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Beppo Levi and the Arithmetic of Elliptic Curves†

Norbert Schappacher and René Schoof

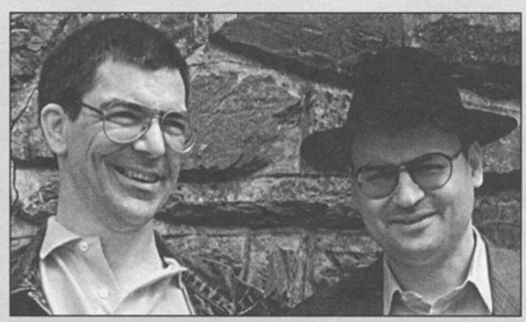
Introduction

Most students of mathematics encounter the name of the Italian mathematician Beppo Levi in integration theory when they learn "Beppo Levi's Lemma" on integrals of monotone sequences of functions. The attribution of this result is historically correct, but it by no means exhausts Beppo Levi's mathematical accomplishments.

Between 1897 and 1909, Beppo Levi (1875–1961) actively participated in all major new mathematical developments of the time. He was a man of great perseverance and energy, with an independent mind and a wide mathematical and philosophical culture. His list of publications includes more than 150 mathematical papers. Apart from his lemma, Beppo Levi is known for his work (at the very beginning of this century) on the resolution of singularities of algebraic surfaces. N. Bourbaki's *Éléments d'histoire des mathématiques* mention Beppo Levi as one of the rare mathematicians to have recognized the Axiom of Choice as a principle used implicitly in set theory, before Zermelo formulated it.

As we shall see below, the role of Beppo Levi set theory seems sometimes overrated. On the other hand, his work on the arithmetic of elliptic curves has not received the attention it deserves. He occupied himself with this subject from 1906 to 1908. His investigations, although duly reported by him at the 1908 International Congress of Mathematicians in Rome, appear to be all but forgotten. This is striking because in this work Beppo Levi anticipated explicitly, by more than 60 years, a famous conjecture made again by Andrew P. Ogg in 1970, and proved by Barry Mazur in 1976.

Shortly before his retirement, Beppo Levi faced a tremendous challenge which he more than lived up to: he was forced to emigrate, and devoted the last 20 years of his long life to building up mathematics in Rosario, Argentina.



René Schoof (l) and Norbert Schappacher (r)

Norbert Schappacher received his training at Bonn, Göttingen, Berkeley, and Paris (Orsay). After several years as a Heisenberg fellow based at the Max-Planck-Institut für Mathematik in Bonn—including a stay at the IAS, Princeton—he was appointed professor at Université Louis-Pasteur, Strasbourg. He works in arithmetic algebraic geometry. Since 1982 he has also written a number of historical papers, both on the political history of mathematics in Nazi Germany, and on aspects of the history of the arithmetic of elliptic curves.

René Schoof obtained his degree at the University of Amsterdam. Since 1990 he has worked in Italy, where he is a professor at the second University of Rome. His main interests are in algebraic number theory and arithmetic algebraic geometry, with particular regard for the computational aspects of these fields.

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In this article we briefly describe Beppo Levi's life and mathematical work, with special emphasis on his forgotten contributions to the arithmetic of elliptic curves. For more detailed biographical information the reader is referred to an extremely well researched article [Coen 1994]; a convenient list of Beppo Levi's publications is in [Terracini 1963, pp. 601–606].[1] Beppo Levi's collected papers are about to be published by the Unione Matematica Italiana.

Family and Student Years

Beppo Levi was born on May 14, 1875 in Torino, Italy, the fourth of 10 children. His parents were Diamantina Pugliese and Giulio Giacomo Levi, a lawyer and author of books in law and political economics. Perhaps the greatest mathematical talent in the family was Beppo's brother Eugenio Elia Levi, who was his junior by 8 years. By the time Eugenio became a "normalista" at the elite Scuola Normale Superiore in Pisa, Beppo was already an active mathematician. He took great interest in the mathematical education of his younger brother, and Eugenio had a brilliant career [Levi 1959–1960]. In 1909 the 26-year-old Eugenio was appointed professor at the university of Genova. He worked in complex analysis and the theory of Lie algebras. The "Levi condition" on the boundary of a pseudoconvex domain and the "Levi decomposition" of a Lie algebra are named after him. In World War I, Eugenio Levi volunteered for the Italian army. He died a captain, 33 years old, when the Italian army was overrun by the Austrians at Caporetto (October 21, 1917). He was the second brother Beppo lost in the war: Decio, an engineer and the last child of Beppo Levi's parents, had been killed on September 15, 1917 at Gorizia.

Beppo enrolled in the university of his home town Torino in 1892, when he was 17. His most influential teachers were Corrado Segre, Eugenio d'Ovidio, Giuseppe Peano, and Vito Volterra. Although he maintained an interest in all the mathematics he learned, he became most closely affiliated with Segre, and thus grew up in the famous Italian school of algebraic geometry. In July 1896 he obtained his degree, the *laurea*, writing his *tesi di laurea* on the variety of secants of algebraic curves, with a view to studying singularities of space curves.

Singularities of Surfaces

While completing his *tesi di laurea*, Beppo Levi was also helping his teacher Corrado Segre proofread Segre's important article "Sulla scomposizione dei punti singolari delle superficie algebriche." There Segre defines the infinitely near multiple points of a singular point on an algebraic surface.

The question arose whether a certain procedure to eliminate singularities would eventually terminate. More precisely: Under which conditions does the sequence of multiplicities of the infinitely near points obtained by successive quadratic transformations $x = x'z'$, $y = y'z'$, $z = z'$, reach 1 after finitely many steps? Segre thought this was the case unless the starting point lies on a multiple component of the surface, and he wanted to deduce this from a result by the Scandinavian geometer Gustaf Kobb. But that result was not correct in full generality, and Segre's corollary was justly criticized by P. del Pezzo. It was the young Beppo Levi who supplied—with a proof that even satisfied Zariski in 1935—a complete solution of the problem, which forms the content of his first publication [Levi 1897a].

So much for the substance of this particular issue between Segre and del Pezzo. But it was only a tiny episode in a ferocious controversy between the two mathematicians. It is probably fortunate that most of this exchange of published notes (four by del Pezzo, the last one being entitled in ceremonial latin "Contra Segrem," and two by Segre) appeared in fairly obscure journals. They were not included in Segre's collected works. We understand that the exchange reflects a personal animosity predating the mathematical issues [Gario 1988]; cf. [Gario 1992].

Beppo Levi continued to work for a while on this subject. He attacked the problem of the resolution of singularities of algebraic surfaces and claimed success.

Beppo Levi's method is that of Segre: to alternate between quadratic transformations (as above) and monoidal transformations of the ambient space, followed by generic projections. This procedure typically creates new, "accidental" singularities of the surface. The main problem then is to control this procreation of singular points, showing that the process eventually terminates. Beppo Levi's solution was published in the paper [Levi 1897b] which appeared in the *Atti dell'Accademia delle Scienze di Torino*, Segre's house paper. It was acknowledged to be correct and complete by E. Picard in the early 1900s, by Severi in 1914, by Chisini in 1921—but *not* by Zariski in 1935. In his famous book on algebraic surfaces [Zariski 1935], Zariski points out gaps in all the proofs for the resolution of singularities of algebraic surfaces that were available at the time, including Levi's. In a dramatic climax, Zariski closes this section of his book with a note added in proof to the effect that Walker's function-theoretic proof (which had just been finished) "stands the most critical examination and settles the validity of the theorem beyond any doubt." See also the remarkable introduction to the article [Zariski 1939].

To to sure this was not the end of the history. . . .

We will not discuss the completeness of Beppo Levi's proof; but we may quote H. Hironaka's [1962] footnote

[1]See also Homenaje a Beppo Levi, *Revista Unión Matematica Argentina* 17 (1955), as well as the special volume *Mathematicæ Notæ* 18 (1962).

about the papers [Levi 1897a, 1897b] from his lecture at the International Mathematical Congress in Stockholm on his famous theorem on the resolution of singularities of arbitrary algebraic varieties (in characteristic 0)—a result for which he obtained the Fields Medal later.

> . . . the most basic idea that underlies our inductive proof of resolution in all dimensions has its origins in B. Levi's works, or, more precisely, in the theorem of Beppo Levi. . . .

Here, Hironaka alludes to the main result of B. Levi's [1897a], which we discussed above.

Let us conclude this section with a slightly more general remark: It is commonplace today to think of the Italian algebraic geometers as a national school which contributed enormously to the development of algebraic geometry, *in spite of* their tendency to neglect formal precision—a tendency which is seen as the reason for the many "futile controversies" which mark this school.[2]

It seems to us that this view is rather biased. An historically more adequate account would have to measure the fundamental change of paradigm introduced in the thirties by Zariski and Weil. One has to try and imagine what it must have been like to think about desingularization without the algebraic concept of normalization. As for the controversies, they do not seem to be a result of formal incompetence so much as of personal temperament and competition; examples of such feuds can be found long after the end of the Italian school of algebraic geometry. Finally, even the very name of "Italian School of Algebraic Geometry" can be misleading, in that it does not bring out the strong European connection of this group of mathematicians.[3]

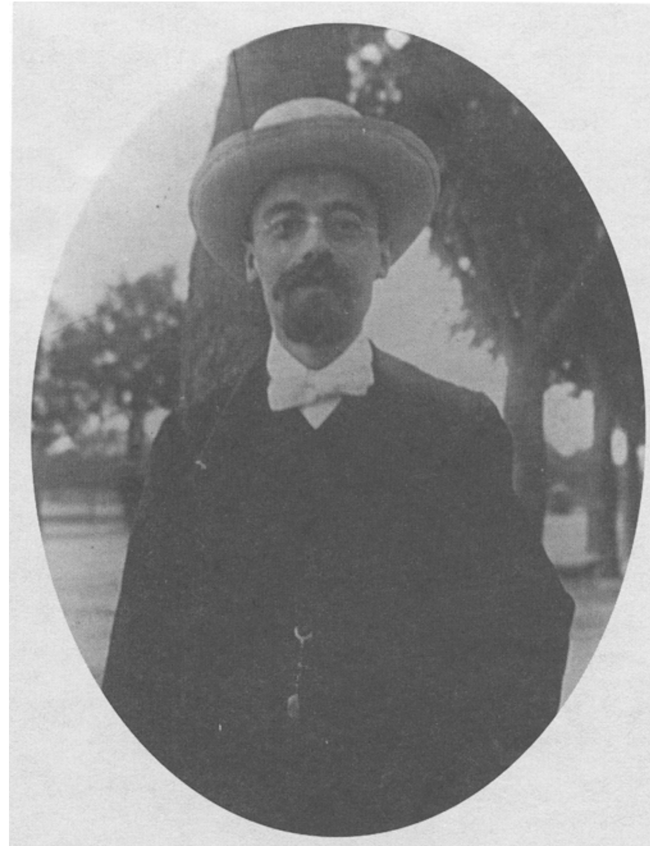

Figure 1. The youthful Beppo Levi.

Axiom of Choice and Lebesgue's Theory of Integration

In spite of his beautiful work on algebraic surfaces, Beppo Levi gave up his assistantship to the chair of Luigi Berzolari at the University of Torino in 1899 (the year that the latter moved to Pavia), and accepted positions at secondary schools in the northern Italian towns of Vercelli, Piacenza, and Torino, and also in faraway places like Bari and Sassari. He probably accepted these somewhat mediocre but better-paid positions in order to contribute to the finances of the family in Torino, after his father's untimely death in 1898. In 1901 Beppo Levi was candidate for a professorship (at Torino); the position was given to Gino Fano, with Beppo Levi ranking third on that occasion.

From this period dates Beppo Levi's contact with early variants of the Axiom of Choice, which earned him a mention in a footnote of Bourbaki's *Eléments d'histoire des mathématiques* [Bourbaki 1974, p. 53]. Thanks to Moss [1979], and in particular to the extremely thorough historical study by Moore [1982, especially §1.8: "Italian Objections to Arbitrary Choices"], we may be historically a little more precise than Bourbaki (and also safely dismiss the apocryphal story told by Abraham Fraenkel in [Fraenkel and Bar-Hillel 1958, p. 48]).

Apparently following the local Torino tradition, which had been started very early by G. Peano, of criticizing uncontrolled applications of arbitrary choices in set theory, Beppo Levi published a criticism of Felix Bernstein's thesis, pointing out a certain partition principle that Bernstein used [Levi 1902]. In this sense Beppo Levi belongs to the prehistory of Zermelo's famous ar-

[2]For instance D. Mumford (Parikh 1990, xxvf): "The Italian school of algebraic geometry was created in the late 19th century by a half dozen geniuses who were hugely gifted and who thought deeply and nearly always correctly about their field. . . . But they found the geometric ideas much more seductive than the formal details of the proofs. . . . So . . . they began to go astray. It was Zariski and . . . Weil who set about to tame their intuition, to find the principles and techniques that could truly express the geometry while embodying the rigor without which mathematics eventually must degenerate to fantasy."—Or Dieudonné [1974, 102f]: "Malheureusement, la tendance, très répandue dans cette école, à manquer de précision dans les définitions et les démonstrations, ne tarda pas à entraîner de nombreuses controverses futiles, ..."

[3]For instance, an obituary notice for Corrado Segre in *Ann. di Mat. Pura et Appl.* (4) 1 (1923), p. 319f. describes the "second phase" of Italian geometry (initiated by Segre and a few Italian colleagues) as a wonderful synthesis of ideas by "Cremona, Steiner, v. Staudt, Plücker, Clebsch, Cayley, Brill, Noether, and Klein."

ticle [Zermelo 1904]. Moore makes it clear, however, that Beppo Levi cannot be said to have already possessed Zermelo's Axiom of Choice. Indeed, Beppo Levi was never ready to admit this axiom in general; see [Moore 1982, §4.7] for a discussion of Beppo Levi's later attempts to regulate the use of this and other problematic principles of set theory. Alternatively, the reader might wish to consult the easily accessible letter to Hilbert [Levi 1923] to get acquainted with Beppo Levi's peculiar idea of *deductive domains.*

A little later, Beppo Levi tried to come to terms with the new theory of integration and measure of Henri Lebesgue. In a letter to Emile Borel postmarked June 1, 1906, Lebesgue writes [Lebesgue 1991, p. 148f]:

> My dear Borel, ... My theorems, invoked by Fatou, are now criticized by Beppo Levi in the *Rendiconti dei Lincei.* Beppo Levi has not been able to fill in a few simple intermediate arguments and got stuck at a serious mistake of formulation which Montel earlier pointed out to me and which is easy to fix. Of course, I began by writing a note where I treated him like rotten fish. But then, after a letter from Segre, and because putting down those interested in my work is not the way to build a worldwide reputation, I was less harsh. . . .

2. Ciò posto, dimostrerò la proposizione seguente:
Se una successione non decrescente di funzioni $f_n(x)$ positive ed integrabili nell'aggregato $\mathfrak{A}$ ha in $\mathfrak{A}$ un limite $f(x)$, e se esiste ed è finito il

$$\lim_{n=\infty} \int_{\mathfrak{A}} f_n(x)\,dx,$$

la funzione $f(x)$ è integrabile in $\mathfrak{A}$ e si ha

$$\int_{\mathfrak{A}} f(x)\,dx = \lim_{n=\infty} \int_{\mathfrak{A}} f_n(x)\,dx. \qquad (2)$$

Si ponga $\lim_{n=\infty} \int_{\mathfrak{A}} f_n(x)\,dx = A.$

Figure 2. Facsimile from page 776 of [Levi 1906a].

Beppo Levi's Lemma.
Let f_n be a nondecreasing sequence of integrable functions on a measurable set E such that $\lim_{n\to\infty} \int_E f_n$ is finite; then $f(x) = \lim_{n\to\infty} f_n(x)$ is finite almost everywhere and is integrable with

$$\int_E f = \lim_{n\to\infty} \int_E f_n .$$

Lebesgue's reply to Beppo Levi's criticism, published in *Rendiconti dei Lincei* [Lebesgue 1906], makes it quite clear who is the master and who is the apprentice in this new field. This somewhat marginal role of Beppo Levi's first papers on integration may explain why his name is often lacking in French accounts of integration and measure theory. Even Dieudonné, in chapter XI (written by himself) of the "historical" digest [Dieudonné, *et al.* 1978], fails to mention Beppo Levi's works altogether. In English- and German-speaking countries however, a course on Lebesgue's theory will usually be the one occasion where the students hear Beppo Levi's name mentioned. His famous Lemma was published in the obscure *Rendiconti del Reale Istituto Lombardo di Scienze e Lettere* [Levi 1906a]. The article provides the proof of a slight generalization of one of Lebesgue's results. The statement of Beppo Levi's Lemma which we give in the inset is a resumé of sections 2 and 3 of [Levi 1906a]. The lemma was quoted (and thereby publicized) by G. Fubini in his important paper in *Rendiconti Acc. dei Lincei* [Fubini 1907], which contains the proof of what mathematicians still know as Fubini's theorem, in the case of a rectangle domain.

Levi's lemma is similar to Fatou's lemma, which coincidentally also dates back to 1906 (*Acta Math.* 30, 335–400). One can thus build up the theory without reference to Beppo Levi's result. It is nevertheless difficult to understand why Dieudonné omits Levi's name from his "historical" account of Lebesgue's theory.

Fubini's theorem for a rectangle, to quote Hawkins [1975, p. 161], "marked a real triumph for Lebesgue's ideas." As Fubini said, the Lebesgue integral "is now necessary in this type of study." In fact, Fubini's theorem had been anticipated by Beppo Levi, albeit without a detailed proof, in a footnote of a very substantial paper on the Dirichlet Principle [Levi 1906b, p. 322]. Here Beppo Levi observed that Pringsheim's careful investigation of double integrals in Riemann's theory of integration carries over to Lebesgue's theory, yielding, in fact, a simpler and more general statement.

This paper has inspired a number of developments in functional analysis and variational calculus. Thus Riesz [1934] derived from its section 7 the idea for an alternative proof (not using separability) of the existence of orthogonal projections onto a closed subspace of a Hilbert space. The proof uses what other authors isolate as "Beppo Levi's inequality"—see for instance [Neumark 1959, §5.2].

Furthermore, a certain class of functions studied in [Levi 1906b] led Nikodym to define the class of what he called (BL)-functions [Nikodym 1933]. This idea was carried further in the study of so-called "spaces of Beppo Levi type" [Deny and Lions 1953].

It is also possible that a remark in [Levi 1906b] inspired some of Lebesgue's later contributions to Dirichlet's Principle. A passage in another letter of Lebesgue to Borel (12 February 1910) seems to suggest this. But the history of Dirichlet's Principle at that time is very intricate, so we do not go into details here.

Elliptic Curves

In December 1906, 10 years after his *laurea,* Beppo Levi was appointed professor for *geometria proiettiva e descrittiva* at the University of Cagliari on the island of

There Is No Elliptic Curve over Q with a Rational 16-Torsion Point

(Beppo Levi's d'après)

Beppo Levi uses the curves given by the equations

$$Y^2(X - Z) - aX^2Y + (a + b)XYZ - bXZ^2 \quad (a, b \in \mathbf{Q})$$

in $\mathbf{P}^2$. They possess the rational points $P_1 = (1:0:0)$, $P_2 = (0:0:1)$, $P_3 = (0:1:0)$, and $P_4 = (1:1:1)$. Each point is on the tangent of the preceding one. The point P_1 has order 8 if we take P_4 as the neutral element of the group of points, i.e., if P_4 is a point of inflection. This means that the Hessian vanishes in P_4, and this is equivalent to

$$b(a^2 - 3a + 2 + b) = 0.$$

The case $b = 0$ corresponds to a degenerate cubic, but all other curves with $b = -a^2 + 3a - 2 = -(a-1)(a-2)$ are smooth curves containing the rational point P_1 of order 8. Conversely, every elliptic curve over $\mathbf{Q}$ with a point of order 8 is of the above form for some $a, b \in \mathbf{Q}$.

Before studying rational points of order 16, we observe that not only does the tangent at P_1 intersect the cubic in P_2, but so does the tangent of the point $P_1' = (1 : -a^2 - 2a : a)$. In terms of the group law we have $P_1' = 5P_1$.

Suppose that Q is a point on the curve of order 16. Without loss of generality we may assume that $-2Q = P_1$, i.e., Q is a point which has P_1 on its tangent. There are three more such points: Q', Q'', and Q'''. They are not necessarily rational, but if Q has rational coordinates, so does at least one other point, say, Q'. Let l_1 be the line joining Q and Q', and let l_2 be the line joining Q'' and Q'''. Beppo Levi views $l_1 \cup l_2$ as a degenerate member of the family fascio, $\mathfrak{F}$ of conics through Q, Q', Q'', and Q'''. He explicitly describes $\mathfrak{F}$ by observing that the partial derivative of the cubic equation with respect to X describes a conic in $\mathfrak{F}$ and so does the equation $YZ = aX^2$ (subtract X times the equation of the first conic from the cubic equation and divide by Y). The conic $l_1 \cup l_2$ is the unique member that passes through P_1'. It is not difficult to compute the equations of l_1 and l_2:

$$Y - a(1 \pm \sqrt{a-1})X \pm \sqrt{a-1}Z = 1 - a.$$

If Q is a rational point, the lines l_1 and l_2 admit equations with rational coefficients, i.e.,

$$a - 1 = c^2 \quad \text{for some } c \in \mathbf{Q}.$$

The line l_1 intersects the cubic in the points P_1, Q, and Q'. The coordinates of $Q = (x : y : z)$ satisfy the equation

$$(c^2 + 1)x^2 + (c^2 + 1)(c - 1)xz + c(c - 1)z^2 = 0.$$

If the point Q is rational, the discriminant of this quadratic equation is a square:

$$d^2 = (c + 1)(c - 1)(c^2 + 1)(c^2 - 2c - 1) \quad \text{for some } d \in \mathbf{Q}.$$

This is Beppo Levi's way of writing down a model of what we know as the modular curve $X_1(16)$ over $\mathbf{Q}$. Note in passing that the genus of $X_1(16)$ is 2. Beppo Levi solves this equation by means of Fermat's method of infinite descent. There are two possibilities. Up to sign, either $c - 1$, $c + 1$, $c^2 + 1$, and $c^2 - 2c - 1$ are all squares in $\mathbf{Q}$, or they are all two times a square. In the first case one easily arrives at the equation

$$X^4 + 1 = 2Y^2,$$

and in the second case at

$$X^4 + 1 = Y^2.$$

Already in 1738 Euler had shown that the first equation only admits the trivial rational solutions $(X, Y) = (\pm 1, \pm 1)$. The analogous fact for the second equation, including the trivial solutions $(0, \pm 1)$, had been known already to Fermat: Its solutions correspond to $(c, d) = (0, \pm 1)$ and $(\pm 1, 0)$. This implies $b = 0$, which yields a degenerate cubic curve. These points together with the two points at infinity therefore account for the six rational cusps of the curve $X_1(16)$.

We conclude that an elliptic curve over $\mathbf{Q}$ cannot have any rational points of order 16.

The Chord and Tangent Process and Points of Finite Order

Counting multiplicities, any line in $\mathbf{P}^2$ meets a nonsingular plane cubic curve $E \hookrightarrow \mathbf{P}^2$ in three points. Thus, given two points P and Q on E, the line ("chord") that joins them intersects E again in a third point R, which may coincide with P or Q. If the points P and Q happen to be equal, the role of "the line that joins them" is played by the tangent to E at $P = Q$. If the curve is defined over a field K, and if P and Q are K-rational points, then so is R. This defines a law of composition $E(K) \times E(K) \to E(K)$, called the *chord and tangent process.*

The flex points of E, i.e., the points where the tangent to E has multiplicity 3, are precisely the points which reproduce themselves under the chord and tangent process. There are exactly nine flex points over any algebraically closed field of characteristic different from 3. But except in trivial cases when the set $E(K)$ is very small, no flex point will be a neutral element for this law of composition.

Now let us suppose that E is an *elliptic curve over* K, i.e., a plane cubic E defined over K with a fixed K-rational point $O \in E(K)$. One may then modify the chord and tangent process into an abelian group law with neutral element O, by defining $P \oplus Q$ to be the point resulting from the chord and tangent process applied to O and R, the latter point being itself the result of the chord and tangent process applied to P and Q. Thus the point R resulting from P and Q by the chord and tangent process is the inverse of $P \oplus Q$ in the group. The proof that this modified law is actually associative, which is the only nonobvious group axiom to check here, is somewhat involved to give in purely geometric terms. Note in passing that, in terms of the group law, the flex points in $E(K)$ are precisely the points annihilated by 3.

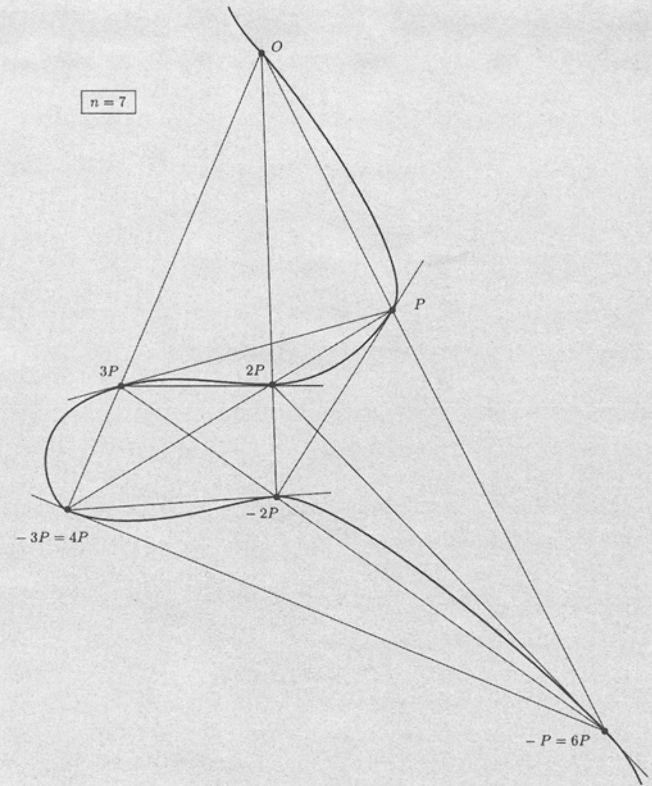

Figure 3. $O = 7P$ is a flex point which may be taken as origin for the group law.

Sardinia. Because of the Italian system of *concorsi*, this somewhat isolated place has been the starting point of quite a number of academic careers. For instance, in the 1960s, the later Fields medallist Enrico Bombieri, Princeton, also was first appointed professor at Cagliari.

At the end of 1906, Beppo Levi was a candidate for the Lobachevsky prize of the Academy of Kazan, on the basis of two papers on projective geometry and trigonometry. In spite of the positive scientific evaluation of the works he only received an "honorable mention" because, it was said, the prize was reserved for contributions to non-euclidean geometry. In fact, the prize was not awarded at all that year [Kazan 1906].

The year 1906 may have been the richest year for Levi's mathematical production. It was probably early that year that he began to work on the arithmetic of cubic curves.

We have seen how Beppo Levi often became acquainted with a new theory by a critical reading of seminal papers. The subject of the Arithmetic of Algebraic Curves had been defined and christened by Henri Poincaré in his momentous research program [Poincaré 1901]. This program is best understood as the attempt to reform the tradition of *diophantine analysis*, whose practitioners were perfectly happy every time a certain class of diophantine equations could be solved explicitly (or shown to be unsolvable) by some trick adapted to just these equations. Now Poincaré proposed to apply some of the notions developed by algebraic geometry during the 19th century.

More precisely, Poincaré's idea was to study smooth, projective algebraic curves over the rational numbers up to birational equivalence. The first birational invariant that comes to mind is of course the *genus*, and a typical first problem studied is then the nature of the set of rational points of a curve of given genus g. Poincaré starts with the case of rational curves, $g = 0$. Their rational points (if there are any) are easily parametrized, and a remarkably complete study of this case (although Poincaré does not

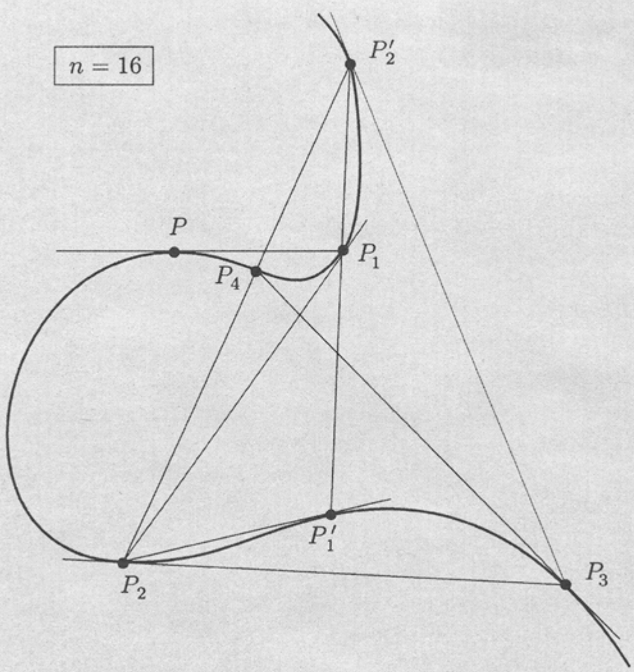

Figure 4. P_4, the fourth tangential of P, is a flex point. In terms of the group law, one has $P_1 = -2P$, $P_2 = +4P$, $P_3 = -8P$, and $P_4 = +16P$.

In Figure 3 for example, one checks that, with the flex point O as origin of the group law, $2P$ is indeed $P \oplus P$, because the line joining P to $-2P$ is tangent to E at P, and $2P$ is the third point of intersection with E of the line through $-2P$ and O. Note that the point $-2P$ is independent of the fixed origin O because it is given in terms of the chord and tangent process alone. $-2P$ is called the (first) tangential of P.

More generally, the set $[P]$ of all points that can be obtained, starting from P, by repeated application of the chord and tangent process can be described in terms of the group law by

$$[P] = \{nP \mid n \in \mathbf{Z}, n \equiv 1 \pmod 3\} \subset \{nP \mid n \in \mathbf{Z}\} =: \langle P \rangle.$$

The cyclic subgroup $\langle P \rangle$ of $E(K)$ can be infinite or finite. The points P where it is finite are the torsion elements or the *points of finite order* of the group $E(K)$. Whether a point has finite order or not is independent of the choice of origin O implicit in the group law. In fact, P is of finite order if and only if $[P]$ is a finite set. One finds $[P] = \langle P \rangle$ if and only if $O \in [P]$. In fact, $O \in [P]$ implies also $-P \in [P]$, in view of the definition of $[P]$.

Every nonsingular plane cubic curve over K which admits a K-rational point is birationally equivalent to one that admits a K-rational flex point. Assume therefore that such a rational flex exists and choose henceforth only flex points as origin for the group law. Suppose $P \in E(K)$ is of order $n < \infty$. Then $\pm nP = O$ is a flex point, which already belongs to $[P]$ if $n \not\equiv 0 \pmod 3$ because then $\pm n \equiv 1 \pmod 3$. Thus $[P] = \langle P \rangle$ in this case. This is the situation illustrated in the Figures 3 and 4 for $n = 7$ and 16. The latter configuration is impossible to realize over the rational number field $K = \mathbf{Q}$, by one of Beppo Levi's results.

In Beppo Levi's terminology, the configuration of Figure 3 is "polygonal," whereas Figure 4 shows "una configurazione arborescente." The difference is that the flex in Figure 4 arises directly as the fourth tangential of P, whereas O in Figure 3 is not the tangential of any other point of the configuration.

interrupt his flow of ideas by references) had in fact been published by Hilbert and Hurwitz in 1890.

The case of curves of genus 1 with a rational point, i.e., of elliptic curves, occupies the bulk of Poincaré's article. Without loss of generality one may assume the elliptic curve is given as a curve of degree 3 in $\mathbf{P}^2$, by a nonsingular homogeneous cubic equation. It is the theory of the rational points of these curves, as sketched by Poincaré, that Beppo Levi is picking up, criticizing, and developing further in 1906.

Any line in $\mathbf{P}^2$ meets a cubic curve E in three points (counting multiplicities). If the line and the curve are both defined over $\mathbf{Q}$, and two of the three points of intersection are rational, i.e., have rational homogeneous coordinates in $\mathbf{P}^2$, then so is the third. This defines a law of composition $E(\mathbf{Q}) \times E(\mathbf{Q}) \to E(\mathbf{Q})$, called the chord and tangent method (see boxed text). Except in rather trivial cases [when the set $E(\mathbf{Q})$ is very small], this does not afford a group structure on the set of rational points. But it may always be turned into an abelian group, essentially by choosing an origin. Neither Poincaré nor Beppo Levi takes this step toward the group structure, and both work with the chord and tangent process itself.[4] Flex points are then special, in that starting from such a point, the method does not lead to any new point.

This basic method of the arithmetic of elliptic curves had been used by Fermat when working with certain diophantine problems, and in particular in some of his proofs by *infinite descent*. Its geometric meaning seems to have been first observed by Newton—see [Schappacher 1990] for a more detailed history of the method.

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[4]Although an abstract notion of group was defined already in H. Weber's article "Die allgemeinen Grundlagen der Galois'schen Gleichungs-Theorie," *Mathematische Annalen* 43 (1893), 521–524, and differently in J.A. de Séguir's *Eléments de la théorie des groupes abstraits*, Paris, 1904, the mathematicians of the time were certainly not trained the way we are today to look for this structure.

During 1906–1908 Levi published four remarkable papers on the subject in *Atti della Reale Accademia delle Scienze di Torino* [Levi 1906–1908]. The first paper is rather general. Beppo Levi avoids the very difficult question whether a plane cubic curve possesses a rational point or not, by assuming, once and for all, that the curves under consideration do. He classifies these elliptic curves up to isomorphism, not only over **C**, but over **Q**.

Generalizing the chord and tangent process, Beppo Levi also considers deducing new rational points on E from given ones by intersecting E with curves over **Q** of degree higher than 1. He knows, as did Sylvester and others before him, that this apparently more general notion of *rational deduction of points* does not yield any more general dependencies than the chord and tangent method.

In the first paper, Levi gives, in particular, a birationally invariant definition of the *rank* with respect to the chord and tangent process of the set of rational points on an elliptic curve over **Q**, under an assumption which amounts to saying that $E(\mathbf{Q})$ is a finitely generated abelian group. He justly criticizes Poincaré for having overlooked that, given two birationally equivalent curves, one may and the other may not have a rational point of inflection—this actually makes Poincaré's notion of rank not birationally invariant!

Beppo Levi's notion of rank does not coincide with what we call today the **Z**-rank of the finitely generated abelian group $E(\mathbf{Q})$: Beppo Levi adds to the free rank the minimum number of points needed to generate the torsion subgroup. (To be precise, the minimality condition that he writes down for his (finite) *basis* of the set of rational points is not strong enough to make the rank uniquely defined; he only asks that a basis be minimal in the sense that none of its points be expressible in terms of the others.)

In a footnote he stresses very explicitly that the assumption of finite generation for $E(\mathbf{Q})$ was not proved (it was established only in 1922, by L.J. Mordell):

> [The finite rank assumption] may be doubtful: either there might exist a cubic curve with a basis consisting of infinitely many rational points: in this case one would say that the rank is infinite; or no basis exists at all in the sense that, for any given set of rational points, one can obtain these points rationally from other points which themselves cannot be obtained rationally from the given set: this would occur if every rational point were on the tangent of another rational point.

Thus Beppo Levi is more explicit than Henri Poincaré, who did not let the possibility of $E(\mathbf{Q})$ not being finitely generated enter into his discussion. Like Beppo Levi's footnote, Mordell's proof of 1922 has two parts: it is shown that the rank cannot be infinite, and then, by means of the theory of heights, it is shown that the second possibility indicated by Levi does not occur.

The last part of Beppo Levi's first note is devoted to elliptic curves all of whose points of order 2 are rational. For these curves Beppo Levi seems to embark on a general 2-descent, but he does not quite conclude it. (A more historically minded study of Levi's notes in the context of the development of the method of descent is given in [Goldstein 1993].)

Ogg's Conjecture

Starting from a given rational point, other rational points on a given elliptic curve may be constructed applying successively the chord and tangent method, but only to the given point or to points constructed in previous steps. As Levi puts it [Levi 1906–1908, §11], usually one would in this way obtain infinitely many rational points, but in certain exceptional cases the procedure "fails" in the sense that one ends up in some kind of a loop. In modern language, this means that the point of departure has finite order in the group $E(\mathbf{Q})$.

Beppo Levi sets out to classify these "failures" of the chord and tangent method; that is, to determine what the structure of the subgroup of points of finite order on an elliptic curve over **Q** can be. The last three papers in the series are devoted to this problem.

Beppo Levi's method is straightforward. He takes a general nonsingular cubic curve and writes down explicitly what the "failure" of the chord and tangent method for a given rational point on the curve means for the coefficients of the equation. The chord and tangent method can of course "fail" in various ways, and each way gives rise to a certain finite configuration of points and lines. Levi distinguishes four types: *configurazioni arborescenti, poligonali, poligonali misti,* and, finally, *configurazioni con punti accidentali.*

Let us translate this into modern terminology: Levi fixes a finite abelian group A and computes under what conditions on the coefficients of the curve the group of rational points admits A as a subgroup. He is aware of the complex analytic theory of elliptic functions and he exploits this as well as the restrictions on the structure of A coming from the fact that the elliptic curve is already defined over **R** (even over **Q**): if A occurs, it is either a cyclic group, or a cyclic group times $\mathbf{Z}/2\mathbf{Z}$. Beppo Levi's *configurazioni arborescenti, poligonali,* and *poligonali misti* correspond, respectively, to subgroups of the form $A = \mathbf{Z}/n\mathbf{Z}$ where n is a power of 2, n is odd, or n is even but not a power of 2. The *configurazioni con punti accidentali* correspond to the groups $A = \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, with n even.

Beppo Levi's method yields explicit parametrizations of elliptic curves with given torsion points; the problem of their existence then comes to solving certain diophantine equations. Sometimes this is very easy. Thus Levi shows that the groups

$$\left.\begin{array}{ll} \mathbf{Z}/n\mathbf{Z} & \text{for } n = 1, 2, \ldots, 10, 12, \\ \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \text{for } n = 2, 4, 6, 8, \end{array}\right\}*$$

all occur infinitely often.

What is more remarkable, Beppo Levi can also show that certain *configurazioni* **do not occur:** the group $A = \mathbf{Z}/n\mathbf{Z}$ does not occur for $n = 14, 16, 20$, and $A = \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ does not occur for $n = 10$ and 12. In these cases he must study some thorny diophantine equations, defining plane curves of genus 1 or 2. He concludes by infinite descent, very much in the spirit of Fermat. In today's jargon of the arithmetic of elliptic curves, the infinite descent involves a 2-descent [Levi 1906–1908, §17].

In some other cases Beppo Levi does not overcome the technical difficulties; for $n = 11$ and $n = 24$ he gives the equations but cannot rule out the existence of unexpected solutions.

The equations Levi finds, say, for the groups $\mathbf{Z}/n\mathbf{Z}$, are equations for the modular curves $X_1(n)$ that parametrize elliptic curves together with a point of order n. The modular curves called today $X_0(N)$ and $X(N)$ and their explicit equations—which we would regard as fairly similar to $X_1(n)$—had actually been studied in the 19th century; see for instance [Kiepert 1888–1890]. There is no indication that Beppo Levi was aware of this connection. He seems to have looked at his equations only the way he obtained them: parametrizing families of elliptic curves with given torsion points. Neither upper half-plane, modular groups, nor modular functions are evident in his work.

The "easy" cases, where examples of elliptic curves with a torsion point of order n abound, are precisely those cases where the genus of $X_1(n)$ is zero. The equation that Beppo Levi finds for $n = 11$ is (what we recognize today as) a $\mathbf{Z}$-minimal equation for the curve $X_1(11)$ of genus 1:

$$Y^2X - Y^2Z - X^2Z + YZ^2 = 0.$$

Its five obvious rational points are all "cusps" of $X_1(11)$. Of course, Beppo Levi does not see these cusps as boundary points of the fundamental domain of $\Gamma_1(11)$ acting on the upper half-plane. For him they are simply the solutions to the equation which correspond to degenerate cubic curves.

At the 1908 International Mathematical Congress in Rome, Beppo Levi reported on his work on elliptic curves [Levi 1909]. There he also explained what he thought would happen for the other values of n: he believed that the above list exhausts all possibilities.

This is how he states it: for the *configurazioni arborescente* he has proved that $\mathbf{Z}/n\mathbf{Z}$ where n is a power of 2 cannot occur for $n = 16$ and therefore n "... cannot contain the factor 2 to a power exceeding 3." For the *configurazioni poligonali* he writes regarding the group $\mathbf{Z}/n\mathbf{Z}$ with n odd: "It is very probable that for $n > 9$ there do not exist any more rational points. . . ." As far as the *configurazioni poligonali misti* are concerned, he remarks that he has shown that $\mathbf{Z}/n\mathbf{Z}$ cannot occur when $n = 20$ and that therefore $n = 5(2^k)$ cannot occur for any $k \geq 2$. He has shown that $\mathbf{Z}/2n\mathbf{Z}$ does not occur for $n = 7$ and "it is probable that they do not exist either for larger odd values of n."

It is not difficult to see that these conjectures already imply that the above list for the groups of the type $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ should be complete: "Such configurations exist for $n = 2, 4, 8, 6$, but not for $n = 12$ and 10; one can argue that they do not exist for larger values of n."

Apart from the group $\mathbf{Z}/24\mathbf{Z}$ which he does not mention, this means precisely that Beppo Levi believed the list(*) to be complete. More than 40 years later T. Nagell made the same conjecture [Nagell 1952a].[5] In our days the conjecture became widely known as Ogg's conjecture, after Andrew Ogg, who formulated it 60 years after Beppo Levi.

The problem studied by Beppo Levi and later by Billing, Mahler, Nagell, and Ogg has been very important in the development of arithmetic algebraic geometry. In the years following 1970, rapid progress was made by invoking the arithmetic theory of modular curves. The cases $n = 11$, 15, and 24 had already been taken care of before the fifties [Billing and Mahler]; [Nagell 1952b]. (It is touching to read Beppo Levi's review of [Billing and Mahler 1940] in *Mathematical Reviews* where he recognizes the equation of $X_1(11)$ that he had already published in 1908 but was unable to solve completely at the time.)

Ogg shows that $n = 17$ does not occur, in his important paper [Ogg 1971] where the connection between this problem and the theory of modular curves is spelled out. Ligozat [1975] and Kubert [1976] take care of several small values of n. In 1973 Mazur and Tate [1973] show that there do not exist rational points of order $n = 13$ on elliptic curves over $\mathbf{Q}$. This exceptional case is eliminated by means of a 19-descent (!), performed in the language of flat cohomology, on a curve of genus 2. Finally in 1976, Barry Mazur proves the conjecture. It is a consequence of his careful study of the modular curves $X_0(n)$, which are closely related to the curves $X_1(n)$. His paper [Mazur 1977] is a milestone. The techniques developed therein are basic in the proof of the main conjecture in Iwasawa theory by Mazur and Wiles [1984], Ribet's reduction [Ribet 1990] reducing Fermat's Last Theorem to the Conjecture of Taniyama, Shimura, and Weil and Wiles' recent proof of this conjecture. The latest development concerning torsion on elliptic curves is Loïc Merel's proof (Spring 1994) of the general boundedness conjecture. This is the statement that the K-rational torsion of any elliptic curve defined over a field K of degree d over $\mathbf{Q}$ is bounded in terms of d alone. Beppo Levi's conjecture is an explicit version of the special case $d = 1$ of this new theorem. Merel's proof builds upon Mazur's work and subsequent refinements by S. Kamienny, combining them with other recent results in the arithmetic of elliptic curves.

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[5] We thank Professor A. Schinzel for bringing this paper to our attention.

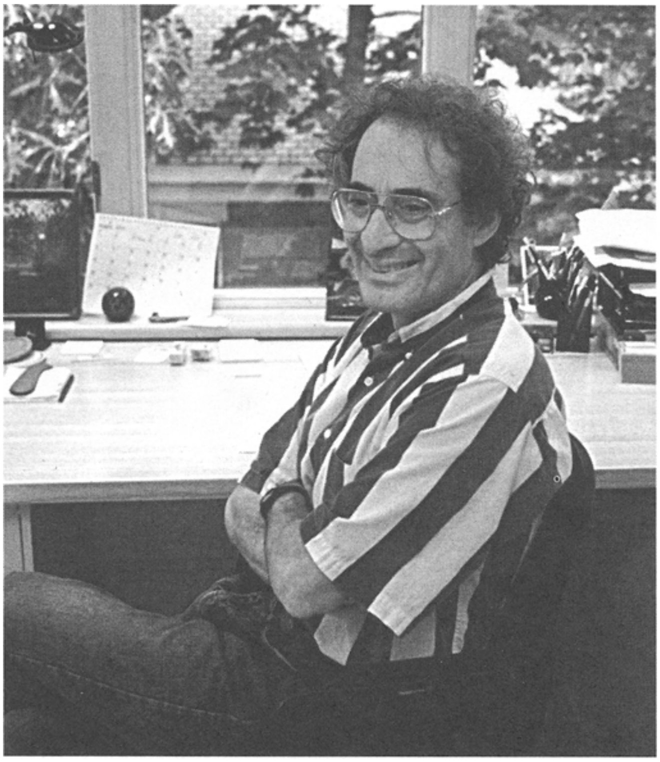

Figure 5. Barry Mazur.

Parma—Bologna—Rosario

Having treated the first 10 years of Beppo Levi's professional life rather extensively, we will be much shorter with the remaining 50 (!) years. This half century from 1908 to the end of the 1950s falls naturally into three periods: almost 20 years in Parma, 10 years at Bologna, and a good 20 years in Argentina.

In 1909 Beppo Levi married Albina Bachi. She was from the town of Torre Pelice in Piemonte, the alpine north west of Italy, as was Beppo Levi. He had started visiting his future in-laws in 1906, the year of his nomination at Cagliari. Three children, Giulio, Laura, and Emilia, came from the marriage. At the end of 1910 the family left Cagliari: Beppo Levi was appointed at the university of Parma, on Italy's mainland. He stayed there until 1928. Among his uninterrupted production (increasingly also on questions of mathematics teaching), there is one remarkable number-theoretic contribution from this Parma period: a paper on the geometry of numbers in *Rendiconti del Circolo Matematico di Palermo* [Levi 1911] where he claimed to give a proof of a conjecture of Minkowski's concerning critical lattices in $\mathbf{R}^n$. However, [Keller 1930] mentions a letter of Beppo Levi in which he acknowledged a gap in his proof. A complete proof of this result was given only in the 1940s [Hajós 1942].

It was in Parma that the Levis lived through World War I and the ensuing political transformation of Italy and of Europe. A reflection of these events—immediately painful for Beppo Levi through the death of two of his younger brothers—can be found in his remarkable speech (11 January 1919) at Parma University for the opening of the academic year 1918–1919, on "Nations and Humanity" [Levi 1919].

Another nonresearch publication of his from roughly the same period is *Abacus from One to Twenty* [Levi 1922], a booklet conceived and illustrated by Beppo Levi himself. It is designed to introduce children to the first numbers and elementary arithmetic operations. In the explanatory notes at the end of the booklet the author sketches his general ideas about the concept of number as being the fundamental example of mathematical representation, distinct from the process of counting, as well as about teaching elementary arithmetic.

In the 1920s, the first decade of Mussolini's rule, Beppo Levi was explicitly antifascist. He signed the Croce manifesto in 1925. Around that time his situation at Parma University became increasingly difficult because more and more disciplines had to be suppressed for budgetary reasons. In the end, the sciences were reduced to chemistry, and Levi was the only professor left at the mathematics department. It therefore came as a great relief when he obtained his transfer to Bologna—a town with a traditionally famous university—at the end of 1928, after all obstacles to his nomination there had finally been overcome.

While in Bologna he held various posts in the Italian Mathematical Society (U.M.I.), and took care of the *Bolletino dell'Unione Matematica Italiana* for many years. It may have been through correspondence related to a paper submitted to this journal that Beppo Levi first entered into contact with a mathematician from Argentina.

In spite of his personal opposition to fascism, Beppo Levi took the oath to fascism in 1931, like most other Italian mathematicians. This oath was generally considered a mere formality; even the church held that it was a legitimate claim by the government for obedience. The mathematician Levi-Cività added a private reservation, and the government showed that it was quite prepared to accept the substance without the form. Of roughly 1200 professors, only 11 refused to sign. The 71-year-old Vito Volterra was one of them. Volterra, by the way, stayed in Italy, where he died in 1940—so the SS car that came to his house in 1943 to deport him had to leave empty. . . .

It was only after the rapprochement between Hitler and Mussolini, in 1938, that Italian fascism adopted some of the racial policies of the Nazis which at that time were building toward their monstrous climax in Germany and German-controlled Europe. Thus Levi-Cività, Beppo Levi, and a total of 90 Italian Jewish scholars lost their jobs in 1938, and most of them had to start looking for a country of refuge.

Beppo Levi was 63 years old when he lost his professorship in Bologna. At age 64 he started as the director of the newly created mathematical institute at the Universidad del Litoral in Rosario, Argentina.

The founding of this institute at Rosario, upstream from Buenos Aires,[6] took place at a time of cultural expansion of several provincial Argentinian cities, mainly Rosario, Córdoba, and Tucumán. A relative prosperity helped in the development of more substantial groups of professionals, mainly lawyers, medical doctors, and engineers, who promoted local cultural activity in these cities and invited leading intellectuals and artists from Buenos Aires to lecture or visit there. These professionals were financially better off, and their clients were richer yet. Societies, orchestras, art galleries, and publishing houses began to emerge in this period in Rosario.

The official opening ceremony of the mathematical institute in Rosario was held in 1940. Lectures were delivered by Cortés Plá, Rey Pastor, and Beppo Levi. These two men had been the key to Beppo Levi's arrival in Argentina. Plá was an engineer who taught physics and had an active interest in the history of science. Plá was a friend and admirer of Rey Pastor, the Spanish mathematician who founded the Argentine mathematical school.

[6]We are grateful to Eduardo L. Ortiz, London, for the information on Argentina contained in the following paragraphs. For the opening of the Rosario institute, see *Publicaciones del Instituto de Matemáticas de la Universidad Nacional del Litoral* 3_5 (1940); cf. E. Ortiz (ed.), *The works of Julio Rey Pastor*, London 1988.

Figure 6. Beppo Levi: the Argentine years.

Figure 7. Beppo Levi (front row, fourth from left) at a meeting of the Argentine Mathematical Union, 1948.

Beppo Levi was extremely active in Rosario, and the new responsibilities which he happily embraced right from the very beginning of the job gave more satisfaction to his life in emigration than one would expect.[7] Apart from organizing and managing the Institute (assisted by Luis A. Santaló) he founded and edited a journal and a book series of his institute. The journal appeared for the first time already in 1939, and as of 1941 was called *Mathematicæ Notæ* (Boletin dej Instituto de Matematica). Roughly one-third of Beppo Levi's publications are in Spanish. These are his papers from the Argentinian period, many of which appeared in *Mathematicæ Notæ.* Beppo Levi continued teaching at Rosario until the age of 84.

In 1956 (shortly before he turned 81) he was awarded the Italian *Premio Feltrinelli.* Unfortunately the official text of the prize committee [Segre 1956] shows a somewhat uncertain appreciation of some of Beppo Levi's works. At the end of the evaluation the committee of this prize for Italian citizens congratulates itself that Beppo Levi has highly honored the name of Italy by his work in Argentina . . .

Beppo Levi died on 28 August 1961, 86 years old, in Rosario, where his institute is now named after him. He was probably the shortest mathematician in our century, with the longest professional activity.

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We would like to thank all colleagues and friends who gave us hints or sent us documents while we were preparing this article. We are particularly indebted to Professor Salvatore Coen, Bologna, for freely sharing his extensive knowledge of Beppo Levi's life and work with us, and for making available some of the less accessible publications of Beppo Levi; and to Dr. Laura Levi, Buenos Aires, for a most interesting correspondence which conveyed a vivid impression of the personality of her father. Thanks to Raymond Seroul, Strasbourg for drawing the figures on his computer.

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[7] This is reflected already in some of the first letters back to relatives in Italy. The same letters also show that his wife found her life rather more difficult in Rosario. We owe this information to Dr. Laura Levi.

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