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CHAPTER 54

DAVID HILBERT, REPORT ON ALGEBRAIC NUMBER FIELDS ('*ZAHLBERICHT*') (1897)

Norbert Schappacher

In this report Hilbert summed up the current state of knowledge in algebraic number theory, at the same time enriching and organising the subject in ways that were to influence developments for decades. However, the reception of the work has been somewhat mixed.

First publication. 'Die Theorie der algebraischen Zahlkörper', *Jahresbericht der Deutschen Mathematiker-Vereinigung, 4* (1897), 175–546.

Later edition. In Hilbert, *Gesammelte Abhandlungen*, vol. 1, Berlin and Heidelberg: Springer, 1932 (repr. 1970), 63–363. [Some modernised spelling, errata worked into the text; further corrections include some indicated in the copy of the *Jahresbericht* that Olga Taussky-Todd used at the Technical University Vienna when working on the *Gesammelte Abhandlungen*. – Thanks to C. Binder for pointing this out.]

French translation. By M.A. Levy as 'Théorie des corps de nombres algébriques', *Annales de la Faculté des Sciences de l'Université de Toulouse* (1909: publ. 1910), 3rd. fasc.

English translation. The theory of algebraic number fields (trans. I.T. Adamson, intro. by F. Lemmermeyer and N. Schappacher), Berlin: Springer, 1998.

Manuscript. None exists, but Hilbert's personal copy with a few annotations is held in his *Nachlass* (Göttingen University Library Archives).

Related articles: Gauss (§22), Dirichlet (§37), Weber (§53), van der Waerden (§70).

1 A REPORT AND ALMOST A TEXTBOOK

This report by David Hilbert (1862–1943) was his first major writing after moving in 1895 to Göttingen University from the university in his home town of Königsberg. At Göttingen, he soon built up a reputation as the leading mathematician of his generation, with massive contributions to various mathematical disciplines; three others are discussed in this volume

Landmark Writings in Western Mathematics, 1640–1940
I. Grattan-Guinness (Editor)

(§55, §57 and §77). He also gave highly influential lecture courses, including in physics from the 1900s, and directed a number of doctoral students which was then unprecedented for a mathematician.

Hilbert's so-called 'Zahlbericht' of 1897 was one of the reports on the state of mathematical disciplines commissioned by the *Deutsche Mathematiker-Vereinigung* (hereafter, 'DMV') which was founded in 1890, during the first years of its existence; the first ten volumes of the *Jahresbericht der DMV* contain thirteen such reports. Hilbert and Hermann Minkowski (1864–1909) were asked on the occasion of the DMV meeting at Munich in September 1893 to write a joint report covering all of number theory. They decided to divide up the work, leaving to Minkowski subjects like continued fractions, quadratic forms, and the geometry of numbers. Both started working on the report in 1894. In the end, only Hilbert's part was completed, on 10 April 1897, but Minkowski did comment on Hilbert's manuscript and read the galley proofs.

Unlike most of the other reports commissioned by the DMV, Hilbert's *Zahlbericht* goes beyond the mere business of stocktaking. It gave a remarkably systematic and lucid treatment of algebraic number theory, thereby firmly establishing this discipline as a major domain of pure mathematics and providing at the same time its principal reference book for more than twenty years after its appearance, and leaving its mark on textbooks in this area until today. Already in a letter of 31 March 1896 Minkowski had predicted that the report would 'certainly be greeted by general applause, and will push Dedekind's and Kronecker's works very much to the background' ([Minkowski, 1973, 80]: compare §53). But it should be noted that Hilbert's own most far-reaching number-theoretic works, where he envisaged general class field theory while studying nothing but the arithmetic of quadratic extensions, appeared only after the *Zahlbericht*, in 1899 and 1902.

The advanced character of the report made it obviously inaccessible for a broader readership. Hilbert taught a course in the winter of 1897–1898 where he emphasized quadratic number fields, and he subsequently encouraged Julius Sommer [Blumenthal, 1935, 398], who had followed these lectures, to write a textbook which dwells on quadratic and cubic fields as an introduction to algebraic number theory [Sommer, 1907]. Similarly, his American doctoral student L.W. Reid (1899 thesis on class number tables for cubic fields) published a strongly example-oriented textbook treating exclusively quadratic extensions [Reid, 1910]. Hilbert contributed to it an introduction where one reads: 'The theory of numbers is independent of the change of fashion and in it one does not see, as is often the case in other departments of knowledge, one conception or method at one time given undue preeminence, at another suffering undeserved neglect'. We will briefly discuss in section 3 below how Hilbert actually chose among the various 'conceptions and methods' that existed in the literature on which he had to report.

2 THE PREFACE: NUMBER THEORY AND ARITHMETISATION

One reason for its great impact, apart from its striking expositional quality, was the fact that Hilbert was able to present current (algebraic) number theory as a leading mathematical discipline in tune with what he saw as the dominating values of the time. In his strong, sweeping preface, he not only recapitulates that number theory through its very origin is marked by the 'simplicity of its foundations, the precision of its concepts, and the purity of its truths', but also lists many interrelations of number theory with various other branches of mathematics, claiming in the end that 'if I am not mistaken, the whole modern development of pure mathematics takes place principally under the badge ('*Zeichen*') of number'. And Hilbert alludes explicitly to the 'arithmetisation' of function theory by Richard Dedekind (1831–1916), Karl Weierstrass and Georg Cantor (§46, §47), and to studies in the axiomatisation of geometry, of which he was soon going to be the champion himself with his 1899 essay on the foundations of geometry (§55).

Hilbert did *not* here allude to Leopold Kronecker (1823–1891), who had been the first to suggest in print (in 1887) a programme of explicitly 'arithmetizing' all of pure mathematics, but with the exclusion of geometry and mechanics [Boniface and Schappacher, 2001, intro.], thereby implying a separation of number theory and analysis from geometry. Hilbert had still faithfully echoed this separation in his 1891 lectures on projective geometry [Toepell, 1986, 21], a separation which Dedekind shared as well, and which can even be considered as being handed down from Gauss. However, in the preface to the *Zahlbericht* he emphasized the similarity of all mathematical disciplines once they are treated 'with that rigour and completeness [...] which is actually necessary'.

As to the style of the *Zahlbericht*, it is meant to reflect the mature state of the theory of algebraic number fields. Hilbert tried to avoid Ernst Kummer's 'formidable computational apparatus, so that here too Riemann's principle be realised according to which the proofs ought to be forced not by calculations, but by pure thought'.

Kronecker's programme of arithmetisation had also been inspired by the desire to have number theory and its genuine methods—which, for Kronecker, were thought to be found essentially in C.F. Gauss—govern pure mathematics. Likewise Hilbert's report, in its own way, consciously and successfully portrays (algebraic) number theory as a model theory for pure mathematics, both in content and in form. It not only came out different in style from all the other reports commissioned by the DMV, but effectively created a new special type of technical mathematical treatise, marked by the exceedingly stringent overall logical organization of virtually all 19th-century literature in algebraic number theory. Hilbert delicately differentiated between *Hilfssätze* (only of momentary importance in the argument at hand), continuously numbered *Sätze*, and *Sätze* whose statements were printed in italics and were supposed to be major starting-points for future developments. With these distinctive literary features, the *Zahlbericht* echoes, from the turn of the 20th century, the role played by Gauss's *Disquisitiones arithmeticae* 96 years earlier (§22). To be sure, the mathematical-historical context in 1897 was very different from the one that Gauss's book had changed so profoundly in 1801, and the very theory of integers in an arbitrary algebraic number field, which constitutes the subject of the *Zahlbericht,* is entirely a creation of the 19th century. Yet, both works represent, each one in its time, major inthronisation rites performed by number theorists for the 'Queen of Mathematics' before the eyes of their mathematical colleagues.

3 DEDEKIND VERSUS KRONECKER, ARITHMETIC VERSUS ALGEBRA

There are several features of the *Zahlbericht* which mark the time when it was written and which may surprise the unsuspecting modern reader. In the 1860s and 1870s, algebraic number theory had been the fairly solitary domain of research of a few individuals, among whom Dedekind in Braunschweig and Kronecker in Berlin stood out as the most visible and influential.

An alternative, completely viable and general approach by Egor Ivanovitch Zolotarev (1847–1878)—his second proposal for an algebraic number theory—was published only in 1880, after the author's death, and was incorrectly thought by both Dedekind and Kronecker to yield as incomplete a theory as Zolotarev's first proposal from his 1874 Russian thesis. This, added to the fact that Zolotarev had been an outsider to the German arithmetical community, was probably why Hilbert did not even mention Zolotarev in his references.

Dedekind developed his ideal-theoretic approach in three subsequent editions (1871–1894) of supplements 10 (or 11) of his edition of Dirichlet's lectures on number theory (§37); it was to become one of the major sources of inspirations for the theory of commutative rings by Emmy Noether (1882–1935) in the 1920s. Kronecker is known to have thought about general algebraic number theory as of the 1850s, and he finally published an extensive account of his attempt at a unified theory for both algebraic number theory and the arithmetic theory of algebraic functions in one or several variables in 1882 [Kronecker, 1882]. This publication also contains numerous hints at the evolution of his ideas, especially in the case of number fields, and their relations to other authors.

Then, in the 1880s and 1890s, energetic younger people were entering the subject—on the one hand Kronecker's pupil Kurt Hensel, and on the other hand Adolf Hurwitz (1859–1919) and Hilbert, both of whom cared little about the methodological preferences of either Dedekind or Kronecker in this area of research. According to Otto Blumenthal, Hilbert told later that once he and Hurwitz went for a walk in Königsberg where 'one of us presented Kronecker's proof for the unique decomposition into prime ideals, the other Dedekind's, and we would find both awful' [Blumenthal, 1935, 397]. In several papers of the mid 1890s, while using Dedekind's notions of (number) field and ideal, Hurwitz defined ideals via finite sets of generators, and used a basically Kroneckerian approach via polynomials in several unknowns to derive the unique decomposition of ideals into prime ideals. This was much to Dedekind's chagrin, who criticized this approach—which he had actually tried and developed himself earlier—as lacking methodological and conceptual purity [Dedekind, 1895]. Hilbert also published on this circle of ideas in 1894, giving a certain priority to Galois number fields; see our comments on Part 2 of the report in the next section.

In the *Zahlbericht,* ideals are defined in Dedekind's style as sets of algebraic integers which are closed under linear combinations with algebraic integer coefficients (art. 4). But both for the uniqueness of decomposition into prime ideals in arbitrary number fields (arts. 5–6), and for the proof that the ramified primes are precisely the divisors of the discriminant (arts. 10–13), Hilbert adopts essentially the Kronecker–Hurwitz method and mentions Dedekind's approach only in a reference.

In her comment of 1930 made for Dedekind's *Gesammelte Werke* [Dedekind, 1895, 58], Noether strongly endorsed Dedekind's criticism of Hurwitz, and she pointed out how long it had taken Dedekind's point of view to enter standard courses and textbooks. She did not mention Hilbert's *Zahlbericht* there, but Olga Taussky-Todd later remembered her criticising it, and claiming that Emil Artin, too, had accused Hilbert of having 'delayed the development of algebraic number theory by decades'. This may very well have been directed at the non-Dedekindian features of the text [Brewer and Smith, 1981, 82, 90].

More generally, Hilbert's *Zahlbericht* makes even less use of unifying notions from abstract algebra than one might have expected from a text written in the last decade of the 19th century. Thus, while the notion of (number) fields and their arithmetic is at the very heart of Hilbert's concept of algebraic number theory, and even though Hilbert does use the word '(Zahl)ring' for orders in algebraic number fields, this does not mean that he employs here parts of our current algebraic terminology; rather than referring to a general algebraic structure, the word 'ring' is used for certain sets of algebraic integers. Even more striking for the modern reader is that Hilbert does not employ general abstract notions from group theory that could have unified the discussions of various situations which we immediately recognize as analogous. For instance, he did not heed Minkowski's advice, given in a letter of 21 July 1896 [Minkowski, 1973, 83] to group together at the beginning of art. 100 all lemmata about finite Abelian groups needed in the proof of the so-called Kronecker–Weber theorem (Satz 131).

Similarly, no formal notion of quotient group is used in the *Zahlbericht,* even though the concept of factor group had been first defined and used by Otto Hölder as early as 1889 and discussed in the second volume of Heinrich Weber's *Lehrbuch der Algebra* of 1896 (§53). Thus, when we would say that 'G/H is cyclic of order h', Hilbert writes elaborate prose such as 'The members of G are each obtained precisely once when we multiply the members of H by $1, g, \ldots, g^{h-1}$ where g is a suitably chosen member of G'; see, for example, Sätze 69, 71 and 75. It is remarkable to note by comparison that the 33-year-old Kronecker, while generalizing Gaussian periods to roots of unity of composite order, encountered subgroups H of $(\mathbf{Z}/m\mathbf{Z})^*$ such that the quotient $(\mathbf{Z}/m\mathbf{Z})^*/H$ is cyclic, and added that this property is 'at the same time so characteristic that it could be used as the definition' of such subgroups ([Kronecker, 1856, 33f]; I thank B. Petri for pointing this out to me).

4 CONTENT AND STRUCTURE

The contents of the *Zahlbericht* are summarised in Table 1. We have already made a few comments on its first Part, which contains the basic arithmetic theory of a general finite extension of the field of rational numbers: integers, ideals, discriminant, units, ideal classes, the relationship of the class number with the residue at $s = 1$ of the zeta-function of the field, *Zahlringe*, that is, orders.

The second Part deals with the decomposition of primes in a Galois extension: decomposition group and inertia group, and the corresponding subfields. This theory had been essentially developed but not published by Dedekind, and later independently worked out and published by Hilbert in 1894. Georg Frobenius and Dedekind in their correspondence of February 1895 vented their anger about the fact that Hilbert had failed to acknowledge Dedekind's priority, even though Dedekind had sent Hilbert an offprint in June 1892 explicitly indicating his unpublished work. But Dedekind never published a complaint about Hilbert like the one he wrote against Hurwitz [Dedekind, 1895]. The exposition of this theory in the *Zahlbericht* (arts. 36–47) follows Hilbert's 1894 paper to a large extent.

Table 1. Summary of Hilbert’s report. 372 pages.

Chs.	**Arts.**	**Thms.**	**Topics**
Preface.			Modern number theory and its role in mathematics.
			Part 1: *Theory of a general number field.*
1	1–3	1–5	Number fields; algebraic integers; norm, different, discriminant, integral basis.
2	4–6	6–16	Ideals; decomposition into prime ideals; forms (in Kronecker’s sense).
3–4	7–13	17–37	Congruences mod. an ideal; the discriminant and its divisors; the fundamental equation and unit form of a number field.
5	14–16	38–41	Relative extensions of number fields.
6	17–21	42–48	Units of a number field.
7	22–29	49–57	Ideal classes; class number and the residue of the zeta-function at $s = 1$; characters of an ideal class.
8–9	30–35	58–66	Classes of forms; orders; modules.
			Part 2: *The Galois number field.*
10–11	36–47	67–80	Decomposition in the presence of the Galois group acting; decomposition and inertia groups and fields; powers of the prime divisors of different and discriminant.
12–13	48–52	81–87	Subfields, densities of primes, and composita.
14	53	89	Class group generated by primes of degree one.
15	54–58	90–94	Relative cyclic extensions.
			Part 3: *The quadratic number field.*
16–20	59–90	95–116	Quadratic and norm residue symbol; genus theory; analytic class number formula. Class fields and complex multiplication *not* treated in the report.
			Part 4: *The cyclotomic field.*
21–22	91–98	117–127	Degree, integral basis, discriminant, decomposition, units and circular units.
23	99–104	128–131	All Abelian number fields are cyclotomic.
24	105–112	132–138	Normal bases and root numbers (‘Gauss sums’).
25	113–115	139–140	Eisenstein’s reciprocity law for lth powers.
26–27	116–124	141–146	Cyclotomic analytic class number formula; cyclotomic theory applied to quadratic fields.

Table 1. *(Continued)*

			Part 5: *Kummer's number field.*
28–30	125–135	147–152	Power and norm residue, local symbols, logarithmic derivatives of units; prime ideals with prescribed characters.
31–34	136–165	153–167	The regular Kummer field; ideal classes, genus theory; lth power reciprocity; product formula for norm residue symbols.
35	166–171		Re-arrangement of the preceding theory of the regular Kummer field, avoiding logarithmic derivatives.
36	172–173	168–169	Fermat's Last Theorem for regular prime exponents.

At the end of the second Part, one finds a series of theorems first stated and proved in this generality in the *Zahlbericht*, and which are remarkable for their later impact: Satz 89–94. In Satz 89, Hilbert gives a *non-analytic* proof for the fact that the ideal class group is *generated* by the classes of prime ideals of degree 1. This theorem and its proof have apparently not received the attention they deserve; it took 80 years to see that the proof had to be completed in a technical point [Washington, 1989].

Hilbert's Satz 90, itself a literal generalization of a slightly more special result and proof of Kummer's, has become a household name since the introduction of Galois cohomology in the 1950s. This reinterpretation—which transforms Hilbert's explicit statement into the triviality of a first cohomology group: $H^1(G, K^*) = 1$—along with the substantial generalisation from cyclic to abelian extensions K/k (and many even more substantial generalizations or analogues in later developments), was first initiated by Noether in her work on what was then called the 'Principal Genus Theorem' [Noether, 1933]. To be sure, she translated into the calculus of cross product algebras; the further translation into Galois cohomology came later [Lemmermeyer, to appear].

Satz 91 on the existence of relative units was to be the first in a series of generalizations of Dirichlet's Unit Theorem; and Sätze 92–94 have been forerunners of important results in class field theory. For slightly more detailed comments on these and other mathematical points, see the introduction to the English translation of the *Zahlbericht* by Lemmermeyer and Schappacher.

The third Part of the *Zahlbericht* deals with quadratic fields. Gauss's genus theory (Satz 100) is treated via the Hilbert symbol, that is, the local norm residue symbol that is the main systematic novelty that Hilbert introduced into the treatment of algebraic number theory: he shifted the emphasis from the question, whether a given element is an lth power, to the question of whether it is the norm of an element in a certain extension of degree l. Quadratic reciprocity (Satz 101) is also couched in terms of Hilbert's symbol. This Part also contains the analytic class number formula in the quadratic case, as well as a discussion of arbitrary orders in quadratic fields and their relation to quadratic forms.

The theory of cyclotomic fields follows suit in the fourth Part, including the theory of circular units, and together with Hilbert's proof of the so-called Kronecker–Weber Theorem (Satz 131) to the effect that every abelian extension of the rational numbers is contained in a suitable cyclotomic field. Hilbert had actually been the first mathematician to have published (in 1896) a complete proof of this conjecture by Kronecker [Neumann, 1981, 125].

Then follows a largely original discussion of normal bases and what Hilbert calls their 'associated root numbers', that is, generalized Gaussian sums. The prime decomposition of Gaussian sums was obtained in fair generality by Ludwig Stickelberger. Hilbert quotes this article, but only in the context of quadratic fields and not in this section where he derives his own results towards the decomposition of root numbers (Satz 133, 134) and never gives more than a special case of Stickelberger's theorem (Satz 138), which was already known to C.G.J. Jacobi and Kummer. Helmut Hasse's incidental complaint about 'Hilbert's inconceivably not giving [Stickelberger's result] in his Zahlbericht' (letter to Harold Davenport, 22 February 1934) indicates how much later number theorists relied on Hilbert's report as a comprehensive reference for the 19th-century literature. The subject of root numbers has developed into an active field of research only in the last 30 to 40 years.

The *Zahlbericht* culminates in the long fifth and last Part on *the Kummer number field*. Hilbert describes it in the preface as

> the theory of those fields which Kummer took as a basis for his researches into higher reciprocity laws and which on this account I have named after him. It is clear that the theory of these Kummer fields is the highest peak reached on the mountain of today's knowledge of arithmetic; from it we look out on the wide panorama of the whole explored domain since almost all essential ideas and concepts [...] find an application in the proof of the higher reciprocity laws.

Concretely, the Kummer field is obtained by adjoining to the rational number field all lth roots of unity and an lth root of an element of this cyclotomic field which is not an lth power. The theory works all the way for regular prime numbers l. It is especially in this Part that Hilbert's struggles with Kummer's formidable 'computational apparatus'. In fact, he does the whole theory twice over: the first time around (essentially arts. 131–165), he defines the local norm residue symbol directly and uses Kummer's device of logarithmic derivatives of circular units to derive its relevant properties at the bad places. The major stepping stone on the way to the general reciprocity law is Eisenstein's reciprocity law which relates a rational to an arbitrary cyclotomic integer. Although this presentation already reduces 'Kummer's computational devices to a small amount' (art. 166), Hilbert then does go on to rearrange the theory 'in a way, completely avoiding those computations' (arts. 166–171). The trick is to use the product formula to recuperate the information needed at the bad places from those at the good ones. Either way, the reciprocity laws are developed along with genus theory for the Kummer fields, and Hilbert treats genus theory via 'characters' defined in terms of suitable local norm residue symbols. This feature as well as several technical improvements account for the difference, and in fact superiority of Hilbert's presentation over Kummer's genus theory.

The *Zahlbericht* ends with a proof of Fermat's 'last theorem' (in a generalized form) for regular prime exponents (art. 172), and other special cases of it (art. 173).

5 LATER REACTIONS

Later commentators have reacted differently to Hilbert's *Zahlbericht* in general and to his treatment of Kummer's achievements in particular. Major number theorists of the following generation like Erich Hecke and Hasse either learned their number theory from the *Zahlbericht* or used it as a standard reference. Even mathematicians like Felix Hausdorff and Hermann Weyl, whose principal research interests were far from number theory, were influenced by it. Hausdorff for instance, in his letter of congratulations to Hilbert's 70th birthday, wrote: 'My preferred dish among all the delicate things you have served us is the Zahlbericht. It is the most lucky blend of past, present, and future (the three dimensions of time, according to Hegel): the perfect command and exposition of the past, the solution of new problems, and the most refined prescience of things to come'.[1]

In his 1922 praise of 'The algebraist Hilbert', Otto Toeplitz (himself not a number theorist) went as far as writing that 'Hilbert has extracted from Kummer's difficult works overflowing with inductive material, which few before him had read, and which only few will now have to read after him and thanks to him, a universe of general facts and theses' [Toeplitz, 1922, 73]. Hasse in 1932 (in Hilbert, *Gesammelte Abhandlungen,* vol. 1, 529) and Emil Artin in 1962 [Artin, 1965, 549] acknowledged, more soberly than Toeplitz, the conceptual simplification and clarification of Kummer's theory obtained by Hilbert. On the other hand, in section 3 above we have mentioned and tried to interpret Noether's criticism of the *Zahlbericht* from the 1930s.

In 1975, André Weil wrote [Kummer, 1975, 1]:

> The great number-theorists of the last century are a small and select group of men. ... Most of them were no sooner dead than the publication of their collected papers was undertaken and in due course brought to completion. To this there were two notable exceptions: Kummer and Eisenstein. Did one die too young and the other live too long? Were there other reasons for this neglect, more personal and idiosyncratic perhaps than scientific? Hilbert dominated German mathematics for many years after Kummer's death. More than half of his famous *Zahlbericht* [...] is little more than an account of Kummer's number-theoretical work, with inessential improvements; but his lack of sympathy for his predecessor's mathematical style, and more specifically for his brilliant use of p-adic analysis, shows clearly through many of the somewhat grudging references to Kummer in that volume.

Even though the polemical evaluation of Hilbert's toiling as 'inessential improvements' clearly reflects Weil's personality, as does the intentional anachronism to speak of 'p-adic methods' in the middle of the 19th century, his opinion is surely best understood in the context of the renaissance of Kummer's ideas and techniques in the wake of the development of Iwasawa theory, which started in the 1960s and continues to this very day. But all

[1] 'Meine Lieblingsspeise unter all den Delikatessen, mit denen Sie uns bewirtet haben, ist der Zahlbericht. Das ist die glücklichste Mischung zwischen Vergangenheit, Gegenwart und Zukunft (den drei Dimensionen der Zeit, nach Hegel): vollendete Beherrschung und Darstellung des bereits Geleisteten, Lösung neuer Probleme, und feinstes Vorgefühl für die kommenden Dinge' (Göttingen University Library Archives, Cod. Ms. Hilbert 452c, Nr. 15, 21 January 1932). I thank Walter Purkert for having communicated this letter.

these fairly recent developments did of course occur on the firm basis of a well-established algebraic number theory (and class field theory), to the consolidation of which no other single publication has contributed more than Hilbert's *Zahlbericht.*

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