Gotthold Eisenstein
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Norbert Schappacher

The early development and untimely death of a genius is a well-known theme in general ("Whom the gods love ..."), and in musical biography in particular, with Mozart and Schubert as outstanding examples. But three geniuses among the nineteenth century mathematicians, whose lives were cut off all too soon, make Mozart's death at close to 37 years, and even Schubert's at hardly less than 32 years, almost seem to have come at a reasonably mature age. The mathematicians are Evariste Galois – who lost his life at 20 in an absurd duel, Niels Henrik Abel – who succumbed to tuberculosis at age 26, and finally Gotthold Eisenstein, whose frail body held out exactly 1000 days longer than Abel's, before giving in to the same disease.

If it is true that Eisenstein died young, this still did not spare him a few miserable years during his life – nor even a dubious afterlife. In fact, few people loved him. The scientific world did bestow fantastic honours on the very young man. But the mathematical community did not keep an adequate memory, either of his person, or of his visionary work. Felix Klein for instance – in his exaggerated endeavor to elevate Riemann beyond comparison – portrayed Eisenstein not only as megalomaniacal and paranoid, but also mathematically as a *Formelmensch*, a man of formulas (see [4]). This was an expression of the Göttingen spirit from the turn of the century, under Hilbert and Klein, a spirit that contrasts sharply with Gauss's unusually favourable attitude towards Eisenstein. (Hilbert, although a little more favourably disposed towards Eisenstein's œuvre than Klein, strove in his grandiose systematic treatment of algebraic number theory, the *Zahlbericht* of 1897, to replace what he called Kummer's calculations in the arithmetic of cyclotomic fields by "Riemann's conceptual method." He was thus proud to show that he could do completely without Kummer's logarithmic derivatives of units, i.e., without a technique that has since come to be a key tool of contemporary arithmetic through the work of Artin, Hasse, Iwasawa, Shafarevich, Coates and Wiles.)

But let us take things in order.

The very fact that Ferdinand Gotthold Max Eisenstein, the first of six children, all born in Berlin, survived childhood can be seen as an exception, for all his siblings were not so lucky. He survived meningitis and other numerous illnesses. He did remain hypochondriac, though, throughout his life.

His parents were Protestants after having converted from Judaism. The father had served for 8 years in the Prussian army, and then tried to be a businessman, apparently with not much success. Eisenstein was very closely attached to his mother who seems to have had a creative pedagogical instinct. For instance, in one

Ferdinand Gotthold Max Eisenstein

of the two autobiograpical sketches of Eisenstein that we have (the one written in August 1843 and reproduced in [3], predating a shorter one from March 1847 [2]; both texts reveal a curious mixture of naive openness and conventional style), he relates that his mother taught him the letters of the alphabet by associating a pictorial meaning to each of them, like a doorway for "O", and a key for "K".

Intellectually precocious, the child showed as much thirst for knowledge as lack of skill and acculturation in practical everyday matters. His enormous mathematical talent was noticed and encouraged by his teachers in secondary school. He started learning calculus from reading Euler and Lagrange on his own when he was about 15 years old. In 1842 he bought his own copy of Gauss's *Disquisitiones Arithmeticæ*, in French translation. Number theory quickly became his favourite interest in mathematics. During his time at the *Friedrich-Werdersches Gymnasium*, from 1840 to 1842, he also attended, as an 18- and then 19-year-old, courses at the University of Berlin, in particular, some given by Dirichlet.

Until 1842, Eisenstein lived either in or not far from Berlin. In the summer of this year, however, before finishing the *Gymnasium*, he went with his mother on the only big voyage of his life: to England, to join his father who was doing business there at the time. The family seems to have toured England, Ireland and Wales quite a bit, the son reading Gauss all the way, and playing the piano when possible. He would have liked to settle in Dublin. He had met Hamilton there, who entrusted him with a mansucript for the Berlin Academy. The family returned to

Berlin in 1843; the parents separated; and the 20 year old Eisenstein obtained his graduation from school, enrolling as a student at the University of Berlin. The following year he stepped into the limelight of science.

When Eisenstein delivered Hamilton's manuscript to the Academy, he also added a paper of his own, on cubic forms. This led to Crelle's interest in the prodigy. What happened next is not only a proof of Eisenstein's unusual talent, but also an amazing illustration of what the Berlin scientific establishment of the time was capable of doing for a young genius.

Volume 27 of *Crelle's Journal* (1844) contains no less than fifteen papers (and one problem) by Gotthold Eisenstein; the following volume 28 (still 1844) another eight (plus another problem), in German, French, and also Latin. Thus, within the first year, the young author had more publications to his credit than years of age.

In March 1844, Eisenstein was invited to meet Alexander von Humboldt. Humboldt was to remain Eisenstein's lifelong devoted benefactor. Through Humboldt, and partly through Crelle, Eisenstein received, as of April 1844, financial aid from the King, and sometimes from the Academy. This allowed him to survive, but since the payments were never turned into a regular salary, it also left him with an increasing feeling of material insecurity, especially after the 1848 upheaval. Humboldt's various attempts over the years to secure a professorship at some university for the young man did not work out.

In June 1844, Gauss expressed high praise for Eisenstein's papers. That same month, Eisenstein went to Göttingen for two weeks to see Gauss, with a letter of recommendation from Humboldt. No other young talent was ever treated as well as Eisenstein by the Göttingen master. When proposing Dirichlet for the order *pour le mérite* in 1845, Gauss noted he would have almost proposed Eisenstein instead. In 1847, Gauss wrote a foreword to a collection of Eisenstein's papers which was published as a book.

In February 1845, Eisenstein received an honorary doctorate from the University of Breslau.

The tide began to turn around the end of 1845 and the beginning of 1846, at first in a psychological sense: Eisenstein was essentially lonely. Kronecker with whom he met almost daily earlier in 1845, left Berlin. Other friends of comparable age who did stay in Berlin seemed to fade away. The friendly correspondence with Max Stern in Göttingen (the first non-converted Jew to hold a professorship in mathematics in Prussia) could not make up for the lack of direct contact.

Then there were professional problems: Jacobi, with whom Eisenstein had an intense relationship with many ups and downs, started a public priority dispute with Eisenstein in early 1846. From Jacobi's point of view, this bang on the head of an overrated young star had a beneficial effect. For Eisenstein it probably felt like the end of youth. Onlookers like Gauss and the astronomer Schumacher discussed in their correspondence possible effects of mathematical genius on human relations. A modern expert like André Weil finds it "incomprehensible that Jacobi, Cauchy, and Eisenstein have each published proofs" of the quadratic reciprocity law which were "virtually identical with" Gauss's sixth proof, "and that they even raised priority questions among each other about this."

Eisenstein's production continued through his death, totalling some 40 mathematical articles and more than 700 printed pages. In 1846, Humboldt succeeded in freeing Eisenstein from military service. In 1847, Eisenstein obtained his *Habilitation*, i.e., the right to teach courses at the University of Berlin. This he did apparently with good success, drawing between 10 and 20 students to his analysis courses, and around 5 to more advanced topics such as elliptic functions.

In 1851 for the first time, his health did not permit him to teach. In this same year, upon Gauss's proposal, Eisenstein was elected (along with Kummer) corresponding member of the Göttingen Academy. The following year, five months before Eisenstein's death, Dirichlet arranged for him to be elected as member of the Berlin Academy, succeeding Jacobi.

Eisenstein's attitude to the republican ideas that erupted in March 1848 is not transparent. In 1850, Schumacher wrote to Gauss that Eisenstein was "very red", quoting alledged violent anti-reactionary statements of Eisenstein's from as late as the end of 1849. Eisenstein did drop in at Republican Clubs before March 1848, and during the street fighting on March 19, 1848, he was found in a house (not his own) from which snipers had fired on the royal troups. He was arrested, severely maltreated, and violently forced to march, with lots of other prisoners, the long way to the Spandau Citadel from which he was released the day after. He wrote down a very detailed description of what happened to him for a collection of similar reports by other victims of the transport to Spandau. This collection was published as a book by Adalbert Roerdansz in 1848 ([5]): "A Martyrium for Freedom," under the motto *facta loquuntur* (the facts will speak). The declared goal of the book was to accuse the military of abuse of power in order to create awareness and thus eventually improve the conduct of the troops. The fact that Eisenstein agreed to cooperate with this more idealistic than revolutionary project, does not really reveal his political ideas. In his report, he depicts himself as a peaceful teacher at the university who had not done anything unlawful and who found himself in the house where he was arrested simply because he was seeking shelter from the fighting.

Looking back from today's vantage, Eisenstein's mathematics appear to us more up-to-date than ever. It is not so much the harvest of theorems, nor the creation of full-fledged theories, but the way of looking at things which amazes us – and which may have also been what impressed Gauss.

Three major themes pervade Eisenstein's work: first, the theory of forms, × where the goal was to generalize Gauss's arithmetic theory of quadratic forms contained in the *Disquisitiones Arithmeticæ*. Already Hilbert in his *Zahlbericht* made the comment, which is commonplace today, that we would have to translate these results into the language of cubic number fields in order to appreciate them.

× in partic. cubic forms

Second, the theory of higher reciprocity laws, i.e., the search for a generalization of quadratic reciprocity which Gauss had also treated in his *Disquisitiones Arithmeticæ*, and which he continued to come back to, giving new proofs of it. In order to tackle the general reciprocity law for m-th power residues, one has to work in the ring $\mathbb{Z}[\zeta_m]$ of m-th roots of unity, and thus Eisenstein was among the first mathematicians to use Kummer's arithmetic theory of ideal numbers for these rings, which was created in the early 1840s. In one of his last papers (from 1850), Eisenstein managed to settle the most general reciprocity law in the special case where one of the two numbers to be compared is a rational, the other a cyclotomic integer. This is still quite far from the goal, but was crucial progress at the time when it was obtained. It took Kummer a decade longer to go all the way, and Hilbert in his *Zahlbericht* still uses Eisenstein's law as a crucial stepping stone for both proofs of the general reciprocity law that he presents.

Incidentally, when working on this *Zahlbericht*, Hilbert and Minkowski deplored that the collected papers of Eisenstein's had not been edited. But they did not take action themselves, and so it fell finally to the Chelsea Publishing Company to publish the two volumes in 1975. There was a second edition already in 1989.

The third big mathematical theme taken up by Eisenstein was the rapidly developing theory of elliptic functions, which Gauss had studied without publishing work on the subject. Following Fagnano, Euler and Legendre, Abel had made major progress, and Jacobi had set up a whole theory of the domain in his *Fundamenta Nova*. In a major paper of 1847: *Genaue Untersuchung der unendlichen Doppelprodukte* ..., Eisenstein developed his own independent analytic theory of elliptic functions, based on the technique of summing certain conditionally convergent series. André Weil made this and Kronecker's subsequent work along these lines the subject of his Princeton Lectures in 1974 which later appeared as a book: "Elliptic Functions according to Eisenstein and Kronecker," [6], which has and continues to have major immediate impact on current research in arithmetic algebraic geometry.

There is another circle of ideas in which Eisenstein's investigations appear to have been visionary from our point of view. Gauss, in the introduction to the seventh chapter on cyclotomy of the *Disquisitiones Arithmeticæ*, mentions that one may develop an analogue of cyclotomy for the lemniscate, i.e., an arithmetic study of the division values of the inverse of the elliptic integral $\int \frac{\mathrm{d}x}{\sqrt{1-x^4}}$. This was actually treated by Gauss himself in unpublished notes, and independently by Abel. Eisenstein developed Abel's theory further in the arithmetic direction, and managed to deduce from it the biquadratic reciprocity law. This may not seem such a big achievement in that this kind of analysis of reciprocity laws in terms of elliptic functions does not go very far towards the general reciprocity for m-th power residues. But Eisenstein's development of the arithmetic of the division of the lemniscate, especially in his long and detailed series of articles from 1850, *Über die Irreductibilität* ..., is startling for the way in which he proceeds. He derives a formula for the multiplication by a prime on the elliptic curve $y^2 = 1 - x^4$ which, through its natural generalization in Kronecker's work, is the direct source of what today is known as the Shimura-Taniyama congruence relation in the theory of complex multiplication. By the way, it is here, in proving the irreducibility of the lemniscatic division equation, that Eisenstein establishes his well-known irreducibility criterion.

These articles *Über die Irreductibilität* ... have also recently led A. Adler [1] to suspect that Eisenstein had some sort of knowledge about Jacobian varieties of Fermat curves.

On a different note, Eisenstein sketched on 7 April 1849, writing into his copy of the *Disquisitiones Arithmeticæ*, a proof of the functional equation for the Dirichlet L-function modulo 4 in the critical strip. It is tempting and plausible to think, as Weil proposed [7], that Riemann received basic ideas for his fundamental paper on the zeta function in conversations with Eisenstein that same month in Berlin.

The final scene shows the 83-year-old Alexander von Humboldt following Eisenstein's coffin at the cemetery at *Blücherplatz*. He had obtained money from the King for Eisenstein to go to Sicily for a cure, but it was already too late. The plague of the nineteenth century had taken yet another distinguished victim.

References

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