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CM Motives and the Taniyama Group

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0. Introduction

0.1. The ring of endomorphisms $\mathrm{End}(A)$ of an elliptic curve A over $\mathbf{C}$ is either isomorphic to $\mathbf{Z}$ or to an order $\mathcal{O}$ in an imaginary quadratic field K. In the latter case, A is said to have complex multiplication by $\mathcal{O}$. The classical theory of complex multiplication describes explicitly the action of $\mathrm{Aut}(\mathbf{C}/K)$ on A—i.e., on its j-invariant—and on the torsion points of A, i.e., on its Tate module. In particular, one shows that A always has a model over the so-called ring class field of $\mathcal{O}$, a certain abelian extension of K, and the action on torsion points (the so-called "reciprocity law") is given in terms of the class field theory of K. Furthermore, the L-function of an elliptic curve with complex multiplication defined over a number field k is seen to be a product of two L-functions of Hecke characters of k with values in K.[1]

The generalisation of this theory of complex multiplication to abelian varieties of higher dimension is due to [**Shimura and Taniyama**, 1961]. An abelian variety $A/\mathbf{C}$ of dimension n (say, A simple) is said to have complex multiplication if its ring of endomorphisms has maximal possible rank over $\mathbf{Z}$, i.e., if it is isomorphic to an order in a field E of degree $[E:\mathbf{Q}] = 2n$. Then E is necessarily a so-called CM-field, i.e., a totally imaginary quadratic extension of a totally real field. In this case, A (together with its endomorphisms) has a model over some number field k, and the action of $\mathrm{End}(A)$ on the tangent space at 0 of A defines a k-linear representation of E which diagonalises over some smallest CM subfield K of k, called the reflex field of E (with respect to A). The Shimura-Taniyama reciprocity law—see [**Shimura**, 1971, Theorem 5.15]—describes the action of $\mathrm{Aut}(\mathbf{C}/K)$ on A and its torsion

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[1] These notions go back to [**Hecke**, 1918, 1920]. But the link with CM elliptic curves is due to [**Deuring**, 1953ff].

points in terms of the class field theory of K. Furthermore, the L-function of A over k is expressed as a product of L-functions of Hecke characters of k with values in E.

In the elliptic curve case K is imaginary quadratic, and since it is easy to analyse the action of the continuous automorphism of complex conjugation, one obtains the complete action of Aut $\mathbf{C}$ on A. Also, if A (but not all its endomorphisms) is defined over a real field k_0 (for instance, $k_0 = \mathbf{Q}$), then it is not difficult to identify the L-function of A over k_0 as one of the two Hecke L-functions occurring as factors of the L-function of A over $k_0 \cdot K$. See [**Deuring** 1953ff].

In the general case however, the action of automorphisms on the abelian variety A which are not trivial on the reflex field K was not known before 1980. And in the case where A (not its complex multiplication) descends to a number field k_0 not containing K, it was not clear in general whether and how the L-function of A over k_0 could be expressed by Hecke L-functions.

0.2. This incompleteness of the "classical" theory of complex multiplication of abelian varieties was not just an esthetic blunder, but represented a serious desideratum in view of the applications to (special points of) Shimura varieties. According to Deligne, Shimura varieties should parametrise motives. This prompted Langlands [**Langlands**, 1979] to look for a motivic formulation of the problem of conjugation of Shimura varieties.

In doing so, Langlands reduces to Shimura varieties of tori and uses Serre's results from [**Serre**, 1968], where a group-theoretic treatment of algebraic Hecke characters and abelian ℓ-adic representations had been given. In particular, Serre had defined the (connected) "Serre group" $\mathcal{S}$, the representations of which may be thought of as "CM-motives over $\overline{\mathbf{Q}}$"—see 1.3 below.[2]

The group scheme corresponding to a (hypothetical) category of CM motives over $\mathbf{Q}$ should be an extension of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$) by $\mathcal{S}$—see 1.2, 1.4 below. In [**Langlands**, 1979], Langlands wrote down explicit cocycles characterising a certain extension of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$) by $\mathcal{S}$ which he called the Taniyama group and which he conjectured to be the group scheme corresponding to "the" category of CM motives over $\mathbf{Q}$.

It seems that it was Milne and Shih who shortly hereafter observed how Langlands's conjectural description of CM motives implied a generalisation of the Shimura-Taniyama reciprocity law to all automorphisms of $\mathbf{C}$—cf. [**Milne, Shih**, 1981 and 1982] and [**Milne**, 1981]. (For the reverse implication, see Milne's contribution "Motives and Shimura varieties" to the present Proceedings.)

0.3. In Langlands's *Märchen*, the category of CM motives was hypothetical and the word "motive" was not given a precise meaning. Soon after the

[2] Serre himself mentions Grothendieck's hypothetical theory of motives as an inspirational background in the introduction to [**Serre**, 1968].

Corvallis conference Deligne made progress on absolute Hodge cycles and the category of motives defined with them—see [**Deligne** 1979, 0.9–0.11], cf. [**Panchishkin** 1993].

More precisely, Deligne was able to show that, on an abelian variety, every Hodge cycle is an absolute Hodge cycle: [**Deligne** (**Milne**), 1982]. Thus, any category of motives for absolute Hodge cycles which is generated by a class of abelian varieties is not only well defined, but in it, we actually have at our disposal essentially all homomorphisms and objects that are expected to exist.

0.3.1. Using this manageable category of motives, Deligne was then able to prove, not later than the summer of 1981 [**Deligne**, 1982], that *Langlands's Taniyama group is isomorphic to the motivic Galois group of the category* $\mathscr{CM}_{\mathbf{Q}}$ of absolute Hodge cycle CM motives over $\mathbf{Q}$ (see §1 below for the precise definition of $\mathscr{CM}_{\mathbf{Q}}$).

This theorem of Deligne solves the two problems mentioned in §0.1 above, to wit: (a) it generalises the Shimura-Taniyama reciprocity law to all automorphisms of $\mathbf{C}$, and (b) it settles the problem about the L-function of a potentially CM abelian variety. See [**Deligne**, 1982, pp. 262/3: Remarques 4 and 2].

0.4. To appreciate (b), recall that, *a priori*, absolute Hodge cycle motives might have some undesirable properties. Specifically, absolute Hodge cycles cannot in general be demonstrated to behave well under reduction. Therefore, we cannot be sure at first that our motives have strictly compatible systems of ℓ-adic representations. So their L-functions are not under control. However, the Taniyama group is related to the Weil group (of $\mathbf{Q}$). Thus, via Deligne's result, the objects of $\mathscr{CM}_{\mathbf{Q}}$ do give finite-dimensional representations of the Weil group. See §5 below for details.

In fact, (b) had essentially been established independently slightly earlier, by H. Yoshida [**Yoshida**, 1981].

0.5. As for the general reciprocity law (a), John Tate (who spent the academic year 1980–81 in Paris) worked out, and partially proved, in the Spring of 1981 a conjecture giving a completely explicit class field theoretic generalisation of the Shimura-Taniyama reciprocity law: see [**Tate**, 1981], a manuscript which, among others, is projected to appear in the Collected Papers of Tate. In our notation of 4.4 below, Tate's conjecture says that $f_E(s, \lambda) = g_E(s, \lambda)$ for all $s \in G_{\mathbf{Q}}$ and for all "CM-types" λ, i.e., for all characters λ of the Serre group associated to abelian varieties of CM type by E.

Everything that Tate formulated and proved was (motivated by, but) logically independent of Langlands's construction and Deligne's work on Langlands's conjecture. In fact, Tate's conjecture and his partial proof of it could have in principle been obtained by Shimura and Taniyama back in the fifties.

However, in the fall of 1981 Deligne gave a complete proof of Tate's statement using the theory of absolute Hodge cycles. More precisely, Deligne showed that all one had to know to derive the full conjecture of Tate from Tate's own partial result was the following: the quantitites $f_E(s, \lambda)$ and $g_E(s, \lambda)$ satisfy a relation $\prod f_E(s, \lambda_i)^{a_i} = 1$ and $\prod g_E(s, \lambda_i)^{a_i} = 1$ $(a_i \in \mathbf{Z})$, whenever the corresponding linear combination of the CM-types λ_i is the trivial character of the Serre group. This follows trivially if we can consistently define $f_E(s, \lambda)$ and $g_E(s, \lambda)$ for any character of the Serre group, not just for CM-types of abelian varieties. [3]

0.6. In the following presentation we do the following: we generalise Tate's formalism from CM abelian varieties to arbitrary CM motives. Here the word "motive" will always refer to the absolute Hodge cycle theory, and "CM motives" are defined as arising from abelian varieties which, over $\mathbf{C}$, have complex multiplication. Note that this implies that we have the corresponding quantitites f and g, for instance for the motives of Fermat hypersurfaces, as well as for motives of CM type obtained from K3 surfaces—see [**Deligne and Milne**, 1982, p. 217].

Now, by generalising Tate's formalism in this way we also retrieve Langlands's construction of the Taniyama group and Deligne's theorem (0.3.1). In fact, Tate's quantities f_E, g_E (generalised to arbritrary CM motives) each characterise a certain extension of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by $\mathcal{S}$, and it is elementary to show (see §§3 and 4 below) that g_E gives the motivic Galois group of Deligne's category $\mathcal{CM}_{\mathbf{Q}}$, and that f_E defines Langlands's Taniyama group! In this way, the proof of Tate's (generalised) conjecture gives, by the same token, the proof of Deligne's theorem on the Taniyama group—see Theorem 4.4 below.

0.7. Deligne's theorems mentioned in 0.3 were first published in [**Deligne et al.**, 1982], where the reader can also find a survey of Langlands's definition of the Taniyama group [**Milne and Shih**, 1981, (I)]. Later treatments can be found in [**Schappacher**, 1988, I.6] and [**Milne**, 1989, chapter I]. See also the brief accounts contained in [**Anderson**, 1986] and [**Blasius**, 1986].

The presentation given here is somewhat analogous to Milne's unpublished manuscript [**Milne**, 1981]. I saw this manuscript for the first time after the first version of this paper was written. It was in fact G. Anderson who had explained these things to me in 1984.

0.8. Thanks are due to the referee and the editor for helpful suggestions, and especially for prodding me to put in more details and explanations. All

[3] Note here the perfect analogy with Deligne's motivic proof of Shimura's monomial relations between periods of CM abelian varieties: see [**Deligne** (**Brylinski**), 1980, **Schappacher**, 1988, Chapter IV].

mistakes or shortcomings still are only mine, of course; but they may have become more easily detectable in this way.

1. The category $\mathscr{CM}_{\mathbf{Q}}$

Let k be a field embeddable into $\mathbf{C}$. Fix an algebraic closure $\overline{k}$ of k, and write $G_k = \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k. The letter σ denotes embeddings $\sigma : k \hookrightarrow \mathbf{C}$.

Throughout this article, the word "motive over k" will refer to the semi-simple Tannakian category $\mathscr{M}_k$ of motives over k for absolute Hodge cycles, cf. [**Deligne**, 1979, 0.9 – 0.11, **Deligne et al.**, 1982, **Panchishkin**, 1993]. Recall the realisation fibre functors built into the theory of absolute Hodge cycles, Betti realisation H_σ, for each embedding $\sigma : k \to \mathbf{C}$, the de Rham realisation H_{dR}, and for each prime number ℓ, the ℓ-adic realisation H_ℓ. If X is a nonsingular projective algebraic variety defined over k, then X defines an object $h(X)$ of $\mathscr{M}_k$, and for each $i \in \mathbf{Z}$ there exists a motive $h^i(X)$, an object of $\mathscr{M}_k$, such that $h(X) = \sum_i h^i(X)$. The various realisations of $h^i(X)$ are, respectively, singular cohomology $H^i([X \times_{k,\sigma} \mathbf{C}](\mathbf{C}), \mathbf{Q})$, algebraic de Rham cohomology $H^i_{\mathrm{dR}}(X/k)$, and ℓ-adic cohomology $H^i_{\text{ét}}(X \times_k \overline{k}, \mathbf{Q}_\ell)$.

A simple abelian variety A over k is said to be of *CM-type* if $\mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{End}_{/k} A$ is a CM-field (i.e., a purely imaginary quadratic extension of a totally real algebraic number field) of degree $2 \cdot \dim A$. An arbitrary abelian variety A over k is said to admit *complex multiplication*, or "*to be CM*", if every simple isogeny factor is of CM-type. Finally, A/k is said to admit *potential complex multiplication*, or for short, "*to be potentially CM*", if $A \times_k L$ is CM for some finite extension L of k.

Define $\mathscr{CM}_k$ to be the smallest full ($\mathbf{Q}$-linear neutralised) Tannakian subcategory of $\mathscr{M}_k$ containing the motives $h^1(A)$ for all abelian varieties A defined over k which are potentially CM.

1.1. LEMMA. *$\mathscr{CM}_k$ contains arbitrary Tate twists $\mathbf{Q}(m)$, $m \in \mathbf{Z}$, and the category $\mathscr{M}_k^0$ of Artin motives over k is a full Tannakian subcategory of $\mathscr{CM}_k$.*

This latter category $\mathscr{M}_k^0$ is by definition the Tannakian subcategory of motives generated by the $h^0(X)$, where X is a variety of dimension zero defined over k—cf. [**Deligne and Milne**, 1982, p. 211, **Panchishkin**, 1993]. In particular, the automorphism group scheme of the fibre functor H_σ (for *any* embedding σ) is, $\underline{Aut}^{\otimes}(\mathscr{M}_k^0, H_\sigma) \cong \mathscr{G}_k$, where the Galois group is viewed as a constant group scheme. This follows from the fact that for X of dimension zero, $X(\overline{\mathbf{Q}})$ is just a finite collection of points with Galois action.

PROOF. First, the cohomology ring of an abelian variety is well known to be the exterior algebra on $H^1(A)$. This holds for every realisation of $h(A)$ and carries over to absolute Hodge cycle motives. Thus, the category $\mathscr{CM}_k$ contains all motives $h^i(A)$ for CM abelian varieties A defined over k.

The claim of the lemma concerning Tate twists follows from the isomorphism of motives $h^2(E) = \mathbf{Q}(-1)$ for any elliptic curve E (with or without CM) which is geometrically irreducible over k. In fact, this follows from the nondegenerate alternating pairing in every realisation, $H^1(E) \times H^1(E) \to H(\mathbf{Q}(-1))$.

To see that every object of $\mathscr{M}_k^0$ occurs in $\mathscr{CM}_k$, note that for every geometrically irreducible abelian variety A/k, one has $h^0(A) = h^0(\operatorname{Spec} k)$. Thus, for any extension L/k, $h^0(A \times_k L) = h^0(\operatorname{Spec} L)$. These latter motives generate $\mathscr{M}_k^0$ as a Tannakian category.

The central object of this survey is the $\mathbf{Q}$ group scheme of tensor automorphisms of the fibre functor H_B on $\mathscr{CM}_{\mathbf{Q}}$:

$$\mathscr{U} = \underline{Aut}^{\otimes}(\mathscr{CM}_{\mathbf{Q}}, H_B).$$

Here we write " B " (for *Betti*) instead of σ because there is only one embedding of $\mathbf{Q}$ into $\mathbf{C}$.

This pro-algebraic group, which determines the category $\mathscr{CM}_{\mathbf{Q}}$ up to equivalence, is the motivic Galois group mentioned in the introduction. It will eventually be identified with the Taniyama group. But before defining the Taniyama group (in §4 below) we will introduce and formalise the finer structures with which $\mathscr{U}$ is naturally equipped.

By the Tannakian formalism, the fully faithful inclusion functor $\mathscr{M}_{\mathbf{Q}}^0 \to \mathscr{CM}_{\mathbf{Q}}$ induces a faithfully flat homomorphism of $\mathbf{Q}$ group schemes $\mathscr{U} \to G_{\mathbf{Q}}$,[4] and the essentially surjective base change functor $\mathscr{CM}_{\mathbf{Q}} \to \mathscr{CM}_{\overline{\mathbf{Q}}}$ induces a closed immersion $\mathscr{U}^\circ \to \mathscr{U}$, where $\mathscr{U}^\circ = \underline{Aut}^{\otimes}(\mathscr{CM}_{\overline{\mathbf{Q}}}, H_\sigma)$ for some fixed embedding $\sigma : \overline{\mathbf{Q}} \to \mathbf{C}$.

These homorphisms, in fact, form an *exact sequence of* $\mathbf{Q}$ *group schemes*

$$1 \to \mathscr{U}^\circ \to \mathscr{U} \to G_{\mathbf{Q}} \to 1. \tag{1.2}$$

PROOF. More generally, for any $s \in G_{\mathbf{Q}}$ the fibre over s in $\mathscr{U}$ may be written $\underline{Isom}^{\otimes}(H_\sigma, H_{\sigma\circ s})$, where we view H_B on $\mathscr{CM}_{\overline{\mathbf{Q}}}$ via σ. The point is that for any $\mathbf{Q}$-algebra R, any automorphism $g \in \mathscr{U}(R) = \underline{Isom}^{\otimes}(H_B \otimes R, H_B \otimes R)$ in the fibre above s and any object M of $\mathscr{CM}_{\mathbf{Q}}$, the identifications

$$\begin{aligned} H_\sigma(M \times \overline{\mathbf{Q}}) \otimes R = H_B(M) \otimes R &\xrightarrow{g_M} H_B(M) \otimes R \\ &= H_{\sigma\circ s}(M \times \overline{\mathbf{Q}}) \otimes R \end{aligned}$$

define homomorphisms in a way functorial in $M \times \overline{\mathbf{Q}}$ and R, and compatible with tensor products. But any object N of $\mathscr{CM}_{\overline{\mathbf{Q}}}$ is a direct factor of an object of the form $M \times \overline{\mathbf{Q}}$ for M defined over $\mathbf{Q}$: in fact, N is defined

[4] Here, as before, $G_{\mathbf{Q}}$ is considered as a constant group scheme.

over a number field, so take the restriction of scalars to $\mathbf{Q}$ and extend back up to $\overline{\mathbf{Q}}$. Therefore, g defines an element of $\mathrm{Isom}(H_\sigma \otimes R, H_{\sigma\circ s} \otimes R)$. This establishes a bijection between the fibre above s and $\underline{Isom}^{\otimes}(H_\sigma, H_{\sigma\circ s})$. Cf. [**Jannsen**, 1990, pp. 52–54, proof of 4.7.e].

1.2.1. REMARK. 1.2 shows that $\mathscr{U}^\circ$ may be identified with the connected component of the identity in $\mathscr{U}$.

1.3. PROPOSITION. *$\mathscr{U}^\circ$ is isomorphic to the (connected) Serre group $\mathscr{S}$.*

We will sketch the proof, at the same time introducing $\mathscr{S}$. For $k = \overline{\mathbf{Q}}$, Deligne's absolute Hodge cycle theorem implies that for all $\sigma : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, the Betti realisation functor H_σ defines an equivalence of Tannakian categories between $\mathscr{CM}_{\overline{\mathbf{Q}}}$ and the subcategory of rational Hodge structures generated by those coming from CM abelian varieties. The identification of $\mathscr{U}^\circ = \underline{Aut}^{\otimes}(\mathscr{CM}_{\overline{\mathbf{Q}}}, H_\sigma)$ with $\mathscr{S}$ comes from an explicit description of those Hodge structures.

1.3.1. *CM-types.* Let $k \subset \mathbf{C}$ be a number field, i.e., a finite extension of $\mathbf{Q}$, and let A be an *abelian variety of CM-type* defined over k. So A is equipped with an isomorphism $E \to \mathbf{Q} \otimes \mathrm{End}_{/k} A$ for some CM-field E of degree $[E:\mathbf{Q}] = 2 \cdot \dim A$. Then every element of E defines an endomorphism of the rational Hodge structure $H^1_B(A)$.[5] But

$$H^1_B(A) \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{\tau: E \hookrightarrow \mathbf{C}} H^1_B(A) \otimes_{E,\tau} \mathbf{C}.$$

Thus, since $\dim_E H^1_B(A) = 1$, for each $\tau: E \hookrightarrow \mathbf{C}$ there exists $n_\tau \in \{0, 1\}$ such that

$$H_B(A) \otimes_{E,\tau} \mathbf{C} \subseteq H^{n_\tau, 1-n_\tau}.$$

This system of Hodge numbers (n_τ) determines the *CM-type of* A which, classically,[6] is simply the half-system of embeddings $T = \{\tau \mid n_\tau = 1\}$. Note that for $c =$ complex conjugation, one has $T \cap cT = \varnothing$, $T \cup cT = \mathrm{Hom}(E, \mathbf{C})$.

Conversely, every such half-system T of embeddings of E satisfying these conditions occurs as the CM-type of an abelian variety of CM-type with CM by E which is defined over some finite extension of k. In fact, over $\mathbf{C}$, if $g = [E:\mathbf{Q}]$, then take $\mathbf{C}^g/\mathscr{O}_E$ with the ring of integers $\mathscr{O}_E$ embedded into $\mathbf{C}^g$ via the direct sum of the elements of T. Like any CM abelian variety, this has a model over $\overline{\mathbf{Q}}$—see footnote 7 for a more precise statement.

1.3.2. LEMMA. *The $\mathbf{Z}$-span in $\mathbf{Z}[\mathrm{Hom}(E, \mathbf{C})]$ of all CM-types (n_τ) is equal to the set of all $(m_\tau) \in \mathbf{Z}[\mathrm{Hom}(E, \mathbf{C})]$ such that $w = m_{c\tau} + m_\tau$ is independent of τ.*

[5] That is the Betti cohomology $H^1(A \times_\sigma \mathbf{C})$, for the fixed inclusion $\sigma : k \hookrightarrow \mathbf{C}$.

[6] See [**Shimura and Taniyama**, 1961, §5.2] or [**Shimura**, 1971, §5.5 B]. There is, however, a difference in normalisation: we work with $H^1(A)$ rather than the classical $H_1(A)$. Recall that $H^{1,0} = H^0(A, \Omega^1) \otimes_k \mathbf{C}$.

PROOF. It is clear that all elements in the $\mathbf{Z}$-span of CM-types have the desired property. Conversely, let the system (m_τ) satisfy the condition of the lemma, and choose $\tau_1 \in \mathrm{Hom}(E, \mathbf{C})$ as well as some CM-type T_1 containing τ_1; then $(m_\tau) - m_1 \cdot T_1$ has coefficient 0 at τ_1. Next, for $\tau_1 \neq \tau_2 \in T_1$, choose a CM-type T_2 containing τ_2, but $c\tau_1$. Then $(m_\tau) - m_1 T_1 - (m_2 + m_1) \cdot T_2$ has coefficient 0 at τ_2. We can continue like this for all elements $\tau_1, \dots, \tau_n \in T_1$, where $n = [E:\mathbf{Q}]$. So we finally correct the given (m_τ), modulo an element from the $\mathbf{Z}$-span of CM-types, to a system (m'_τ) which involves only $c\tau_1, \dots, c\tau_n$ and still satisfies the condition that $w' = m'_{c\tau} + m'_\tau$ is independent of τ. It follows that $(m'_\tau) = w' \cdot cT_1$, and the lemma is proved.

1.3.3. Another way to conceive of CM-types, which paves the way to the Serre group, is to view them as algebraic homomorphisms between k^* and E^*—cf. [**Deligne**, 1977, §5.1].

Consider the $k \otimes_{\mathbf{Q}} E$ module of holomorphic differentials $\Omega^1_{/k}(A)$. The homomorphism

$$\det_k(1 \otimes \cdot\,; \Omega^1_{/k}(A)) : E^* \to k^*$$

is in fact a homomorphism of $\mathbf{Q}$-algebraic groups

$$R_{E/\mathbf{Q}}\mathbf{G}_m \to R_{k/\mathbf{Q}}\mathbf{G}_m.$$

The link with the system of Hodge numbers (n_τ) is simply given by $\det_k(1 \otimes x\,; \Omega^1_{/k}(A)) = \prod_{\tau:E\to\mathbf{C}} \tau(x)^{n_\tau}$ (remember that $k \subset \mathbf{C}$).

The analogous algebraic homomorphism in the opposite direction

$$\det_E(\cdot \otimes 1\,; \Omega^1_{/k}(A)) : R_{k/\mathbf{Q}}\mathbf{G}_m \to R_{E/\mathbf{Q}}\mathbf{G}_m \tag{1.3.4}$$

corresponds to what is classically called *the dual* (or *reflex*) *CM-type* of the one given by the (n_τ). More precisely, this dual homomorphism will in general be extended up from proper CM subfields of k. The smallest CM subfield K of k to which this dual CM-type descends is called the *reflex field* of the initial CM-type of E.

If we write the dual type 1.3.4 in the form $(n^\#_\sigma)$, where σ ranges over the embeddings of k into $\mathbf{C}$, then K is the smallest CM subfield of k which admits a CM-type (n^*_ϕ) such that $n^\#_\sigma = 1 \Leftrightarrow n^*_\phi = 1$ for $\phi = \sigma_{|K}$. With this notation, it is the CM-type (n^*_ϕ) on the reflex field that is classically called the dual of (n_τ).

Alternatively, the reflex field $K \subset \overline{\mathbf{Q}}$ is simply the fixed field of the stabiliser in $G_{\mathbf{Q}}$ of the initial CM-type (n_τ). Cf. [**Shimura and Taniyama**, 1961, §8.3].

1.3.5. For any CM field E, let $\mathcal{S}_E$ be the biggest $\mathbf{Q}$-algebraic quotient of $R_{E/\mathbf{Q}}\mathbf{G}_m$ through which all CM-types factorise. In other words, the character group $X(\mathcal{S}_E)$ is the group of all mappings $x \mapsto \prod_{\tau:E\to\mathbf{C}} \tau(x)^{m_\tau}$ for (m_τ) such that $w = m_{c\circ\tau} + m_\tau$ is independent of τ. This last condition makes sense for any number field L instead of E and defines $\mathcal{S}_L$ in general. The characters of $\mathcal{S}_L$ are necessarily extended up from CM subfields of L, via the norm.

1.3.6. DEFINITION. $\mathcal{S} = \varprojlim_L \mathcal{S}_L = \varprojlim_{E\ \mathrm{CM}} \mathcal{S}_E$ (with inverse limits taken with respect to the norm maps) is called the (connected) Serre group.

By 1.3.2, all characters of $\mathcal{S}_E$ defined over $\overline{\mathbf{Q}}$ are generated by the types of abelian varieties A defined over $\overline{\mathbf{Q}}$ of CM-type by E. The CM-type clearly characterises the rational Hodge structure $H^1_B(A)$ of A. By Deligne's theorem on absolute Hodge cycles on abelian varieties [**Deligne** (**Milne**), 1982], the type therefore characterises the motive $h^1(A)$ over $\overline{\mathbf{Q}}$. An integral linear combination of types corresponds to the corresponding E-linear tensor product of motives M in $\mathcal{CM}_{\overline{\mathbf{Q}}}$ (which admit coefficients in E such that $H_B(M)$ is a one-dimensional E vector space).

Furthermore, the norm map $N_{E'/E}$ corresponds to the extension of coefficients $\otimes_E E'$ on motives with an E action. So passing from characters of $\mathcal{S}_E$ to those of $\mathcal{S}$ has the effect of not having to worry about the fact that, in general, a product of two characters λ_1, λ_2 with fields of values E may have values generating a proper subfield E_0 of E, in which case $\lambda_1 \otimes \lambda_2$ corresponds to a motive with coefficients in E which is obtained from another motive, with coefficients in E_0, by extension of the field of coefficients.

Summing up we conclude that via $M \mapsto H_B(M)$, *the category* $\mathcal{CM}_{\overline{\mathbf{Q}}}$ *is equivalent to the category of finite-dimensional representations of* $\mathcal{S}$. This shows 1.3.

1.3.7. An alternative description of the Serre group is as follows.

DEFINITION. (i) A rational Hodge structure is CM if it is polarisable and its Mumford-Tate group is abelian.

(ii) $\mathcal{S}$ is the affine group scheme corresponding to the Tannakian category of CM Hodge structures, with the forgetful fibre functor.

EXAMPLE. If A is an abelian variety of CM-type (1.3.1), then $H^1_B(A)$ is a CM Hodge structure.

We do not go into this point of view here. See for instance [**Schappacher**, 1988, I 6.1]; cf. [**Milne**, 1989, p. 294ff]. The fact that this approach gives the same pro-torus $\mathcal{S}$ as described above is one way to see that the CM Hodge structures are precisely those obtained as Betti realisations of objects in $\mathcal{CM}_{\overline{\mathbf{Q}}}$.

We now describe some finer properties of the extension 1.2.

1.4. Just as for the Serre group $\mathcal{S}$, the whole of $\mathcal{U}$ is also an inverse limit of algebraic groups coming from finite levels, and in fact, the whole exact sequence 1.2 is the inverse limit of the following:

$$(1.4.1) \qquad 1 \to \mathcal{S}_E \to \mathcal{U}_E \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \to 1.$$

Here E runs over finite CM fields Galois over $\mathbf{Q}$, E^{ab} is the maximal abelian extension of E in $\overline{\mathbf{Q}}$, and $\mathcal{U}_E$ is the affine group scheme corresponding to the full Tannakian subcategory of $\mathscr{CM}_{\mathbf{Q}}$ generated by the objects that admit coefficients in E. Equivalently[7] $\mathscr{U}_E$ is generated (as a Tannakian category) by $R_{L/\mathbf{Q}}h^1(A)$, with A an abelian variety of CM type by E, defined over some algebraic number field $L \subset E^{\mathrm{ab}}$. For $E \hookrightarrow E'$ the map $\mathscr{U}_{E'} \to \mathscr{U}_E$ is induced by extension of coefficients $\otimes_E E'$ on objects with coefficients in E.

1.5. Since $\mathscr{S}$ is a (pro-)torus, the left action of $\mathscr{U}$ on $\mathscr{S}$ by conjugation (i.e., $x \mapsto uxu^{-1}$) factors through $G_{\mathbf{Q}}$. This defines a left Galois action on $\mathscr{S}$ which, on characters $\lambda \in X^*(\mathscr{S})$, transports to the left action of $G_{\mathbf{Q}}$ by left translation: $\lambda^s(x) = \lambda(s^{-1}(x))$—see [**Deligne**, 1982, §(B)].

If λ factors through $\mathscr{S}_E$ with E Galois over $\mathbf{Q}$ and λ corresponds to $(n_\tau)_\tau$, then λ^s corresponds to $(n_{\tau s})_\tau$.

1.5*. There is also the "usual" left Galois action on $\mathscr{S}$ and its characters $\lambda \mapsto s \circ \lambda$. In terms of the other notation, $s \in G_{\mathbf{Q}}$ takes $(n_\tau)_\tau$ to $(n_{s^{-1}\circ\tau})_\tau$.

For example, if λ is the CM-type of an abelian variety A/k as in 1.3.1, then the conjugate abelian variety A^s, together with the conjugate isomorphism $E \to \mathrm{End}(A^s)$, has CM-type $s \circ \lambda$. Writing λ as (n_τ), the set $T = \{\tau \mid n_\tau = 1\}$ is transformed into sT.

1.6. For each prime ℓ, the absolute Galois group $G_{\mathbf{Q}}$ acts on the ℓ-adic realisation $H_\ell(M)$ of an object M of $\mathscr{CM}_{\mathbf{Q}}$. Choose an embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}$. Then in view of the comparison isomorphism $H_\ell(M) \cong H_B(M) \otimes \mathbf{Q}_\ell$, each $s \in G_{\mathbf{Q}}$ gives an automorphism of the fibre functor $H_B \otimes \mathbf{Q}_\ell$, and therefore, by definition, an element of $\mathscr{U}(\mathbf{Q}_\ell)$. This defines a continuous homomorphism

$$G_{\mathbf{Q}} \xrightarrow{\varepsilon_\ell} \mathscr{U}(\mathbf{Q}_\ell).$$

Artin motives carry a rational $G_{\mathbf{Q}}$ action on their Betti realisation. This implies that ε_ℓ is a continuous splitting on the $\mathbf{Q}_\ell$-rational points of the map $\mathscr{U} \to G_{\mathbf{Q}}$—but note that it is not a homomorphism of $\mathbf{Q}_\ell$-algebraic groups.

Putting together all finite places ℓ (and writing the finite adèles of a number field k as $k_{\mathbf{A}_f}$), we obtain a splitting

$$G_{\mathbf{Q}} \xrightarrow{\varepsilon} \mathscr{U}(\mathbf{Q}_{\mathbf{A}_f}). \tag{1.6.1}$$

Analogous splittings exist for the sequences 1.4.1 as well; they will be written

$$\mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \xrightarrow{\varepsilon_E} \mathscr{U}_E(\mathbf{Q}_{\mathbf{A}_f}).$$

The splitting ε is the limit of the splittings ε_E.

[7] Here we need to know that abelian varieties of CM type by E, defined initially, say over $\overline{E}$, admit a model over E^{ab}. This follows from E being Galois over $\mathbf{Q}$: it contains the reflex field of any of its CM-types, and the result follows, for instance, from the consideration of the moduli fields of suitable structures without automorphisms; see [**Shimura**, 1971, p. 130, p. 216].

We will briefly denote the exact sequence 1.2, equipped with all the extra structures described above, by the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{S} & \longrightarrow & \mathscr{U} & \longrightarrow & G_{\mathbf{Q}} & \longrightarrow & 1 \\ & & & & & & \| & & \\ & & & & \mathscr{U}(\mathbf{Q}_{\mathbf{A}_f}) & \xleftarrow{\varepsilon} & G_{\mathbf{Q}} \,. & & \end{array} \tag{1.7}$$

2. Taniyama extensions and their invariant

2.0. Definition. A *Taniyama extension* is an affine group scheme $\mathscr{V}$ over $\mathbf{Q}$ that fits into an exact sequence

$$1 \to \mathscr{S} \to \mathscr{V} \to G_{\mathbf{Q}} \to 1\,, \tag{2.0.1}$$

which is the inverse limit of exact sequences indexed by CM fields E, as in the sequence (1.4.1), and such that the Galois action induced by the sequence on $\mathscr{S}$ is as described in §1.5. Furthermore, a Taniyama extension by definition admits ℓ-adic splittings ε_ℓ, for every ℓ, as in 1.6. As a shorthand notation for Taniyama extensions we will use diagrams such as 1.7.

The aim of this section is to characterise Taniyama extensions by a cocycle-like invariant.

Given a CM field $E \subset \mathbf{C}$ Galois over $\mathbf{Q}$, the map $\mathscr{V}_E(E) \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ from the analogue of (1.4.1) for $\mathscr{V}$ is surjective, because S_E splits over E, so that $H^1(E, S_E) = 0$ by Hilbert 90. Choose any set-theoretic splitting α_E of this surjection. Recall that ε_E is the splitting (as in (1.6.1)) of the surjection $\mathscr{V}_E(\mathbf{Q}_{\mathbf{A}_f}) \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ which is implicit in the Taniyama extension $\mathscr{V}$. Then for $s \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$, define

$$\beta_E(s) = \alpha_E(s)^{-1} \cdot \varepsilon_E(s) \ (\mathrm{mod}\ S_E(E)) \in S_E(E_{\mathbf{A}_f})/S_E(E)\,. \tag{2.1}$$

2.2. Proposition. (i) *The map* β_E *does not depend on the choice of the splitting* α_E.

(ii) *For all* $s, t \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$, *we have* $\beta_E(st) = t^{-1}(\beta_E(s)) \cdot \beta_E(t)$.

(iii) *The system of maps* β_E, *for* E *ranging over CM algebraic number fields Galois over* $\mathbf{Q}$, *characterises the Taniyama extension* $\mathscr{V}$ (*i.e., the exact sequence* 2.0.1, *inverse limit of the sequences that serve to define* β_E, *with all additional structures*) *up to unique isomorphism.*

Proof. (i) If α_E and α'_E are two E-rational splittings, then for all $s \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$, one has $\beta_E(s) \cdot \beta'_E(s)^{-1} = \alpha_E(s)^{-1} \cdot \alpha'_E(s) \in \mathscr{S}_E(E)$.

(ii) Dropping the subscript E from the notation momentarily, we find that

$$\begin{aligned} \beta(st) \cdot [t^{-1}(\beta(s)) \cdot \beta(t)]^{-1} &= \alpha(st)^{-1}\varepsilon(st)\varepsilon(t)^{-1}\alpha(t)\alpha(t)^{-1}\varepsilon(s)^{-1}\alpha(s)\alpha(t) \\ &= \alpha(st)^{-1}\alpha(s)\alpha(t) \in \mathscr{S}_E(E)\,. \end{aligned}$$

(iii) is straightforward—cf. the more detailed discussion in §2 (in particular, Proposition 2.7) of **[Milne and Shih**, 1982], where our β_E corresponds to $(\bar{b})^{-1}$.

2.2.1 REMARK. In [**Milne and Shih**, 1982] it is also proved that maps

$$\beta_E : \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \to S_E(E_{\mathbf{A}_f})/S_E(E)$$

satisfying 2.2.(ii) come from an extension

$$1 \to \mathcal{S}_E \to \mathcal{V}_E \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \to 1$$

if and only if (i) their values are invariant under $\mathrm{Gal}(E/\mathbf{Q})$ and (ii) β_E lifts to a map $b : \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \to S_E(E_{\mathbf{A}_f})$ such that $b(st)^{-1}t^{-1}(b(s))b(t)$ is locally constant.

Now for any character $\lambda : \mathcal{S}_E \to \mathbf{G}_m$ define a finite idèle class of E as follows:

$$\gamma_E(s, \lambda) = \lambda(\beta_E(s)) \in E^*_{\mathbf{A}_f}/E^*. \tag{2.3}$$

Then the following properties are easily checked.

2.4. COROLLARY. *For all automorphisms $s, t \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ and all characters $\lambda, \lambda' \in X(\mathcal{S}_E)$, one has*

(i) $\gamma_E(s, \lambda)\gamma_E(s, \lambda') = \gamma_E(s, \lambda \cdot \lambda')$;

(ii) $\gamma_E(t, \lambda)^s = \gamma_E(t, s \circ \lambda)$;

(iii) $\gamma_E(st, \lambda) = \gamma_E(s, \lambda^t)\gamma_E(t, \lambda)$. [8]

(iv) *Let $E \subset E'$ be a finite extension of CM fields, both Galois over $\mathbf{Q}$.* [9] *Then $\gamma_{E'}(s, \lambda \circ N_{E'/E}) = \gamma_E(s, \lambda)$.*

(v) *The system of maps γ_E, for E ranging over CM algebraic number fields Galois over $\mathbf{Q}$, characterises the Taniyama extension $\mathcal{V}$ up to unique isomorphism.*

3. The invariant of $\mathcal{U}$

For the rest of the paper, $\overline{\mathbf{Q}}$ will denote the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, and E will range over CM fields contained in $\overline{\mathbf{Q}}$ that are Galois extensions of $\mathbf{Q}$.

3.1. DEFINITION. For every CM field E Galois over $\mathbf{Q}$, let $g_E(s, \lambda)$ be the invariant $\gamma_E(s, \lambda)$ of the previous section for the Taniyama extension $\mathcal{U}$ of 1.2.

To determine g_E explicitly, let M be a motive in $\mathcal{CM}_{\overline{\mathbf{Q}}}$ with coefficients in E and of rank 1 over E. Then for any $s \in G_{\mathbf{Q}}$, the conjugate motive

[8] Recall from 1.5, 1.5* the notations $t \circ \lambda$, resp., λ^t for the two actions of $t \in G_{\mathbf{Q}}$ on characters of $\mathcal{S}_E$.

[9] By Hilbert 90, $E^*_{\mathbf{A}_f}/E^* \hookrightarrow E'^*_{\mathbf{A}_f}/E'^*$.

M^s is well defined in $\mathscr{CM}_{\overline{\mathbf{Q}}}$ and is also a motive of rank 1 over E. Fix bases

$$\theta : E \xrightarrow{\simeq} H_B(M), \qquad \xi : E \xrightarrow{\simeq} H_B(M^s).$$

Then there exists a finite idèle $a \in E^*_{\mathbf{A}_f}$ such that the following diagram commutes:

$$\begin{array}{ccc} E_{\mathbf{A}_f} & \xrightarrow{\theta\otimes\mathbf{Q}_{\mathbf{A}_f}} & H_B(M)\otimes_E E_{\mathbf{A}_f} \\ \cdot a \downarrow & & \downarrow s \\ E_{\mathbf{A}_f} & \xrightarrow{\xi\otimes\mathbf{Q}_{\mathbf{A}_f}} & H_B(M^s)\otimes_E E_{\mathbf{A}_f}. \end{array} \tag{3.1.1}$$

Recall that $\mathscr{S}_E$ is the group scheme associated to CM motives defined over $\overline{\mathbf{Q}}$ with coefficients in E. So let $\lambda : \mathscr{S}_E \to \mathbf{G}_m$ be the character corresponding to the motive M. Then the finite idèle class $a \cdot E^* \in E^*_{\mathbf{A}_f}/E^*$ depends only on λ and on s.

3.2. Proposition. $g_E(s, \lambda) = a \cdot E^*$.

Proof. It suffices—possibly after enlarging E—to treat the case where the given character λ of $\mathscr{S}_E$ is the restriction of an E-rational representation $\rho : \mathscr{U}_E \to \mathrm{GL}_E(V)$ with $\dim_E V = 1$. Then $g_E(s, \lambda) = \lambda(\beta_E(s)) = \rho(\alpha(s))^{-1} \cdot \rho(\varepsilon_E(s))$. But $\rho(\alpha(s)) \in E^*$, and the proposition follows from the definition of ε_E.

As a consequence, these finite idèle classes $a \cdot E^*$ have the formal properties of the γ_E recorded in 2.4 above. They also satisfy the following lemma for all characters λ, where c denotes complex conjugation, w is the weight of the (CM) Hodge structure given by λ (cf. 1.3.2 ff above), and Ψ is the cyclotomic character. If $s \in \mathbf{G}_{\mathbf{Q}}$ and if $\zeta \in \overline{\mathbf{Q}}^*$ is a root of unity, then $\Psi(s) \in \hat{\mathbf{Z}}$ satisfies $\zeta^s = \zeta^{\Psi(s)}$.

3.3. Lemma. (i) $g_E(c, \lambda) = 1$.
(ii) $g_E(s, \lambda)^{1+c} = \Psi(s)^{-w} \cdot E^*$.

Proof. (i) Complex conjugation is defined rationally on $H_B(M)$. In other words, $\varepsilon_E(c) \in \mathscr{U}(\mathbf{Q}) \subset \mathscr{U}(\mathbf{Q}_{\mathbf{A}_f})$. So (i) follows from 2.2 and 2.3.

(ii) Use 2.4 (ii). By 1.3.2, $\lambda \cdot (c \circ \lambda) = N_{E/\mathbf{Q}}{}^w$. This norm corresponds to the Tate motive $E(-1)$ with coefficients in E because the Hodge structure of $\mathbf{Q}(-1)$ is pure of type $(1, 1)$. The ℓ-adic realisation of $E(-1)$ is the dual of $E \otimes \varprojlim \mu_{\ell^n}$—which explains the minus sign in the exponent of Ψ.

4. Tate's invariant and Langlands's construction

The commutative diagram preceding theorem 3.2 specialises to the situation of the Shimura-Taniyama reciprocity law when $M = h_1(A)$, with A as in §1.3.1, provided s fixes the reflex field of A. In this case, Shimura and

Taniyama provide a class field theoretic description of the finite idèle class $a \cdot E^*$. A generalisation of this reciprocity law to all s, with $M = h_1(A)$, was found by Tate in [**Tate**, 1981]. We will now present Tate's approach, generalising it to arbitrary rank 1 motives M, and explain how this vindicates Langlands's construction of the Taniyama group. When we specialise our formalism to abelian varieties, the formulas are not literally the same as in Tate's original paper—essentially because we represent an abelian variety A by the motive $h^1(A)$.

4.1. For any CM field $E \hookrightarrow \overline{\mathbf{Q}}$ that is Galois over $\mathbf{Q}$, choose some system of representatives

$$v : \mathrm{Gal}(E/\mathbf{Q}) \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}), \qquad v(\tau)_{|E} = \tau$$

in such a way that for all $\tau \in \mathrm{Gal}(E/\mathbf{Q})$ one has $v(c\tau) = v(\tau c) = v(\tau)c$, where c denotes complex conjugation.

Let λ be a character of S_E as above, lift it via $R_{E/\mathbf{Q}}\mathbf{G}_m \to S_E$ to write it in the form $\lambda = (n_\tau)_{\tau \in \mathrm{Gal}(E/\mathbf{Q})}$. Then given $s \in G_{\mathbf{Q}}$, the following formula defines an element of $\mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ which is independent of the choice of v:

$$V_E(s, \lambda) = \prod_{\tau \in \mathrm{Gal}(E/\mathbf{Q})} \left(v(\tau) \cdot (s^{-1}{}_{|E^{\mathrm{ab}}}) \cdot v(\tau s^{-1})^{-1}\right)^{n_\tau}. \tag{4.1.1}$$

4.1.2. *Notation.* Normalise the reciprocity map $r_E : E^*_{\mathbf{A}} \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ of global class field theory for E to be the reciprocal of the classical Artin map. Note that since E is totally imaginary, r_E factorises through $E^*_{\mathbf{A}_f}/E^*$. Our normalisation implies, in particular, that $r_{\mathbf{Q}}(\Psi(s)) = s_{|\mathbf{Q}^{\mathrm{ab}}}$ for all $s \in G_{\mathbf{Q}}$ with the cyclotomic character Ψ defined as in 3.3. Also, as in 3.3, write the weight of λ as w.

4.2. PROPOSITION. *For s, λ as above there exists a unique idèle class $f_E(s, \lambda) \in E^*_{\mathbf{A}_f}/E^*$ satisfying the following two identities:*

(i) $r_E(f_E(s, \lambda)) = V_E(s, \lambda)$,

(ii) $f_E(s, \lambda)^{1+c} = \Psi(s)^{-w} \cdot E^*$.

The proof is a straightforward generalisation of Tate's proof for the same result in case λ is a CM-type. We need

4.2.1. LEMMA. *The quotient group $\ker(r_E)/E^*$ is uniquely divisible and complex conjugation c acts trivially on it. In particular, $1 + c$ acts bijectively on $\ker(r_E)/E^*$.*

We quote the proof of the lemma from [**Tate**, 1981]—cf. [**Lang**, 1983, chapter 7, Lemma 2.1]:

If U is a subgroup of finite index in $\mathscr{O}_E^*$, then the group in question is isomorphic to $\overline{U}/U$, where $\overline{U}$ is the closure of U in $\widehat{\mathscr{O}_E}^*$. By a theorem of Chevalley [**Chevalley**, 1951; **Artin and Tate**, 1967], $\overline{U}$ is isomorphic to

$\varprojlim (U/U^n)$. On taking U in the real subfield E_0 of E and torsion free, the lemma follows because U is isomorphic to a product of $\mathbf{Z}$'s, and $\hat{\mathbf{Z}}/\mathbf{Z}$ is uniquely divisible.

PROOF OF 4.2. First, r_E is surjective, so there exists $a \in E^*_{\mathbf{A}_f}$ such that $r_E(a) = V_E(s, \lambda)$. Then

$$r_E(a^{1+c}) = r_E(a) \cdot cr_E(a)c^{-1} = V_E(s, \lambda) \cdot V_E(s, c \circ \lambda).$$

The latter identity follows from the fact that $c^2 = 1$, that c commutes with $\mathrm{Gal}(E/\mathbf{Q})$ (because E is a CM field), and by taking $v' = c \circ v \circ c$ instead of v to define V_E: $cv(\tau)c \cdot c(s^{-1}{}_{|E^{\mathrm{ab}}})c \cdot cv(\tau s^{-1})^{-1}c = v'(\tau c) \cdot (s^{-1}{}_{|E^{\mathrm{ab}}}) \cdot v'(\tau c s^{-1})$.

Next we check that $V_E(s, \lambda) \cdot V_E(s, c \circ \lambda) = \mathrm{Ver}_{E/\mathbf{Q}}(s^{-1})^w = \mathrm{Ver}_{E/\mathbf{Q}}(s)^{-w}$, where $\mathrm{Ver}_{E/\mathbf{Q}}$ is the transfer map from $G_{\mathbf{Q}}$ to G_E^{ab}. It is given by the formula $\mathrm{Ver}_{E/\mathbf{Q}}(s) = \prod_{\tau \in \mathrm{Gal}(E/\mathbf{Q})} v(s\tau)^{-1} \cdot (s_{|E^{\mathrm{ab}}}) \cdot v(\tau)$. So we have to show that

$$\prod_{\tau \in \mathrm{Gal}(E/\mathbf{Q})} (v(\tau) \cdot (s^{-1}{}_{|E^{\mathrm{ab}}}) \cdot v(\tau s^{-1})^{-1}) = \prod_{\tau \in \mathrm{Gal}(E/\mathbf{Q})} v(s^{-1}\tau)^{-1} \cdot (s^{-1}{}_{|E^{\mathrm{ab}}}) \cdot v(\tau).$$

This is done by first relabelling τ as $\tau^{-1}s$, and then passing from v to $v'(\tau) = v(\tau^{-1})^{-1}$; the product is invariant under these substitutions.

By class field theory, $\mathrm{Ver}_{E/\mathbf{Q}} \circ r_{\mathbf{Q}} = r_E \circ i$, where i is the natural embedding of $\mathbf{Q}^*_{\mathbf{A}}$ into $E^*_{\mathbf{A}}$. Applying both sides to $\Psi(s)$, we see that $a^{1+c}\Psi(s)^w \in \ker(r_E)$.

But the structure of $\ker(r_E)/E^*$ has been studied in the lemma. It follows that one can correct a to satisfy conditions (i) and (ii) and that this correction is unique modulo E^*.

While the requirement 4.2 (ii) matches 3.3 (ii), the following list of properties of the f_E is reminiscent of 2.4. All are easily checked for V_E instead of f_E and follow from this using 4.2.

4.3. PROPOSITION. *For all automorphisms $s, t \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ and all characters $\lambda, \lambda' \in X(\mathcal{S}_E)$, one has*

(i) $f_E(s, \lambda) f_E(s, \lambda') = f_E(s, \lambda \cdot \lambda')$.

(ii) $f_E(t, \lambda)^s = f_E(t, s \circ \lambda)$.

(iii) $f_E(st, \lambda) = f_E(s, \lambda^t) f_E(t, \lambda)$.

(iv) *Let $E \subset E'$ be a finite extension of CM fields. Then* $f_{E'}(s, \lambda \circ N_{E'/E}) = f_E(s, \lambda)$.

(v) $f_E(c, \lambda) = 1$.

We can now formulate the central result presented in this chapter.

4.4. THEOREM (Tate-Deligne). $f_E = g_E$. *In words, the class field theoretic invariant f_E also comes from a Taniyama extension and in fact from one that is isomorphic to the motivic Galois group $\mathcal{U}$ of the category $\mathcal{CM}_{\mathbf{Q}}$.*

For the proof, define $e_E(s, \lambda) = g_E(s, \lambda) \cdot f_E(s, \lambda)^{-1}$.

4.4.1. PROPOSITION. (i) $e_E(s, \lambda)e_E(s, \lambda') = e_E(s, \lambda \cdot \lambda')$.

(ii) $e_E(t, \lambda)^s = e_E(t, s \circ \lambda)$.

(iii) $e_E(st, \lambda) = e_E(s, \lambda^t)e_E(t, \lambda)$.

(iv) *Let* $E \subset E'$ *be a finite extension of CM fields. Then* $e_{E'}(s, \lambda \circ N_{E'/E}) = e_E(s, \lambda)$.

(v) *If* λ *is the trivial character, then* $e_E(s, \lambda) = 1$.

(vi) $e_E(c, \lambda) = 1$.

(vii) *If* λ *is the CM-type of an abelian variety as in* 1.3.1 *and if* $s \circ \lambda = \lambda$ *(i.e., if* s *fixes the reflex field of this CM-type: see* 1.5^**), then* $e_E(s, \lambda) = 1$.

All these properties, except the last one, are immediate consequences of what we know about f_E, g_E. As for (vii), it is but a reformulation of the Shimura-Taniyama reciprocity law: [**Shimura**, 1971, Theorem 5.15], cf. [**Shimura and Taniyama**, 1961, §13]. Here is how one checks (vii):

Let $\{\phi_1^*, \dots, \phi_n^*\}$ be the reflex type of λ on the reflex field K (see (1.3.4)). The Shimura-Taniyama reciprocity law says precisely that for a finite idèle $x \in K^*_{\mathbf{A}_f}$ such that $r_K(x) = s$, one has

$$\begin{aligned} g_E(s, \lambda) &= \det{}_E\left(x \otimes 1, \Omega^1(A)\right)^{-1} \\ &= \prod_i \phi_i^*(x)^{-1} \in E^*_{\mathbf{A}_f}/E^*. \end{aligned}$$

(The inverse comes in because we not only have the opposite sign convention for the reciprocity map from Shimura, but also work with $h^1(A)$ instead of its dual.)

To check that $f_E(s, \lambda)$ gives the same value, Tate decomposes the original type T of E given by λ into its orbits under the left action of G_K. This gives a disjoint union $T = \bigcup T_j$. Then $V_E(s, \lambda) = \prod F_j(s)$, where $F_j(s) = \prod_{\tau \in T_j} \left(v(\tau) \cdot (s^{-1}{}_{|E^{\mathrm{ab}}}) \cdot v(\tau s^{-1})^{-1}\right)$.

Following Tate, fix j temporarily, choose $w_j \in G_{\mathbf{Q}}$ such that $w_{j|E} \in T_j$, and let $L = K^{w_j^{-1}} \cdot E$ so that $G_L = w_j^{-1} G_K w_j \cap G_E$. By the basic functorial properties of Artin's reciprocity law, we see that the following diagram is commutative, where the vertical arrows are the respective Artin maps.

$$\begin{array}{ccccccc} K^*_{\mathbf{A}_f} & \xrightarrow{\text{incl}} & L^{w_j *}_{\mathbf{A}_f} & \xrightarrow{w_j^{-1}} & L^*_{\mathbf{A}_f} & \xrightarrow{\text{norm}} & E^*_{\mathbf{A}_f} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G_K^{\mathrm{ab}} & \xrightarrow{\text{Ver}} & w_j G_L^{\mathrm{ab}} w_j^{-1} & \xrightarrow{w_j^{-1} \cdot w_j} & G_L^{\mathrm{ab}} & \xrightarrow{\text{incl}_*} & G_E^{\mathrm{ab}}. \end{array}$$

If we denote by G_j the composite of the maps in the top row, we see that $\prod_i \phi_i^*(x) = \prod_j G_j(x)$. In order to prove (vii), i.e., $r_E(g_E(s, \lambda)) = V_E(s, \lambda)$, it therefore suffices to show that the composite of the maps in the bottom row of our diagram is just $s \mapsto F_j(s^{-1})$. This follows in the same way as we checked 4.2 (ii).

Theorem 4.4.1 is proved. Theorem 4.4 now follows from

4.4.2. PROPOSITION (Deligne). *Any family of maps* e_E *satisfying all the properties listed in Proposition* 4.4.1 *is trivial:* $e_E(s, \lambda)$ *is a principal idèle for all* s, λ.

We start to prove this proposition by showing that *the idèle classes* $e_E(s, \lambda)$ *all have square* 1. This is the part of the theorem already proved by Tate in [**Tate**, 1981].

4.4.3. LEMMA. (i) *If* λ *is the CM-type of an abelian variety as in* 1.3.1 *and if* $\lambda^s = \lambda^t$ *and* $s \circ \lambda = t \circ \lambda$ *(in other words, if* s *and* t *act alike on the reflex field of* λ*), then* $e_E(s, \lambda) = e_E(t, \lambda)$.

(ii) *For all* s *and all characters* λ, $e_E(s, \lambda)^c = e_E(s, \lambda)$.

(iii) *For all* s *and all* λ, $e_E(s, \lambda)^2 = 1$.

(iv) *For all* s *and all* λ, *there exists a representative* $e \in e_E(s, \lambda)$, $e \in E^*_{\mathbf{A}_f}$, *such that* $e^c = e$ *and* $e^2 = 1$.

PROOF. (i) We use 4.4.1 (iii) and (vii):

$$\begin{aligned} e_E(s, \lambda) &= e_E(tt^{-1}s, \lambda) = e_E(t, \lambda^{t^{-1}s})e_E(t^{-1}s, \lambda) \\ &= e_E(t, \lambda) \cdot 1. \end{aligned}$$

(ii) First, we assume that λ is as in part (i). By 4.4.1 (ii), $e_E(s, \lambda)^c = e_E(s, c \circ \lambda) = e_E(s, \lambda^c)$, because c commutes with every automorphism of a CM field. Now $e_E(s, \lambda^c) = e_E(sc, \lambda)$ by 4.4.1 (iii) and (vi), and $e_E(sc, \lambda) = e_E(cs, \lambda)$ by 4.4.3 (i), again because c commutes with every automorphism of a CM field. Thus, applying 4.4.1 (iii) and (vi) once more, we find that $e_E(s, \lambda)^c = e_E(cs, \lambda) = e_E(s, \lambda)$. Finally, for general λ, the claim follows from 4.4.1 (i) (recall Lemma 1.3.2).

The statements 3.3 (ii) and 4.2 (ii) imply that $e_E(s, \lambda)^{1+c} = 1$. So (ii) implies (iii).

As for (iv), start with some representative $e \in e_E(s, \lambda)$, $e \in E^*_{\mathbf{A}_f}$. By part (ii), $e^{1-c} \in E^*$. By Hilbert 90, we may therefore correct e by an element of E^* to achieve $e^{1-c} = 1$. In other words, we may (and do) assume that e is an idèle of the totally real subfield F of E. By part (iii), $e^2 \in E^* \cap F_{\mathbf{A}}{}^* = F^*$ — an element that is locally a square at all finite places. By class field theory, it is the square of an element of F^*. Therefore, we can adjust e to satisfy (iv).

This proves Lemma 4.4.3.

We now conclude the proof of Proposition 4.4.2, by following Deligne's argument contained in a letter from Deligne to Tate dated 8 October 1981. The substance of this letter is also recorded in [**Lang**, 1983, chapter 7, §4].

4.4.4. There exists some CM field E Galois over $\mathbf{Q}$ and some CM-type λ on E such that $e_E(s, \lambda) = 1$ for all $s \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$. Indeed, take for instance $E = \mathbf{Q}(\sqrt{-1})$, and let λ be the CM-type of the abelian variety of dimension 1 given by $y^2 = x^3 - x$. Then $e_E(s, \lambda) = 1$, for s fixing E (which in this case is equal to its own reflex field), is a special case of the Shimura-Taniyama reciprocity law — so special a case in fact, that it was essentially established already in [**Eisenstein**, 1850]. On the other hand, $e_E(c, \lambda) = 1$ by 4.4.1 (vi). So $e_E(s, \lambda) = 1$ for all s by 4.4.3 (i).

4.4.5. We now reduce Proposition 4.4.2 to the case $\overline{\lambda^c} = \overline{\lambda}$, where $\lambda \mapsto \overline{\lambda}$ denotes reduction modulo 2 on the character group: $X^*(\mathcal{S}_E) \to X^*(\mathcal{S}_E) \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z}$.

Indeed, by 4.4.1 (i) it suffices to prove that $e_E(s, \lambda) = 1$ for all s and all CM-types λ on E. Now given E_1, E_2 two CM fields Galois over $\mathbf{Q}$, and for $i = 1, 2$ CM-types λ_i on E_i, then by 4.4.1 (iv), we may compare the $e_{E_i}(s, \lambda_i)$ on a suitable common overfield; in other words, we may (and do) assume that $E_1 = E_2$. Then $\lambda = \lambda_1 \lambda_2^{-1}$ has weight $w = 0$, and therefore satisfies $\overline{\lambda^c} = \overline{\lambda}$. Then $e_E(s, \lambda) = 1$ implies that $e_{E_1}(s, \lambda_1) = e_{E_2}(s, \lambda_2)$. This proves 4.4.2 in view of 4.4.4.

4.4.6. It now remains to show $e_E(s, \lambda) = 1$ for all λ such that $\overline{\lambda^c} = \overline{\lambda}$.

Note that $e_E(s, \lambda)$ depends only on $\overline{\lambda}$ because we know from 4.4.3 that $e_E(s, \lambda^2) = e_E(s, \lambda)^2 = 1$.

It is convenient to switch to additive notation: write λ as $(n_\tau)_{\tau \in \mathrm{Hom}(E, \overline{\mathbf{Q}})}$ or as $\sum n_\tau \tau$. Considering $\overline{\lambda}$ amounts to reading the n_τ modulo 2, and our hypothesis on λ says that for all τ, we have $n_\tau \equiv n_{c\tau} \pmod 2$. It follows that over $\mathbf{Z}/2\mathbf{Z}$, $\overline{\lambda}$ is a linear combination of characters of the form $\tau - c\tau = \tau - \tau c$ for $\tau \in \mathrm{Hom}(E, \overline{\mathbf{Q}})$.

Given such a $\tau - \tau c$, choose $t \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ such that $\tau = t \circ \mathrm{id}_E$. Then the formula $e_E(st, (\mathrm{id}_E - c)) = e_E(s, (\mathrm{id}_E - c)^t) \cdot e_E(t, (\mathrm{id}_E - c)) = e_E(s, \tau - \tau c) \cdot e_E(t, \mathrm{id}_E - c)$ shows that it suffices to verify that $e_E(s, \mathrm{id}_E - c) = 1$ for all s.

To do this, choose a representing idèle $e \in F^*_{\mathbf{A}_f}$ for $e_E(s, \mathrm{id}_E - c)$ as in 4.4.3 (iv). Everywhere locally it is ± 1, and we have to show that the sign is everywhere the same—say $+1$ after multiplying globally with $\pm 1 \in F^*$. For this it is enough to prove that, given any two distinct finite places v_1, v_2 of F, the signs of e at v_1 and v_2 agree. Let F' be a totally real quadratic extension of F that is inert at v_1 and v_2, and let E' be the composite of F' and E. Then $e_E(s, \mathrm{id}_E - c) = e_{E'}(s, (\mathrm{id}_E - c) \circ N_{E'/E}) = e_{E'}(s, N_{E'/E} \circ (\mathrm{id}_{E'} - c)) = N_{E'/E}(e_{E'}(s, \mathrm{id}_{E'} - c))$. Representing $e_{E'}(s, \mathrm{id}_{E'} - c)$ by a finite idèle e' of F' as in 4.4.3 (iv), we see that, up to a global sign, $e = N_{F'/F} e'$. But $e'_{v_i} = \pm 1$, and the norm takes this to the square, so $e_{v_i} = +1$, which is what we wanted to show.

This completes the proof of Proposition 4.4.2 and thereby of Theorem 4.4.

4.4.7. Remark. We saw that 4.4.1 (vii) is equivalent to the Shimura-Taniyama reciprocity law. Thus, Proposition 4.4.2 implies a generalisation of this law which describes the action of all of $G_{\mathbf{Q}}$ on any CM abelian variety. This was mentioned in the introduction. Some details concerning the Galois action on additional data of the abelian variety (polarisation type) are worked out explicitly in [**Tate**, 1981] and [**Lang**, 1983, chapter 7].

In view of §2, Theorem 4.4 implies that for all E and s as above, there exists a class $f_E(s) \in S_E(E_{\mathbf{A}_f})/S_E(E)$ satisfying $f_E(s, \lambda) = \lambda(f_E(s))$ for all λ. Following Milne [**Milne**, 1989, pp. 305–307], we now give a direct construction of the elements $f_E(s)$ from a slight reformulation of Langlands's construction of the Taniyama group. This provides additional insight into Tate's invariant $f_E(s, \lambda)$, and via 4.4, a new perspective on the Taniyama extension $\mathscr{U}$ associated with CM motives. It is actually this point of view that is going to give us control over the L-functions of CM motives—see §5 below.

4.5. Theorem. *For every CM field E and every $s \in G_{\mathbf{Q}}$, there exists a unique, explicit class $f_E(s) \in S_E(E_{\mathbf{A}_f})/S_E(E)$ such that for all λ, one has $f_E(s, \lambda) = \lambda(f_E(s))$.*

Here is Langlands's approach in a nutshell (cf. [**Milne**, 1989, loc. cit.]):

Let E be a CM field Galois over $\mathbf{Q}$, and let $W^f_{E/\mathbf{Q}}$ be the quotient of the Weil group of $E/\mathbf{Q}$ by the image of the kernel of r_E under the inclusion $E^*_{\mathbf{A}}/E^* \hookrightarrow W_{E/\mathbf{Q}}$. Then we have the following commutative diagram with surjective vertical arrows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E^*_{\mathbf{A}_f}/E^* & \longrightarrow & W^f_{E/\mathbf{Q}} & \longrightarrow & \mathrm{Gal}(E/\mathbf{Q}) & \longrightarrow & 1 \\
 & & r_E \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathrm{Gal}(E^{\mathrm{ab}}/E) & \longrightarrow & \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) & \longrightarrow & \mathrm{Gal}(E/\mathbf{Q}) & \longrightarrow & 1.
\end{array}$$

Given $s \in G_{\mathbf{Q}}$, let $\hat{s} \in W^f_{E/\mathbf{Q}}$ be any element mapping to $s_{|E^{\mathrm{ab}}} \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$. Furthermore, for all $\tau \in \mathrm{Gal}(E/\mathbf{Q})$ choose representatives $w(\tau) \in W^f_{E/\mathbf{Q}}$. Let λ and (n_τ) be as in §4.1.

4.5.1. Lemma. $f_E(s, \lambda) = \prod_{\tau\in\mathrm{Gal}(E/\mathbf{Q})}(w(\tau)\hat{s}^{-1}w(\tau s^{-1})^{-1})^{n_\tau}$.

For the proof, one has to check that the formula gives elements satisfying the two conditions of Proposition 4.2. This is easy. (If $\lambda = N_{E/\mathbf{Q}}$, then the formula defines the transfer map from $W^f_{E/\mathbf{Q}}$ to the subgroup $E^*_{\mathbf{A}_f}/E^*$.)

Now in order to get the class $f_E(s)$ from $f_E(s, \lambda)$, use the canonical cocharacter $\mu_E : \mathbf{G}_m \to \mathscr{S}_E$. It is canonical over $\mathbf{C}$ and defined over E which is considered as a subfield of $\mathbf{C}$. Its dual map $\check{\mu}$ on the character groups sends a character of $\mathscr{S}_E$ given by $(n_\tau)_{\tau\in\mathrm{Hom}(E,\mathbf{C})}$ to the character of $\mathbf{G}_m$ given by n_{id}. Let $\varphi \in \mathrm{Gal}(E/\mathbf{Q})$ operate on μ_E in such a way that $(\mu_E^\varphi)\check{}\,((n_\tau)_\tau) = \check{\mu}_E\,((n_{\varphi\circ\tau})_\tau)$.

Define

$$f_E(s) = \prod_{\tau\in\mathrm{Gal}(E/\mathbf{Q})} \mu_E^\tau(w(\tau)\hat{s}^{-1}w(\tau s^{-1})^{-1}). \tag{4.5.2}$$

Then Theorem 4.5 follows immediately from Lemma 4.5.1.

At this point it is an exercise to check independently of Theorem 4.4 that $f_E(s, \lambda)$ is indeed the invariant of a Taniyama extension—cf. end of proof of Proposition 2.2. Following Langlands, we call the Taniyama extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{S} & \longrightarrow & \mathscr{T} & \longrightarrow & G_{\mathbf{Q}} & \longrightarrow & 1 \\ & & & & & & \| & & \\ & & & & \mathscr{T}(\mathbf{Q}_{\mathbf{A}_f}) & \xleftarrow{\varepsilon} & G_{\mathbf{Q}} & & \end{array} \tag{4.5.3}$$

charcterised by the invariants $f_E(s, \lambda)$ **The Taniyama Group**. Theorem 4.4 then says that the Taniyama extensions $\mathscr{U}$ and $\mathscr{T}$ are uniquely isomorphic.

To end this section we show the relationship between the Taniyama group and the Weil group. It was suggested to Langlands by Casselman and will be used in §5 to control the L-functions of motives in $\mathscr{CM}_{\mathbf{Q}}$.

4.5.4. PROPOSITION (Langlands). *For every CM field E Galois over $\mathbf{Q}$, there is a homomorphism $\phi_E : W_{E/\mathbf{Q}} \to \mathscr{T}_E(\mathbf{C})$ making the diagram below commutative:*

$$\begin{array}{ccccccccc} & & & & W_{E/\mathbf{Q}} & = & W_{E/\mathbf{Q}} & & \\ & & & & \phi_E \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathscr{S}_E(\mathbf{C}) & \longrightarrow & \mathscr{T}_E(\mathbf{C}) & \longrightarrow & \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) & \longrightarrow & 1. \end{array}$$

4.5.5. COROLLARY. *There exists a commutative diagram*

$$\begin{array}{ccccccccc} & & & & W_{\mathbf{Q}} & = & W_{\mathbf{Q}} & & \\ & & & & \phi \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathscr{S}(\mathbf{C}) & \longrightarrow & \mathscr{T}(\mathbf{C}) & \longrightarrow & G_{\mathbf{Q}} & \longrightarrow & 1. \end{array}$$

SKETCH OF THE PROOF. The corollary follows by passing to the limit. To prove the proposition, Langlands simply checks (with slightly different normalisations) that the 2-cocycle[10] describing the extension $1 \to \mathscr{S}_E(E) \to \mathscr{T}_E(E) \to \mathrm{Gal}(E^{ab}/\mathbf{Q}) \to 1$,

$$d_{s,t} = \tilde{f}_E(st) \cdot t^{-1}(\tilde{f}_E(s))^{-1} \cdot \tilde{f}_E(t)^{-1},$$

becomes trivial after inflation to $W_{E/\mathbf{Q}}$ with values in $\mathscr{S}_E(\mathbf{C})$. Here we denote by $\tilde{f}_E(s)$ some representative in $\mathscr{S}_E(E_{\mathbf{A}_f})$ of the class $f_E(s)$. So in the notation of 2.1, we are working with some $\alpha_E(s)^{-1} \cdot \varepsilon_E(s)$ instead of $\beta_E(s)$. We may arrange these representatives such that the resulting map $\tilde{f}_E : \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \to \mathscr{S}_E(E_{\mathbf{A}_f})$ is locally constant.

[10] Actually, in our somewhat unusual normalisation, we do not really get a 2-cocycle. In fact, in the notation of 2.1, we have $d_{s,t} = \alpha(st)^{-1}\alpha(s)\alpha(t)$, rather than the usual $\alpha(s)\alpha(t)\alpha(st)^{-1}$.

Lift $\tilde{f}_E$ to the Weil group to get the following commutative diagram. (Note that the composite of the maps in the bottom row is f_E.)

$$\begin{array}{ccccc} W_{E/\mathbf{Q}} & \xrightarrow{\hat{f}_E} & \mathcal{S}_E(E_{\mathbf{A}}) & \longrightarrow & \mathcal{S}_E(E_{\mathbf{A}})/\mathcal{S}_E(E) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) & \xrightarrow{\tilde{f}_E} & \mathcal{S}_E(E_{\mathbf{A}_f}) & \longrightarrow & \mathcal{S}_E(E_{\mathbf{A}_f})/\mathcal{S}_E(E). \end{array}$$

Then $\hat{f}_E(\hat{s}\hat{t}) \cdot t^{-1}(\hat{f}_E(\hat{s}))^{-1} \cdot \hat{f}_E(\hat{t})^{-1} \in \mathcal{S}_E(E) \subset \mathcal{S}_E(E_{\mathbf{A}})$ lifts $d_{s,t}$. For any infinite place v of E, let $\hat{f}_v$ be the local component at v. Then $\hat{s} \mapsto \hat{f}_v(\hat{s})$ trivialises $d_{s,t}$ over $E_v = \mathbf{C}$.

4.5.6. REMARK. The map ϕ of Proposition 4.5.4 is unique up to (reversed) 1-cocycles of $W_{E/\mathbf{Q}}$ with values in $\mathcal{S}_E(\mathbf{C})$.

5. CM motives and L-functions

One of the main applications that A. Weil drew from his newly defined "Weil groups" in [**Weil**, 1951b, section VI] was to the L-functions of their representations: via a generalisation of R. Brauer's induction theorem, they may be decomposed into Hecke L-functions "mit Grössencharakteren". In other words, the Grothendieck group of finite-dimensional continuous complex representations of $W_{\mathbf{Q}}$ is generated by representations of the form $\mathrm{Ind}_{L/\mathbf{Q}}\chi$, for algebraic number fields L and quasi-characters χ of $L^*_{\mathbf{A}_f}/L^*$.

Not all such quasi-characters will factor through $W_{\mathbf{Q}} \xrightarrow{\phi} \mathcal{T}(\mathbf{C})$ (where $\mathcal{T}$ is the Taniyama group). An obvious necessary condition that they do is that they define a (CM) Hodge structure (which would be their restriction to $\mathcal{S}(\mathbf{C})$). In other words, at most the *algebraic Hecke characters* (Weil's "quasi-characters of type A_0") will be visible in $\mathcal{CM}_{\mathbf{Q}}$—and in fact they all are.

5.1. THEOREM. *For an algebraic number field L, let ${}_L\mathcal{T}$ be the preimage of $G_L \subset G_{\mathbf{Q}}$ in $\mathcal{T}$ with respect to* 4.5.3. *Then the characters* $\mathrm{Hom}({}_L\mathcal{T}, \mathbf{C}^*)$ *are naturally identified with the algebraic Hecke characters of L.*

There are several ways to prove this theorem. We may, for instance, use 4.4, and prove 5.1 by constructing explicitly, for every algebraic Hecke character χ of L with values in an algebraic number field E, a motive $M(\chi)$ in $\mathcal{CM}_{\mathcal{L}}$ with coefficients in E whose λ-adic realisations—for finite places λ of E—are just the one-dimensional λ-adic G_k-representations attached to χ. We refer to [**Schappacher**, 1988, Chapter I, §4] for a detailed explanation of how to build up such motives $M(\chi)$ from abelian varieties of CM type.

Sticking with $\mathcal{T}$ rather than $\mathcal{U}$ one can show that the group $\tilde{\mathcal{S}}_L = \varprojlim \mathcal{S}_m$, constructed by Serre for every number field L in order to accomodate all algebraic Hecke characters of L [**Serre**, 1968], is isomorphic to the subquotient ${}_L\mathcal{T}_L$ of the Taniyama group, i.e., to the group at level L in the inverse system of quotients ${}_L\mathcal{T}_E$ the limit of which is

$$1 \to \mathcal{S} \to {}_L\mathcal{T} \to G_L \to 1.$$

See [**Langlands**, 1979, p. 224]; cf. [**Deligne**, 1982, §(E)]. We do not give the details here.

There are a few important consequences of Theorem 5.1 (and of Brauer's induction theorem applied to $\mathcal{T} \otimes \mathbf{C}$)[11] which are worth stressing, cf. also [**Anderson**, 1986, p. 181].

5.2. COROLLARY. *For every object M of $\mathcal{CM}_{\mathbf{Q}}$ with coefficients in some number field E, the system of λ-adic Galois representations $(H_\lambda(M))_\lambda$ (where λ ranges over the finite places of E) is strictly compatible.*

5.3. COROLLARY. *For each object M of $\mathcal{CM}_{\mathbf{Q}}$, there exist algebraic number fields $L_1, \dots, L_r$, integers $m_1, \dots, m_r$, and for each $i = 1, \dots, r$, an algebraic Hecke character χ_i of L_i such that*

$$L(M, s) = \prod_{i=1}^{r} L_{L_i}(\chi_i, s)^{m_i}.$$

5.5. THEOREM. *If M, M' are two objects of $\mathcal{CM}_{\mathbf{Q}}$, if ℓ is a prime number and $H_\ell(M)$, $H_\ell(M')$ the ℓ-adic representations of the motives, then the natural map*

$$\mathrm{Hom}_{\mathcal{CM}_{\mathbf{Q}}}(M, M') \otimes \mathbf{Q}_\ell \hookrightarrow \mathrm{Hom}_{G_{\mathbf{Q}}}(H_\ell(M), H_\ell(M'))$$

is surjective.

Proof. Let $\mathcal{L}$ be the affine group scheme over $\mathbf{Q}_\ell$ corresponding to the $\mathbf{Q}_\ell$-linear neutral Tannakian category of ℓ-adic representations π of $G_{\mathbf{Q}}$ which are *potentially locally algebraic*, in the sense of [**Serre**, 1968, III.3.3], i.e., possibly after restricting to a subgroup G_L of finite index in $G_{\mathbf{Q}}$, π factors through $\mathrm{Gal}(L^{\mathrm{ab}}/L)$, admits a conductor, and is given by a certain character π_{alg} of $\mathcal{S}_L$ on principal ideals generated by numbers congruent to 1 modulo the conductor. Mapping π to π_{alg} (over some sufficiently large number field L) induces a morphism $\mathcal{S} \otimes \mathbf{Q}_\ell \to \mathcal{L}$. On the other hand, if π_{alg} is trivial, then π is of finite order. This defines $\mathcal{L}$ as an extension of $G_{\mathbf{Q}}$ by $\mathcal{S} \otimes \mathbf{Q}_\ell$, and 5.5 follows from

5.5.1. LEMMA. *The extension*

$$1 \to \mathcal{S} \otimes \mathbf{Q}_\ell \to \mathcal{L} \to G_{\mathbf{Q}} \to 1$$

is isomorphic to

$$1 \to \mathcal{S} \otimes \mathbf{Q}_\ell \to \mathcal{U} \otimes \mathbf{Q}_\ell \to G_{\mathbf{Q}} \to 1.$$

[11] It applies because $\mathcal{T} \otimes \mathbf{C}$ is the inverse limit of $\mathbf{C}$-algebraic groups $\mathcal{T}_E \otimes \mathbf{C}$ whose connected components are tori; to wit $\mathcal{S}_E \otimes \mathbf{C}$.

PROOF. See [**Deligne**, 1982, §(D)]. The morphism $\mathscr{L} \to \mathscr{U} \otimes \mathbf{Q}_\ell$ is induced by mapping an E_λ-rational (with $\mathbf{Q}_\ell \subseteq E_\lambda$) representation ρ of $\mathscr{U} \otimes E_\lambda$ to $G_{\mathbf{Q}} \xrightarrow{\varepsilon_\ell} \mathscr{U}(\mathbf{Q}_\ell) \hookrightarrow \mathscr{U}(E_\lambda) \xrightarrow{\rho} \mathrm{GL}(V)$. One checks that $\rho_{|\mathscr{S}\otimes\mathbf{Q}_\ell}$ and $(\rho \circ \varepsilon_\ell)_{\mathrm{alg}}$ give the same representation of $\mathscr{S}_L$ by reducing to the case of a CM type (CM types generate all possible representations by Lemma 1.3.2 above). Then the morphism $\mathscr{L} \to \mathscr{U} \otimes \mathbf{Q}_\ell$ is trapped in a mapping between both extensions with identity on the left $(\mathscr{S} \otimes \mathbf{Q}_\ell)$ and on the right $(G_{\mathbf{Q}})$.

5.6. COROLLARY. *If two objects M, M' of $\mathscr{CM}_{\mathbf{Q}}$ have the same L-function, then they are isomorphic.*

This follows from 5.5 and the semisimplicity of our category.

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