periods could be computed in terms of special values of  $L^*(\chi, s)$ . If one could not only construct the motive  $M(f)$  in  $\mathcal{M}_{\mathbb{Q}}(E_0)$  - with  $E_0$  the field generated by the Fourier coefficients of  $f$  - but also show that, since  $f$  comes from  $\chi$ ,  $M(f)$  lies in  $\mathcal{M}_{\mathbb{Q}}^{\text{av}}$ , and in fact in  $\mathcal{CM}_{\mathbb{Q}}$ , then theorem I, 5.1 would allow to compare  $M(f)$  and  $M(\chi)$ , and thereby yield II, 2.1 for  $\chi$  (or closely related characters), provided certain non vanishing results are available, about the special values of  $L(\chi, s)$  mentioned before.

This hypothetical "modular proof" of II, 2.1 seems a long way off at the moment - cf. Oda's work [Od 1], [Od 2]. However, it provides the romantic background for what we actually prove in this chapter: First of all, we consider only the case that  $K$  is imaginary quadratic. In this case, recent observations of U. Jannsen's, in connection with his more general theory of mixed motives, easily give us the actual motive  $M(f)$  whose realizations were already described in [DP] § 7 - this is discussed in § 1 below. Then, after introducing the theta series for Hecke characters of  $K$  we do prove, in § 2, that  $M(f)$  lies in  $\mathcal{CM}_{\mathbb{Q}}$ . But in order to do so, we have to use II, 2.1 for Hecke characters of  $K$  - in this case the theorem was first proved in [GS], [GS'].

#### 1. Motives for modular forms

Let  $k \geq 0$  and  $N \geq 1$  be integers. Let  $f(z) = \sum_{n \geq 1} a_n q^n$  ( $q = e^{2\pi iz}$ ) be a newform on  $\Gamma_0(N)$  of weight  $k+2$  with character  $\epsilon$ , which is an eigenfunction for the Hecke operators  $T_p$ ,  $p$  prime,  $p \nmid N$ :

$$T_p f = a_p f; \quad a_1 = 1.$$

Put  $E_0 = \mathbb{Q}(a_n | (n, N) = 1) \hookrightarrow \mathbb{C}$ . It is known that  $E_0$  is a number field of finite degree over  $\mathbb{Q}$ .

1.1 Theorem [Eichler-Shimura-Deligne-Jannsen]

There exists a motive  $M(f)$  in  $\mathcal{M}_{\mathbb{Q}}(E_0)$  of rank two over  $E$  such that

$$L_N^*(M(f), s) := \left( \prod_{p \nmid N} \det_{E_0} (1 - F_p \cdot p^{-s} | H_{\ell}(M(f)))^{-\tau} \right)_{\tau} = \left( \sum_{\substack{n=1 \\ (n, N) = 1}} a_n^{\tau} n^{-s} \right)_{\tau},$$

where  $\tau \in \text{Hom}(E_0, \mathbb{C})$ ,  $\ell \neq p$  prime,  $F_p$  is a geometric Frobenius element at  $p$ , and  $\text{Re } s \gg 0$ .

The proof of this theorem is indicated in [Ja], Cor. 1.4, building upon [DR] and [DP] § 7. Let us sketch very briefly how one can show that the realizations written in [DP], 7.6 actually are realizations of a motive in  $\mathcal{M}_{\mathbb{Q}}$ , by using a somewhat different argument - which, however, was also suggested to me by Jannsen.

1.1.1 Write  $A_0 = Y_1(N)$ , and  $\bar{A}_0 = X_1(N)$  the modular curves without, and with cusps. Suppose  $N \geq 3$  and denote by  $\pi_1 : A_1 \rightarrow A_0$  the universal elliptic curve. Put

$$A_k = \underbrace{A_1 \times_{A_0} \dots \times_{A_0} A_1}_{k \text{ factors}}$$

if  $k \geq 2$ . Choose a smooth compactification  $\bar{A}_k$  of  $A_k$  - cf., for instance [DR], 5.5. Let  $Z$  be a desingularization of  $\bar{A}_k \setminus A_k$ . Then, by [DH III], Cor. 8.2.8, one has in the diagram

$$\begin{array}{ccccc} H_c^{k+1}(A_k, \mathbb{Q}) & \xrightarrow{\beta} & H^{k+1}(\bar{A}_k, \mathbb{Q}) & \xrightarrow{\alpha} & H^{k+1}(\bar{A}_k \setminus A_k, \mathbb{Q}) \\ & & \searrow \tilde{\alpha} & & \downarrow \\ & & & & H^{k+1}(Z, \mathbb{Q}) \end{array}$$

that

$$H_!^{k+1}(A_k, \mathbb{Q}) := \text{Im}(\beta) = \ker(\alpha) = \ker(\tilde{\alpha}) .$$

As  $H^{k+1}(\overline{A}_k)$  and  $H^{k+1}(Z)$  are honest regard motives, and  $\tilde{\alpha}$  comes from an absolute Hodge cycle  $\tilde{\alpha} : H^{k+1}(\overline{A}_k) \rightarrow H^{k+1}(Z)$ , its kernel, too, defines a motive in  $\mathcal{M}_{\mathbb{Q}}$ , inside of which one now continues to cut out the desired submotive:

1.1.2 By Lieberman 's trick - see [DR], 5.3 -, preferably modified by taking the part where  $[m_1] \times \dots \times [m_k]$  act as  $m_1 \dots m_k$ , for sufficiently many collections of integers  $(m_1, \dots, m_k)$ , one obtains a motive with realizations

$$H_!^1(A_o, (R^1\pi_{k,*}\mathbb{Q})^{\otimes k}) .$$

1.1.3 Next, take invariants under the action of the symmetric group  $S_k$ , and finally pass to the submotive of

$$H_!^1(A_o, \text{Sym}^k(R^1\pi_{k,*}\mathbb{Q})) = H^1(\overline{A}_o, j_*\text{Sym}^k(R^1\pi_{k,*}\mathbb{Q}))$$

annihilated by the kernel of the homomorphism of the Hecke algebra  $\mathbb{T}$ ,

$$\begin{array}{lcl} \mathbb{T} & \rightarrow & E_o \\ T_p & \rightarrow & a_p . \end{array}$$

This produces the realizations described in [DP], 7.6.

1.1.4 In case  $N$  was 1 or 2, or if we want to construct motives for modular forms on more general congruence subgroups, one has to close the construction of  $M(f)$  by passing to the invariants under a finite subgroup.

It is fairly clear that, at every stage, we have only applied absolute Hodge cycles in cutting out the next smaller motive. But we leave the details to the reader.

The following problem seems to be unsolved.

1.2 Problem. Show that for "generic"  $f$  of weight  $k+2 \geq 3$  - and in particular for  $\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$  ? -  $M(f)$  does not lie in  $\mathcal{M}_{\mathbb{Q}}^{\text{av}}(E_0)$ .

1.2.1 For  $k = 0$ , that is, if  $f$  has weight two,  $M(f)$  "is" essentially the abelian variety attached to  $f$  by Shimura - see, e.g., [Shim], Thm. 7.14. So,  $M(f) \in \mathcal{M}_{\mathbb{Q}}^{\text{av}}$  for  $k = 0$ .

## 2. CM modular forms

2.0 Let  $K$  be an imaginary quadratic field, embedded  $K \xrightarrow{1} \mathbb{C}$  in a fixed way, and write  $-D$  the discriminant of  $K$ . Let  $\chi$  be an algebraic Hecke character of  $K$ , of conductor  $\hat{\epsilon}$ , with infinity type  $w \cdot 1$ , for some  $w \geq 1$ . Denote by  $E \supset K$  the number field generated by the values of  $\chi$ . Write the theta series  $f$  attached to  $\chi$ , and an embedding  $\tau : E \hookrightarrow \mathbb{C}$ :

$$f^{\tau}(z) = \sum_{\substack{\mathfrak{a} \\ (\mathfrak{a}, \hat{\epsilon})=1}} \chi^{\tau}(\mathfrak{a}) q^{N\mathfrak{a}} = \sum_{n \geq 1} a_n^{\tau} q^n,$$

where  $\mathfrak{a}$  runs over all integral ideals of  $K$  prime to  $\hat{\epsilon}$ , and

$a_n = \sum_{N\mathfrak{a}=n} \chi(\mathfrak{a})$  - thus,  $a_1 = 1$ . - By Hecke, [He], n<sup>o</sup> 23, 27,  $f^{\tau}(z)$  is, for each  $\tau$ , a newform of weight  $w+1$  (i.e.,  $k = w-1$ , in the notation of § 1) on  $\Gamma_0(N)$ , with  $N = D \cdot N\hat{\epsilon}$ , and character  $\epsilon$  given on prime numbers  $p \nmid N$  by

$$\epsilon(p) = \left(\frac{-D}{p}\right) \frac{\chi(p \cdot \sigma_K)}{p^w}.$$

2.1 As in 1, call  $E_0$  the field  $\mathbb{Q}(a_n \mid (n, N) = 1)$ , and write  $\mathbb{Q}(\epsilon)$  the field generated by the values of  $\epsilon$ .

2.1.1 Lemma (i)  $E = K \cdot E_0$

(ii)  $E_0 \supset \mathbb{Q}(\epsilon)$ .

The proof is left to the reader.

Three cases occur, as far as the constellation  $E_0 \subset E$  is concerned:

2.1.2  $E_0$  is totally real.

This is the same as saying that  $E$ , which is naturally a CM field, has  $E_0$  as its maximal totally real subfield. Also, like in 2.1.1 (i), one sees that 2.1.2 occurs if and only if  $\chi$  is equivariant with respect to complex conjugation, i. e.,

$$(2.1.2)' \quad \chi(\bar{a}) = \overline{\chi(a)},$$

for all  $(a, Nf) = 1$ .

Note also that  $\epsilon = 1$  implies 2.1.2, and that 2.1.2 in turn yields  $\epsilon^2 = 1$  (but not necessarily  $\epsilon = 1$ , as is shown by the example of  $\chi$  with (2.1.2)' of conductor (1).)

2.1.3  $E = E_0$ .

By 2.1.1 (i), if  $E_0$  is not totally real,  $E_0$  has to be a CM field. But NOT 2.1.2 does not imply 2.1.3 because it may also be that:

2.1.4  $E \neq E_0$  and  $E_0$  is not totally real.

As an example (pointed out to me by J. Tilouine), take  $\chi$  satisfying (2.1.2)', and  $\nu$  a Dirichlet character of  $\mathbb{Q}$  such that  $K \not\subset \mathbb{Q}(\nu(Na) \mid a \text{ ideal of } K)$ . Then, for the twist  $\chi \cdot (\nu \circ N_{K/\mathbb{Q}})$ , one is in case 2.1.4.

2.2 Proposition Let  $f$  be as in 2.0 and  $E_0$  as defined in 2.1. Let  $M(f)$  be the motive attached to  $f$  by 1.1. Then there is a natural embedding

$$K \otimes_{\mathbb{Q}} E_0 \hookrightarrow \text{End}_K(M(f) \times K),$$

inducing on  $1 \otimes E_0$  the coefficient structure of  $M(f) \in \mathcal{M}_{\mathbb{Q}}(E_0)$ , and such that, for every idempotent  $e$  of  $K \otimes E_0$  with  $e(K \otimes E_0) \cong E$ , the direct factor  $e(M(f) \times K)$  of  $M(f) \times K$  is a motive either for  $\chi$  or for the complex conjugate  $\bar{\chi}$ , in the sense of I, 3.3.

Proof. We shall essentially generalize Shimura's proof of theorem 1 in [Shi E]. (For the broader perspective of this method cf. [Shi F], [Ri 1], [Mom], and [Ri 2].)

First assume that the conductor  $f$  of  $\chi$  satisfies:

$$(2.2.0) \quad D|f \quad \text{and} \quad \bar{f} = f .$$

In analogy to the definition of the Hecke operators, the double class  $\Gamma_1(N) \delta \Gamma_1(N)$ , with

$$\delta = \begin{pmatrix} 1 & \frac{1}{D} \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Q}) ,$$

induces an algebraic correspondence on  $X_1(N) \times X_1(N)$ . This can be lifted to  $A_K$  and closed up in  $\bar{A}_K$ , and actually induces an (absolute-Hodge-) endomorphism  $\Delta$  of the direct sum  $\bigoplus_{\lambda} M(f_{\lambda})$ , with  $\lambda$  running through all algebraic Hecke characters of  $K$  defined modulo  $\mathfrak{f}$ , defined over  $\mathbb{C}$ . To see this, note that for any such  $\lambda$ , the corresponding theta series

$$f_{\lambda}(z) = \sum_{(\alpha, f) = 1} \lambda(\alpha) \cdot q^{N\alpha}$$

can be written

$$f_{\lambda}(z) = \sum \lambda(b)^{-1} F_b(z) ,$$

where  $b$  varies over a fixed system of integral ideals of  $K$  representing the ray classes of ideals prime to  $f$  modulo principal ideals  $(\alpha)$  generated by elements  $\alpha \equiv 1 \pmod{f}$  in  $K^*$ , and where

$$F_b(z) = \sum_{\substack{(\alpha) \\ \alpha \in b}} \alpha^w \cdot q^{N(\alpha) / N(b)} .$$

Thus, by the Shimura isomorphism,  $H_{\text{DR}}(\bigoplus_{\lambda} M(f_{\lambda})/\mathbb{C})$  is generated by the  $F_b$ 's (and their antiholomorphic counterparts); but on them  $\Delta$  acts via

$$(2.2.1) \quad \Delta_{\text{DR}}(F_b)(z) = \sum_{j=1}^r F_b|_{w+1}(\delta\gamma_j) = r \cdot (e^{2\pi i/D})^{\frac{N\alpha}{N'b}} \cdot F_b,$$

if  $\Gamma_1(N)\delta\Gamma_1(N) = \bigcup_{j=1}^r \Gamma_1(N)\delta\gamma_j$ . Here, as  $D|f$ , by 2.2.0,

$N\alpha/N'b$  is independent of the choice of  $\alpha \in b$ ,  $\alpha \equiv 1 \pmod{f}$ .

2.2.2 Note that, for the case 2.2.0, it would have been sufficient to let  $\lambda$  above vary over the characters of precise conductor  $f$ . But, for future reference, we throw in all  $\lambda$  defined modulo  $f$  - understanding that, for them, the motive denoted  $M(f_\lambda)$  is, for once, not the motive  $M(f_\lambda)$  constructed in § 1, but rather the motive obtained by the procedure sketched in 1.1.1-3 for  $f_\lambda$ , considered as an eigenform on  $\Gamma_1(N)$  - where it is not a newform. The proper motive for  $f_\lambda$  in the sense of § 1 is found inside the current  $M(f_\lambda)$  as the invariants under the finite group which is the quotient of  $\Gamma_1(N)$  by the group  $\Gamma_1(M)$  on which  $f_\lambda$  "is new" - cf. 1.1.4.

2.2.3 From 2.2.1, we see that we can embed  $\mathbb{Q}(e^{2\pi i/D})$  into  $\text{End}/\mathbb{C}(\oplus_\lambda M(f_\lambda))$  by sending  $e^{2\pi i/D}$  to  $r^{-1} \cdot \Delta$ . The imaginary quadratic field  $K \subset \mathbb{Q}(e^{2\pi i/D})$  then induces endomorphisms which stabilize  $M(f) \subset \oplus M(f_\lambda)$ , as is checked again on the de Rham realization, using 2.2.1. Next, it is easy to see that this embedding

$$K \hookrightarrow \text{End } M(f)$$

is defined over  $K$ .

Moreover, the commutator of the matrices  $\begin{pmatrix} 1 & 1/D \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ 0 & 1/p \end{pmatrix}$  lies in  $\Gamma_1(N)$  if the prime  $p$  splits completely in  $\mathbb{Q}(\mu_D)$ . As  $T_p$  is 0 on  $M(f)$  for all  $p$  that stay prime in  $K$ , we see that the action of  $K$  on  $M(f) \times K$  we defined commutes with the action of  $E_0$ .

2.2.4 Thus we know - in case 2.2.0 - that  $M(f) \times K$  is of rank 1 over  $K \otimes E_0$ . Since the actions of  $K$  and  $E_0$  come from algebraic correspondences, we see that the one dimensional Galois representations of  $e(M(f) \times K)$  form a strictly compatible system of  $E$ -rational  $\lambda$ -adic representations - where  $\lambda$  now denotes finite

places of  $E$ . In view of the L-function of  $M(f)$  over  $\mathbb{Q}$ , the Hecke character defined by  $M(f) \times K$  - see I, 1.4 - has to be either  $\chi$  or  $\bar{\chi}$ .

2.2.5 If  $f$  is arbitrary, we consider  $\chi$  as a character defined modulo  $D \cdot f \cdot \bar{f}$ . It occurs then as one of the imprimitive characters  $\lambda$  of the above argument, and 2.2.3 and 2.2.4 show that " $M(f)$ " =  $M(f_\lambda)$  - in the sense 2.2.2 - has the required  $K \otimes E_0$  structure. Now, this  $K$ -action clearly commutes with the finite group  $G$  such that  $M(f_\lambda)^G$  is the proper motive  $M(f)$  of the newform  $f$ .

q.e.d.

2.3.0 Let  $\psi$  be any Hecke character of  $K$ , with values in some CM field  $E'$ , of conductor  $f'$ , with infinity type 1, such that  $L^*(\psi, 1) \in (E' \otimes \mathbb{C})^*$ . The existence of such  $\psi$  is most easily deduced from the fact that the modular symbols generate the first homology of the modular curves - see [Shi M], theorem 2. Using this argument, we have already passed to the newform  $g(z) = \sum \psi(\mathfrak{a}) q^{N\mathfrak{a}} = \sum b_n q^n$  on  $\Gamma_0(N')$ ,  $N' = D \cdot N f'$ , associated to  $\psi$  as  $f$  is to  $\chi$  in 2.0. Write  $E'_0$  the field generated by the Fourier coefficients of  $g$ .

2.3.1 Let  $\mu$  be the finite order character of  $K$  such that

$$\chi = \mu \cdot \psi^w,$$

and  $\mathbb{Q}(\mu)$  its field of values. By 2.2, the motive  $M(g) \times K$  has a natural  $K \otimes E'_0$  action. Calling  $\tilde{E}$  the composite  $E_0 \cdot E'_0 \cdot \mathbb{Q}(\mu)$  - in some fixed algebraic closure of  $\mathbb{Q}$  - we get the motives with coefficients in  $K \otimes_{\mathbb{Q}} \tilde{E}$  defined over  $K$ :

- $\tilde{M}(\mu) = M(\mu) \otimes_{\mathbb{Q}(\mu)} (K \otimes \tilde{E})$
- $\tilde{M}(g) = (M(g) \times K) \otimes_{K \otimes E'_0} (K \otimes \tilde{E})$
- $\tilde{M}(f) = (M(f) \times K) \otimes_{K \otimes E_0} (K \otimes \tilde{E})$

They are all of rank 1 over  $K \otimes \tilde{E}$ . - For the next theorem we suppose that the  $K$  actions on  $M(f) \times K$  and  $M(g) \times K$  have been normalized so that, for every idempotent  $\tilde{e}$  of  $K \otimes \tilde{E}$  - cf. 2.2 - , the factor  $\tilde{e}(\tilde{M}(f))$  is a motive for  $\chi$  (with values in  $\tilde{E} \cdot E$ ) if and only if  $\tilde{e}(\tilde{M}(g))$  is a motive for  $\psi$  (with values



in  $\tilde{E} \cdot E$  ) .

2.4 Theorem. There is an isomorphism of motives with coefficients in  $K \otimes \tilde{E}$  , defined over  $K$  ,

$$\tilde{M}(\mu) \otimes_{K \otimes \tilde{E}} \tilde{M}(g)^{\otimes_{K \otimes \tilde{E}} w} \cong \tilde{M}(f) .$$

Proof. First, note that, if  $w = 1$  , then  $\tilde{M}(f)$  and  $\tilde{M}(g)$  both lie in  $\mathcal{M}_K^{\text{av}}$  - cf. 1.2.1 - , and I, 5.1 gives us the isomorphism of the theorem. - For  $w \geq 2$  , we shall construct this absolute Hodge correspondence via the relation between periods and L-values: By 1.1 and the construction of  $f$  and  $g$  , we have

$$L_N^*(\tilde{M}(f), s) = \left( \sum a_n^\tau n^{-s} \right)_\tau : \tilde{E} \hookrightarrow \mathbb{C} = L_{Df, \tilde{E}}^*(\chi, s)$$

$$L_{N'}^*(\tilde{M}(g), s) = \left( \sum b_n^\tau n^{-s} \right)_\tau : \tilde{E} \hookrightarrow \mathbb{C} = L_{Df', \tilde{E}^*}^*(\psi, s) ,$$

where, on the right, we have written the L-functions of  $\mathbb{Q}$  § 6 for the characters  $\chi$  and  $\psi$  , considered as taking values in  $\tilde{E}$  , deleting the Euler factors above  $N$  , resp.  $N'$  . Such Euler factors, taken at critical integers  $s$  , lie in  $\tilde{E}^*$  , and will therefore be disregarded in the argument that follows.

Since  $w \geq 2$  , it is well-known that no component of  $L^*(\chi, w)$  vanishes. Also,  $L(\psi^\tau, 1) \neq 0$  , for all  $\tau$  , by construction. Therefore, as  $\chi = \mu \cdot \psi^w$  , it follows from II, 2.1 - which, for  $K$  imaginary quadratic, was already proved in [GS] and [GS'] - , using II, 1.8.1/3 and II, 1.7.12 (iv), that

$$(2.4.0) \quad L^*(\tilde{M}(\mu) \otimes \tilde{M}(g)^{\otimes w}, w) = L^*(\tilde{M}(f), w) \in (\tilde{E} \otimes \mathbb{C})^* ,$$

up to a factor in  $(\tilde{E} \otimes 1)^*$  .

Now, as Deligne points out - [DP], 7.6 - the motive  $M(f)$  is constructed in such a way that

$$L^*(\tilde{M}(f), w) = c^+(R_{K/\mathbb{Q}} \tilde{M}(f)(w)) \in (\tilde{E} \otimes \mathbb{C})^* / \tilde{E}^* .$$

Similarly, the analogous relation for the left hand side of 2.4.0 follows from [DP], 7.2 - or from the observation that  $\tilde{M}(\mu) \otimes \tilde{M}(g)^{\otimes w}$  is in  $\mathcal{M}_K^{\text{av}}$ ; cf. the first sentence of this proof.

As  $K$  is imaginary quadratic, 2.4.0 then implies via II, 1.7.3-6 - note that  $M(f)$  is of Hodge type  $(w,0) + (0,w)$  so that II, 1.7 applies! - that

$$p(\tilde{M}(\mu) \otimes \tilde{M}(g)^{\otimes w}) = p(\tilde{M}(f)) .$$

Thus we have shown that the two motives to be compared in 2.4 are of rank 1 over  $K \otimes \tilde{E}$ , defined over  $K$ ; they are motives for the same algebraic Hecke character - to wit,  $\mu \cdot \psi^w = \chi$  - , in the sense of I, 3.3, extended to our situation where the coefficient algebra may be a product of fields; and they have the same periods  $p$ . Since we are in a rank 1 situation, this is sufficient to physically construct an absolute Hodge correspondence between them establishing their isomorphism: all it comes really down to is choosing bases - i. e., each time a non trivial element - for the various realizations of the two motives.

q.e.d.

2.4.1 Remark. Richard Pink, in an unpublished note, has shown that the absolute Hodge correspondence we just constructed is actually an algebraic cycle, in the special case where  $K = \mathbb{Q}(\sqrt{-4})$ ,  $\psi$  is the Hecke character of the elliptic curve  $y^2 = 4x^3 - 4x$  - cf. I, 7.5 - , or, in other words, of  $X_0(32)$ , and  $\chi = \psi^2$ .

2.4.2 Corollary .  $M(f)$  "lies in"  $\mathcal{CM}_{\mathbb{Q}}(E_0)$ , in the sense that it is isomorphic in  $\mathcal{M}_{\mathbb{Q}}(E_0)$  to an object of  $\mathcal{CM}_{\mathbb{Q}}(E_0)$ ; or again, that  $M(f)$ , viewed as a representation of the motivic Galois group, is equivalent to the inflation of a representation of the Taniyama group.

Via I, 5.1, this corollary implies that, for an idempotent  $e$  of  $K \otimes E_0$  as in 2.2, the motive  $e(M(f) \times K)$  is isomorphic to one of the standard motives - see I § 4 -  $M(\chi)$  or  $M(\bar{\chi})$ .

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ALPHABETICAL LIST OF SYMBOLS AND CONCEPTS

The following is mainly a list of notation in alphabetical order, according to the usual english transliteration of the symbols. - Are also listed important concepts used in the text which do not have an established or easily guessed symbol attached to them, and which do not occur in one of the headings of the table of contents.

A.

A	abelian variety (or elliptic curve) with complex multiplication (CM)	I,1;III,1
$a_D$	element of $\mathbb{B}_{\mathbb{Q}(\sqrt{-D})}^0$ such that $J_D = J_{\mathbb{Q}(\sqrt{-D})}(a_D)$	0,8.3
	affine group scheme	I,2.3.1
$\mathcal{A}\#$	category of arithmetic Hodge structures	I,7.3.1
$A_K, \bar{A}_K$	Kuga-Sato variety, compactified	V,1.1.1
$\alpha$	adelic splitting of Taniyama group	I,6.5
	Anderson's theory of Jacobi sums	0,8.2-4;I,7; II,4.1-3
$\tilde{A}_p$	reduction of A at p	I,1.3
$\text{Aut}^{\otimes} \omega$	tensor automorphisms of fibre functor	I,2.3

B.

$\mathbb{B}, \mathbb{B}^0, \mathbb{B}_K,$	free abelian group on $\mathbb{Q}/\mathbb{Z} \setminus \{0\}$ , and	0,8.2.1-2
$\mathbb{B}_K^0, \mathbb{B}(p)$	certain subgroups thereof	
	Brauer's induction lemma (generalized, applied to $\mathbb{C}$ )	I,6.5.8

C.

$C_{AH}^p(X/K)$	absolute Hodge cycles of codim $p$ on $X$ , defined over $K$	I,2.1
$\chi$	Hecke character (of $K$ with values in $E$ )	0,1
$\chi_A$	idèle character associated to $\chi$	0,5
$\chi_{D,p}$	$D$ -th power residue symbol modulo $p$	0,8.1
$\chi_\lambda$	$\lambda$ -adic idèle class character of $\chi$ 1-dim. $\lambda$ -adic Galois representation	0,5 I,1.3-5
$\chi^T$	Größencharaktere associated to $\chi$ "Chowla-Selberg-formula" (= Lerch's formula)	0,6 III,1.4.4; III,2;III,3.3
$\mathcal{E}_K$	category of motives over $K$ generated by potentially CM abelian varieties	I,4
$\widetilde{\mathcal{E}}_K$	category of Anderson's ulterior motives corrigenda	I,7.3.3 II,1.7.11; II,3.1.1 II,3.4
$c^\pm(\chi)$	periods of $\chi$	II,1.8
$c^\pm(M)$	Deligne's periods of the motive $M$ critical integer	II,1.6 II,2.0
$\text{Crit}_K(\underline{a})$	(half of the) critical integers for $J_K(\underline{a})$	II,4.2
$\mathcal{E}_K^\psi$	category of the $h(X)$ 's, $X$ a variety over $K$ cyclotomic character: see $\psi$	I,2.2

D.

$D$	-discriminant of $\mathbb{Q}(\sqrt{-D})$	0,8.1;III,1
$\mathfrak{d}$	discriminant of extension of number fields	II,1.4.8



$D^\pm$	fudge factor in comparison of $p$ and $c^\pm$ Deligne's rationality conjecture	II,1.7.3-12 II,2.1; V, Introd.
$\delta(K'/K, \sigma)$	fudge factor for $p(M)$ under restriction of scalars	II,1.4.2-8
$\Delta_\mu$	fudge factor in refined "Chowla-Selberg-formula"	III,1.3.2; III,1.4.4-5; III,1.5.6; III,3.2
$\delta^\pm$	fudge factor in comparison of $p$ and $c^\pm$	II,1.7.3-12
$\det_E(M)$	determinant motive	II,1.1.2
$D_\sigma$	fudge factor of $p(M)$ under restriction of scalars	II,1.4.2-8
$\langle \rangle$	representative in $\mathbb{Q}$ of $\mathbb{Q}/\mathbb{Z}$	0,8.1.4
<u>E.</u>		
$E, E', E_0$	fields of values/coefficients (always of finite degree over $\mathbb{Q}$ )	0; I; II; V
$E, E_m, E_m$	"Euler's" arithmetic Hodge structures	I,7.3.2; I,7.4.2; I,7.4.5
$\text{End}_K A$	endomorphisms defined over $K$ of $A$	I,1.1
$\epsilon$	- adelic splitting of $S_K$ - Dirichlet character of $\mathbb{Q}(\sqrt{-D})$ - character of theta series $f$	0,7.4 0,8.1; III,1 V,2.0
<u>F.</u>		
$f$	newform on $\Gamma_0(N)$ of weight $> 1$	V
$\mathfrak{f}$	conductor or defining ideal	0,1
$f_E$	Tate's class field theoretic "cocycle"	I,6.4.2; I.6.5
$\text{Frob } \mathfrak{p}$	geometric Frobenius element	I,1.2

G.

$\Gamma, \Gamma(\underline{a})$	gamma function/product of special values thereof	II,4.0; II,4.2.3
$g_E$	Tate's "cocycle" derived from reciprocity law for CM motives	I,6.3.1; I,6.5
$g_K(\underline{a}, \#)$	Jacobi sum	0,8.2.3
$g_p$	Gauss sum map	0,8.2.2
$G(\sigma)$	motivic Galois group (relative to $H_\sigma$ )	I,2.3.4

H.

H	Hilbert class field of $\mathbb{Q}(\sqrt{-D})=K$	III,1
	H/K-curves	III,1.5.2
$H_A, H_{A^f}, H_{DR}, H_\ell, H_\sigma$	various cohomology theories	I,2.1
$h_D$	class number of $\mathbb{Q}(\sqrt{-D})$	0,8.1;0,8.4.3; III,1.0
	height function (Faltings')	III,2
$\#od_{\mathbb{Q}}$	category of rational Hodge structures	I,2.4.2; I,6.0-1
$h(X)$	the motive of a variety	I,2.2

I.

$I, I_\sigma$	comparison isomorphism	I,2.1.1; I,7.3.1; II,1.1.1; II,1.2.3; II,1.6.3
	invariance lemma of periods under extension of the field of definition	II,3.2.7

J.

$J_D$	basic Jacobi sum Hecke character of $\mathbb{Q}(\sqrt{-D})$	0,8.1;0,8.3
$J_K(\underline{a})$	general Jacobi sum Hecke character	0,8.2.4

K.

K	field of definition (of finite degree over $\mathbb{Q}$ )	passim
	Karoubian envelope	I,2.2
	$\bar{K}/K$ -forms of rank-1 motives	II,3.5
	Kummer's series for $\log \Gamma$	III,2

L.

$L(A/K, s)$	(false) Hasse-Weil L-function of an abelian variety $A$ over $K$	I,1.7
$\Lambda(\chi^T, s)$	Hecke L-function with $\Gamma$ -factors	0,6
$L(\chi^T, s)$	Hecke L-function	0,6
	Legendre's conjecture	III,3.1.1
	Legendre's period relation generalized	II,1.5.4
	lemniscatic arc length	I,7.5;V,2.4.1
$L^*(A/K, s)$	$\mathbb{E}\mathbb{O}\mathbb{C}$ -valued L-function of $A$ with complex multiplication	I,1.7
$L^*(\chi, s)$	$\mathbb{E}\mathbb{O}\mathbb{C}$ -valued array of Hecke L-functions	0,6
$L^*(M/K, s)$	$\mathbb{E}\mathbb{O}\mathbb{C}$ -valued L-function of a motive in $\mathcal{M}_K(E)$	I,6.5.9
$L_{\text{usual}}(A/K, s)$	Hasse-Weil L-function of an abelian variety $A$ over $K$	I,1.7

M.

$M(\underline{a})$	motive for special Jacobi sum Hecke character	I,7.1
$M(\chi)$	(standard) motive for the Hecke character $\chi$	I,4-5
$M(f)$	motive for newform $f$	V,1.1
$\mathcal{M}_K$	category of motives over $K$	I,2.2
$M_K(\underline{a})$	motive for Jacobi sum Hecke character	I,7.2
$\mathcal{M}_K^{av}$	category of motives obtained from abelian varieties	I,2.4.2
$\mathcal{M}_K^-$	false category of motives over $K$	I,2.2
$\mathcal{M}_K^+$	false category of effective motives over $K$	I,2.2
$\mathcal{M}_K(E)$	category of motives over $K$ with coefficients in $E$	I,3
$\mathcal{M}_K^0$	category of Artin motives over $K$	I,2.4.1
	modular proof of Deligne's rationality conjecture (hypothetical)	V,Intro.
	monomial period relations (Shimura)	II,4.4.3;IV,1
	motivic Galois group: see $G(\sigma)$	
$MT(A), MT(V)$	Mumford-Tate group of an abelian variety/a rational Hodge structure	I,2.3.3; I,6.0.2
$\mu$	- cocharacter of a rational Hodge structure - Hecke character of finite order (= "Dirichlet character")	0,7.3.4; I,6.0-1 II,3;III,1; IV,2

N.

$n(\sigma, \tau)$	Hodge coefficients of a CM-structure	0,4;I,1.7; I,4.2;I,5.1; I,6.1.5
$n_1, n_c$	coefficients of $T_D$	0,8.1.3;III,1
	number of CM-type	0,3

P.

$p(\chi)$	period of the Hecke character $\chi$	II,1.8
	period conjecture of Gross	IV,2.1.7-9
$p(M)$	period of the motive $M$	II,1.1
	pseudoabelian envelope	I,2.2
$\psi$	the cyclotomic character	0,8.2.1

Q.

	$\mathbb{Q}$ -curves	I,7.5; III,1.5.4
$\mathbb{Q}(\pm 1), \mathbb{Q}(n),$ $\mathbb{Q}_{\text{DR}}(n)$	powers of Tate motive, and its realizations	I,2.1
	quotients of L-values generating abelian extensions	II,3.3.0

R.

	"reflex motive" of Blasius	II,2.2
	reflex principles	IV,1.2.11
$\mathcal{R}ep_K(G)$	category of finite dimensional representations of an affine group scheme $G$	I,2.3
	"root numbers" (of Dirichlet characters) as twisting factors	II,3.4

S.

$\mathcal{S}$	$= R_{\mathbb{C}/\mathbb{R}} \mathbb{C}_m$	I,6.0
	semi-simple (tannakian) category	I,2.2
	the Serre group: see $Z, Z_K$	
$S_K = \varprojlim_f S_{K,f}$	Serre's group	0,7.2-4

$St_K$	Stickelberger ideal	0,8.4
	strong Weil curve for $\Gamma_0(27)$	I,7.5
	strong Weil curve for $\Gamma_0(32)$	I,7.5;V,2.4.1
<u>T.</u>		
T	- infinity type - CM-type of abelian variety	0,1-3 I,1.6
$\tau, \tau_K, \tau_{E'K^{\tau}E}$	Taniyama group, and subquotients thereof	I,6.4-5
$\tilde{\tau}$	group scheme of $\tilde{\mathcal{E}}_M$	I,7.3.3
T	Hecke algebra	V,1.1.3
	tannakian category (neutralized)	I,2.2;I,2.3.1
	Tate twist/Tate motive: see $Q(n)$	
$T_D$	infinity type of $J_D$	0,8.1.5; 0,8.3;III,1
	tensor category	I,2.2
$\theta_K$	map used in writing Stickelberger elements	0,8.2.1
$T_{\ell}(A)$	Tate module of $A$	I,1.1
$T_p$	Hecke operator	V,1
	transcendence theory	I,1.4;II,4.0; III,3.1.1
<u>U.</u>		
$u$	group scheme for $(\mathcal{E}_{M_Q}, H_B)$	I,6.3;I,6.5

V.

$V_E$	generalized "half-transfer" of Tate	I,6.4.0; II,3.2.8
$V_\ell(A)$	$= T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$	I,1.1

W.

$W$	arithmetic Hodge structure	I,7.3.1
$w, \tilde{w}$	weight	0,3;0,7.3.4; I,6.0-1

X.

$X_m^n$	Fermat hypersurface	I,7.1;II,4.1
$X_1(N)$	closed modular curve	V,1.1.1

Y.

$Y_1(N)$	open modular curve	V,1.1.1
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Z.

$Z = \varprojlim Z_K$	the Serre group	0,7;I,6.1.4
$Z_{K,f}$	finite level of the Serre group	0,7.1