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Regulators in Analysis, Geometry and Number Theory

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Preface

This book is an outgrowth of the Workshop on "Regulators in Analysis, Geometry and Number Theory" held at the Edmund Landau Center for Research in Mathematical Analysis of The Hebrew University of Jerusalem in 1996. During the preparation and the holding of the workshop we were greatly helped by the director of the Landau Center: Lior Tsafriri during the time of the planning of the conference, and Hershel Farkas during the meeting itself. Organizing and running this workshop was a true pleasure, thanks to the expert technical help provided by the Landau Center in general, and by its secretary Simcha Kojman in particular. We would like to express our hearty thanks to all of them.

However, the articles assembled in the present volume do not represent the proceedings of this workshop; neither could all contributors to the book make it to the meeting, nor do the contributions herein necessarily reflect talks given in Jerusalem. In the introduction, we outline our view of the theory to which this volume intends to contribute. The crucial objective of the present volume is to bring together concepts, methods, and results from analysis, differential as well as algebraic geometry, and number theory in order to work towards a deeper and more comprehensive understanding of regulators and secondary invariants.

Our thanks go to all the participants of the workshop and authors of this volume. May the readers of this book enjoy and profit from the combination of mathematical ideas here documented.

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Introduction

A. Reznikov and N. Schappacher

The theory of regulators, of which this volume presents various recent highlights, is best described as the border area where number theory leaves its original habitat within the domain of abstract algebra in order to rub shoulders with analysis and geometry, in particular, with differential geometry. The unsuspecting algebraist may react to such an alliance with distrust; Sylvester for instance, in one of his formulations beyond emulation, once scorned the unintuitive "recourse to concepts drawn from reticulated arrangements, as in the applications of geometry to arithmetic made by Dirichlet and Eisenstein."[1]

Today's number theorists or arithmetic algebraic geometers, however, who at any rate are used to seeing boundaries between mathematical (or physical) theories lose their traditional significance, seem to welcome the theory of regulators above all precisely because it awards analysis, at least conjecturally, an even more serious right to residence than the mere definition of L-functions would imply. More precisely, L-functions made their appearance in the history of mathematics towards the end of the first half of the 19th century, in parallel with work by Dirichlet (Dirchlet L-functions, in the case of quadratic characters), Eisenstein (Eisenstein's double series, which today we relate to L-functions of elliptic curves with complex multiplication), and Riemann's zeta function (whose investigation by Riemann was probably inspired also by Eisenstein's musings about the functional equation of one of Dirichlet's L-functions).[2]

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[1] J.J. Sylvester, Math. Papers, vol. III, p. 344: "On certain ternary cubic-form equations" (1879/80).

[2] See A. Weil, On Eisenstein's Copy of the *Disquisitiones*; *in:* Algeraic Number Theory—in honor of Kenkichi Iwasawa (Coates, Greenberg, Mazur, Satake, eds.); Advanced Studies in Pure Mathematics **17**, Academic Press 1989, 463–469.

During the 20th century, the growing awareness of *arithmetic* algebraic geometry initiated by Poincaré, Weil, and Hasse led to the common interpretation of the zeta and L-functions in number theory as gadgets that use analysis simply as an expedient to store all various local data about a geometric object X defined over a number field K, gathered by looking at the reduction of X at the different places of K, into a single mathematical entity. But already the so-called analytic class number formula (which goes back to the 19th century), i.e., the occurrence of the unit regulator in the residue at 1 (or the derivative at 0) of the Dedekind zeta function of an algebraic number field, pointed to a genuinely "global" nature of L-functions. This perspective was increasingly developed in the second half of the 20th century, first at the central point of the (conjectured) functional equation in the conjecture of Birch and Swinnerton-Dyer, and then, following the pioneering work of S. Bloch, at all integer points in the far-reaching conjectures of Beilinson,[3] and their more recent refinement due to Bloch and Kato.[4]

Let us look at this first and principal strand of the historical development of the arithmetic theory of regulators in a little more detail.

The arithmetic of regulators and L-values

In the analytic class number formula (rewritten using the functional equation), the first nonvanishing derivative at $s = 0$ of the Dedekind zeta function $\zeta_F(s)$ of an algebraic number field F is expressed as a rational multiple of the regulator of F, which itself is a nonvanishing determinant of logarithms of absolute values of fundamental units of the ring of integers $\mathfrak{o}_F$. The work of Bloch and Beilinson started from the basic observation that $\mathfrak{o}_F^* = K_1(\mathfrak{o}_F)$, and then proceeded

- to establish a general numerology associating a certain part of a higher K-group (or "motivic cohomology group") to any given pair (M, n) of a motive M and an integer n—the latter may be conveniently normalized with respect to the (in general only conjectural) functional equation of the L-function,

- to define a regulator map on this motivic cohomology group which generalizes the logarithm of the absolute value of units in the number field case, and allows us to form a determinant which is then conjectured to equal, up to a rational multiple, the first nonvanishing term in the Taylor expansion around $s = n$ of the L-function of M.

Beilinson defined his general regulators via Chern class maps on higher K-groups with values in the corresponding Deligne cohomology. This is quite satisfying

[3] See the volume edited by M. Rapoport, N. Schappacher, and P. Schneider: Beilinson's Conjectures on L-values, Oberwolfach Proceedings April 1986, *Perspectives in Mathematics* **4** (Academic Press) 1988. The conjectures had been proposed in: A.A. Beilinson, Higher regulators and values of L-functions, *J. Soviet Math.* **30** (1985), 2036–2070.

[4] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, Grothendieck Festschrift, vol. I, *Progress in Mathematics* **86**; Birkhäuser, Boston, Basel, 1990, 333–400.

from a functorial point of view but usually inaccessible to explicit computations if only because our knowledge of K-groups is severely limited.

In the special case of noncritical values $s = n$ of the Dedekind zeta function $\zeta_F(s)$ of an algebraic number field F, however, the rational K-groups are known as a consequence of our knowledge of the cohomology of discrete arithmetically defined groups.[5] This is the general context of the contribution to this volume of **Blasius and Rogawski**.

Furthermore, A. Borel was able to determine the first nonvanishing coefficient in the Taylor expansion of $\zeta_F(s)$ at $s = 1-n$ as the covolume of the corresponding "Borel" regulator map, defined by integrating a specific differential form against homology cycles coming from K-theory.[6] It is a nontrivial affair to check that Beilinson's regulator coincides, up to a rational factor, with Borel's regulator.[7]

Soon after, it was realized independently by Deligne and Scholl[8] that the conjectures of Birch and Swinnerton-Dyer, Deligne, and Beilinson concerning special values of L-functions can all be treated rather uniformly via the (partly hypothetical) theory of mixed motives. In this framework, Beilinson's regulator map reappears as a realization functor for mixed motives.

The later refinement of Beilinson's conjectures due to Bloch and Kato, which amounts to an indirect determination of the ratio between the L-value (resp. derivative) and the regulator up to ± 1, was represented at the Jerusalem workshop in particular by **Peter Schneider**'s two survey talks on Kato's refinement of the Bloch-Kato conjectures—but it is not present as such in these proceedings. Instead, the articles in the present volume reflect

- variants of this main strand of ideas, such as the generalizations of Zagier's polylogarithm conjecture;

- other recent imports of differential geometry into arithmetic, in particular Arakelov Theory.

Polylogarithms

It was D. Zagier who, having previously investigated the interrelations of volumes of hyperbolic manifolds, the dilogarithm, and special values of the Riemann zeta function, guessed from numerical experiments a conjectural expression for all

[5] See A. Borel, Stable real cohomology of arithmetic groups, *Ann. Sci. ENS* **7** (1974) 235–272.

[6] See the beautiful exposition in A. Borel, Cohomologie de SL_n et valeurs de fonctions zêta, *Ann. Scuola Normale Superiore* **7** (1974), 613–636.

[7] See the chapter by Rapoport in the volume edited by Rapoport, Schappacher, Schneider quoted in footnote 3, as well as H. Esnault, On the Loday symbol in the Deligne-Beilinson cohomology, *K-theory* **3** (1989), 1–28.

[8] See: A.J. Scholl, Remarks on special values of L-functions; *in:* L-functions and Arithmetic (J.H. Coates, M.J. Taylor, editors), Cambridge Univ. Press 1991, 373–392; as well as: C. Deninger, A.J. Scholl, The Beilinson Conjectures; in the same Durham proceedings, pp. 173–209, in particular the appendix to this article.

(noncritical) integral special values of Dedekind zeta-functions as linear combinations of certain polylogarithms. Thus Zagier's conjecture, like Beilinson's, also predicts explicitly (in the number field case) the transcendental part of $\zeta_F(n)$, for all noncritical n. Several nontrivial special cases of Zagier's conjectures were settled in 1993 in groundbreaking work by A. Goncharov, which uses Borel's works quoted above.

In his contribution to this volume, **Alexander Goncharov** attacks a new case via the Aomoto (and the classical) trilogarithm: the value $\zeta_F(4)$.

The general formal relation between Zagier's polylogarithm conjecture and Beilinson's conjectures (specialized to the case at hand) is given by Deligne's and Beilinson's motivic interpretation of Zagier's conjecture. A key notion here is that of a (motivic) variation of Tate-Hodge structures on $\mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$. In a more down-to-earth way, one may observe that the values of all higher polylogarithms at roots of unity in $\mathbf{G}_m \setminus \{1\} = \mathbf{P}^1 \setminus \{0, 1, \infty\}$ lie in the image of Beilinson's regulator map from K-theory to Deligne's cohomology of cyclotomic fields.

With this general formalism in mind, one may say that "the goal of the theory of polylogarithms is to give an explicit description of motivic cohomology of algebraic varieties and of regulator maps. For example, the K-theory groups of a field F have a natural γ-filtration and the motivic cohomology of $\mathrm{Spec}(F)$ is equal to $H^i_{\mathcal{M}}(\mathrm{Spec}(F), \mathbf{Q}(n)) = \mathrm{gr}^n_\gamma K_{2n-i}(F)_{\mathbf{Q}}$. One would like to construct a natural complex representing $\mathbf{R}\mathrm{Hom}(\mathbf{Q}(0), \mathbf{Q}(n))$ in the category of mixed (Tate) motives over F, with cohomology groups $H^i_{\mathcal{M}}(\mathrm{Spec}(F), \mathbf{Q}(n))$.

For $n = 2$ the answer is given by the Bloch-Suslin complex (in degrees 1 and 2): $\delta\colon \mathbf{Q}[F^* - \{1\}]/R_2 \to \bigwedge^2 F^*_{\mathbf{Q}}$, where $\delta([x]) = x \wedge (1-x)$ and R_2 is generated by

$$[x] - [1/y] + [(1-x)/(1-xy)] - [1/(1-xy)] + [(1-y)/(1-xy)],$$

as $\mathrm{Coker}(\delta) = K_2(F)_{\mathbf{Q}}$ and $\mathrm{Ker}(\delta) = K_3^{\mathrm{ind}}(F)_{\mathbf{Q}}$.

The relations in R_2 come from the functional equation of the dilogarithm function Li_2 ($\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} z^n/n^k$) and its single-valued version $D_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$. For $F = \mathbf{C}$ the map $\mathbf{Q}[\mathbf{C}^* - \{1\}] \to \mathbf{R}$ sending $[x]$ to $D_2(x)$ factors through $\mathbf{Q}[\mathbf{C}^* - \{1\}]/R_2$ and its restriction to $\mathrm{Ker}(\delta) = K_3^{\mathrm{ind}}(\mathbf{C})_{\mathbf{Q}} \to \mathbf{R}$ coincides with the Borel regulator (up to a scalar)."[9]

Strong computational evidence for a conjecture of Goncharov's concerning the weight 4 part of the Quillen K-theory of fields, and thereby indirect support for Zagier's conjecture about Dedekind zeta-functions at $s = 4$ is presented in **Herbert Gangl**'s contribution to this book.

Replacing $\mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$ in the above by a punctured elliptic curve leads to a theory of (mixed motivic sheaves associated to the) so-called "elliptic polylogarithms." As functions these are simply Kronecker-Eisenstein-Lerch series, as-

[9]Quoted from the featured review no. 98d:11073 by J. Nekovář of the article: J. Wildeshaus, On an elliptic analogue of Zagier's conjecture, *Duke Math. J.* **87** (1997), 355–407; in: Mathematical Reviews 1998. For the functional equations satisfied by polylogarithms, see J. Oesterlé, Polylogarithmes, *Séminaire Bourbaki* **762**, (1992–93).

sociated to the elliptic curve in question, i.e., the same kind of functions which Bloch already discovered as the right substitutes of the logarithm in his seminal construction of a regulator on $K_2(E)$ for an elliptic curve E.[10]

The contribution to this volume which runs along these lines and aims at the greatest generality is **Andrey Levin**'s article, whereas the articles by **Spencer Bloch** and **Jörg Wildeshaus** are more immediately inspired by the one classical case which had been the focus of attention already at the very beginning of Beilinson's conjectures in the work of Bloch of the late 1970s, as well as in the first experimental forays into the elliptic analogue of Zagier's polylogarithm conjecture performed in 1991 by Don Zagier and Henri Cohen: the value at $s = 2$—or, equivalently (given the functional equation), the derivative at $s = 0$—of the L-function of an elliptic curve defined over $\mathbf{Q}$.[11]

Inroads of differential geometry

In the spirit of the definition given at the beginning of this introduction, Arakelov theory, i.e., the introduction of hermitian geometry at the infinite places with a view to having differential geometry contribute to a perfect analogy between number fields and function fields, belongs to the theory of regulators. The same is true, and for the same reason, for the even more daring attempt to find in classical analysis and topology the tools for a truly satisfactory treatment of the Euler factors at infinity of the zeta and L-functions. This latter line of research is represented here by **Christopher Deninger**'s article.

The contributions to this volume, by **Hélène Esnault**, **Kai Köhler**, **Klaus Künnemann**, and **Vincent Maillot**, and by **John Lott** are situated along the following line of development.

The hyperbolic volume, from the "regulator" point of view, is the imaginary part of the Cheeger-Chern-Simons class. It is an invariant in $H^{2i-1}(\mathrm{BGL}(\mathbf{C}), \mathbf{R})$. So for a compact manifold M and a representation $\rho : \pi_1(M) \longrightarrow \mathrm{GL}(\mathbf{C})$, one obtains a hyperbolic volume class $\mathrm{vol}(\rho) \in H^{2i-1}(M; \mathbf{R})$. If M is a compact hyperbolic three-manifold and ρ is the natural representation, this gives back the classical hyperbolic volume $\mathrm{vol}(M)$ of M.

The central importance of the numerical volume $\mathrm{vol}(M)$ as an invariant of hyperbolic manifolds follows from the theorem of Wang-Gromov: *For any bound B, there are only a finite number of hyperbolic manifolds of dimension ≥ 4 with volume bounded by B.*

The situation changes dramatically in the case of three-manifolds: according to Thurston, the set of volumes of hyperbolic three-manifolds is a nondiscrete well-

[10]See the volumes *Motives* (Jannsen, Kleiman, Serre, editors), Proceedings of Symposia in Pure Mathematics **55**, AMS 1994, in particular the contributions by Beilinson & Levin, and by Goncharov. See also J. Wildeshaus, Realizations of polylogarithms, Springer Lecture Notes in Math. **1650**, 1997, as well as the Duke article of the same author mentioned in the previous footnote.

[11]As background, see also A. Goncharov and A. Levin, Zagier's Conjecture on $L(E, 2)$, *Inventiones Math.* **132** 1998, 393–432, as well as K. Rolshausen and N. Schappacher, On the second K-group of an elliptic curve, *Crelle* **495** (1998), 61–77.

ordered subset of the reals. Despite some strong conjectures due to Milnor, the arithmetic nature of this subset remains mysterious at the moment.

The connection of these volumes with regulators was established by Beilinson who indicated a proof of the fact that, in the case of flat bundles over an algebraic variety, the hyperbolic volume class (or, equivalently, the Borel regulator), is the imaginary part of the Bloch-Beilinson regulator in Deligne cohomology. Using this and vanishing results for the hyperbolic volume invariant $\mathrm{vol}(\rho)$, the conjecture of Bloch that all regulators of flat bundles over projective varieties are torsion has recently been settled by Reznikov.

Ever since its invention in 1973 by Cheeger, Chern and Simons,[12] the *Chern-Simons invariant* has played an increasing role in geometry, topology and mathematical physics.

Formally speaking, it is a cohomology class $\mathrm{ChS}_i \in H^{2i-1}(\mathrm{BGL}^\delta(\mathbf{C}), \mathbf{C}/\mathbf{Z})$. For any manifold M, and a representation of the fundamental group $\rho : \pi_1(M) \to GL(\mathbf{C})$, one gets a class $\mathrm{ChS}(\rho) \in H^{2i-1}(M, \mathbf{C}/\mathbf{Z})$. The representation may be viewed as a flat bundle $\mathcal{E}$ over M, so that $\mathrm{ChS}(\rho)$ may be interpreted as a secondary class, attached to the (torsion) Chern class $c_i(\mathcal{E})$. The fundamental property of the Chern-Simons class which makes it so interesting is rigidity, that is, one has $\mathrm{ChS}(\rho_t) = \mathrm{const}$, in any continuous family of representations ρ_t.

The Chern-Simons invariant plays a central role in low-dimensional topology, since it provides a frame for a set of invariants of three-manifolds which is sufficient to prove the infinite generation of the homology sphere bordism group, as has become clear after the work of Floer, Fintushel, Stern and Furuta.

The connection to regulators has emerged from the ground-breaking work of Bloch and Beilinson. In particular, for flat bundles on algebraic varietes, the Chern-Simons class is claimed to map to the Chern class in Deligne cohomology.

For an affine variety V, one is led to replace the finite dimensional Lie group $\mathrm{GL}_n(\mathbf{C})$ by the so-called current-group $\mathrm{GL}_n(\mathbf{C}[V])$. One then defines cohomology classes similar to the Chern-Simons class. These give rise to regulators in $\mathrm{Hom}(K_i^{alg}(V), \mathbf{C}/\mathbf{Z})$.[13]

The Ray-Singer analytic torsion is a fundamental invariant of a flat bundle $\mathcal{E}$ over a compact manifold M. It is by definition the value at 0 of the zeta-function of the twisted Laplace operator acting on the sections of $\mathcal{E}$. By a well-known theorem of Cheeger and W. Müller, the analytic torsion coincides with the combinatorial torsion, which can help to compute this invariant. The theorem and the yoga around it was recently generalized considerably by Bismut.

Calculations of the analytic torsion are central in Witten's evaluation of volumes of moduli spaces. And this evaluation in turn involves special vaues of so-called Witten zeta-functions at positive integers which, when suitably normalized, are essentially integers. On the other hand, one may derive divisibility results about these values, via group cohomology (multiplicative transfer). The results are parallel to the classical von Staudt theorem, thereby building another bridge from these analytic theories to number theory.

[12] S.-S. Chern, J. Simons, Characteristic classes and geometric invariants; *Annals Math.* **100** (1974); J. Cheeger, J. Simons, Characteristic classes and secondary invariants, in: *Geometry and Topology* (Alexander, Harer, editors), Springer LNM 1167, 1986.

[13] See A. Reznikov: Homotopy of Lie algebras and higher regulators (preprint 1993); Characteristic classes in symplectic topology (preprint 1994).