Goldstein · Schappacher · Schwermer

The Shaping of Arithmetic after C. F. Gauss's Disquisitiones Arithmeticae

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With 36 Figures

Springer

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In memoriam

MARTIN KNESER
1928 – 2004

Foreword

In 1998, the editors convinced themselves that it was the right time to take stock of recent research concerning the modern history of number theory, and to evaluate in its light our comprehension of the development of this discipline as a whole. One issue at stake was to bring together historiographical results coming from different disciplines and linguistic domains which, we felt, had remained too often unaware of each other.

We organized two meetings at the *Mathematisches Forschungsinstitut Oberwolfach*: first a small RIP-workshop held June 14–19, 1999, among historians of number theory and historians of related topics, and then a larger conference which took place June 17–23, 2001, two hundred years after the publication of Carl Friedrich Gauss's *Disquisitiones Arithmeticae*. The latter brought together historians and philosophers of mathematics with number theorists interested in the recent history of their field. Two further meetings, organized by one of us in Vienna and Zürich the following years, continued our venture.

Two concrete projects arose from these activities. One concerned the creation of resources, for scholars and students: we initiated a bibliography of secondary literature on the History of Number Theory since 1800.[1]

The present volume is the second result of our work. It aims at answering the question, already raised during the first workshop, on the role of Gauss's *Disquisitiones Arithmeticae* in the definition and evolution of number theory. This role is here appraised in a comparative perspective, with attention both to the mathematical reception of the treatise, and to its role as a model for doing mathematics. The volume is the result of a collective work. Although all authors have kept their proper voices, they have also accepted quite a bit of editorial interference with a view to making the volume as coherent as possible. We have nonetheless left room for original analyses and results, including newly discovered documents.

1. A preliminary version of this bibliography has been kindly put on line by Franz Lemmermeyer on a website hosted by the University of Heidelberg (http://www.rzuser.uni-heidelberg.de/~hb3/HINTbib.html).

During its rather long elaboration, the present book has greatly profited from the help of many individuals and institutions which we here gratefully acknowledge: the *Mathematisches Forschungsinstitut Oberwolfach* and its director at the time of the meetings, Matthias Kreck; the *Erwin-Schroedinger International Institute for Mathematical Physics* at Vienna; the ETH at Zürich, and special encouragement provided by Urs Stammbach; the CIRM at Luminy, which gave us a week's refuge for our editorial work in the summer of 2003; the *Laboratoire de mathématiques de l'Université Paris-Sud*, the *Institut de mathématiques de Jussieu*, as well as the *Institut de la recherche mathématique avancée* at Strasbourg which actively supported our joint work for the preparation of this book. Special thanks go to the *Abteilung Handschriften und Alte Drucke der Niedersächsischen Staats- und Universitätsbibliothek Göttingen*, and in particular to Jürgen Rohlfing, for the expert collaboration which made so many documents available to us, some of them as high quality scans. We also express our sincere gratitude to Springer-Verlag and their associated staff at Heidelberg and Leipzig, especially to Joachim Heinze, who believed in this project at an early stage. Our warmest thanks go to Frazer Jarvis for linguistic work on the texts, and to Jim Ritter for constant technical and moral support.

And last but not least, we thank all the participants of the Oberwolfach meetings who have shared their insights and knowledge with us for the benefit of the project: besides the authors of this book, Leo Corry, Hélène Gispert, Jeremy Gray, Ralf Haubrich, Helmut Koch, Martina Schneider, Takase Masahito, Erhard Scholz, Urs Stammbach, and Hans Wussing.

One of the participants at the 2001 conference was Martin Kneser. Despite his serious illness, his intense, visible passion for number theory and its history was a challenging inspiration to all of us, historians and mathematicians alike. Martin Kneser died on February 16, 2004. We dedicate this book to his memory.

July 2006

Catherine Goldstein
Norbert Schappacher
Joachim Schwermer

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Editions of Carl Friedrich Gauss's *Disquisitiones Arithmeticae*

The *Disquisitiones Arithmeticae* has been omitted from the list of references of the individual chapters: we list underneath its various editions. Throughout this book, passages from Gauss's *Disquisitiones Arithmeticae* are referred to only by the article number. The title of Gauss's work is routinely abbreviated as "D.A." For all works, a mention of [Author 1801a] refers to the item "Author. 1801a" in the bibliography, a mention of [Author 1801/1863] refers to the 1863 edition in this item.

1801. *Disquisitiones Arithmeticae*. Leipzig: Fleischer. Repr. Bruxelles: Culture et civilisation, 1968. Repr. Hildesheim: Olms, 2006. Rev. ed. in *Werke*, vol. 1, ed. Königliche Gesellschaft zu Göttingen [E. Schering]. Göttingen: Universitäts-Druckerei, 1863; 2nd rev. ed., 1870; repr. Hildesheim: Olms, 1973.

http://gallica.bnf.fr
http://dz-srv1.sub.uni-goettingen.de/cache/toc/D137206.html

1807. *Recherches arithmétiques*. French transl. A.-C.-M. Poullet-Delisle. Paris: Courcier. Repr. Sceaux: Gabay, 1989.

http://gallica.bnf.fr

1889. *Arithmetische Untersuchungen*. German transl. H. Maser. In *Untersuchungen über höhere Arithmetik*, pp. 1–453. Berlin: Springer. Repr. New York: Chelsea, 1965; 2nd ed., 1981.

http://dz-srv1.sub.uni-goettingen.de/cache/toc/D232699.html

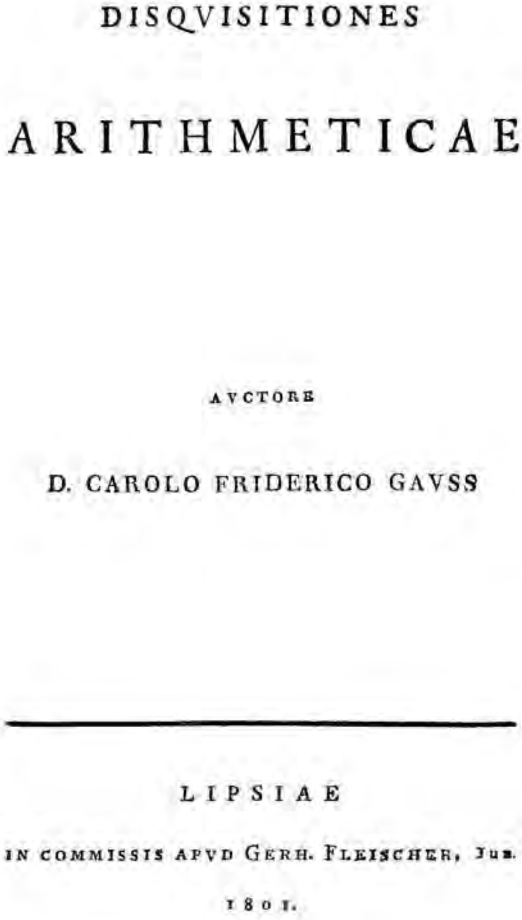
1959. *Arifmetičeskie issledovaniya*. Russian transl. V. B. Dem'yanov. In *Trudi po teorii čisel* [Works on number theory], ed. I. M. Vinogradov, B. N. Delone, pp. 7–583. Moscow: Academy of Sciences USSR.

1966. *Disquisitiones Arithmeticae*. English transl. A. A. Clarke. New Haven: Yale University Press. Rev. ed. W. C. Waterhouse. New York: Springer, 1986.

1995. *Disquisitiones Arithmeticae*. Spanish transl. H. Barrantes Campos, M. Josephy, A. Ruiz Zùñiga. Colección Enrique Pérez Arbelaez 10. Santa Fe de Bogotá: Academia Colombiana de Ciencias Exactas, Fisicas y Naturales.

1995. *Seisuu ron*. Japanese transl. Takase Masahito. Tokyo: Asakura-Shoten.

1996. *Disquisicions aritmètiques*. Catalan transl. G. Pascual Xufré. Barcelona: Institut d'Estudis Catalans, Societat Catalana de Matemàtiques.



DISQVISITIONES

ARITHMETICAE

AVCTORE

D. CAROLO FRIDERICO GAVSS

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1801.

Fig. I. Title page of *Disquisitiones Arithmeticae*, 1801 edition
(Private copy)

Part I

A Book's History

Welche Wichtigkeit Gauß: *Disquisitiones Arithmeticae für die Entwicklung der Mathematik gehabt haben, darüber existiert wohl kein Zweifel. Es ist ein Werk, das ungefähr in der Mathematik dieselbe Stellung einnimmt, wie die Kritik der reinen Vernunft von* Kant *in der Philosophie.*

Carl Itzigsohn, March 23, 1885


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SCIENCES.

Recherches arithmétiques, par M. C. Fr. Gauss (de Brunswick), traduites par A. C. M. Poullet-Delisle, professeur de mathématiques au Lycée d'Orléans. — Paris 1807.

ANTIQUITÉS CELTIQUES

Fig. I.1. A newspaper review of the *Disquisitiones Arithmeticae*
Gazette nationale, ou le Moniteur universel, March 21, 1807

I.1

A Book in Search of a Discipline (1801–1860)

CATHERINE GOLDSTEIN and NORBERT SCHAPPACHER

Carl Friedrich Gauss's *Disquisitiones Arithmeticae* of 1801 has more than one claim to glory: the contrast between the importance of the book and the youth of its author; the innovative concepts, notations, and results presented therein; the length and subtlety of some of its proofs; and its role in shaping number theory into a distinguished mathematical discipline.

The awe that it inspired in mathematicians was displayed to the cultured public of the *Moniteur universel ou Gazette nationale*[1] as early as March 21, 1807, when Louis Poinsot, who would succeed Joseph-Louis Lagrange at the Academy of Sciences six years later, contributed a full page article about the French translation of the *Disquisitiones Arithmeticae*:

> The doctrine of numbers, in spite of [the works of previous mathematicians] has remained, so to speak, immobile, as if it were to stay for ever the touchstone of their powers and the measure of their intellectual penetration. This is why a treatise as profound and as novel as his *Arithmetical Investigations* heralds M. Gauss as one of the best mathematical minds in Europe.[2]

1. This French newspaper, created by Charles-Joseph Panckoucke in the first months of the French Revolution, had the goal of informing its readers of administrative, political, and cultural events and of promoting French achievements. It opened its columns regularly to reviews of books recommended by the *Institut national des sciences et des arts.*

2. *Gazette nationale ou Le Moniteur universel* 80 (1807), 312: *La doctrine des nombres malgré leurs travaux [antérieurs] est restée, pour ainsi dire, immobile ; comme pour être dans tous les tems, l'épreuve de leurs forces et la mesure de la pénétration de leur esprit. C'est pourquoi Monsieur Gauss, par un ouvrage aussi profond et aussi neuf que ses* Recherches arithmétiques *s'annonce certainement comme une des meilleures têtes mathématiques de l'Europe.*

A long string of declarations left by readers of the book, from Niels Henrik Abel to Hermann Minkowski, from Augustin-Louis Cauchy to Henry Smith, bears witness to the profit they derived from it. During the XIX$^{\text{th}}$ century, its fame grew to almost mythical dimensions. In 1891, Edouard Lucas referred to the *Disquisitiones Arithmeticae* as an "imperishable monument [which] unveils the vast expanse and stunning depth of the human mind,"[3] and in his Berlin lecture course on the concept of number, Leopold Kronecker called it "the Book of all Books."[4] In the process, new ways of seeing the *Disquisitiones* came to the fore; they figure for instance in the presentation given by John Theodore Merz in his celebrated four-volume *History of European Thought in the Nineteenth Century*:

> Germany … was already an important power in the Republic of exact science which then had its centre in Paris. Just at the beginning of the nineteenth century two events happened which foreboded for the highest branches of the mathematical sciences a revival of the glory which in this department Kepler and Leibniz had already given to their country. … The *first* was the publication of the 'Disquisitiones Arithmeticae' in Latin in 1801.[5] … [Gauss] raised this part of mathematics into an independent science of which the 'Disquisitiones Arithmeticae' is the first elaborate and systematic treatise.… It was … through Jacobi, and still more through his contemporary Lejeune-Dirichlet … that the great work of Gauss on the theory of numbers, which for twenty years had remained sealed with seven seals, was drawn into current mathematical literature… The seals were only gradually broken. Lejeune-Dirichlet did much in this way, others followed, notably Prof. Dedekind, who published the lectures of Dirichlet and added much of his own.[6]

Gauss's book (hereafter, we shall often use the abbreviation "the D.A." to designate it) is now seen as having created number theory as a systematic discipline in its own right, with the book, as well as the new discipline, represented as a landmark of German culture. Moreover, a standard history of the book has been elaborated. It stresses the impenetrability of the D.A. at the time of its appearance and integrates it into a sweeping narrative, setting out a continuous unfolding of the book's content, from Johann Peter Gustav Lejeune-Dirichlet and Carl Gustav Jacob Jacobi on.

In this history modern algebraic number theory appears as the natural outgrowth of the discipline founded by the *Disquisitiones Arithmeticae*. Historical studies have accordingly focused on the emergence of this branch of number theory, in particular on the works of Dirichlet, Ernst Eduard Kummer, Richard Dedekind, Leopold Kronecker, and on the specific thread linking the D.A. to the masterpiece of algebraic number theory, David Hilbert's *Zahlbericht* of 1897. In addition, they have also explored the fate of specific theorems or methods of the D.A. which are relevant for number theorists today.

Yet a full understanding of the impact of the *Disquisitiones Arithmeticae*, at

3. [Lucas 1891], p. vi: *Ce livre, monument impérissable dévoile l'immense étendue, l'étonnante profondeur de la pensée humaine.*
4. [Kronecker 1891/2001], p. 219: *das Buch aller Bücher.*
5. The second event alluded to by Merz is the computation of Ceres's orbit, also by Gauss.
6. [Merz 1896–1914], vol. I, pp. 181, 181–182 (footnote), 187–188 and 721.

all levels, requires more than just a "thicker description"[7] of such milestones; it requires that light be shed on other patterns of development, other readers, other mathematical uses of the book – it requires a change in our questionnaire. We need to answer specific questions, such as: What happened to the book outside Germany? What were the particularities, if any, of its reception in Germany? Which parts of it were read and reworked? And when? Which developments, in which domains, did it stimulate – or hamper? What role did it play in later attempts to found mathematics on arithmetic?

Such questions suggest narrower foci, which will be adopted in the various chapters of the present volume. In this first part, however, we take advantage of the concrete nature of our object of inquiry – a book – to draw a general map of its tracks while sticking closely to the chronology. That is to say, instead of going backwards, seeking in the *Disquisitiones Arithmeticae* hints and origins of more recent priorities, we will proceed forwards, following Gauss's text through time with the objective of surveying and periodizing afresh its manifold effects.[8]

But let us start, first, at the beginning of all beginnings…

1. The Writing and the Architecture of the *Disquisitiones Arithmeticae*

Gauss began to investigate arithmetical questions, at least empirically, as early as 1792, and to prepare a number-theoretical treatise in 1796 (i.e., at age 19 and, if we understand his mathematical diary correctly, soon after he had proved both the constructibility of the 17-gon by ruler and compass and the quadratic reciprocity law). An early version of the treatise was completed a year later.[9] In November 1797, Gauss started rewriting the early version into the more mature text which he would give to the printer bit by bit. Printing started in April 1798, but proceeded very slowly for technical reasons on the part of the printer. Gauss resented this very much, as his letters show; he was looking for a permanent position from 1798. But he did use the delays to add new text, in particular to sec. 5 on quadratic forms, which had roughly doubled in length by the time the book finally appeared in the summer of 1801.[10]

7. The reference is to Gilbert Ryle and Clifford Geertz, in particular [Geertz 1973].

8. We have systematically tracked mentions of the D.A. in the main nineteenth-century journals, and then in the complete works – if published – of the mathematicians encountered. For want of space (in the text or in the margin …), only part of the evidence used to establish our main claims can be presented here.

9. Parts of the manuscript, known as *Analysis residuorum*, were published posthumously in Gauss's *Werke*; other parts were identified as such in 1980 by Uta Merzbach in different German archives; see [Merzbach 1981], also for a global comparison of the early version with the printed book; for detailed comparisons of specific parts, in particular sec. 2, see [Bullynck 2006a] and [Bullynck 2006b], appendices A and B. See also [Schlesinger 1922], sec. III.

10. Basic data on the genesis of the *Disquisitiones* can be derived from Gauss's mathematical diary, [Gauss 1796–1814], and from his correspondence. Our quick summary here is based on [Merzbach 1981]. What exactly Gauss found in his predecessors and how he was influenced by them remains a difficult question, in spite of his own historical

1.1. The First Sections: Congruences to the Fore

Let us skim through the contents of the *Disquisitiones Arithmeticae* as they appeared in 1801.[11] Although it may make somewhat tedious reading, we think it useful to indicate the full variety of matters treated by Gauss. The 665 pages and 355 articles of the main text are divided unevenly into seven sections. The first and smallest one (7 pp., 12 arts.) establishes a new notion and notation which, despite its elementary nature, modified the practice of number theory:

> If the number a measures[12] the difference of the numbers b, c, then b and c are said to be *congruent according to a*; if not, *incongruent*; this a we call the *modulus.* Each of the numbers b, c are called a *residue* of the other in the first case, a *nonresidue* in the second.[13]

The corresponding notation $b \equiv c \pmod{a}$ is introduced in art. 2. The remainder of sec. 1 contains basic observations on convenient sets of residues modulo a and on the compatibility of congruences with the arithmetic operations. To consider numbers or equations up to a given integer was not new with Gauss.[14] His innovation was to turn this occasional computational device into a topic in its own right.

Section 2 (33 pp., 32 arts.) opens with several theorems on integers including the unique prime factorization of integers (in art. 16), and then treats linear congruences in arts. 29–37, including the Euclidean algorithm and what we call the Chinese remainder theorem. At the end of sec. 2, Gauss added a few results for future reference which had not figured in the 1797 manuscript, among them: (i) properties

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remarks. We do not go into it here, referring for a first orientation and survey of Gauss's obvious predecessors, in particular Euler, Lagrange and Legendre, to [Weil 1984]; the less-expected tradition of German arithmetical textbooks and the more general impact of Lambert's and Hindenburg's works are explored in the original thesis of Maarten Bullynck, [Bullynck 2006b].

11. The reader is invited to go back and forth between our rough summary and Gauss's original detailed table of contents which we reproduce on the double page 10–11. In the present section, expressions in quotation marks, with no explicit reference attached, are the English translations of key words from this Latin table of contents. The table is copied from the 1801 edition of the D.A. except that, for the sake of readability, we have modified the letters "u" and "v" according to current Latin spelling. Other surveys of the book are proposed in [Bachmann 1911], [Rieger 1957], [Neumann 1979–1980], [Bühler 1981], chap. 3, [Neumann 2005].

12. Along with this classical Euclidean term "to measure" (*metiri*), which, as well as "modulus" (small measure), reminds us of the additive flavour of Euclidean division, Gauss also used "to divide" (*dividere*) as of sec. 2, art. 13, in the context of a product of natural integers. This diversity of expressions was not always maintained in translations.

13. Our translation of the opening paragraph of D.A., art. 1: *Si numerus a numerorum b, c differentiam metitur, b et c* secundum *a* congrui *dicuntur, sin minus,* incongrui*: ipsum a* modulum *appellamus. Uterque numerorum b, c, priori in casu alterius* residuum*, in posteriori vero* nonresiduum *vocatur.*

14. Gauss acknowledged this fact in the footnote to art. 2, noticing that Legendre had used a simple equality in such situations, and pleading at the same time for his own, unequivocal notation. Other authors are discussed in [Bullynck 2006b], appendix A.

of the number $\varphi(A)$ of prime residues modulo A (arts. 38–39); (ii) in art. 42, a proof that the product of two polynomials with leading coefficient 1 and with rational coefficients that are not all integers cannot have all its coefficients integers;[15] and (iii) in arts. 43 and 44, a proof of Lagrange's result that a polynomial congruence modulo a prime cannot have more zeros than its degree.[16]

Section 3 (51 pp., 49 arts.) is entitled "On power residues." As Gauss put it, it treats "geometric progressions" $1, a, a^2, a^3, \ldots$ modulo a prime number p (for a number a not divisible by p), discusses the "period" of a modulo p and Fermat's theorem, contains two proofs for the existence of "primitive roots" modulo p, and promotes the use of the "indices" of $1, \ldots, p-1$ modulo p with respect to a fixed primitive root, in analogy with logarithm tables.[17] After a discussion, in arts. 61–68, of n^{th} roots mod p from the point of view of effective computations, the text returns to calculations with respect to a fixed primitive root, and gives in particular in arts. 75–78 two proofs – and sketches a third one due to Lagrange – of Wilson's theorem, $1 \cdot 2 \cdots (p-1) \equiv -1 \pmod{p}$. The analogous constructions and results for an odd prime power are discussed in arts. 82–89, the exceptional case of the powers of $p = 2$ in arts. 90–91. Finally, integers n for which there exists a primitive root modulo n are characterized in art. 92.

Section 4 (73 pp., 59 arts.), "On congruences of degree 2," develops a systematic theory of "quadratic residues" (i.e., residues of perfect squares). It culminates in the "fundamental theorem" of this theory, from which "can be deduced almost everything that can be said about quadratic residues,"[18] and which Gauss stated as:

> If p is a prime number of the form $4n+1$, then $+p$, if p is of the form $4n+3$, then $-p$, will be a [quadratic] residue, resp. nonresidue, of any prime number which, taken positively, is a residue, resp. nonresidue of p.[19]

Gauss motivated this *quadratic reciprocity law* experimentally, gave the general statement and formalized it in tables of possible cases (arts. 131 and 132), using the notation aRa', resp. aNa', to mean that a is a quadratic residue, resp. nonresidue, modulo a'.[20] He also gave here the first proof of the law, an elementary one by

15. This is one of several results known today as "Gauss's lemma."
16. See [Bullynck 2006a], for a closer study of sec. 2 in comparison to Gauss's earlier manuscript of the D.A.
17. In modern terms, the period is the order of the element a in the multiplicative group $(\mathbf{Z}/p\mathbf{Z})^*$, Fermat's theorem states that this order divides $p-1$, a primitive root is a generator of the group and the index of an element is the corresponding exponent with respect to the chosen generator.
18. D.A., art. 131: … *omnia fere quae de residuis quadraticis dici possunt, huic theoremati innituntur.*
19. Our translation of D.A., art. 131: *Si p est numerus primus formae* $4n+1$*, erit* $+p$*, si vero p formae* $4n+3$*, erit* $-p$ *residuum vel non residuum cuiusuis numeri primi qui positive acceptus ipsius p est residuum vel non residuum.* The supplementary theorems about the quadratic residue behaviour of -1 and 2 are treated in parallel.
20. Today one usually sees this quadratic reciprocity law written in terms of Legendre's symbol. It is defined, for any integer a and p a prime number not dividing a, by

induction.[21] A crucial nontrivial ingredient (used in art. 139) is a special case of a theorem stated in art. 125, to the effect that, for every integer which is not a perfect square, there are prime numbers modulo which it is a quadratic nonresidue.[22]

1.2. Quadratic Forms

The focus changes in sec. 5 of the D.A., which treats "forms and indeterminate equations of the second degree," mostly binary forms, in part also ternary. With its 357 pp. and 156 arts., this section occupies more than half of the whole *Disquisitiones Arithmeticae*. Leonhard Euler, Joseph-Louis Lagrange, and Adrien-Marie Legendre had forged tools to study the representation of integers by quadratic forms. Gauss, however, moved away from this Diophantine aspect towards a treatment of quadratic forms as objects in their own right, and, as he had done for congruences, explicitly pinpointed and *named* the key tools. This move is evident already in the opening of sec. 5:

> The form $axx + 2bxy + cyy$,[23] when the indeterminates x, y are not at stake, we will write like this, (a, b, c).[24]

Gauss then immediately singled out the quantity $bb - ac$ which he called the "determinant"[25] – "on the nature of which, as we will show in the sequel, the prop-

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$\left(\frac{a}{p}\right) = \pm 1 \equiv a^{\frac{p-1}{2}} \pmod p$, that is, 1 if a is quadratic residue modulo p, -1 if not; see [Legendre 1788], p. 186, and D.A., art. 106, for the last congruence. Given distinct odd prime numbers p, q, the quadratic reciprocity law then says that $\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{q}{p}\right)$. Notwithstanding Dirichlet's criticism (published after Gauss's death) of the D.A. as lacking a notational calculus for quadratic residues, [Dirichlet 1889–1897], vol. 2, p. 123, one may point out that Gauss's formulation stresses the normalization of $\pm p$, a phenomenon that would recur with higher reciprocity laws; see [Neumann 2005], p. 308.

21. In the late 1960s, John Tate "lifted directly from the argument which was used by Gauss in his first proof of the quadratic reciprocity law" his determination of the second K-group of the field of rational numbers $K_2(\mathbf{Q})$; see [Milnor 1971], p. 102. More generally, Gauss's argument is seen to provide an inductive procedure to determine successively the local factors at all primes p of a given Steinberg symbol on $\mathbf{Q}^* \times \mathbf{Q}^*$, and the decomposition of the universal continuous Steinberg symbol with values in $\{\pm 1\}$ is tantamount to the reciprocity law. See [Milnor 1971], p. 101–107; cf. [Tate 1971], § 3.
22. This follows easily from the reciprocity law; Gauss does not even bother to give the details. The difficulty is to prove it *directly* (arts. 125–129) in a special case which then makes the proof by complete induction of the reciprocity law work: that every $\pm p$, for a prime p of the form $4n + 1$, is a quadratic nonresidue modulo some prime $q < p$.
23. Gauss's convention that the coefficient of the mixed term be even is original and its advantages and drawbacks have been much in debate; see J. Schwermer's chap. VIII.1.
24. D.A., art. 153: *Formam* $axx + 2bxy + cyy$*, quando de indeterminatis* x, y *non agitur, ita designabimus* (a, b, c). In [Kronecker 1891/2001], p. 235, this move is heralded as the first time in history that a system of three discrete quantities was introduced.
25. Nowadays called, sometimes up to a constant, the *discriminant* of the form. For the various normalizations and names of this quantity in the XIX[th] century, see [Dickson 1919–1923], vol. 3, p. 2. We will usually employ Gauss's word in this chapter.

erties of the form chiefly depend"[26] – showing that it is a quadratic residue of any integer primitively represented[27] by the form (art. 154).

The first part of sec. 5 (arts. 153–222, 146 pp.) is devoted to a vast enterprise of a finer classification of the forms of given determinant, to which the problem of representing numbers by forms is reduced. Gauss defined two quadratic forms (art. 158) to be equivalent if they are transformed into one another under substitutions of the indeterminates, changing (x, y) into $(\alpha x + \beta y, \gamma x + \delta y)$, for integral coefficients $\alpha, \beta, \gamma, \delta$, with $\alpha\delta - \beta\gamma = +1$ or $= -1$.[28] Two equivalent forms represent the same numbers. If $\alpha\delta - \beta\gamma = +1$, the equivalence is said to be "proper," if $\alpha\delta - \beta\gamma = -1$, "improper." While integral invertible substitutions were already used by Lagrange, this finer distinction is due to Gauss and greatly exploited by him. After generalities relating to these notions and to the representation of numbers by forms – in particular (art. 162), the link between the problem of finding *all* transformations between two, say, properly equivalent forms, when one is known, and the solutions of the equation $t^2 - Du^2 = m^2$, where D is the determinant of the forms and m the greatest common divisor of their coefficients – the discussion then splits into two very different cases according to whether the determinant is negative or positive. In each case, Gauss showed that any given form is properly equivalent to a so-called "reduced" form (art. 171 for negative, art. 183 for positive discriminants), not necessarily unique, characterized by inequalities imposed on the coefficients.[29] The number of reduced forms – and thus also the number of equivalence classes of forms – of a given determinant is finite. Equivalence *among reduced forms* is studied – in particular, the distribution of the reduced forms of given positive determinant into "periods" of equivalent reduced forms, art. 185 – and a general procedure is given to determine if two binary quadratic forms with the same determinant are (properly or improperly) equivalent and to find all transformations between them. Using this, Gauss settled the general problem of representing integers by quadratic forms (arts. 180–181, 205, 212), as well as the resolution in integers of quadratic equations with two unknowns and integral coefficients (art. 216). The first half of sec. 5 closes with a brief historical reminder (art. 222).

The classification of forms also ushers the reader into the second half of sec. 5, entitled "further investigations on forms." Art. 223 fixes an algorithm to find a good representative for every (proper equivalence) class of quadratic forms of a given determinant. Representing classes by reduced forms avoids working with the infinite classes abstractly, just as Gauss never worked with our field $\mathbf{Z}/p\mathbf{Z}$, the elements of which are infinite sets of integers, but with conveniently chosen residues modulo p.

26. D.A., art. 154: *Numerum* $bb - ac$, *a cuius indole proprietates formae* (a, b, c) *imprimis pendere, in sequentibus docebimus,* determinantem *huius formae vocabimus.*

27. I.e., which can be written as $axx + 2bxy + cyy$, for two *coprime* integers x and y.

28. Gauss also handled the general case of arbitrary substitutions with integral coefficients transforming a form into another one which is then said to be "contained" in the first.

29. A reduced form (A, B, C) of determinant $D < 0$ is such that $A \leq 2\sqrt{-D/3}$, $B \leq A/2$, $C \geq A$. A reduced form of determinant $D > 0$ is such that $0 \leq B < \sqrt{D}$, $\sqrt{D} - B \leq |A| \leq \sqrt{D} + B$.

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In art. 226, certain classes are grouped together into an "order" according to the divisibility properties of their coefficients.[30] There follows (arts. 229–233) a finer grouping of the classes within a given order according to their "genus." Gauss showed that, for every odd prime divisor p of the determinant of a form (with coprime coefficients), integers prime to p that can be represented by the form (and thus by all forms of its class) are either all quadratic residues, or all nonresidues modulo p: recording this information, as well as similar information at $p = 2$ for even discriminants, defines the "character" of the form (or of the class of the form). Classes with the same character are put into the same genus. The principal genus for a determinant D is that of the principal form $(1, 0, -D)$.[31]

In art. 235, Gauss defined a form $F(X, Y) = AXX+2BXY+CYY$ to be a "composite" of $f(x, y) = axx + 2bxy + cyy$ and $f'(x', y') = a'x'x' + 2b'x'y' + c'y'y'$, if $F(B_1, B_2) = f(x, y) \cdot f'(x', y')$, for certain transformations of the indeterminates $B_i(x, y; x', y')$, bilinear, as we would say, in x, y and x', y'. While this definition generalizes time-honoured relations like the following,[32] with $F = f = f'$:

$$(xx' - Nyy')^2 + N(xy' + yx')^2 = (x^2 + Ny^2) \cdot (x'^2 + Ny'^2),$$

the generality of the concept allowed Gauss to enter uncharted territory, for instance, to check – by elaborate computations – formal properties like the commutativity and associativity of the operation, as far as it is defined on the level of forms (arts. 240–241). In the end, the concept yields a multiplicative structure on the set of orders (art. 245) and of genera, with the principal genus acting like a neutral element (arts. 246–248), and indeed of classes (art. 239 and art. 249) of the same determinant.[33]

This rich new structure gave Gauss a tremendous leverage: to answer new questions, for instance, on the distribution of the classes among the genera (arts. 251–253); to come back to his favourite theorem, the quadratic reciprocity law, and derive a second proof of it from a consideration of the number of characters that actually correspond to genera of a given discriminant (arts. 261–262); to solve a long-standing conjecture of Fermat's (art. 293) to the effect that every positive integer is the sum of three triangular numbers. For this last application, as well as for deeper insight into the number of genera, Gauss quickly generalized (art. 266 ff.) the basic theory of reduced forms, classes etc., from binary to ternary quadratic forms. This

30. It is with explicit reference to this terminology that Richard Dedekind would later introduce the notion of order into algebraic number theory in [Dedekind 1930–1932], vol. 1, pp. 105–158.

31. See also §2 of F. Lemmermeyer's chap. VIII.3 below. Such classificatory schemes, then part and parcel of the natural sciences, already existed in mathematics, with variants, see Hindenburg's classification in [Bullynck 2006b], pp. 259–260. Note, however, that Gauss put classes below genera and orders.

32. [Weil 1986]. Cf. the blackboard in [Weil 1979], vol. III, p. ii.

33. This particularly difficult theory of the composition of forms has been reformulated several times by Gauss's successors; two different perspectives, emphasizing different aspects of its history and of its current relevance, are proposed in chaps. II.2 and II.3 below, by H.M. Edwards and by D. Fenster and J. Schwermer.

gave him in particular explicit formulae for the number of representations of binary quadratic forms, and of integers, by ternary forms, implying especially that every integer $\equiv 3 \pmod 8$ can be written as the sum of three squares, which is tantamount to Fermat's claim.[34] Sec. 5 closes (arts. 305–307) by open-ended reflections on the analogy between the multiplicative structure of the prime residue classes modulo an integer and of the classes of quadratic forms.[35]

Sec. 5 sometimes displays, and often hides, a tremendous amount of explicit computations performed by Gauss,[36] of numbers of classes, genera, or representations. To mention just one striking example of such extensive computations, which had an intriguing long term history, Gauss observed that any given "classification," that is, any given pair of numbers, one for the number of genera (which Gauss wrote as a Roman numeral) and one for the number of classes contained in a single genus (Arabic numeral), is realized by at most finitely many negative determinants:

> It seems beyond doubt that the sequences written down do indeed break off, and by analogy the same conclusion may be extended to any other classification. For instance, since in the whole tenth thousand of determinants there is none corresponding to a class number less than 24, it is highly probable that the classifications I.23, I.21 etc.; II.11; II.10 etc.; IV.5; IV.4; IV.3; IV.2 stop already before -9000, or at least that they contain extremely few determinants beyond -10000. However, *rigorous* proofs of these observations appear to be most difficult.[37]

34. The entry in Gauss's mathematical diary about this problem is the only one accompanied by Archimedes's exclamation "EYPHKA"; see [Gauss 1796–1814], July 10, 1796.
35. This as well as the composition of orders and genera alluded to above would provide one of the sources for the later development of the abstract concept of group, see [Wussing 1969], I, § 3.3, and [Wussing 2001].
36. Examples relative to the composition of forms are displayed in H.M. Edwards's chap. II.2 below, who argues that such computations play a crucial role in Gauss's conception of a well-founded theory. See also Gauss's addition to art. 306 at the end of the 1801 edition and the tables in [Gauss 1863/1876], pp. 399–509. Gauss discussed how much numerical material on quadratic forms ought to be published *in extenso* in an 1841 letter to H. C. Schumacher, translated in [Smith 1859–1865], §119. Cf. [Maennchen 1930] and [Neumann 1979–1980], p. 26.
37. Our translation of D.A., art. 303: *Nullum dubium esse videtur, quin series adscriptae revera abruptae sint, et per analogiam conclusionem eandem ad quasuis alias classificationes extendere licebit. E.g. quum in tota milliade decima determinantium nullus se obtulerit, cui multitudo classium infra 24 responderit: maxime est verisimile, classificationes I.23, I.21 etc.; II.11; II.10 etc.; IV.5; IV.4; IV.3; IV.2 iam ante* -9000 *desiisse, aut saltem perpaucis determinantibus ultra* -10000 *comprendere. Demonstrationes autem* rigorosae *harum observationum perdifficiles esse videntur.* Indeed, for one of the simplest constellations of numbers of classes and genera (corresponding to "orders of class number one" in imaginary quadratic fields, in Richard Dedekind's terminology of 1877), the proof that the list of determinants found by Gauss (art. 303) is actually complete was given by Kurt Heegner only in 1954 – and at first not accepted – by a method which subsequently would greatly enrich the arithmetic of elliptic curves. A book on Heegner by H. Opolka, S.J. Patterson, and N. Schappacher is in preparation.

1.3. Applications

Explicit calculations had evidently been part and parcel of number theory for Gauss ever since he acquired a copy of [Lambert 1770] at age 15, and launched into counting prime numbers in given intervals in order to guess their asymptotic distribution.[38] In these tables, Johann Heinrich Lambert made the memorable comment:

> What one has to note with respect to all factorization methods proposed so far, is that primes take longest, yet cannot be factored. This is because there is no way of knowing beforehand whether a given number has any divisors or not.[39]

The whole D.A. is illustrated by many non-trivial examples and accompanied by numerical tables. Section 6 (52 pp., 27 arts.) is explicitly dedicated to computational applications. In the earlier part of sec. 6, Gauss discussed explicit methods for partial fraction decomposition, decimal expansion, and quadratic congruences. Its latter part (arts. 329–334) takes up Lambert's problem and proposes two primality tests: one is based on the fact that a number which is a quadratic residue of a given integer M is also a quadratic residue of its divisors and relies on results of sec. 4; the second method uses the number of values of $\sqrt{-D}$ mod M, for $-D$ a quadratic residue of M, and the results on forms of determinant $-D$ established in sec. 5.

The final Section 7 on cyclotomy (74 pp., 31 arts.) is probably the most famous part of the *Disquisitiones Arithmeticae*, then and now, because it contains the conditions of constructibility of regular polygons with ruler and compass. After a few reminders on circular functions – in particular (art. 337), the fact that trigonometric functions of the angles kP/n, for a fixed integer n and for $k = 0, 1, 2, \dots n-1$, where $P = 2\pi$ denotes "the circumference of the circle," are roots of equations of degree n – Gauss focused on the prime case and the irreducible[40] equation

$$X = 0, \quad \text{where } X = \frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x + 1\ ;\ n > 2 \text{ prime},$$

which his aim is to "decompose *gradually* into an increasing number of factors in such a way that the coefficients of these factors can be determined by equations of as

38. See Gauss's letter to Johann Franz Encke of December 24, 1849 in [Gauss 1863/1876], pp. 444–447. In [Biermann 1977], pp. 7–18, it is established that the page of Gauss's mathematical diary which follows the last entry of July 9, 1814, records the dates when Gauss counted prime numbers in certain intervals. On the influence of Lambert's and Hindenburg's tables on Gauss's sec. 6, see [Bullynck 2006b]. See also [Maennchen 1930], in particular pp. 27–35.

39. [Lambert 1770], pp. 29–30: *Was übrigens bey allen bißher erfundenen Methoden, die Theiler der Zahlen aufzusuchen, zu bemerken ist, besteht darinn, daß man bey Primzahlen am längsten aufsuchen muß, und zuletzt doch nichts findet, weil man nicht voraus weiß, ob eine forgegebene Zahl Theiler hat oder nicht.* Lambert went on to propose Fermat's Little Theorem as a first necessary criterion for primality.

40. D.A., art. 341. The word "irreducible" was established a few decades later. Cf. O. Neumann's chap. II.1 below.

low a degree as possible, until one arrives at simple factors, i.e., at the roots Ω of X."[41] Art. 353 illustrates the procedure for $n = 19$, which requires solving two equations of degree three and one quadratic equation (because $n - 1 = 3 \cdot 3 \cdot 2$); art. 354 does the same for $n = 17$ which leads to four quadratic equations ($n - 1 = 2 \cdot 2 \cdot 2 \cdot 2$).[42]

All roots of $X = 0$ are powers r^i of one of them, but to solve the equation, Gauss replaced the natural sequence of the exponents i, that is $1, 2, \ldots, n - 1$, by the more efficient bookkeeping provided by sec. 3:

> But in this [natural] form of the roots there is presented no means of distributing them into cyclical periods, nor even of ascertaining the existence of such periods or of determining their laws. It was the happy substitution of a geometrical series formed by the successive powers of a primitive root of n in place of the arithmetical series of natural numbers, as the indices [i.e., exponents] of r, which enabled [Gauss] to exhibit not merely all the different roots of the equation $\frac{x^n-1}{x-1}$, but which also made manifest the cyclical periods which existed among them. Thus if α was a primitive root of n and $n - 1 = mk$, then in the series $r, r^\alpha, r^{\alpha^2}, \ldots, r^{\alpha^{k-1}}, \ldots, r^{\alpha^{mk-1}}$ the m successive series which are formed by the selection of every k^{th} term, beginning with the first, the second ... are periodical.[43]

Complementary results on the auxiliary equations, i.e., those satisfied by the sums over all the roots of unity in a given period, are given in art. 359, applications to the division of the circle in the final arts. 365 and 366. As a byproduct of his resolution of $X = 0$, Gauss also initiated a study of what are today called "Gauss sums," i.e., certain (weighted) sums of roots of unity, like the sum of a period, or of special values of circular functions. For instance, he proved (art. 356) that, for an odd prime n and an integer k not divisible by n,

$$\sum_R \cos \frac{kRP}{n} - \sum_N \cos \frac{kNP}{n} = \pm\sqrt{n} \quad \text{or} \quad 0,$$

according to whether $n \equiv 1$ or $n \equiv 3 \bmod 4$. Here, R varies over the quadratic residues, N over the quadratic non-residues modulo n.[44]

1.4. The Disquisitiones Arithmeticae as a System

In the preface of the D.A., Gauss explicitly restricted the objects of arithmetic to be the rational integers; he wrote:

41. Our translation of D.A., art. 342: *Propositum disquisitionum sequentium ... eo tendit, ut X in factores continuo plures* GRADATIM *resolvatur, et quidem ita, ut horum coëfficientes per aequationes ordinis quam infimi determinentur, usque dum hoc modo ad factores simplices sive ad radices Ω ipsas perveniatur.*

42. For Gauss's early annoucements of these results and details on the case of the 17-gon, see [Reich 2000]. The impact on the theory of equations is discussed in O. Neumann's chap. II.1.

43. [Peacock 1834], p. 316. Gauss described his view in his letter to Christian Gerling of January 6, 1819; see [Gauss & Gerling 1927/1975], p. 188.

44. The sign of $\sqrt{n}$ depends on whether k is or is not a quadratic residue modulo n, but Gauss did not succeed in proving this fact in the D.A. See S.J. Patterson's chap. VIII.2 below.

> The theory of the division of the circle … which is treated in sec. 7 does not belong *by itself* to arithmetic, but its *principles* can only be drawn from higher arithmetic.[45]

In sec. 7 itself, he promised that the "intimate connection" of the topic with higher arithmetic "will be made abundantly clear by the treatment itself."[46] There is of course the technical link mentioned above, that is, the bookkeeping of the roots of unity via sec. 3. But the "intimate connection" that Gauss announced goes further and also concerns the systemic architecture of the treatise.

Despite the impressive theoretical display of sec. 5, one cannot fully grasp the systemic qualities of the D.A. from the torso that Gauss published in 1801. At several places in the D.A. and in his correspondence a forthcoming volume II is referred to. The only solid piece of evidence we have is what remains of Gauss's 1796–1797 manuscript of the treatise. This differs from the structure of the published D.A. in that it contains an (incomplete) 8th chapter (*caput octavum*), devoted to higher congruences, i.e., polynomials with integer coefficients taken modulo a prime and modulo an irreducible polynomial.[47] Thus, according to Gauss's original plan, sec. 7 would not have been so conspicuously isolated, but would have been naturally integrated into a greater, systemic unity. The division of the circle would have provided a model for the topic of the *caput octavum*, the theory of higher congruences; it would have appeared as part of a theory which, among many other insights, yields two entirely new proofs of the quadratic reciprocity law.[48]

The treatise would thus have come full circle in several respects: beginning with ordinary congruences and ending with higher congruences; encountering various periodic structures along the way: prime residues, periods of reduced quadratic forms of positive discriminant, classes of quadratic forms which are all multiples of one class, cyclotomic periods and their analogues mod p; and proving quadratic reciprocity four separate times in the process.

That scientificity ought to be expressed by way of a system was a widespread idea in Germany in the second half of the XVIIIth century. Lambert, whose works were well represented in Gauss's library, wrote, besides his scientific *œuvre*, several philosophical texts developing this idea, at least two of which Gauss owned personally.[49] German idealist philosophers from Immanuel Kant to Georg Wilhelm Friedrich Hegel, for instance Johann Gottlieb Fichte, Friedrich Wilhelm Joseph von

45. Our translation of D.A., *praefatio: Theoria divisionis circuli …, quae in Sect. VII tractatur,* ipsa *quidem* per se *ad Arithmeticam non pertinet, attamen eius* principia *unice ex Arithmetica Sublimiori petenda sunt.*

46. D.A., art. 335: *tractatio ipsa abunde declarabit, quam intimo nexu hoc argumentum cum arithmetica sublimiori coniunctum sit.*

47. We summarize in this paragraph G. Frei's analysis, in chap. II.4 below, of the *caput octavum* and its importance for the economy of the whole treatise that Gauss originally planned.

48. Gauss published them later independently; see G. Frei's chap. II. 4 below.

49. Maarten Bullynck has drawn our attention to [Lambert 1764] and [Lambert 1771]; see [Bullynck 2006b], p. 278. Unfortunately, the dates of acquisition for these items are not known.

Schelling, Karl Leonhard Reinhold, and Jakob Friedrich Fries, cultivated various systemic programmes. Starting with Fichte, a system with a circular instead of linear architecture – returning to its initial point which thereby receives its higher justification – is called upon to provide a self-justifying foundation for the unfolding of self-consciousness. With Hegel this would become the unfolding of reason; in his first philosophical publication, which appeared in the same year as Gauss's D.A., Hegel wrote that "the method of the system, which may be called neither analytical nor synthetical, is realized most purely if it appears as the development of reason itself."[50]

The systemic design of Gauss's original plan for his arithmetic fits those ambient ideas remarkably well.[51] It makes it possible, for instance, to appreciate the four proofs of the quadratic reciprocity law originally planned for the treatise in a dual way: deducing a theorem at a certain place of the systemic development endows it with a specific theoretical meaning;[52] on the other hand, the various proofs of the same result connect these theoretical frameworks into a system which is not simply a deduction of increasingly complicated theorems from initial axioms. In Gauss's words,

> It is the insight into the marvellous interlinking of the truths of higher arithmetic which constitutes the greatest appeal of these investigations.[53]

Another systemic cyclicity is created precisely by the already mentioned recurrence of periodic structures throughout the treatise. Gauss himself insisted on the analogy between what we call cyclic components of class groups and the multiplicative structure of residues modulo a prime number:

> The proof of the preceding theorem will be found to be completely analogous to the proofs of arts. 45, 49, and the theory of the multiplication of classes actually has a very great affinity in every respect with the argument of sec. 3.[54]

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50. [Hegel 1801], p. 35: *Am reinsten gibt sich die weder synthetisch noch analytisch zu nennende Methode des Systems, wenn sie als eine Entwicklung der Vernunft selbst erscheint.* For a general orientation about the philosophical ideas alluded to in this paragraph, see [Ritter, Gründer 1998], art. "System," pp. 835–843.

51. In spite of Gauss's philosophical interests – e.g., he is said to have read Kant's *Critique of Pure Reason* several times; see [Dunnington 1955], p. 315; cf. J. Ferreirós's chap. III.2 and J. Boniface's chap. V.1 below – we have no evidence of a direct and conscious inspiration; later mentions of Hegel by Gauss are rather critical; see for instance [Gauss & Schumacher 1862], vol. 4, n° 944, p. 337. A reference to Gauss's idea of science as a system in the not very reliable biographical essay [Waltershausen 1856], p. 97, suggests only a banal deductive structure.

52. From the philosophical point of view, cf. [Hartmann 1972], p. 106: "The point easily lost sight of is that the [systemic] methodological structure provides a new meaning to categories that already have a meaning."

53. Our translation of [Gauss 1817], p. 160: *Dann ist gerade die Einsicht in die wunderbare Verkettung der Wahrheiten der höhern Arithmetik dasjenige, das einen Hauptreiz dieses Studiums ausmacht, und nicht selten wiederum zur Entdeckung neuer Wahrheiten führt.*

54. Our translation of D.A., art. 306: *Demonstratio theor. praec. omnino analoga invenietur*

Gauss thus drew the attention of the reader to the fact that sec. 3 was not only instrumental for decomposing the cyclotomic equation in sec. 7 but also linked the theory of forms to the rest. He also significantly called "irregular" a determinant whose principal genus was not cyclic, i.e., not constituted by the multiples of a single class of forms.

Half a century later, the mathematician Ernst Eduard Kummer reflected upon a suitable system for "the more recent mathematics," and concluded that it should not be linear but

> rather like the system of the universe; its goal would be to give not just the deduction of the mathematical truths, but an insight into all the essential relations among them.[55]

As mentioned above, the subject of the *Disquisitiones Arithmeticae* was natural numbers and Gauss's proofs were anchored in intricate computations, both formal (as in the sec. 5) and numerical, ultimately based on integers. The tension between this anchorage of the book and the striving towards a wider theoretical scope, as illustrated in the last section of the D.A., will be a recurring theme in what follows. It explains why the question of the reception of the book is tightly linked to the shaping of number theory as a specific discipline.

2. The Early Years of the *Disquisitiones Arithmeticae*

Gauss's own impressions of the early reception of the *Disquisitiones Arithmeticae* are scattered in his correspondence. A letter of June 16, 1805, to Antoine-Charles Marcel Poullet-Delisle, his French translator,[56] summarizes them well:

> It is for me as sweet as it is flattering that the investigations contained in my Work, to which I devoted the best part of my youth, and which were the source of my sweetest pleasures, have acquired so many friends in France; a fate quite different from what

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demonstrationibus in arts. 45, 49, reveraque theoria multiplicationis classium cum argumento in Sect. III tractato permagnam undique affinitatem habet. Cf. the note on D.A., art. 306.IX: "Démonstration de quelques théorèmes concernant les périodes des classes des formes binaires du second degré," [Gauss 1863/1876], pp. 266–268, where Gauss used, of course informally, the word "group" (*groupe*) referring to all classes of forms of given determinant.

55. [Kummer 1975], vol. 2, p. 697: *die neuere Mathematik … wird sich erst später ihr eigenthümliches System schaffen, und zwar wol nicht mehr nur ein in einer Linie fortlaufendes, dessen Vollkommenheit allein darin liegt, dass das Folgende überall durch das Vorhergehende begründet werde, sondern ein dem Weltsysteme ähnlicheres, dessen Aufgabe es sein wird, über die blosse Begründung der mathematischen Wahrheiten hinausgehend, eine allseitige Erkenntnis der wesentlichen Beziehungen derselben zueinander zu geben.* In [Kummer 1975], vol. 2, p. 687, the parallel is made explicit between Hegel's principle of the systemic "self-interpretation of content" (*Sichselbstauslegen des Inhalts*) and the system required for the new mathematics since Gauss.

56. On his life, see [Boncompagni 1882].

> they found in Germany where a taste for the most difficult parts of pure mathematics is the property of a very small number of persons.[57]

The *Disquisitiones Arithmeticae* had in fact been mentioned at the French Academy at least as early as January 1802:

> Citizen Legendre communicates a geometrical discovery, made in Germany by M. Charles Frédéric Bruce [sic], from Brunswick, and published by him in his work entitled *Disquisitiones arithmeticae*, Leipsik, 1801,[58]

and was commented upon very positively from all quarters.[59] The project of a French translation was supported by arguably the most prominent mathematician of the time, Pierre-Siméon Laplace, and on May 31, 1804, Joseph-Louis Lagrange wrote to Gauss:

> Your *Disquisitiones* have put you at once among the first mathematicians, and I consider the last section as one which contains the most beautiful analytic discovery made in a long time. Your work on planets will moreover have the merit of the importance of its topic.[60]

The beginning of this praise is often quoted, but taken in its entirety, the citation provides important clues about the early reception of the D.A. First, attention focused on the last section, the resolution of $x^n - 1 = 0$ through auxiliary equations and its consequences for the constructibility of regular polygons; this is the part of the book which borders both on the general theory of equations and on geometry. Second, Gauss's innovation was described as "analytical." Finally, number theory (and more generally pure mathematics) was considered a subsidiary subject compared to astronomy or mathematical physics.

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57. Letter published by Ernest Fauque de Jonquières in 1896, *Comptes rendus de l'Académie des sciences* 122, p. 829: *Il m'est aussi doux que flatteur que les recherches contenues dans mon Ouvrage, auxquelles j'avais dévoué la plus belle partie de ma jeunesse, et qui ont été la source de mes plus douces jouissances, aient acquis tant d'amis en France; sort bien inégal à celui qu'elles ont trouvé en Allemagne où le goût pour les parties plus difficiles des mathématiques pures n'est la propriété que d'un fort petit nombre de personnes.* In the letter, Gauss also expressed his hopes to publish the sequel of the D.A., a project he described as delayed for lack of time and printer.

58. *Procès verbaux de l'Académie des sciences*, registre 114, vol. II, séance du 6 pluviôse an 10 (26 janvier 1802), p. 457: *Le Citoyen Legendre communique une découverte géométrique, faite en Allemagne par M. Charles Frédéric Bruce, de Brunswick, et publiée par lui dans son ouvrage intitulé* Disquisitiones arithmeticae*, Leipsik, 1801.*

59. Gauss's fame in France was decisive for his connection to Alexander von Humboldt (then in Paris), and for establishing on the German scene, through Humboldt, a place for himself and, afterwards, for other number theorists; see H. Pieper's chap. III.1 in this volume.

60. [Lagrange 1867–1892], vol. 14, p. 299: *Vos Disquisitiones vous ont mis tout de suite au rang des premiers géomètres et je regarde la dernière section commme contenant la plus belle découverte analytique qui ait été faite depuis longtemps. Votre travail sur les planètes aura de plus le mérite de l'importance de son objet.* "Geometer" (*géomètre*) remains a standard terminology for "mathematician" in French during the XIX[th] century.

This challenges the disciplinary position of the *Disquisitiones Arithmeticae* both in its mathematical and cultural aspects. Two situations typified the first generation of its readers:[61] either their involvement with the book, even if significant and fruitful, occupied only a small part within all their mathematical activities, or they themselves occupied a marginal position in the mathematical community. Augustin-Louis Cauchy is a good example of the first category, and Sophie Germain of the second.[62] And Gauss himself, after all, turned to astronomy and geodesy to secure a position.

This does not mean, however, that the reading of the D.A. at the time was restricted to one section, nor that it remained superficial and did not lead at all to innovative work. Sophie Germain displayed in her papers a thorough knowledge of all the sections: those on congruences of course which she used freely in her work on Fermat's Last Theorem as well as in her new proof of the quadratic residue behaviour of the prime 2, but also the difficult sec. 5, and specifically the composition of forms and the theory of ternary forms.[63] In his memoir on symmetric functions, presented to the *Institut* on November 30, 1812, Cauchy relied on concepts and a notation which he borrowed directly from the D.A., such as the idea of an adjoint and the notion of determinant, to prove that if a function of n quantities takes less than p distinct values, where p is the greatest prime divisor of n, then it can take at most 2 values, and furthermore to develop his theory of combinations and of determinants, seen with hindsight as key steps towards the development of group theory.[64]

However, as Lagrange's quote suggested, the D.A. was first of all taken up for its treatment of $x^n - 1$ and therefore mostly in treatises on algebra. For instance, Sylvestre François Lacroix included in the third edition of his *Complément des Élemens d'algèbre* (1804) a discussion of Gauss's results in the section on binomial equations.[65] Lagrange's second edition of his *Traité de la résolution des équations*

61. Different aspects of this early reception have been documented in [Neumann 1979–1980], [Reich 1996], [Reich 2000], [Goldstein 2003], and [Neumann 2005].
62. On the shifting professional status of number theory and number theorists, and the characteristics of the craft at different moments, see [Goldstein 1989].
63. See for instance Bibliothèque Nationale de France, Manuscripts f.fr 9118, pp. 40–41, 86; f.fr. 9114, pp. 92–94. On Germain's work, see [Edwards 1977], pp. 61–65, and [Laubenbacher, Pengelley 1998].
64. See [Cauchy 1815], where Cauchy – perhaps significantly – refers to Gauss's "Recherches analytiques" [sic]; cf. [Belhoste 1991], pp. 32–35. Between 1813 and 1815, Cauchy also published, in [Cauchy 1813–1815], his proof of Fermat's general claim to the effect that each natural number is the sum of no more than n n-gonal numbers. Gauss had shown in passing how to reduce the case $n = 4$ (already proved by Lagrange) to his theorem for $n = 3$; see end of D.A., art. 293. Cauchy then managed to reduce the cases $n \geq 5$ to those proven by Gauss; the most original elements of this proof, however, seem quite independent of the D.A.
65. [Lacroix 1804], p. 92: *M. Gauss, dans un ouvrage très-remarquable, intitulé Disquisitiones Arithmeticae, a fait voir que* toute équation à deux termes, dont l'exposant est un nombre premier, peut être décomposée rationnellement en d'autres équations dont les degrés sont marqués par les facteurs premiers du nombre qui précède d'une unité ce

numériques de tous les degrés [Lagrange 1770/1808] also includes, as a final note XIV, a simplification of D.A., art. 360 rendering "the consideration of auxiliary equations superfluous,"[66] thanks to what are called "Lagrangian resolvents," i.e., sums such as $r + \alpha r^a + \alpha^2 r^{a^2} + \cdots + \alpha^{\mu-2} r^{a^{\mu-2}}$, where α is a $(\mu - 1)^{\text{th}}$, r a μ^{th} root of unity, and a a primitive root modulo μ.

Gauss's treatment of the cyclotomic equation was mentioned many times, in various countries, during the first decades of the century: Peter Barlow, a mathematics teacher at the Royal Military Academy, Woolwich, better known for his chromatic telescope lens, his work on magnetism and as a royal commissioner for railways, devoted a chapter to it in his book on Diophantine analysis, [Barlow 1811]. Nikolai Ivanovič Lobačevskii, who had been introduced to Gauss's work by his teacher (and Gauss's friend) Martin Bartels at Kazan, published a note on it.[67] Charles Babbage included the D.A. among the small list of works he recommended to the members of the newly created Cambridge Analytical Society, mentioning in particular "that celebrated theorem of Gauss on the resolution of the equation of $x^n - 1 = 0$."[68] Hegel used it as a famous example in his *Wissenschaft der Logik* in a discussion of analytical versus synthetical proofs.[69] In April 1818, Poinsot explained to the Academy his ideas to develop simultaneously the theory of the resolution of $x^n - 1 = 0$ and that of $x^n - 1 = \mathfrak{M}(p)$, in Legendre's notation.[70] Mindful of a general view of the world and the sciences in which the notion of "order" (*ordre*) played a key role, Poinsot saw Gauss's reindexation of the roots of the cyclotomic equation as a paradigm of the fact that order was the natural source of the properties of numbers, and algebra the proper domain to express it, see [Poinsot 1819–1820].

These variegated allusions to "analytic" and "analysis" redirect our attention to the problem of the disciplinary landscape in which the *Disquisitiones Arithmeticae*

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nombre premier. ..., *mais pour le démontrer il faut recourir à des propriétés des nombres, que je ne pourrai faire connaître qu'à la fin de cet ouvrage.* Indeed, on pp. 294–315, the required parts of D.A., sec. 3 are explained, and then applied to the discussion of $x^{17} - 1 = 0$ and, in less detail, of $x^{19} - 1 = 0$.

66. [Poinsot 1808/1826], p. xviii: *de sorte que sa méthode rend superflue cette considération des équations auxiliaires.* Lagrange used Gauss's sec. 7 to "reduce the resolution of binomial equations to the same principle as that of cubic and quartic equations"; see [Lagrange 1867–1892], vol. 14, p. 300.

67. See [Neumann 2005], p. 313.

68. [Babbage 1813/1989], p. 32.

69. [Hegel 1812–1816], vol. 2, p. 325. Hegel concluded that Gauss's method to resolve $x^m - 1 = 0$ (which he called "one of the most important extensions of analysis in recent times") is synthetic, not analytic (in the Kantian, philosophical sense), because it uses the residues modulo m and primitive roots, "which are not data of the problem itself."

70. This notation was used in [Legendre 1798/1808] for what Gauss wrote as $x^n \equiv 1$ (mod p) (in other terms, $x^n - 1$ is a multiple of p). Barlow, in [Barlow 1811] replaced Gauss's triple bar by another symbol of his own device. On the other hand, Christian Kramp, a professor of mathematics in the then French towns of Strasbourg and Cologne, used Gauss's congruence notation in his *Élémens d'Arithmétique Universelle*, published in 1808; see [Cajori 1928–1929], vol. 2, p. 35.

was to be situated.[71] In the preface of the D.A., Gauss directly addressed this issue: as said above, he *defined* the domain of his book as that of general investigations of integers (and of fractions as expressed by integers),[72] more precisely their advanced part, as opposed to the elementary part which deals with the writing of numbers and the usual operations. Moreover, according to Gauss, this domain entertains with indeterminate (or Diophantine) analysis roughly the same relation that universal analysis – which investigates general quantities – entertains with ordinary algebra, i.e., the theory of algebraic equations. In other words, arithmetic provides the general theoretical framework for the investigation of equations in integers or fractions. Gauss thus claimed quite an important status for his domain, *parallel* to analysis and rich in its own applications, as illustrated by secs. 6 and 7 of the D.A.

However, this was markedly different from the usual contemporary point of view;[73] Legendre for instance, in the preface to his *Essai sur la théorie des nombres*, stated:

> I shall not distinguish the Theory of Numbers from Indeterminate Analysis, and I consider these two parts as making up one single branch of Algebraic Analysis.[74]

The report on the mathematical sciences presented to the emperor Napoléon on February 6, 1808 by Jean Baptiste Joseph Delambre, Secretary of the Academy, adopted a very similar view: the presentation moves from geometry to algebra, number theory, and calculus – the three of them seen as analytical investigations – and then on to mechanics, astronomy, physics and geography.[75] Barlow also referred to number theory as a "branch of analysis" in the preface of [Barlow 1811].

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71. One has to keep in mind that 1801 was in the middle of a transitional period for mathematics where one passes from a vision of analysis as a global approach to problems, opposed to the synthetic one associated with Euclid's geometry, to a vision of analysis as a specific mathematical domain dealing with functions and limits. The last decades of the XVIII$^{\text{th}}$ century saw the triumph of algebraic analysis, as promoted for example by Lagrange. See [Jahnke 1990], in particular chaps. 4–8.
72. See also J. Boniface's chap. V.1 below.
73. This does not mean that Gauss's point of view had no precedent. One may think of Pierre Fermat, for instance, advocating an autonomous theory of integers (opposed to general quantities) in his 1657 challenge, see *Œuvres de Pierre Fermat*, ed. P. Tannery, C. Henry, vol. 2, p. 334. Paris: Gauthier-Villars, 1894.
74. [Legendre 1798/1808], p. xi: *Je ne sépare point la Théorie des Nombres de l'Analyse indéterminée et je regarde ces deux parties comme ne faisant qu'une seule et même branche de l'Analyse algébrique.*
75. [Delambre 1810]. About 30 pages are allotted to analysis, as opposed to some 230 pages for the rest. Analysis gets about half the room devoted to astronomy, in accordance with Lagrange's point of view expressed in his 1804 letter. Gauss is mentioned several times, as one of the very rare foreigners in this report which concentrates mainly on the achievements of French scientists. Representing him as "one of the best minds in Europe," but also as a successor of Lagrange and Legendre, i.e., at the same time as European and as an heir and participant of French culture, is quite characteristic of the late French Enlightenment; see [Goldstein 2003].

Notwithstanding the attention paid to the D.A., the disciplinary *status quo* remained unchanged, as textbooks show. In Barlow's treatise for instance, theoretical arithmetic, including that inherited from Gauss's book, only serves as prolegomena to the solution of families of indeterminate equations. Legendre did not adopt Gauss's congruence notation, nor did he distinguish congruences as a topic worthy of a separate treatment. He did present a proof of the quadratic reciprocity law as well as a whole chapter on cyclotomic equations, after Gauss, in the second edition of his *Essai sur la théorie des nombres*, but commented:

> One would have wished to enrich this Essay with a greater number of the excellent materials which compose the work of M. Gauss: but the methods of this author are so specific to him that one could not have, without extensive detours and without reducing oneself to the simple role of a translator, benefited from his other discoveries.[76]

Thus, the *Disquisitiones Arithmeticae* had at first a strong effect, but in a direction *opposite* to that of establishing number theory as an autonomous research discipline, as defined by Gauss in the preface of the D.A., that is, focused on integers and congruences; the book contributed to, and sometimes even launched, developments in algebra (perceived as a branch of analysis), in the theory of equations and in the study of linear transformations, determinants, and substitutions, towards the theories of groups and of invariants.[77] This partial fusion with other domains could even be perceived as a valorization of number theory, in that this would overcome its isolation. In his report to the British Association for the Advancement of Science "on the recent progress and present state of certain branches of analysis," delivered at Cambridge in 1833, George Peacock, after an analysis of sec. 7 of the D.A., which "gave an immense extension to our knowledge of the theory and solution of such binomial equations" ([Peacock 1834], p. 316), comments on Poinsot's investigations on the cyclotomic equation and its analogue modulo p, mentioned above:

> These views of Poinsot are chiefly interesting and valuable as connecting the theory of indeterminates with that of ordinary equations. It has, in fact, been too much the custom of analysis to consider the theory of numbers as altogether separated from that of ordinary algebra. The methods employed have generally been confined to the specific problem under consideration and have been altogether incapable of application when the known quantities employed were expressed by general symbols and not by specific numbers. It is to this cause that we may chiefly attribute the want of continuity in the methods of investigation which have been pursued, and the great confusion which has been occasioned by the multiplication of insulated facts

76. [Legendre 1798/1808], Avertissement, p. vi: *On aurait désiré enrichir cet Essai d'un plus grand nombre des excellens matériaux qui composent l'ouvrage de M. Gauss : mais les méthodes de cet auteur lui sont tellement particulières qu'on n'aurait pu, sans des circuits très étendus, et sans s'assujétir au simple rôle de traducteur, profiter de ses autres découvertes.*

77. For instance, Jacobi in his paper on Pfaff's theory referred to the D.A. for the notion of determinant, see [Jacobi 1882–1891], vol. 4, p. 26. However, given that the D.A. was often not the only source of a mathematical development in those areas, gauging its precise effect is a delicate question.

and propositions which were not referable to, nor deducible from any general and comprehensive theory.[78]

We will see that the following phase would be characterized by an even greater entwinement around the *Disquisitiones Arithmeticae* of several disciplinary orientations. Paradoxically enough, this would help to consolidate the status of number theory, particularly in Germany.

3. The *Disquisitiones* as the Core for Arithmetic Algebraic Analysis

Because of the already-established fame of the book, and because it did not require many prerequisites, the generations of mathematicians born after 1801 often read the D.A. early on: the Norwegian Niels Henrik Abel, born in 1802, studied it in his first year of university, the Italian Angelo Genocchi, born in 1817, perused it while he was still in law school, and the Englishman Arthur Cayley, born in 1821, quoted it in his first important paper on determinants presented to the Cambridge Philosophical Society. The publication of shorter articles focusing on new results, instead of books or long memoirs, began to spread as normal mathematical activity: these papers are strewn with references to the *Disquisitiones Arithmeticae* in the second quarter of the XIX$^{\text{th}}$ century, and witness engagement with several specific questions arising from the first sections. A small industry developed for example around the determination of primitive roots modulo a prime p, accompanied sometimes by the publication of extensive tables. Guglielmo Libri and especially Victor-Amédée Lebesgue also investigated at some length the number of solutions of various types of congruences, in particular $a_1x_1^m + \ldots + a_kx_k^m \equiv a \bmod p$, for $p = hm + 1$.[79]

However, the reception of the *Disquisitiones Arithmeticae* during the 1820s did not simply result from a closer, more systematic study of the book, in the continuation of the first decades: two new factors directly intervened in the 1820s, which would at the same time consolidate and diversify a fledgling domain of research emerging from the *Disquisitiones Arithmeticae*. One was cultural and institutional, the other had to do with mathematical content and technique.[80]

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78. [Peacock 1834], p. 322. We are indebted to M.J. Durand-Richard for drawing our attention to this text.

79. See [Lebesgue 1837]. Lebesgue occupied several professorships in provincial towns and published numerous articles, mostly on number theory, including a new proof of the quadratic reciprocity law. He illustrates well the development of research activities and publications outside the main centres, i.e., outside Paris and far from the Academy in the case of France. Characteristically, he also published, in the same issue of the journal, "astronomical theses" whose real topic was in fact substitutions of forms. His counting of solutions for congruence equations is mentioned (rather dismissively) in André Weil's seminal article [Weil 1975], vol. 1, [1949b], which culminated in the statement of the general "Weil Conjectures" (and referred to art. 358 of the D.A. as the origin of the question).

80. For more details on the two aspects, see respectively Parts III and IV of this book.

3.1. Mathematics in Berlin

The first aspect was the conscious renewal of German, and specifically Prussian, mathematics in the aftermath of the wars against Napoléon and the founding of Berlin University.[81] The values associated with mathematics as of the late 1820s and the corresponding efforts to hire adequate professors allowed Gauss's D.A. to play a prominent role in this development. As Abel wrote to Christoffer Hansteen on December 5, 1825: "The young mathematicians in Berlin and, as I hear, all over Germany almost worship Gauss; he is the epitome of all mathematical excellence."[82] Gauss's letter of recommendation for Dirichlet, for instance, explicitly stressed number theory as a significant topic on which to judge the value of a mathematician, as well as the patriotic aspect of the hiring.[83] August Crelle's foundation of the *Journal für die reine und angewandte Mathematik* in 1826 provided the new discipline with a crucial organ in Germany. A comparison of the papers published in Crelle's journal to those in, say, Joseph Gergonne's *Annales de mathématiques* or the memoirs of the Paris Academy in those years illuminates the new domains touched upon.

This orientation would be reinforced by the lectures offered on number theory at Berlin University and the production of textbooks. Ferdinand Minding, for example, *Privatdozent* at Berlin, published in 1832 an introduction to higher arithmetic, which presents a shortened and expurgated version of the D.A. focusing on the basic parts of the various sections, with the notable exception of sec. 7. Minding dropped the more delicate part of Gauss's theory of forms (genera, composition), but added things expected from the perspective of a textbook, for instance, linear Diophantine equations or continued fractions. He did, however, identify the quadratic reciprocity law as "the most remarkable theorem of higher arithmetic,"[84] and, in a historical endnote, stressed rigour:

> Gauss's *Disquisitiones Arithmeticae* offers a presentation of arithmetic conducted with ancient rigour and distinguished by new discoveries. Among other things an excellent merit of the work lies in the rigorous proof of the reciprocity theorem. Since then the science has been enriched by a non-negligible number of new proofs and results, some of which are to be found in the memoirs of various learned societies, others in mathematical journals, and especially in Crelle's Journal for mathematics.[85]

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81. This foundation goes back to 1810 and Wilhelm von Humboldt's neo-humanist reform (see [Vierhaus 1987]); but its strong impact on the development of mathematics mostly started after Alexander von Humboldt's return from Paris to Berlin in 1827. See [Biermann 1988], chap. 3.
82. Quoted from [Ore 1957], p. 91.
83. This letter to Encke and the hiring of number theorists is discussed in H. Pieper's chap. III.1, §2, below.
84. [Minding 1832], p. 53. We heartily thank Ms. Bärbel Mund at the Göttingen university library for making Gauss's copy of this little book accessible to us. In the author's announcement of his book in Crelle's *Journal* (vol. 7, 1831, pp. 414–416), Minding described pure number theory as both a necessary foundation for algebra (whose basic notion is that of number) and a paradigm for a rigorously developed, autonomous branch of mathematics.
85. [Minding 1832], p. 198: *Eine in antiker Strenge durchgeführte und durch neue Entdek-*

A comparison with Legendre's or Barlow's books mentioned above also reveals a change of focus: Minding's treatment makes congruences an important topic *per se*, instead of a mere prelude to the study of specific Diophantine equations.

3.2. Biquadratic Residues

The second new factor of change was paradoxically brought about by new developments in analysis, specifically analysis involving continuity. It eventually gave rise to an active, largely international, integrated domain of research, which blossomed in the middle of the century: for short, we shall refer to it as *arithmetic algebraic analysis*. It was rooted in XVIII$^{\text{th}}$ century algebraic analysis, onto which the *Disquisitiones Arithmeticae* grafted a potent arithmetical shoot which grew to be its central stem for a while. At least three new developments in analysis gave substance to arithmetic algebraic analysis: (1) the acceptance of complex numbers and Gauss's publications on biquadratic residues between 1825 and 1832; (2) the integration of Fourier analysis into number theory; and (3) the theory of elliptic functions. We shall discuss them in turn, although they were actually tightly interwoven in much of the production of those years.

Gauss's investigations on biquadratic (that is, quartic) residues[86] led him to complex numbers and their claim to enter the subject matter of higher arithmetic. Apparently conceived a few years after the publication of the D.A., his ideas on extending the domain of arithmetic were published only after 1825.[87] The goal of this work was to state and prove a biquadratic reciprocity law; while his first

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kungen ausgezeichnete Darstellung der Arithmetik geben die disquisitiones arithmeticae *von Gauß… Unter andern macht der strenge Beweis des Satzes der Reciprocität ein vorzügliches Verdienst dieses Werkes aus. Seitdem ist die Wissenschaft durch eine nicht unbedeutende Anzahl neuer Beweise und Resultate bereichert worden, welche sich theils in den Denkschriften verschiedner gelehrten Gesellschaften, theils in mathematischen Zeitschriften, und namentlich in Crelle's Journal für Mathematik befinden.*

86. He presented his results to the Göttingen Academy in two installments, the first in 1825, the second in 1831, and announced both presentations in the *Göttingische gelehrte Anzeigen* a few days later, some time before the papers themselves were published, see [Gauss 1863/1876], pp. 65–92, pp. 93–148, pp. 165–168, and pp. 169–178, respectively; an English translation of the second self-announcement is to be found in [Ewald 1996], vol. 1, pp. 306–313. During the first decades of the century, Gauss had also completed the D.A. on several points; see [Gauss 1863/1876].

87. In [Gauss 1863/1876], pp. 165–168, Gauss dated the beginning of his work on cubic and biquadratic residues to 1805. This is compatible with material evidence about two of the early notes on cubic residues published in [Gauss 1900], pp. 5–11. However, his mathematical diary records on February 15, 1807: "Beginning of the theory of cubic and biquadratic residues" (*Theoria Residuorum cubicorum et biquadraticorum incepta*), and on October 23, 1813: "The foundation of a general theory of biquadratic residues, searched for during almost seven years with the greatest effort but always in vain, we have finally and happily discovered on the same day that a son was born to us" (*Fundamentum theoriae residuorum biquadraticorum generalis, per septem propemodum annos summa contentione sed semper frustra quaesitum tandem feliciter deteximus eodem die quo filius nobis natus est*); see [Gauss 1796–1814].

communication handled the results on -1 and 2 as biquadratic residues for any prime of the form $4n + 1$, the second got as far as stating the biquadratic law. However, its most original feature was elsewhere:

> As easily as all such special theorems are discovered by induction, so difficult it is to find a general law for these forms in the same way, even though several common features are obvious. And it is even more difficult to find the proofs of these theorems. … One soon recognizes that totally new approaches are necessary to enter this rich domain of higher arithmetic, … that for the true foundation of the theory of biquadratic residues the field of higher arithmetic, which before had only extended to the real integers, has to be extended to also include the imaginary ones and that exactly the same right of citizenship has to be given to the latter as to the former. As soon as one has understood this, that theory appears in a totally new light, and its results acquire a most surprising simplicity.[88]

Gauss then considered what are now called "Gaussian integers" $a+bi$ (with $i^2 = -1$ and rational integers a, b), and extended to them the concepts and results of arithmetic *as defined in the Disquisitiones Arithmeticae*: the units ± 1, $\pm i$; (complex) prime numbers and congruences; Fermat's Little Theorem; the quadratic reciprocity law, etc. In Gauss's words:

> Almost all the investigations of the first four sections of the *Disquisitiones Arithmeticae* find, with a few modifications, their place also in the extended arithmetic.[89]

He was then able to state a quartic reciprocity law for Gaussian integers.[90]

While we are now used to interpreting Gaussian integers arithmetically, as an instance of algebraic numbers,[91] the legitimacy of complex numbers inside analysis

88. [Gauss 1863/1876], pp. 170–171: *So leicht sich aber alle dergleichen specielle Theoreme durch die Induction entdecken lassen, so schwer scheint es, auf diesem Wege ein allgemeines Gesetz für diese Formen aufzufinden, wenn auch manches Gemeinschaftliche bald in die Augen fällt, und noch viel schwerer ist es, für diese Lehrsätze die Beweise zu finden. … Man erkennt demnach bald, dass man in dieses reiche Gebiet der höhern Arithmetik nur auf ganz neuen Wegen eindringen kann, … dass für die wahre Begründung der Theorie der biquadratischen Reste das Feld der höhern Arithmetik, welches man sonst nur auf die reellen ganzen Zahlen ausdehnte, auch über die imaginären erstreckt werden, und diesen das völlig gleiche Bürgerrecht mit jenen eingeräumt werden muss. Sobald man diess einmal eingesehen hat, erscheint jene Theorie in einem ganz neuen Lichte, und ihre Resultate gewinnen eine höchst überraschende Einfachheit.*

89. [Gauss 1863/1876], p. 172: *Fast die sämmtlichen Untersuchungen der vier ersten Abschnitte der* Disquisitiones Arithmeticae *finden mit einigen Modificationen, auch in der erweiterten Arithmetik ihren Platz.* Gauss also completed his purely arithmetical presentation with a geometric one, interpreting complex numbers as points in the plane, see §§ 38, 39 and pp. 174–178.

90. For two distinct prime Gaussian integers α and β which are primary, that is, congruent to 1 modulo $2+2i$, one has $\left[\frac{\alpha}{\beta}\right]\left[\frac{\beta}{\alpha}\right] = (-1)^{\frac{N\alpha-1}{4}\frac{N\beta-1}{4}}$, where $\left[\frac{\alpha}{\beta}\right]$ is the quartic analogue of the Legendre symbol (which takes the four values, ± 1, $\pm i$) and $N\alpha$ designates the norm of the Gaussian integer α; see [Lemmermeyer 2000], chap. 6. As in the D.A., Gauss did not use any Legendre-like symbol.

91. Several authors would later describe the inclusion of the Gaussian integers as the initial

was still very much in debate during these decades and contemporaries first perceived Gauss's move as establishing links inside analysis. The fact that Gauss announced a proof of the biquadratic reciprocity law, but never published it, provided additional incentive to work on reciprocity laws for small degrees, in particular cubic, quartic and sextic.[92] Jacobi was the first to make a proof of the biquadratic law at least semi-public through Johann Georg Rosenhain's notes of his 1836–1837 Königsberg lectures: "Gauss advertises these theorems very much in that they occupied him particularly, and they are indeed of the highest importance."[93] The proof relied on cyclotomy: Jacobi's point of departure was Gauss's work on biquadratic reciprocity as well as the D.A., reading the latter very much from the point of view of sec. 7. Thus, his emphasis is different from Minding's:

> Number Theory in its present state consists of two big chapters, one of which may be called the theory of the solution of pure equations, the other the theory of quadratic forms. Here I will deal mainly with the first part whose discovery we owe to Gauss.[94]

This description is interesting because it does not mention congruences as a part by itself; on the contrary, it suggests that congruences are a common theme underlying both "chapters," in agreement with the increasing importance of reciprocity laws as the core of number theory, and also with what we can guess about Gauss's original plan of his treatise. It also fits well with the idea of arithmetic algebraic analysis, the equations and forms (as algebraic core) providing the intermediary step between the arithmetical topic (congruences) and analytic tools, as we shall now see.

3.3. Infinite Series

Gauss apparently had long had his own ideas on the proper discipline of analysis. In 1812, he identified limiting processes as the "true soil on which the transcendental functions are generated."[95] This marks a cautious distance from the tradition of algebraic analysis, and Gauss would implicitly confirm this distance later in his

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step in this direction; see for instance [Sommer 1907], p. i: *Seitdem Gauß die Arithmetik durch Aufnahme der komplexen Zahlen $a + b\sqrt{-1}$ erweitert hat, ist eine großartige Theorie der allgemeinen algebraischen Zahlen entstanden*. For a recent example of the same perspective, see the beginning of [Neukirch 1999]. Cf. chap. I.2 below.

92. For detailed overviews, we refer to [Smith 1859–1865], §§28–38, and [Lemmermeyer 2000], chaps. 6–8. See also §3 of C. Houzel's chap. IV.2 below. We will briefly discuss Eisenstein below, who gave altogether five proofs of biquadratic reciprocity.

93. [Jacobi 1836–1837], 35th course, p. 221: *Gauß preist diese Theoreme sehr an, indem sie ihn besonders beschäftigten, u. sie sind in der That von der größten Wichtigkeit.* Handwritten copies of these lectures circulated in Germany; see [Jacobi 1881–1891], vol. 6, 2nd footnote on pp. 261–262. Henry Smith, however, had apparently no access to them when he wrote his report in the 1860s; see [Smith 1859–1865], p. 78.

94. [Jacobi 1836–1837], 1st course, p. 5: *Die Zahlentheorie auf ihrem jetzigen Standpunkte zerfällt in zwei große Kapitel, von denen das eine als die Theorie der Auflösung der reinen Gleichungen, das andere als die Theorie der quadratischen Formen bezeichnet werden kann. Ich werde hier hauptsächlich von dem ersten Theile handeln, dessen Erfindung wir Gauß verdanken.*

95. [Gauss 1866], p. 198: ... *überhaupt die Annäherung an eine Grenze durch Operationen,*

positive reaction to Enno H. Dirksen's voluminous *Organon* [Dirksen 1845], a book which, while obviously rooted in this tradition, stresses "transcendental determinations" (*transzendente Bestimmungsformen*), phenomena lying outside the range of algebra:[96]

> I procrastinated from one week to the next, and it is only now that I have found the time to familiarize myself with the tendency of your work. It is of the kind that I have always held in high esteem. Already very early, a good deal more than 50 years ago, I considered everything I found in books on infinite series very unsatisfactory and abhorrent to the true mathematical spirit, and I recall that I made an attempt, in 1793 or 1794, to develop the basic concepts in a more satisfactory way which, as far as I can remember, was … quite similar to yours.[97]

Infinite series and limits are central to the second aspect of arithmetic algebraic analysis at the time: the introduction of functions of a real variable and Fourier analysis as tools for higher arithmetic. The main actor here was Peter Gustav Lejeune-Dirichlet who, around 1821, left Germany for Paris to study higher mathematics with the *Disquisitiones Arithmeticae* under his arm. The topics and the spirit of Dirichlet's first papers – starting with the $n = 5$ case of Fermat's Last Theorem,[98] and divisors of forms, then calculus and Fourier series – show the strong influence of his Paris mathematical environment. But even there, the traces of his involvement with the D.A. are already obvious from the frequent explicit references to specific articles of it, very similar to the way early-modern authors referred to Euclid's *Elements*. He also reacted very quickly to Gauss's publications on biquadratic reciprocity, studying divisors of quartic forms in his 1828 dissertation *ad veniam docendi*, and completing Gauss's statements on *quadratic* reciprocity for Gaussian integers in 1832.

Back in Berlin from 1829 (professor at Berlin University from 1831), Dirichlet devoted numerous articles to arithmetical questions, typically beginning with some reference to Gauss. His German career indeed demonstrates the new possibilities given specifically to number theorists in Prussia: Gauss recommended him for this

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die nach bestimmten Gesetzen ohne Ende fortgesetzt werden – dies ist der eigentliche Boden, auf welchem die transcendenten Functionen erzeugt werden.

96. [Jahnke 1990], p. 413, with reference to [Dirksen 1845], p. 44.

97. See the letter of Gauss to Dirksen of November 5, 1845 in [Folkerts 1983–1984], p. 73: *habe ich von einer Woche zur anderen procrastinirt, und erst jetzt habe ich dazu kommen können, mich etwas näher mit der Tendenz Ihres Werkes bekannt zu machen. Es ist eine solche die mir von jeher sehr werth gewesen ist. Schon sehr früh, das ist vor weit mehr als 50 Jahren, war mir alles was ich über unendliche Reihen in Büchern fand sehr unbefriedigend, und vom ächten mathematischen Geiste abhorrirend, und ich erinnere mich, daß ich etwa im Jahre 1793 oder 1794 einen Versuch anfing die Grundbegriffe auf eine genügendere Art zu entwickeln, die so weit mein Gedächtniß reicht mit Ihrem Wege … viel Ähnlichkeit hatte.*

98. Fermat's Last Theorem had been proposed as a prize subject by the Paris Academy for the year 1818. As is well-known, Gauss placed the general, theoretical development of higher arithmetic above this individual result. To the news about the prize communicated to him by Olbers, he replied on March 21, 1816 that this isolated statement had little interest for him; see [Gauss & Olbers 1900/1976], part 1, p. 629.

reason.[99] On July 37, 1837, Dirichlet announced at the Academy of Berlin a proof of the statement that "every infinite arithmetic progression whose first term and difference have no common divisor contains infinitely many primes." This fact had been noticed and studied by Legendre – whose arguments were criticized in one of the appendices of the D.A. – but Dirichlet commented:

> It was only after I left completely the path taken by Legendre that I hit upon a totally rigorous proof of the theorem on arithmetic progressions. The proof that I found … is not purely arithmetical but relies in part on the consideration of continuously varying quantities.[100]

Despite the difference in topics, we would like to underline the striking parallel with Gauss's 1831 description of his procedure for biquadratic reciprocity; Gauss also stressed how he was led outside the traditional framework of number theory in order to obtain satisfactory proofs.

To complete his proof on the distribution of primes, Dirichlet needed to establish another result which was interesting for its own sake and which illustrates well the mixture of arithmetic, algebra and analysis at work in this research area: he gave a formula to compute *a priori* the number of classes of quadratic forms of a given determinant. More specifically, for a prime q of the form $4n + 3$, say, he showed first, making Legendre's symbol explicit, that

$$\sum \frac{1}{n^s} \cdot \sum \left(\frac{n}{q}\right) \frac{1}{n^s} \cdot \left(\sum \frac{1}{n^{2s}}\right)^{-1} = \sum \frac{2^\mu}{m^s},$$

where m varies over the odd positive numbers having only quadratic residues of q as prime factors, and μ is the number of distinct prime divisors of the corresponding m. Then he identified the right-hand sum as

$$\sum \frac{1}{(ax^2 + 2bxy + cy^2)^s} + \sum \frac{1}{(a'x^2 + 2b'xy + c'y^2)^s} + \dots,$$

where the quadratic forms in the denominators vary over a complete system of representatives, up to proper equivalence, of forms of determinant $-q$ and where the sums are taken over all integers x and y which make the value of the considered form odd and prime to q. Using results both of the D.A.[101] and of Joseph Fourier's *Théorie de la chaleur* to evaluate the sums for s close to 1, Dirichlet obtained the number h of classes of quadratic forms of discriminant $-q$ to be

$$h = 2\left[1 - \left(\frac{2}{q}\right)\frac{1}{2}\right] \frac{\sum b - \sum a}{q},$$

99. See H. Pieper's chap. III.1 below. Note also that Dirichlet published a large number of his papers in Crelle's *Journal*.

100. [Dirichlet 1889–1897], vol. 1, p. 316: *Erst nachdem ich den von Legendre eingeschlagenen Weg ganz verlassen hatte, bin ich auf einen völlig strengen Beweis des Theorems über die arithmetische Progression gekommen. Der von mir gefundene Beweis … ist nicht rein arithmetisch, sondern beruht zum Teil auf der Betrachtung stetig veränderlicher Grössen.*

101. The paper "Sur l'usage des séries infinies dans la théorie des nombres," [Dirichlet 1889–1897], vol. 1, pp. 357–374, for instance contains 23 references to articles of the D.A.

with a and b varying respectively over the quadratic residues and non-residues of q which are smaller than q. For instance, Gauss's relation for $q \equiv 3 \bmod 4$, proved up to the sign in art. 356 of the D.A., and including the precise sign in [Gauss 1863/1876], pp. 9–45, 155–158,

$$\sum \sin \frac{2\pi an}{q} - \sum \sin \frac{2\pi bn}{q} = \left(\frac{n}{q}\right)\sqrt{q}$$

reduces the evaluation of $\sum \left(\frac{n}{q}\right)\frac{1}{n}$ to that of classical Fourier series of the type $\sum \frac{\sin nz}{n}$, for $0 < z < 2\pi$.

In a letter to Gauss of September 9, 1838, Dirichlet commented:

> I would almost like to conjecture that my method bears some analogy with the investigations alluded to in the final remark of the *disq. arith.*, in particular because you say about these investigations that they throw light on several parts of analysis, and my method is so intimately connected with the remarkable trigonometric series that represent discontinuous functions and whose nature was still completely mysterious at the time of the publication of the *disq. arith.*[102]

Dirichlet favoured this kind of *rapprochement* between the different branches of mathematics:

> The method I use seems to merit some attention above all because of the link it establishes between infinitesimal analysis and higher arithmetic.[103]

For a positive determinant, the formula for the number of classes contains a regulator, involving logarithms and a solution of the Pell equation. He wrote:

> [The expression of the law] is of a more composite nature, and somehow mixed, because, besides the arithmetic elements on which it depends, it contains others which have their origin in certain auxiliary equations appearing in the theory of binomial equations, and therefore belonging to Algebra. The last result is particularly remarkable and offers a new example of these hidden relations that a deep study of

102. [Dirichlet 1889–1897], vol. 2, p. 382: *Ich möchte fast vermuthen, dass meine Methode mit den in der Schlussbemerkung der* disq. arith. *angedeuteten Untersuchungen einige Analogie hat, besonders deshalb, weil Sie von Ihren Untersuchungen sagen, dass sie über mehrere Theile der Analysis Licht verbreiten, und meine Methode in so innigem Zusammenhange mit den merkwürdigen trigonometrischen Reihen steht, welche discontinuierliche Funktionen darstellen, und deren Natur zur Zeit des Erscheinens der* disq. arith. *noch ganz unaufgeklärt war.* Later, Dirichlet would find alternative analytic means to handle the question.

103. [Dirichlet 1889–1897], vol. 1, p. 360: *La méthode que j'emploie me paraît surtout mériter quelque attention par la liaison qu'elle établit entre l'Analyse infinitésimale et l'arithmétique transcendante.* To the French readers to whom this particular paper is addressed, Dirichlet added his hope of attracting in this way the attention of mathematicians who were not *a priori* interested in number theory.

mathematical analysis allows us to discover between what appears to be completely disparate questions.[104]

He expressed such priorities not only in the production of new results, but also in his simplifications of Gauss's proofs, through continued rereading of the D.A., esp. in the 1840s: for instance, he would comment on his simplification of the theory of binary quadratic forms with positive determinant by saying that "the characteristic feature of this method is that it brings irrational numbers into the circle of our ideas."[105] The intervention of analysis reveals links and, paradoxically enough for an advanced subject, it simplifies and democratizes the *Disquisitiones Arithmeticae*:

> My work may also contribute to the advancement of science in establishing on new grounds and closer to the elements beautiful and important theories which until now had been accessible only to the small number of geometers who were capable of the concentration needed in order not to lose the thread of thought in a long series of computations and of very complicated reasonings.[106]

3.4. Elliptic Functions

We now turn to point (3) mentioned at the beginning of § 3.2: the theory of elliptic functions. At the beginning of sec. 7 of the D.A., Gauss put this section, and indeed higher arithmetic as a whole, in a much wider perspective by mentioning "many other transcendental functions" besides the circular functions, to which the methods and results of sec. 7 could be extended. But he gave only one example of such functions: "those which depend on the integral $\int dx/\sqrt{1-x^4}$,"[107] and never published the

104. [Dirichlet 1889–1897], vol. 1, p. 536: *[L'expression de la loi] est d'une nature plus composée et en quelque sorte mixte, puisque, outre les éléments arithmétiques dont elle dépend, elle en renferme d'autres qui ont leur origine dans certaines équations auxiliaires qui se présentent dans la théorie des équations binômes, et appartiennent par conséquent à l'Algèbre. Ce dernier résultat est surtout remarquable et offre un nouvel exemple de ces rapports cachés que l'étude approfondie de l'Analyse mathématique nous fait découvrir entre les questions en apparence les plus disparates.*

105. [Dirichlet 1863], § 72. We quote here J. Stillwell's English translation. Dirichlet associated to such a form $ax^2 + 2bxy + cy^2$ the roots of the equation $ax^2 + 2bx + c = 0$ and derived most of the facts about the reduction of the forms from these roots, and more specifically from their expansions as continued fractions.

106. [Dirichlet 1889–1897], vol. 1, p. 414: *Mon travail pourra encore contribuer à l'avancement de la science en établissant sur de nouvelles bases et en rapprochant des éléments, de belles et importantes théories qui n'ont été jusqu'à présent à la portée du petit nombre de géomètres capables de la contention d'esprit nécessaire pour ne pas perdre le fil des idées dans une longue suite de calculs et de raisonnements très composés.* The context is that of arts. 234 ff in the D.A.

107. D.A., art. 335: *Ceterum principia theoriae ... non solum ad functiones circulares, sed pari successu ad multas alias functiones transscendentes applicari possunt, e.g. ad eas quae ab integrali* $\int \frac{dx}{\sqrt{(1-x^4)}}$ *pendent.* In a letter to Schumacher (who would be the adressee of Jacobi's first notes on elliptic functions in 1827) dated September 17, 1808, Gauss called the functions which are not reducible to circular or logarithmic functions a "magnificent

"big treatise" (*amplum opus*) on these functions promised in 1801.[108] In a letter dated February 8, 1827, Jacobi tested Gauss on this announcement: "The application of higher arithmetic to the division of the elliptic transcendents is promised in the *Disquisitiones*; oh, may the promise be kept!"[109]

When Jacobi himself turned to elliptic functions,[110] he first studied transformations between various elliptic integrals, rather than the immediate analogue of sec. 7, i.e., the division of a single elliptic integral or function. But this division had then just been settled by Abel, who had also been inspired by Gauss's announcement in the D.A.[111] The story of Abel's and Jacobi's rival and parallel development of the theory of elliptic functions is well-known.[112] It was characterized by inverting elliptic integrals classified in [Legendre 1811] and [Legendre 1825–1828], such as $u = \int_0^{\varphi} \sqrt{(1-x^2)(1-\kappa^2x^2)}^{-1} dx$; by recognizing the double periodicity of the resulting complex functions $u \mapsto \varphi(u, \kappa)$; and by studying certain algebraic equations arising from the theory such as the *division equation* and the *modular equation*. The division equation of order $m > 1$ for a given elliptic function φ is that whose roots are the m^2 values $\varphi(\beta)$ such that $\varphi(m\beta) = \varphi(\alpha)$, for a given α; generalizing Gauss's sec. 7 from a cyclic to a bicyclic situation, Abel showed that it could be solved by radicals of rational expressions in $\varphi(\alpha)$. The *modular equation* is the one linking the $\sigma(n)$ possible moduli λ's to a given modulus κ, for which there exist transformations $y = U(x)V(x)^{-1}$, with relatively prime polynomials U, V, where U has odd degree

golden treasure" (*herrliche Goldgrube*), and gave as specific examples those relating to the rectification of the ellipse and the hyperbola; [Gauss & Schumacher 1860], vol. 1, n$^{\text{o}}$ 2, p. 3.

108. See [Schlesinger 1922], secs. III–VI, for the most complete published survey of Gauss's unpublished papers on elliptic functions. See also C. Houzel's chap. IV.2 below as well as [Houzel 1978]. Moreover, Jacobi would recognize formulae related to the theory of elliptic functions in Gauss's article establishing the sign of Gauss sums; see [Jacobi 1836–1837], 35$^{\text{th}}$ course, p. 221: *die Formen enthält, welche auch in der Theorie der elliptischen Funktionen vorkommen.*

109. [Jacobi 1881–1891], vol. 7, p. 400: *Die Anwendung der höheren Arithmetik auf die Theilung der elliptischen Transzendenten ist in den Disquisitiones versprochen; o würde doch das Versprechen erfüllt!* Gauss's reply to this, if any, is not known. At the time of the letter, Abelian functions were not yet public knowledge.

110. See his letter to Schumacher of June 13, 1827 in [Jacobi 1881–1891], vol. 1, pp. 31–48, and his subsequent publications as well as the letters to Legendre in [Jacobi 1881–1891], vol. 1, pp. 185–461.

111. [Schlesinger 1922], p. 183–184. See also [Abel 1881], vol. 1, pp. 263–388, no. 21: *Le procédé par lequel nous allons effectuer cette résolution est entièrement semblable à celui qui est dû à M.* Gauss *pour la résolution de l'équation* $\theta^{2n+1} - 1 = 0$.

112. See [Abel 1902], [Königsberger 1904], as well as the sources in [Abel 1881], [Jacobi 1881–1891], vols. 1 and 7. Dirichlet's obituary of Jacobi, in particular [Jacobi 1881–1891], vol. 1, pp. 7–18, offers a remarkably well written informal account. See also [Houzel 1978] and Houzel's chap. IV.2 below.

n and V degree $n-1$, such that $\dfrac{dy}{\sqrt{(1-x^2)(1-\lambda^2x^2)}} = \dfrac{dx}{M \cdot \sqrt{(1-x^2)(1-\kappa^2x^2)}}$ for an appropriate number M.[113] Linking M and κ gives rise to yet another algebraic equation of the same degree, Jacobi's *multiplier equation*.[114]

Let us explain why we see these developments as characteristic of a new domain of research, for which we have coined the name *arithmetic algebraic analysis*.

First, just as Jean-Baptiste le Rond d'Alembert, in his article on Diophantus in the *Encyclopédie* in 1784, found it natural to underline how useful Diophantus's method of solving his number problems was for the transformation of integrals, the new complex analytic theory of elliptic functions was at first derived directly from arithmetico-algebraic properties of the integrals at hand (with additional inspiration provided by Euler's treatment of the trigonometric and exponential functions), but *not* as the theory of a special type of complex functions within an existing general function theory. The double periodicity of the inverse functions, for instance, was deduced by Abel by defining very meticulously his function $\varphi(\alpha)$, first on a real interval, then on a purely imaginary one via the substitution $\alpha \mapsto \alpha i$, and finally on all complex numbers via the addition theorem; see [Abel 1881], vol. 1, pp. 263–388, secs. 1–5. We note that the same is true for the infinite series introduced by Dirichlet, of the general type $\sum \frac{a_n}{n^s}$, whose construction at first directly reflected arithmetical and algebraic properties.

Second, just like sec. 7 of the D.A., the new analysis of elliptic functions could be claimed by the theory of algebraic equations as well as by arithmetic.[115] Thus, the cyclotomic equations, the division equations of elliptic functions, and the modular equation all functioned as crucial model cases which oriented the general theory. Now, in all these cases, the roots come indexed in a way which permits linear operations on them by the integers taken with respect to a suitable modulus N. Évariste Galois is often described as having created a general, abstract theory which he then also applied to special classes of equations.[116] But we think that theory and examples were much more closely linked at the time, and that the examples we mentioned informed Galois about what he had to formulate in the general theory.

113. See the first half of [Jacobi 1829]. Here, $\sigma(n)$ denotes the sum of all the divisors of n.

114. [Jacobi 1881–1891], vol. 1, p. 261.

115. Concerning relations between the theory of elliptic functions and geometry, Abel's construction of the division of the lemniscate is geometric in precisely the same sense as is Gauss's result of sec. 7. Jacobi's paper, in 1828, concerning Poncelet's closure theorem could be interpreted as a link with more recent geometrical works, see [Jacobi 1881–1891], vol. 1, pp. 277–293 and [Bos, Kers, Oort, Raven 1987]. But most of the time the geometry aimed at within the field during this period was more elementary, as in Kummer's paper on quadrilaterals with rational sides and diagonals, where the geometrical setting is only a gloss on the key issue, the link between elliptic functions and traditional Diophantine analysis; see [Kummer 1975], vol. I, pp. 253–273.

116. For instance, [Kiernan 1971], p. 89, writes about Galois's *second mémoire* ([Galois 1962], pp. 129–147): "This paper was, in Galois' mind, not a development of his theory, but rather an application of it to a particular class of equations."

Sec. 7 of the D.A., or "*la méthode de M. Gauss*," as Galois says, and its elliptic analogues, are recurring signposts in Galois's writings. When he introduced his "number-theoretic imaginaries,"[117] i.e., the solutions of $x^{p^\nu} \equiv x \pmod p$, his chief application was to "the theory of permutations, where one constantly has to vary the form of the indices," and he showed how this indexing of the roots of an equation of prime power degree allowed one to recognize its solvability by radicals.[118] His mathematical testament, written to Auguste Chevalier on May 29, 1832, starts very plainly: "I have done several new things in analysis. Some concern the theory of equations, the others the functions given by integrals."[119] But it exactly delineates our domain: the last item in the first group of results was the determination of the groups of modular equations.

Let us note in passing that Abel's general approach to algebraic equations – contrary to Galois's – aimed at making explicit "the most general form of the solutions" to equations of a given type. This appears not to be linked to the D.A., but rather to the tradition of algebraic analysis. Among Abel's followers on this point, however, we find Jacobi and Kronecker who in their work would link this approach to arithmetic inspirations from the D.A.[120]

In the further development of Galois theory, the parallel between the general theory and the special examples continued to be evident for a while. Enrico Betti, for example, is famous for the first systematic account of Galois's theory in 1852 which established the model of organizing the material, with an abstract part on substitutions preceding the application to algebraic equations, see [Betti 1903], pp. 31–80. But he followed this up by a paper focused on the division and modular equations of

117. The expression *les imaginaires de la théorie des nombres* is from Galois's letter to Chevalier; see [Galois 1962], p. 175. Note that Galois adopted in 1830 the approach to higher congruences that the young Gauss had systematically avoided in his *caput octavum* back in 1797; see § 4.2 of G. Frei's chap. II.4 below, with reference to Gauss's § 338.

118. [Galois 1962], p. 125: *C'est surtout dans la théorie des permutations, où l'on a sans cesse besoin de varier la forme des indices, que la considération des racines imaginaires des congruences paraît indispensable. Elle donne un moyen simple et facile de reconnaître dans quel cas une équation primitive est soluble par radicaux.* In today's parlance, if one interprets the symmetric group S on p^ν letters as the group of permutations of the field $\mathbf{F}_{\mathbf{p}^\nu}$, a subgroup of S is solvable if and only if it is affine, i.e., if all its permutations are of the form $x \mapsto ax+b$. We recall incidentally that Poinsot – who wanted to treat in parallel the study of congruences and of equations – already advocated in his 1808 analysis of the 2nd edition of Lagrange's *Traité des équations* keeping to a theoretical description of the operations involved, in the presence of overwhelming numerical complexity: *Par la nature du problème, la longueur des calculs croit avec une telle rapidité que la question ne peut plus être aujourd'hui de chercher la formule, mais simplement de prédire la suite des opérations qui y conduirait à coup sûr.*

119. [Galois 1962], p. 173: *J'ai fait en analyse plusieurs choses nouvelles. Les unes concernent la théorie des équations ; les autres, les fonctions intégrales.* On relating Galois to the D.A., see also O. Neumann's chap. II.1 below.

120. For Abel and Kronecker, see [Petri, Schappacher 2004], §§ 1.4, 1.5, 2.1. Jacobi presented the "true form of the surds of the equation $x^p = 1$ which has never been given before" in his applications of cyclotomy to number theory, [Jacobi 1881–1891], vol. 6, pp. 254–274.

elliptic functions where he proved Galois's claim concerning the modular equation, see [Betti 1903], pp. 81–95. When he subsequently came back to the general subject of his 1852 article, it was to show (among other things) how to derive the substance of Gauss's method of sec. 7 "directly from the decomposition of the corresponding substitutions."[121] Again with reference to the modular equation, Kronecker noted in 1862 the

> altogether singular circumstance … that the progress of algebra and of its methods depends crucially on the material of equations which is delivered from outside, or if I may say so, on the variety of algebraic phenomena provided by the further development of analysis.[122]

An even later echo of the original unity may be seen in the strong presence of linear substitutions, and the treatment of the modular equation, in [Jordan 1870].

Another telling illustration of the influence of elliptic functions on the theory of algebraic equations is provided by Gotthold Eisenstein's well-known irreducibility criterion, which he introduced to show the irreducibility of the division equations of the lemniscatic elliptic function, and which he also used to give another proof of the irreducibility of $X^{p-1} + \cdots + X + 1$ first shown in D.A., art. 342.[123]

This being said, explicit references to the D.A. in the context of algebraic equations tended to evaporate after 1850 outside Germany. For instance, Joseph-Alfred Serret, in the first edition (1849) of his influential *Cours d'algèbre supérieure*, based on his lectures at the Sorbonne and devoted to the algebraic resolution of equations and "incident questions related to it," discussed congruences in the 23$^{\text{th}}$ and 24$^{\text{th}}$ lessons, cyclotomy in the 26$^{\text{th}}$ and 27$^{\text{th}}$. The 25$^{\text{th}}$ presents "curious and useful theorems" derived from the principles explained before, for instance the decomposition of an integer as a sum of four squares; the name of Gauss is attached here to the notation of congruence and to the solvability of the cyclotomic equations, and the

121. [Betti 1903], p. 123: *La decomposizione dei gruppi complessi di grado non primo forma la sostanza del metodo di Gauss per le equazioni binomie ; io ho mostrato come essa deriva direttamente dalla decomposizione delle sostituzioni che loro appartengono…*

122. [Kronecker 1895–1930], vol. 4, p. 213: *Es liegt dieß an einem ganz eigenthümlichen Umstande, welcher bei den in Rede stehenden Gleichungen auftritt und welcher wiederum zeigt, daß … der Fortschritt der Algebra und ihrer Methoden wesentlich durch das ihr von Außen herzugebrachte Material an Gleichungen bedingt ist oder, wenn ich mich so ausdrücken darf, durch die Mannigfaltigkeit algebraischer Phänomene, welche die Analysis in ihrer weiteren Entwickelung darbietet.*

123. See his letter to C.F. Gauss, August 18, 1847, [Eisenstein 1975], vol. 2, pp. 845–855, and [Eisenstein 1975], vol. 2, pp. 542–544. The footnote in [Lemmermeyer 2000], p. 254, led us to Theodor Schönemann's priority claim in [Schönemann 1850], and to [Schönemann 1846], p. 100, where a marginally more general criterion is derived in the context of higher congruences modulo p^2. This independent development of Gauss's unfulfilled promise of a sec. 8 on higher congruences, started in [Schönemann 1845], was continued in particular by Richard Dedekind and would finally merge, at the very end of the XIX$^{\text{th}}$ century, with the line of thought initiated by Galois's imaginaries and expanded in [Serret 1849/1854], pp. 343-370, into a theory of finite fields; see G. Frei's chap. II.4.

D.A. is proposed more specifically next to Legendre's *Théorie des nombres* as a general reference for the 25th lesson. Five years later in the second edition, this section has totally changed, and the arithmetical theorems are replaced by Galois's theory of imaginaries;[124] a supplementary note about quadratic reciprocity mentions only Legendre as its discoverer and then Jacobi as the author of the specific proof given in the book; see [Serret 1849/1854], pp. 533–537.

The conjunction of the theory of elliptic functions with arithmetic is not restricted to the D.A. Diophantine analysis is touched upon in Kummer's paper on quadrilaterals with rational sides and diagonals mentioned above (footnote 115) and is briefly promoted in very general terms by Jacobi in 1835; see [Jacobi 1881–1891], vol. 2, pp. 51–55. Jacobi was in fact actively promoting the introduction of elliptic functions "into all parts of mathematical analysis,"[125] blurring at the same time the direct connection with the *Disquisitiones Arithmeticae*. As far as it continued the D.A., however, the impact of elliptic functions went deeper and revolved around three interrelated topics: reciprocity laws, class numbers, and complex multiplication – we shall meet them again in the remainder of this chapter. These investigations, combined with the use of complex numbers and Dirichlet's analytical methods, constituted a solid ground for expanding Gauss's ideas, knit together by multiple links into what appeared to be a unified enterprise, though disruptive factors were also present.

In the summer of 1848, for his second term of teaching at Berlin University, Eisenstein offered two lecture courses which, taken together, would have constituted an introduction to arithmetic algebraic analysis: one on "the integral calculus as source of transcendental functions," and another one "explaining Gauss's *Disquisitiones Arithmeticae*, with special investigations on the divisions of the circle."[126]

We have already alluded in § 3.1 to Jacobi's proof of the biquadratic reciprocity law from his Königsberg lectures. One of Eisenstein's proofs is sketched in sec. 3

124. The Gaussian roots of Galois's work are not mentioned in Serret's lectures. In the subsequent restructured editions of this mathematical bestseller, after 1866, properties of integers that are "necessary for the theory of the algebraic resolution of equations," in particular a study of congruences and higher congruences, are the topic of a third section (with several chapters); Gauss's name, like many others, is episodically mentioned on specific points with no particular emphasis.

125. See his little 1831 paper presenting an application to continued fractions, [Jacobi 1881–1891], vol. 1, pp. 329–331, in particular p. 329: *j'ai avancé que les fonctions elliptiques doivent entrer dans toutes les parties de l'analyse mathématique et contribuer essentiellement à leur progrès.*

126. [Eisenstein 1975], vol. 2, p. 902: *Integralrechnung als Quelle der transcendenten Functionen; Erläuterung der Disquisitiones Arithmeticae von Gauss mit speciellen Untersuchungen über die Kreisteilungen.* The second did not find enough interested students, though, so Eisenstein taught instead a class on the *einfachsten Principien der Mechanik.* For a useful quick survey of Eisenstein's life and work, which mentions in particular a few important papers – for instance, on cubic forms, and on ternary quadratic forms – that we do not go into here in spite of their direct connection with the D.A., see Weil's review of Eisenstein's *Mathematische Werke*; e.g., in [Weil 1979], vol. 3, pp. 398–402.

of C. Houzel's chap. IV.2 below. All five of his proofs can pass for showcase illustrations of arithmetic algebraic analysis. But some also contain special aspects which foreshadow the end of this amalgamated research field. For example, in his paper entitled "Applications of Algebra to Higher Arithmetic," after having neatly derived the classical quadratic reciprocity from the polynomial expression of sin pv in the variable $x = \sin v$, and the biquadratic law from an elliptic analogue, Eisenstein remarked:

> Maybe some readers do not approve of using circular or elliptic functions in arithmetic arguments; but one has to observe that these functions enter here only *symbolically*, so to speak, and that it would be possible to eliminate them altogether without changing the substance and the basis of the proof.[127]

He went on to replace, for instance, the division values of the sine function by any suitable geometric division of an arbitrary closed curve only assumed to be symmetric with respect to both coordinate axes.[128]

If arithmetic was allowed to distance itself from analysis here, Eisenstein later rewrote this proof within a long text where analysis clearly takes the lead, for he introduced there the circular and elliptic functions via his own original method of series summation.[129] Eisenstein was not the only author searching for a new analytical foundation of the theory initiated by Abel and Jacobi. The 1830s and 1840s witness a growing interest in elliptic functions, with publications by Joseph Liouville, Alfred Serret, William Roberts, Arthur Cayley, Joseph Raabe, Ludwig Schläfli, and Königsberg colleagues and students of Jacobi, like Friedrich Richelot, Ludwig Adolf Sohncke, and Christoph Gudermann (whose 1839–1840 course on elliptic functions was followed by the young Karl Weierstrass). Many of these authors belong to the tradition of arithmetic algebraic analysis (Richelot's doctoral dissertation was about $x^{257} - 1 = 0$), but they also began to propose alternative constructions of the functions on a purely analytic basis. From the beginning of the 1830s on, Jacobi would use the theory of theta functions as a foundation for the whole theory of elliptic functions in his lectures.[130] In long series of papers in Crelle's *Journal* starting in the late 1830s, Gudermann tried to build a systematic theory of elliptic functions on the

127. [Eisenstein 1975], vol. 1, p. 297: *Il se pourrait qu'on n'approuvât pas l'usage des fonctions circulaires et elliptiques dans les raisonnements arithmétiques ; mais il y a à observer que ces fonctions n'y entrent que d'une manière pour ainsi dire* symbolique*, et qu'il serait possible de les en chasser complètement sans détruire la substance et le fond des démonstrations.*

128. In his 1852 *Théorie des nombres*, focused on Diophantine problems, Eugène Desmarest complained that Gauss had mixed up arithmetic with analysis. We do not know whether Eisenstein reacted to somebody's opinion with his remark, or if it was just a truly Gaussian reflection on the specific merits of various proofs. Smith pointed out that Eisenstein also gave his first proof of the biquadratic law (via cyclotomy, like Jacobi's) "a purely arithmetical form" when he presented it a second time; see [Smith 1859–1865], p. 81, with reference to [Eisenstein 1975], vol. 1, pp. 141–163.

129. [Eisenstein 1975], vol. 1, pp. 357–478. Cf. [Weil 1976].

130. See his *Theorie der elliptischen Funktionen aus den Eigenschaften der Thetareihen abgeleitet*, published from notes taken by C.W. Borchardt in [Jacobi 1881–1891], vol. 1,

basis of expansions in infinite series. In France from 1847, Liouville and Hermite started to work out a general theory of doubly periodic complex functions.[131]

In the same way, Dirichlet later gave alternative proofs of some of his results, using simple calculus, see [Dirichlet 1863], § 103. While his formulas involving Gauss sums had at first tightly linked Fourier series, residues and classes of forms, these new proofs, although technically more direct, loosened the ties between analytical and arithmetical aspects. After the middle of the century, this emancipation of analytical techniques from their algebraic roots, as much as the wish to eliminate analysis from number-theoretical proofs, would contribute to tearing arithmetic algebraic analysis apart.

4. … And Pastures New

While keeping close to their connections with the *Disquisitiones Arithmeticae*, the investigations we described have the potential to launch autonomous lines of development. This phenomenon may also operate at a finer scale than those met in the preceding section. We will demonstrate it for a single, but remarkable, example of research which, while building directly on the D.A. and on developments discussed in the last section, ushered in several new lines of thought independent of Gauss's known and unknown work.

The mathematicians coming together here were Ernst Eduard Kummer (born in 1810), and his younger colleagues Charles Hermite, Gotthold Eisenstein, and Leopold Kronecker (all three born between December 1822 and December 1823). They all had studied the *Disquisitiones Arithmeticae* early on[132] and knew the works of the preceding generation we have just discussed. Complex numbers, elliptic functions (Kummer alone would hardly ever use them in his own work), and Dirichlet's series were part of their resources, and served in turn as filters for their reading of the D.A. and of Gauss's later work. Our point is that this wealth of resources would lead to differing uses of the D.A., and in due course to diverging developments, even though, for a few decades, these perspectives would still be seen as complementary to rather than exclusive of one another, and the main actors would continue to see themselves as taking part in the construction of a vast field of research combining arithmetic, algebra, and function theory.

The impulse for all the far-reaching and diverging developments we are about to sketch in this section was given by a programmatic 5-page note written by Jacobi, [Jacobi 1839]. Not much is proved there; results obtained are alluded to, as was still allowed in those days, and the paper is really about how to look at things,

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pp. 497–538, and Weierstrass's editorial comment thereon in [Jacobi 1881–1891], vol. 1, p. 545. In his *Fundamenta nova*, on the contrary, Jacobi had deduced his theory from the properties of elliptic functions, which were defined by inverting elliptic integrals.

131. See [Belhoste 1996].

132. Kronecker studied the *Disquisitiones Arithmeticae* under Kummer's guidance; see [Kronecker 1891/2001], p. 219. Eisenstein bought the French translation in 1842 and read it while travelling with his family in the British Isles; see [Ullrich 2001], p. 205. As for Hermite, see chap. VI.1 below.

more specifically at "complex prime numbers." It starts by recalling Gauss's work on biquadratic reciprocity which called for generalizing the D.A. from rational to Gaussian integers, introducing "complex numbers of the form $a+b\sqrt{-1}$ as modules [of congruences] or divisors." In perfect coherence with the idea of arithmetic algebraic analysis, Jacobi speculated:

> I do not believe that arithmetic alone has led to such an arcane idea, but that it was drawn from the study of the elliptic transcendents, namely that special type which gives the rectification of the lemniscatic arc. For in the theory of multiplication and division of the lemniscatic arc, the complex numbers of the form $a + b\sqrt{-1}$ play precisely the role of ordinary integers.[133]

After an allusion to the analogous connection between cubic residues, complex integers $\frac{a+b\sqrt{-3}}{2}$ built from cubic roots of unity, and other elliptic functions, Jacobi turned to quadratic forms with Gaussian integer coefficients, thus resuming in a new way a theme alluded to before: the bringing together of what Jacobi saw as the two main parts of number theory, cyclotomy and the theory of forms.

He showed for instance that a Gaussian integer which divides the form $yy - \sqrt{-1}\,zz$ can be represented by it. Such is the case for any prime $p = aa+bb = 8n+1$ because, by Gauss's theory, $\sqrt{-1}$ is a quadratic residue of $a + b\sqrt{-1}$. From this it follows that $a+b\sqrt{-1} = \phi(\alpha)\phi(\alpha^5)$, where $\phi(\alpha)$ is a real linear combination of the powers of an 8^{th} root of unity α. Also $a - b\sqrt{-1} = \phi(\alpha^3)\phi(\alpha^7)$. This shows that any prime $p = 8n + 1$ is the product of four complex numbers built from 8^{th} roots of unity; the three ways in which one can order the four factors in two pairs give the three different representations of the prime p as $a^2 + b^2$, $c^2 + 2d^2$, $e^2 - 2f^2$, the main point being that all three now flow, as Jacobi put it, "from a common source." He then announced identical results for primes of the form $12n + 1$ with 12^{th} roots of unity. More generally, Jacobi had previously noticed that a prime $p = 1+\lambda n$ can usually be represented in different ways as a product of two complex numbers[134] and had manipulated products and quotients of such complex numbers with surprising effects. He commented, allowing us a glimpse of the state of the art concerning "complex prime numbers" a few years before Kummer's work:

> A closer consideration ... convinced me that the complex factors of the prime number p are in general themselves composite, so that if one decomposes them into true complex prime numbers, the factors that make up the denominator will cancel one

133. [Jacobi 1839], p. 275: *Ja ich glaube nicht, dass zu einem so verborgenen Gedanken die Arithmetik allein geführt hat, sondern dass er aus dem Studium der elliptischen Transcendenten geschöpft worden ist, und zwar der besonderen Gattung derselben, welche die Rectification von Bogen der Lemniscata giebt. In der Theorie der Vervielfachung und Theilung von Bogen der Lemniscata spielen nämlich die complexen Zahlen von der Form* $a+b\sqrt{-1}$ *genau die Rolle gewöhnlicher Zahlen.* (That Gauss was indeed aware of some such connection is supported by the last entry (July 9, 1814) of his mathematical diary [Gauss 1796–1814], a document discovered by Paul Stäckel only in 1898).

134. For instance, $p = 8n + 1$ can be written as the product $(a + b\sqrt{-1})(a - b\sqrt{-1})$, or as $(c + \sqrt{-2}d)(c - \sqrt{-2}d)$ or as $(e + \sqrt{2}f)(e - \sqrt{2}f)$.

by one against the prime factors of the numerator.[135]

Jacobi thought that these "true complex prime numbers" might be precisely what he had just obtained for $p = 8n + 1$ or $p = 12n + 1$; he ended his paper by announcing similar results for primes $p = 5n + 1$ (relative to the 5th roots of unity), and hoped for a proof of higher reciprocity laws.[136]

These still mysterious and unproved statements clearly challenged younger mathematicians: we shall examine how four of them, equipped in particular with their different experience of the D.A., rose to the challenge.

4.1. Hermite's Minima of Forms

A French translation of Jacobi's article appeared in 1843, and at this point Hermite entered the scene. After a previous exchange with Jacobi about Abelian functions, Hermite wrote him in 1847 a letter proposing a proof of the decomposition of a prime $5m + 1$ (resp. $7m + 1$) into complex factors built from the 5th (resp. 7th) roots of unity. Hermite took up Jacobi's allusion to elliptic transcendents rather than the link with reciprocity laws; he even mentioned as his immediate starting point Jacobi's theorem that there is no complex analytic function with three independent periods.

The decompositions of prime numbers were deduced from Hermite's celebrated theorem on the minima of quadratic forms which he had proved by closely following Gauss's discussion of ternary forms in the D.A.[137] For a prime $p = 5N + 1$, for instance, Hermite considered linear forms

$$\varphi(\xi) = Nx_0 + (\xi - a)x_1 + (\xi^2 - a^2)x_2 + (\xi^3 - a^3)x_3 + (\xi^4 - a^4)x_4,$$

where ξ is a primitive 5th root of unity and a an integer different from 1, which verifies the congruence $a^5 \equiv 1 \bmod N$. For all integral values of the indeterminates x_i, the product $\mathcal{F} = \varphi(\xi)\varphi(\xi^2)\varphi(\xi^3)\varphi(\xi^4)$ is an integer multiple of N, say MN. Hermite then cleverly associated a quadratic form with real coefficients to this product of

135. [Jacobi 1839], p. 279: *Eine genaue Betrachtung … führte mich zu der Ueberzeugung, dass diese complexen Factoren der Primzahl p im Allgemeinen selbst wieder zusammengesetzt sein müssen, so dass, wenn man sie in die wahren complexen Primzahlen auflöst, die complexen Primzahlen, welche die Factoren des Nenners bilden, gegen die Primfactoren des Zählers sich einzeln aufheben lassen.*

136. Cauchy also established a few of these results, and announced the possibility of proving higher reciprocity laws, in a memoir presented to the Paris Academy on May 31, 1830. Its publication was, however, delayed for ten years as Cauchy left France after the 1830 Revolution; see [Belhoste 1991], chaps. 9 and 10. A shorter version had already appeared in 1829. It ends with the remark that Jacobi told Cauchy he also had obtained the same results with basically the same approach. Via cyclotomy, Cauchy had also deduced results on the decomposition of primes in complex quadratic domains: "M. Jacobi's research on the quadratic forms of prime numbers," Cauchy wrote in [Cauchy 1840], "and one must say as much of mine, may be considered as offering new developments of M. Gauss's beautiful theory."

137. Hermite's response to Jacobi's 1839 note and his derivation of Jacobi's statement are studied in detail in C. Goldstein's chap. VI.1 below.

linear forms: his result on the minima of forms applied to this quadratic form shows that there exist integers $(x_0, \dots, x_4)$ such that the product $\mathcal{F}$ is strictly smaller than $2N$, and thus equals N. This provides the decomposition of N as a product of four complex numbers built from 5^{th} roots of unity.

Hermite's further programme was to study algebraic complex numbers in general and to classify them in the spirit of Lagrange's and Gauss's classification of binary quadratic forms. But pursuing his own direction, he wanted to base this study on that of n-ary quadratic forms, and keep close to elliptic functions. This would lead him and a number of his followers to enter more deeply into general invariant theory and to promote general complex functions as a leading topic.

4.2. Kummer's Ideal Numbers

Meanwhile, Kummer's reaction to Jacobi's statements was more directly linked to reciprocity laws. His first mathematical works of the 1830s had mainly dealt with differential equations and series.[138] He turned seriously to number-theoretical questions in the following decade, in connection with his appointment at the University of Breslau. His letter to his friend and former pupil Kronecker on January 16, 1842, again documents the role of number theory in the Berlin sphere of influence:

> Since I noticed during my last stay in Berlin that the Breslau affair could get serious, I sat down at home and worked very hard in order to elaborate something like a habilitation thesis, and I started with something completely new to me, the cubic residues of the prime numbers $6n + 1$.[139]

In this letter, Kummer mentioned only the D.A. and Dirichlet. A month later, however, he had begun reading Jacobi's work on cubic residues[140] and, following Jacobi even further, getting involved with complex numbers and, with less conviction, elliptic functions. Until the mid-1860s, number theory, and more specifically, the study of "numbers built from roots of unity," would be at the centre of his activities.

Kummer's focus in this early work was what we now call Gauss sums for cubic residues,

$$\sum_0^{p-1} \cos \frac{2\alpha k^3 \pi}{p}, \quad \sum_0^{p-1} \cos \frac{2\beta k^3 \pi}{p}, \quad \sum_0^{p-1} \cos \frac{2\gamma k^3 \pi}{p},$$

for p a prime of the form $3n + 1$, α denoting a cubic residue, β and γ representatives of the two kinds of cubic non-residues. These sums are the roots of the cubic equation $z^3 = 2pz + pt$, where t is a normalized solution of $4p = t^2 + 27u^2$. Kummer's

138. Including his well-known paper on the hypergeometric series. An exception was a small 1835 paper on Fermat's Last Theorem for even exponents.

139. [Kummer 1975], vol. 1, p. 46: *Seit ich bei meiner letzten Anwesenheit in Berlin merkte, es könne mit Breslau Ernst werden, so setzte ich mich zu Hause hin und arbeitete sehr fleißig um so etwas wie eine Dissertation zur Habilitirung zu arbeiten, und ich fing bei etwas mir ganz neuem an, nämlich bei den Cubischen Resten der Primzahlen* $6n + 1$.

140. [Kummer 1975], vol. 1, p. 51: "I had also striven from the very beginning for a cubic reciprocity law," which Jacobi stated. (*Einem Reciprocitätsgesetze für cubische Reste habe ich ebenfalls ganz anfangs nachgestrebt.*)

aim was to determine completely which sum corresponds to which root, in analogy with Gauss's determination of the sign of the quadratic Gauss sums.[141] He failed,[142] but did find results allowing him to compute the Gauss sums up to $p = 100$, using in particular analytic techniques in Dirichlet's style.

Studying our paper of reference [Jacobi 1839] would redirect his research. During the autumn of 1844, Kummer obtained a rigorous proof of Jacobi's statement for 5 and extended it to 7. The proofs are based on a study of "complex numbers built from 5^{th} roots of unity" (respectively 7^{th} roots of 1), that is, complex numbers of the form $f(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4$, for a primitive 5^{th} root of unity α and integer coefficients a_i (or the analogue for 7); for simplicity, we shall call such numbers "5-cyclotomic (or 7-cyclotomic) numbers." The key point of Kummer's proof is that the norm $Nf(\alpha) = f(\alpha)f(\alpha^2)f(\alpha^3)f(\alpha^4)$ allows one to define a Euclidean division on 5-cyclotomic numbers, as was the case for Gaussian integers; adapting the Euclidean algorithm then shows that a prime $p = 5n + 1$ is the norm of a 5-cyclotomic integer, that is a product of four 5-cyclotomic numbers, as desired (with analogous results for 7). Kummer never published these results,[143] but this very same year, he devoted an article to related matters, fittingly published in a volume dedicated by Breslau University to the tercentenary celebration of the University of Königsberg, i.e., Jacobi's university. Here, Kummer studied λ-cyclotomic numbers, this time for an arbitrary prime λ: he showed in particular that the norm $Nf(\alpha) = f(\alpha)f(\alpha^2)\cdots f(\alpha)^{\lambda-1}$, for α a λ^{th} root of unity, is congruent to 0 or 1 modulo λ and gave an explicit criterion, based on computations with ordinary integers, to decide if one complex number divides another. But, above all, he produced the first example of a number having *different* decompositions into $\lambda - 1$ *irreducible* cyclotomic factors, for $\lambda = 23$. The stakes are clearly indicated in a communication to the Berlin Academy, on March 26, 1846:

> However, I have noticed that, even if $f(\alpha)$ can in no way be decomposed into complex factors, it yet does not have the true nature of a complex prime number, because it usually lacks the first and most important property of prime numbers: that is, that the product of two prime numbers is not divisible by any prime number different from them. These numbers $f(\alpha)$ thus have the nature of composite numbers even though they are not decomposable into complex factors; but the factors are then not actual, but *ideal complex numbers*.[144]

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141. See [Kummer 1975], vol. 1, pp. 143–144 and p. 145–163. For the normalizations of Gauss sums, and further results, see S.J. Patterson's chap. VIII.2 below. Kummer's point of departure is closely related to art. 358 of the D.A.

142. It follows from [Heath-Brown, Patterson 1979] that it is not possible to determine the argument of cubic Gauss sums through local informations at a finite number of places (that is, by a finite number of congruences).

143. Kummer's hitherto lost manuscript has been recovered by Reinhard Bölling; see his commentary and transcription in chap. IV.1 below.

144. [Kummer 1975], vol. 1, p. 203: *Ich habe nun aber bemerkt, daß, wenn auch $f(\alpha)$ auf keine Weise in complexe Factoren zerlegt werden kann, sie deshalb noch nicht die wahre Natur einer complexen Primzahl hat, weil sie schon gewöhnlich der ersten und wichtigsten*

From then on, Kummer steadily elaborated his theory of ideal complex numbers, which he defined by a set of divisibility properties, in fact a set of congruences. They could be "multiplied" (more exactly: multiplicatively composed) and they provided a substitute for the decomposition of primes into ordinary complex numbers expected by Jacobi. The construction of the theory, its use in the decomposition of primes, the further applications to the proofs of higher reciprocity laws and of Fermat's Last Theorem for regular primes,[145] have been well documented and analyzed,[146] In particular, H. Edwards has reconstructed Kummer's heavy reliance on the "periods" used by Gauss in sec. 7 (which, as sums of roots of $x^\lambda - 1$, are in particular λ-cyclotomic numbers) to establish the properties of divisibility which his theory required; see [Edwards 1977], chap. 4. Kummer referred to the *Disquisitiones Arithmeticae* quite often: for instance, in a synthesis published in French in 1851, he referred to art. 52 (giving the multiplicative structure of the residues modulo a prime number), art. 306 (on cyclic properties of the classes of forms), and art. 358 on cubic residues and equations. The D.A. serves both as a model and as a source of problems to be taken up afresh. Another feature very much in the style of the D.A. is the use of induction and of numerical examples. Sometimes relying on already existing, extensive tables, like Jacobi's *Canon Arithmeticus*, the examples are highly non-trivial and thus do not serve as mere illustrations but, as in the D.A., as inspirations for the understanding of phenomena and the clarification of laws.[147]

Mathematically, Kummer decisively contributed to placing higher reciprocity laws at the center of attention. Reporting in 1850, through Dirichlet, to the Academy on his recent achievements, he said:

> Through my investigations on the theory of complex numbers and its applications to the proof of Fermat's Last Theorem … I succeeded in discovering the general reciprocity laws for arbitrarily high power residues, which are to be regarded, according to the present state of number theory, as the main task and the summit of this science.[148]

Eigenschaft der Primzahlen ermangelt: nämlich, daß das Product zweier Primzahlen durch keine von ihnen verschiedene Primzahl theilbar ist. Es haben vielmehr solche Zahlen $f(\alpha)$, wenn gleich sie nicht in complexe Factoren zerlegbar sind, dennoch die Natur der zusammengesetzten Zahlen; die Factoren aber sind alsdann nicht wirkliche, sondern ideale complexe Zahlen. Notice the distinction introduced here between what we call today "irreducibility" and "primality."

145. Kummer defined an equivalence relation among ideal complex numbers associated to λ-cyclotomic numbers (more or less saying that two ideal numbers are equivalent if they differ multiplicatively by a usual cyclotomic number); the number of equivalence classes h_λ is finite. A prime λ is regular if it does not divide the class number h_λ. Kummer characterized this condition by an effective test in terms of Bernoulli numbers.

146. See [Edwards 1975–1977], [Edwards 1977], chaps. 4 and 5, [Edwards 1980], [Neumann 1981], and [Haubrich 1992], chap. 3.

147. See [Edwards 1977].

148. [Kummer 1975], vol. 1, p. 346–347: *Bei meinen Untersuchungen über die Theorie der complexen Zahlen und den Anwendungen derselben auf den Beweis des Fermatschen Lehrsatzes … ist es mir gelungen die allgemeinen Reciprocitätsgesetze für beliebig hohe*

Kummer also made available a stock of images and stories concerning number theory. His national German tenor is quite pronounced in this context. In official discourses, but sometimes also in his mathematical papers and reviews, he would insist on what he saw as the German mathematical tradition. For instance, in his announcement of the first volume of Jacobi's *Mathematische Werke*, he described the changing scene of the 1820s like this:

> It was also at that time that Lejeune-Dirichlet returned from France … to Germany, and we proudly count him completely as one of us; for it is the German genius which pulled him back to his fatherland and which gives his works their admirable depth. … As Germans we are certain that we now have the creative force of the *Geist* on our side.[149]

Kummer was also at the origin of some famous anecdotes, like the description of Dirichlet's never putting the D.A. back on the shelf.[150] As we saw in Merz's quote at the beginning of this chapter,[151] later commentators would take their clue from such sources, even if they cut out the explicitly nationalistic component.

Kummer's work on ideal numbers ties together ideas coming particularly from the first sections of the D.A. on congruences and from the last on cyclotomy. Remembering Hermite's approach to Jacobi's problem, it is interesting to understand whether and how Kummer dealt with the sec. 5 on forms. On the one hand, this section operates as a decisive source of inspiration for the key question of equivalence among ideal numbers:

Potenzreste zu entdecken, welche nach dem gegenwärtigen Stande der Zahlentheorie als die Hauptaufgabe und die Spitze dieser Wissenschaft anzusehen sind. One may also recall his famous statement that Fermat's Last Theorem is a curiosity rather than a focal point of science. (*Der Fermatsche Satz ist zwar mehr ein Curiosum als ein Hauptpunkt der Wissenschaft*); see [Kummer 1975], vol. 1, p. 281.

149. [Kummer 1975], vol. 2, p. 695: *Damals kehrte auch Lejeune-Dirichlet aus Frankreich … nach Deutschland zurück: und wir rechnen ihn mit Stolz ganz zu den Unserigen; denn es ist der deutsche Genius, welcher ihn in sein Vaterland zurückgezogen hat, und welcher seinen wissenschaftlichen Arbeiten ihre bewunderungswürdige Tiefe verleiht. … Wir [Deutschen] sind dessen gewiss, dass wir jetzt die schöpferische Macht des Geistes auf unserer Seite haben.* Another example is provided by his 1856 speech at a Leibniz ceremony.

150. See [Dirichlet 1889–1897], vol. 2, p. 315, where Kummer also states that the D.A. had a "much more important influence on Dirichlet than his other, Parisian, studies." (*Dieses hat auf seine ganze mathematische Bildung und Richtung einen viel bedeutenderen Einfluß ausgeübt als seine anderen Pariser Studien.*)

151. Compare for instance Merz with [Kummer 1875], vol. 2, p. 695, where Kummer writes: " the blossoming of the mathematical sciences in Germany dates back to the beginning of the century, when Gauss first appeared on the scene with his *Disquisitiones Arithmeticae*. It towered so much above everything done before in this discipline that at first only very few of the best mathematicians were capable of understanding it. … for a long time he remained in splendid isolation … until about 1826 a new life began in this science for our fatherland, for which Crelle's *Journal* was created as its chief organ. It was then that Jacobi started his investigations on elliptic functions."

> The general investigation of ideal complex numbers has the greatest analogy with the section "on the composition of forms," treated in such a difficult way by Gauss, and the main results which Gauss proved for quadratic forms starting in art. 234 are also valid for the composition of general ideal complex numbers.[152]

However, on the other hand, Kummer perceived his theory of ideal numbers as capable of throwing light in turn on the complicated concept of proper equivalence in Gauss's D.A. Elsewhere, ideal numbers helped him to find the key to the question of irregular determinants.[153] That is, while Hermite's solution was pushing him towards the study of higher forms in order to solve, among others, questions inherited from algebraic numbers and reciprocity laws, Kummer's solution was to develop "ideal complex numbers" to tackle them all. Both the alternative and Kummer's position are very explicit in his 1851 synthesis:

> If one considers the coefficients of a complex number as indeterminates, the norm will represent a homogeneous form of a certain degree, of the kind decomposable into linear factors. The theory of complex numbers amounts in essence to the theory of these forms, and thus belongs to one of the most beautiful branches of higher arithmetic… [But] the discussion of these forms seems to us less simple than that of the complex numbers themselves, which are their factors, their elements so to speak, and of which the analogy with integers is striking.[154]

4.3. Eisenstein and Kronecker on Complex Multiplication

In 1844, Gotthold Eisenstein wrote from Berlin to his friend Moritz Stern in Göttingen: "I am writing up the residues of the 8$^{\text{th}}$, 12$^{\text{th}}$, and also 5$^{\text{th}}$ powers, which are finished."[155] Four years later, he would explain that he had not pursued for some time the investigation of decompositions into complex primes, the main theme of

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152. [Kummer 1975], vol. 1, p. 209: *Die allgemeine Untersuchung über die idealen complexen Zahlen hat die größte Analogie mit dem bei* Gauß *sehr schwierig behandelten Abschnitte:* De compositione formarum*, und die Hauptresultate, welche* Gauß *für die quadratischen Formen pag. 337 sqq. bewiesen hat, finden auch für die Zusammensetzung der allgemeinen idealen complexen Zahlen Statt.*

153. See his 1853 article on this issue, [Kummer 1975], vol. 1, pp. 539–545. Edwards convincingly shows how Kummer's innovative work can be interpretated as a conservative move with respect to the D.A., [Edwards 1977], pp. 152–154: "The Gaussian notion of proper equivalence is something which needs to be saved from [its] appearance of artificiality" and ideal numbers might be this saviour.

154. [Kummer 1975], vol. 1, p. 363, 366: *Si l'on prend les coefficients du nombre complexe pour des indéterminés, la norme représentera une forme homogène d'un certain degré, du genre de celles qui sont décomposables en facteurs linéaires. La théorie des nombres complexes revient, au fond, à la théorie de ces formes, et à cet égard, elle fait partie d'une des plus belles branches de l'Arithmétique supérieure… [Mais] la discussion de ces formes nous paraît moins simple que celle des nombres complexes eux-mêmes, qui en sont les facteurs, les éléments pour ainsi dire, et dont l'analogie avec les nombres entiers est frappante.*

155. [Eisenstein 1975], vol. 2, p. 793: *Die Reste der 8*$^{\text{ten}}$*, 12*$^{\text{ten}}$*, und auch 5*$^{\text{ten}}$ *Potenzen, welche fertig sind, arbeite ich jetzt aus.*

Jacobi's seminal note [Jacobi 1839], because of tensions with Jacobi. However, in Eisenstein's hands, Jacobi's note would help to kindle the *arithmetic theory of complex multiplication* of elliptic integrals and functions.

An elliptic integral like $u = \int_0^{\varphi} \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2x^2)}}$, or its inverse function, an elliptic function like $\varphi(u,\kappa) = \mathrm{sinam}(u,\kappa)$ (in Jacobi's notation), admits multiplications by rational integers in the sense that, for every integer m, $\mathrm{sinam}(mu,\kappa)$ is a rational function of $\mathrm{sinam}(u,\kappa)$ and its derivative, in the same way that $\sin(mu)$ is a polynomial in $\sin(u)$ and $\cos(u)$.[156] They are said to have *complex multiplication* if there exist multiplications besides those by rational integers. This happens precisely when the ratio of the two basic periods of the elliptic function is an imaginary quadratic irrationality $\sqrt{-n}$, and the transformations in question are then multiplications by complex integers of the form $a + b\sqrt{-n}$, or $a + b\frac{(1+\sqrt{-n})}{2}$, depending on n, with a and b integers.

Examples of such elliptic integrals and functions presented themselves from the very beginning: the integral $\int \frac{dx}{\sqrt{1-x^4}}$ measuring the lemniscatic arc has complex multiplication by Gaussian integers; other elliptic integrals such as $\int \frac{dx}{\sqrt{1\pm x^3}}$ admit complex multiplications by numbers of the form $a + b\frac{(1+\sqrt{-3})}{2}$.[157] Eisenstein first studied elliptic functions with complex multiplication by the third (or sixth) roots of unity in the context of cubic reciprocity.[158] But while such examples of complex multiplication had been available for some time, their arithmetic theory – albeit inspired by Abel's[159] and Jacobi's works – only took shape after Kummer had introduced

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156. For odd m, the derivative of $\mathrm{sinam}(u,\kappa)$ does not intervene. These functions (or the integrals) admit also more general *transformations*, linking $\varphi(\frac{au+b}{cu+d},\kappa_1)$ and $\varphi(u,\kappa)$, for different moduli κ_1 and κ, where a, b, c, d are integers with $ad - bc = n > 0$, the *order* of the transformation. Such transformations for integrals were alluded to in §3.4 above. In modern parlance, these transformations are the isogenies between the lattices, or the elliptic curves, associated to elliptic functions. Multiplication by m is obviously a particular case of transformation, with $\kappa_1 = \kappa$ and order m^2.

157. See for instance [Gauss 1796–1814], September 9, 1796; cf. [Gauss 1900], pp. 93–95. Only these two types of complex multiplication are hinted at in [Jacobi 1839]; they are those for which the complex multipliers are generated by a root of unity.

158. In the first volume of [Eisenstein 1975], this theme occurs on pp. 80; 89–94 (this 1844 paper picks up [Jacobi 1839] explicitly, and also envisages a possible generalization from elliptic to Abelian functions with complex multiplication by general rings of cyclotomic integers); 111; 389–394; and 454–461 (this last part containing a parallel treatment of complex multiplication by the 4th and the 6th roots of 1). The tension with Jacobi alluded to above is mentioned in [Eisenstein 1975], vol. 2, pp. 506–510, see also [Lemmermeyer 2000], pp. 270–275 and the footnote 74 of H. Pieper's chap. III.1 below.

159. Beside his resolution of the division equation of the lemniscate mentioned above, Abel remarked in 1828 that the equations for the moduli κ of elliptic integrals with given complex multiplications are solvable by radicals, see [Abel 1881], vol. 1, p. 425–426, and § 1 of C. Houzel's chap. IV.2 below. Note that even Gauss's posthumously published papers did not explicitly anticipate the arithmetic theory of complex multiplication.

ideal numbers in the wake of Jacobi's paper.

A big stride was taken in a long and leisurely article published (in three parts) by Eisenstein in 1850, again in Crelle's journal; see [Eisenstein 1975], pp. 536–619. It starts with Eisenstein's irreducibility criterion, applied to the division equation for the inverse function φ of the integral measuring the lemniscatic arc; see §2 above. But the whole paper revolves around formulae such as

$$\varphi(mt) = \frac{\varphi(t)^{N(m)} + m \cdot P}{1 + m \cdot Q} \tag{1}$$

$$F(k)^{N(n)} = F(nk) + nT \tag{2}$$

The first line states that, for a Gaussian prime number $m \equiv 1 \pmod{2+2i}$, there exist P, Q, polynomials in $\varphi(t)$ with Gaussian integer coefficients, such that (1) holds; the second states that, for a Gaussian prime number $n \equiv 1 \pmod{2+2i}$, and different from m, and for F any polynomial with Gaussian integer coefficients, there exists T, a polynomial with Gaussian integer coefficients, such that (2) holds, when F and T are applied to the roots of the m^{th} division equation of φ; in both formulas, N denotes the norm.

Eisenstein read these identities in two ways: on the one hand, he took them as higher congruences (mod m), resp. (mod n), in the spirit of Gauss's unpublished eighth section[160] and Schönemann's theory, involving the natural, but fundamental, exponentiation ($\varphi(t)^{N(m)}$ and $F(k)^{N(n)}$) which we today view as the Frobenius automorphism with respect to the Gaussian integers modulo m, resp. n. On the other hand, Eisenstein recognized that Kummer's then recent invention of ideal numbers for the cyclotomic integers could be imitated precisely for the algebraic integers generated over the Gaussian numbers by the m^{th} division values of φ,[161] putting into practice his conviction, expressed to Stern in 1844, that the lemniscatic function plays with respect to the complex numbers exactly the same role as the circular and exponential for the real theory.[162]

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160. Like other readers of the D.A., Eisenstein wondered what exactly Gauss had intended to present there; see for instance a note in the margin of his copy of the D.A., art. 62 (in [Ullrich 2001], pp. 208–209, the editor erroneously corrects "Section VIII" into "Section VII"). In his 1850 article, [Eisenstein 1975], vol. 1, footnote on p. 550, he suggested that Gauss might also have wanted to allow certain infinite series to occur in higher congruences. See also the footnote on pp. 559–560, where he suggested improving on Kummer's theory of complex numbers from the point of view of higher congruences. At least today, it is not hard to prove the inductive observation recorded in the last entry of Gauss's mathematical diary using Eisenstein's formulae; cf. [Schappacher 1997], §6. But we have no evidence that anything like this was realized at the time.

161. [Eisenstein 1975], vol. 1, pp. 574–575, where he wrote in particular that before learning of Kummer's theory, he had found these properties only in a rather clumsy form.

162. [Eisenstein 1975], vol. 2, p. 797: *die Lemniscaten Funktionen, welche in Bezug auf die complexen Zahlen genau dieselbe Rolle spielen, als die Sinus für die reelle Theorie.* For an ordinary integer m, the roots of the cyclotomic equation over the rationals, $\frac{x^m-1}{x-1}$ are the m^{th} roots of unity, that is $e^{2ik\pi/m}$, i.e., the division values of the exponential.

The end of his paper presents applications to the theory of 8^{th} power residues. Eisenstein developed the Euclidean algorithm for the domain of 8^{th} roots of 1 and derived Jacobi's decomposition of prime numbers $N(m) = p = 8\lambda + 1$ in this domain (where m denotes a Gaussian prime number). He considerably refined the analysis of the factors of this decomposition with the help of polynomial expressions in the m-division values of the lemniscatic function φ and the 8^{th} roots of 1, which behave under permutations of the division values in a way closely analogous to the behaviour of certain resolvents studied by Kummer while trying to find actual complex numbers, in larger domains, which would represent the ideal factors found in the decomposition of primes in cyclotomic fields.[163]

Given Eisenstein's adoption of Kummer's ideal numbers, the continuity with Kronecker is particularly striking. In 1853, Kronecker conceived of what is now called the Kronecker-Weber Theorem – in his terms,[164] "the roots of any Abelian equation with integer coefficients can be expressed as rational functions of roots of unity" – as a direct application of precisely that same article of Kummer on ideal prime numbers that Eisenstein had transposed to the lemniscatic context. Kronecker thought at first that ideal numbers were needed to prove it, and he also generalized his statement immediately to equations abelian *over the Gaussian numbers*, linking them here to the lemniscatic theory.[165] But Kronecker did not stop there. "The subject itself," as he put it, pushed him to study not just the division equations of a particular elliptic function (the analogue over the Gaussian integers of the cyclotomic equation over the rational numbers), but "the arithmetic properties of those moduli" κ for which complex multiplication occurs.[166]

> The elliptic functions for which complex multiplication occurs are situated … between the circular functions and the other elliptic functions. … thus the values of the moduli of that special type of elliptic functions are characterized as limit values by the fact that only for them does the modular equation have multiple roots. Furthermore, while for circular functions there is only multiplication, and for general elliptic functions both multiplication and transformation, for that particular type of elliptic functions, transformation loses in part its peculiar character and turns into a sort of multiplication by ideal numbers. Indeed, as, for an integer p which is represented by the principal form $x^2 + ny^2$ of determinant $-n$, one of the transformations of order p is the multiplication by $x + y\sqrt{-n}$, i.e., $\sin^2 \mathrm{am}\,(x + y\sqrt{-n})u$ is expressed as rational function of $\sin^2 \mathrm{am}\, u$ and κ, one of the transformations of order q, where q is represented by one of the other forms of determinant $-n$, gives a transformed function: $\sin^2 \mathrm{am}(\mu \cdot u, \lambda)$ expressed as a rational function of $\sin^2 \mathrm{am}(u, \kappa)$ and κ, where λ is one of the other moduli for which multiplication by $\sqrt{-n}$ occurs, and μ belongs to a certain value of $\sqrt{-n}$ (mod. q) and thus represents an ideal factor

163. See [Kummer 1975], vol. I, pp. 211–251.
164. Kronecker coined this terminology "Abelian equation" in that paper. Today it means that the Galois group of the equation is commutative; but see O. Neumann's chap. II.1 below for the evolution of Kronecker's usage.
165. [Kronecker 1895–1930], vol. 4, pp. 3–11; [Petri, Schappacher 2004], pp. 234–239 and pp. 252–255.
166. [Kronecker 1895–1930], vol. 4, p. 209.

of q. These multipliers: μ are explicit algebraic irrationalities and it is in many ways remarkable that this gives a first example where analysis provides the irrationalities for the representation of ideal numbers.[167]

Extending Kummer's usual cyclotomic frame of reference, Kronecker here, in 1858, thought of ideal numbers of the imaginary quadratic domain of complex multiplication. He remarked that the various classes are represented by (functions of) the various moduli for which complex multiplication by $\sqrt{-n}$ occurs, the so-called "singular moduli."[168] Kronecker came up on the one hand with a list of formulae involving class numbers of binary quadratic forms with negative discriminants, and on the other with an explicit resolution, related to the genera of corresponding quadratic forms, of the modular equations in the case of complex multiplication.[169] For instance, he stated that, if $n \equiv 3 \bmod 4$, $\phi(n)$ the sum of divisors of n which are greater than $\sqrt{n}$, $\psi(n)$ of the remaining divisors, one has $2F(n) + 4F(n-2^2) + 4F(n-4^2) + \ldots = \phi(n) - \psi(n)$, where $F(m)$ denotes the number of classes of binary quadratic forms which are either properly primitive (i.e., $(a, 2b, c)$ are coprime) of determinant $-m$ or a multiple of such forms, the sum on the left being stopped when $n - i^2 = 0$ or < 0. Such formulae would in turn be taken up by Hermite, which nicely closes up our circle of papers inspired by Jacobi's 1839 article, [Jacobi 1839].

167. [Kronecker 1895–1930], vol. 4, p. 181: *Die elliptischen Functionen, für welche complexe Multiplication stattfindet, stehen ihren wesentlichen Eigenschaften nach zwischen den Kreisfunctionen einerseits und den übrigen elliptischen Functionen anderseits. … so werden auch die Werthe der Moduln jener besonderen Gattung von elliptischen Functionen dadurch als Grenzwerthe charakterisirt, daß nur für diese … die Modulargleichungen gleiche Wurzeln enthalten. Während ferner für die Kreisfunctionen nur Multiplication, für die allgemeinen elliptischen Functionen aber Multiplication und Transformation stattfindet, verliert die Transformation bei jener besondern Gattung elliptischer Functionen zum Theil ihren eigenthümlichen Charakter und wird selbst eine Art Multiplication, indem sie gewissermaaßen die Multiplication mit idealen Zahlen darstellt. Wie nämlich für eine Zahl p, welche sich durch die zur Determinante* $-n$ *gehörige Hauptform* $x^2 + ny^2$ *darstellen läßt, eine der Transformationen pter Ordnung die Multiplication mit* $x + y\sqrt{-n}$ *d.h. die Darstellung von* $\sin^2 \mathrm{am}\,(x + y\sqrt{-n})u$ *als rationale Function von* $\sin^2 \mathrm{am}\, u$ *und* κ *gewährt, so ergiebt eine der Transformationen qter Ordnung, wenn q durch eine der übrigen zur Determinante* $-n$ *gehörigen Formen darstellbar ist, eine transformirte Function:* $\sin^2 \mathrm{am}(\mu \cdot u, \lambda)$ *ausgedrückt als rationale Function von* $\sin^2 \mathrm{am}(u, \kappa)$ *und* κ*, in welcher* λ *einer der andern Moduln ist, für welche Multiplication mit* $\sqrt{-n}$ *stattfindet, in welcher ferner* μ *zu einem bestimmten Werthe von* $\sqrt{-n}$ (mod. q) *gehört und geradezu die Stelle eines idealen Factors von q vertritt. Diese Multiplicatoren:* μ *sind explicite algebraische Irrationalitäten und es ist in vielfacher Hinsicht bemerkenswerth, daß hier ein erstes Beispiel gegeben ist, in welchem die Analysis die Irrationalitäten zur Darstellung idealer Zahlen gewährt.*

168. This sentence is compatible with the language of the 1850s and 1860s as well as with modern presentations of the theory; see for instance [Serre 1967], last theorem in §1.

169. See [Kronecker 1895–1930], vol. 4, pp. 185–195, cf. [Smith 1859–1865], §§ 130–137, and § 2 (cf. also § 6) of C. Houzel's chap. IV.2 below. An explicit appeal to D.A., art. 227, occurs for instance in [Kronecker 1895–1930], vol. 4, p. 210; see also p. 237.

Kronecker, however, would come back to complex multiplication in a long series of papers from the 1880s,[170] and directly take up Eisenstein's work, and in particular the formulae (1) and (2), in the very context of the quote given above (footnote 167). His "main goal" (*Hauptzielpunkt*) then was to establish – which he does by giving three different proofs – the following vast generalization of (1) and (2)[171] :

$$(-1)^{\frac{1}{2}(n-1)}\sqrt{\lambda}\,\sin\mathrm{am}\,(\mu u,\lambda) \equiv \left(\sqrt{\kappa}\,\sin\mathrm{am}(u,\kappa)\right)^n \pmod{\mu}.$$

Commenting on this formula, Kronecker stressed the simultaneous presence of transformation (indicated by the two moduli: κ and λ) and multiplication (indicated by the multiplier μ).[172] This comment is actually presented as a praise of Jacobi's notation, underscoring the markedly traditional style of this whole series of notes. Contrary to modern reflexes,[173] Kronecker appreciated this formula, not in algebro-geometric terms, but as a quintessential result of arithmetic algebraic analysis: a congruence derived from an algebraic relation involving analytic functions. Already in the long quote above we saw him stress the role of analysis as being able to deliver to arithmetic what is hard or impossible to come by in a purely arithmetic way.[174]

5. In Search of a Discipline

By the middle of the XIX[th] century, the *Disquisitiones Arithmeticae* had left its imprint on several areas of mathematical research: in higher arithmetic and in the theory of equations, of course, but also, for instance, in all those domains where determinants or substitutions were used. It had also found new readers: mathematicians who had studied it closely, often early in their careers, who would devote an important part of their professional activities to developing it, and of whom several occupied key positions on the mathematical scene.

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170. They fill almost one third of vol. 4 of Kronecker's Collected Papers.

171. [Kronecker 1895–1930], vol. 4, pp. 389–471, formula (64), p. 439. The letter n replaces what was called q in the long quote above.

172. [Kronecker 1895–1930], vol. 4. This two-sidedness is one reason to reject the one-sided interpretation of *Kronecker's Jugendtraum* given in Hilbert's 12[th] problem; see Helmut Hasse's *Zusatz 35* in [Kronecker 1895–1930], vol. 5, pp. 510–515. Cf. [Schappacher 1998].

173. Looked at from the second half of the XX[th] century, the formula appears as an ancestor of both the Shimura-Taniyama and the Eichler-Shimura congruence relations; see [Vlăduţ 1991], part I, chaps. 3, 4. Shimura and Taniyama alluded explicitly to Kronecker's formula in [Shimura, Taniyama 1961], sec. 13, p. 110.

174. This is also a recurring thought in Kronecker's lectures [Kronecker 1901]. At the same time, as is well-known, Kronecker forcefully propagated a general arithmetic which was to actually contain analysis, and a foundational view of mathematics where rigour ultimately should be based only on natural integers, see chap. I.2 and §2.2 of B. Petri's and N. Schappacher's chap. V.2 below. At least the technical parts of his papers on elliptic functions seem unaffected by this creed.

5.1. A Research Field

More specifically, we have called arithmetic algebraic analysis the domain of research directly connected with the D.A. that knit together reciprocity laws, series with arithmetical interpretations, elliptic functions and algebraic equations. We argue that it constituted a (research) *field*, in the sense that "all the people who are engaged in [this] field have in common a certain number of fundamental interests, viz., in everything that is linked to the very existence of the field," and that one can uncover "the presence in the work of traces of objective relations … to other works, past or present, [of the field]."[175]

As we have noticed, its main actors were indeed linked by a dense communication network, both personal and mathematical. Their published papers would meet with prompt reactions; quite a number of these papers were in fact excerpts of letters adressed to another mathematician working in the domain. An interesting characteristic feature was the production of new proofs of the central results, a phenomenon of which we have seen several instances.[176]

A main motto was that of the unity of this specific area and of the strive toward unity of the mathematicians working in it. It is expressed for instance by Gauss himself in his preface to Eisenstein's *Mathematische Abhandlungen* in 1847:

> The higher arithmetic presents us with an inexhaustible store of interesting truths, of truths, too, which are not isolated, but stand in a close internal connexion, and between which, as our knowledge increases, we are continually discovering new and sometimes wholly unexpected ties.[177]

At Hermite's jubilee in 1892, Poincaré, discussing how Hermite "illuminated with a new light the admirable edifice raised by Gauss," added that the merit of his discoveries was increased by the "care that [Hermite] always took to make clearly evident the mutual support that all these sciences, apparently so diverse, provide to each other."[178] In the same way, Kummer emphasized that Dirichlet's "mind [was]

175. [Bourdieu 1976/2002], p. 115: *tous les gens qui sont engagés dans un champ ont en commun un certain nombre d'intérêts fondamentaux, à savoir tout ce qui est lié à l'existence même du champ*, and p. 116: *Un des indices les plus sûrs de la constitution d'un champ est … la présence dans l'œuvre de traces de relations objectives … aux autres œuvres, passées ou contemporaines [du champ].*

176. This could go beyond a deliberate attempt to prove a statement published without proof, as in the case of Jacobi's paper discussed above, § 4, or to provide a new perspective on a celebrated result such as the reciprocity law. Since similar results were elaborated independently by different persons, the feeling of an unconscious convergence or repetition of ideas was sometimes expressed: on April 6, 1853, Hermite wrote to Dirichlet about the transformations of an indefinite ternary form into itself, a subject that he had discussed in Berlin with both Eisenstein and Dirichlet: "[it seems] a law of my destiny that all I do in arithmetic is to rediscover some of the discoveries that you have made a long time ago" (Nachlass Dirichlet, Staatsbibliothek zu Berlin-Preussischer Kulturbesitz-Handschriftenabteilung).

177. Quoted from the English translation offered by Smith at the beginning of his report, [Smith 1859–1865], part I, p. 228.

178. [Hermite 1893], p. 6: *vous éclairiez d'une lumière nouvelle l'admirable édifice élevé*

always striving toward unity."[179]

This motto was supported by an actual circulation of concepts and methods; for instance, Dirichlet's analytic techniques were used by Kummer, ideal numbers by Eisenstein and Kronecker, while the study of forms along the lines of the section 5 of the D.A. was extended by Dirichlet and Hermite to forms with Gaussian integers as coefficients; Liouville, Charles Joubert, and Hermite took up Kronecker's relations on the number of classes of quadratic forms. Yet another famous connection between algebraic equations and elliptic functions was established in various ways by Hermite, Kronecker, and Francesco Brioschi: the analytic solution of the general quintic equation via suitable modular or multiplier equations.[180] It was also reinforced by translations and texts of a historical or biographical nature.[181] An example of this tight intertwinement is offered for instance by Jules Houël's letter to Dirichlet of April 30, 1857:

> I just finished the translation of your beautiful memoir *Vereinfachung der Theorie der binären quadratischen Formen von positiver Determinante*, the simplicity of which I admire all the more as I am just studying for the first time section 5 of the D.A. M. Lebesgue, who has volunteered to go over my translation, will send it to M. Liouville in a few days. I also translated and submitted to M. Liouville your obituary address on Jacobi which contains an interesting history of the contemporary development of mathematics, a development of which a considerable part is due to that great man.[182]

Specific media played an important role in this process: we have seen how that of

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par Gauss.... Le prix de vos découvertes est encore rehaussé par le soin que vous avez toujours eu de mettre en évidence l'appui mutuel que se prêtent les unes aux autres toutes ces sciences en apparence si diverses. Poincaré specifically mentioned number theory, algebraic forms, elliptic functions, and modular equations.

179. [Dirichlet 1889–1897], vol. 2, p. 327: *In seinem überall zur Einheit strebenden Geiste konnte er diese beiden Gedankensphären nicht neben einander bestehen lassen.* The two "spheres of thought" are analysis and number theory.

180. See [Petri, Schappacher 2004], §§ 3, 4, and the literature and sources cited there.

181. Cf. [Bourdieu 1976/2002], pp. 116–117: *Un des indices les plus surs de la constitution d'un champ est, avec la présence dans l'oeuvre de traces de relations objectives ... aux autres oeuvres, passées ou contemporaines [du champ], l'apparition d'un corps de conservateurs de vies ... ou des œuvres. ... Et un autre indice du fonctionnement en tant que champ est la trace de l'histoire du champ dans l'oeuvre.* In the case at hand, the mathematicians themselves served as their own historians and biographers.

182. Nachlass Dirichlet, Staatsbibliothek zu Berlin, Preussischer Kulturbesitz, Handschriftenabteilung: *Je viens de terminer la traduction de votre beau Mémoire, intitulé: "Vereinfachung der Theorie der binären quadratischen Formen von positiver Determinante," dont j'admire d'autant plus la simplicité que je suis en train d'étudier pour la première fois la section V des Disq. arithm. M. Lebesgue, qui veut bien se charger de revoir ma traduction, l'enverra sous peu de jours à M. Liouville. J'ai également traduit et remis à M. Liouville l'Eloge de Jacobi, qui renferme une histoire intéressante du développement contemporain des mathématiques, développement dont une part bien considérable revient à ce grand homme.* Note that no more than 4 out of 26 pages of this obituary deal with works by Jacobi which lie outside of arithmetic algebraic analysis.

Crelle's *Journal* was recognized by several mathematicians; Genocchi's proof of the reciprocity law published as a memoir of the Belgian Academy in 1852, on the other hand, passed unnoticed until it was recalled in the Paris *Comptes rendus de l'Académie des sciences* thirty years later.[183]

However, we think that it would be misleading to interpret this situation as the establishment of a *discipline* based on the D.A., in the sense of an "object-oriented system of scholarly activities."[184] Indeed, the developments of different parts of the D.A. provided different key objects on which mathematicians could focus their investigations: congruences for some, algebraic integrals for others; ideal numbers or forms; integers or elliptic functions, etc. To each of them in turn divergent key problems were associated, from reciprocity laws to classification issues. While most references to the D.A. at that time would extol the quality of its proofs – their ancient rigour, in Minding's terms – and while some of these proofs constituted technical models to emulate and even mimic, Gauss's demonstrations were also criticized on several grounds, sometimes for their length and complexity, on other occasions for their synthetic nature. The very activity of proof analysis – reflecting on existing proofs in order to fathom their mechanism and simplify their presentation – which sometimes could effectively gather mathematicians of various orientations around the same statement, was not central for all. According to Jacobi,

> [Dirichlet] alone, not I, nor Cauchy, nor Gauss knows what a completely rigorous mathematical proof is, but we know it only from him. When Gauss says he *proved* something, I take it to be very likely, when Cauchy says it, one may bet as much for or against, when Dirichlet says it, it is *certain.* For myself, I prefer not to get involved in these subtleties.[185]

183. On Genocchi's 1852 memoir and Kronecker's interest in it, see A. Brigaglia's chap. VII.1.
184. [Guntau, Laitko 1987], p. 26: *gegenstandsorientiertes System wissenschaftlicher Tätigkeiten*. We owe this reference to Ralf Haubrich who, at the 2001 Oberwolfach Conference, suggested characterizing a mathematical discipline by a list of *internal* elements such as its subject matter, its core concepts and theorems, its systematization, its proof system, the mathematical values advocated in evaluating its results, etc. A word of caution may be appropriate here: Thomas Kuhn's description of a "disciplinary matrix," in the Postscript to the second edition of *The Structure of Scientific Revolutions*, could appear to be very similar; however, Kuhn's conception is deliberately anchored in the analysis of communities and groups of practitioners, which, by themselves and by their very existence, delineate the characteristics of this "matrix." For that matter, the cyclotomic equation, with its links to circular functions on the one hand, and to primitive roots modulo a prime on the other, is a perfect *paradigm* in Kuhn's sense (that is, a shared key-example), linking (although perhaps tacitly) the practitioners of arithmetic algebraic analysis. But we want to use the word "discipline" here in the more restricted sense indicated above, just as we have used above the sociologically better defined "field" (*champ*) instead of Kuhn's "community."
185. See the letter of Jacobi to A.v. Humboldt, December 21, 1846, [Jacobi & Humboldt], p. 99: *Er allein, nicht ich, nicht Cauchy, nicht Gauss weiß, was ein vollkommen strenger mathematischer Beweis ist, sondern wir kennen es erst von ihm. Wenn Gauss sagt, er habe etwas* bewiesen, *ist es mir sehr wahrscheinlich, wenn Cauchy es sagt, ist ebensoviel pro als contra zu wetten, wenn Dirichlet es sagt, ist es* gewiß. Ich *lasse mich auf diese*

The close relation to the D.A. is of course a kind of trade-mark for all these works; but, as we have pointed out on several occasions, different sections, or even articles of the D.A., were privileged and pondered upon by different authors. We have also indicated moves towards giving a more prominent profile and greater independence to formerly amalgamated components of arithmetic algebraic analysis.

5.2. An Academic Discipline

Yet another perspective deserves consideration. The *Disquisitiones Arithmeticae* is a research monograph, but during the first half of the XIX$^{\text{th}}$ century, research results tended rather to be published in shorter papers, whereas the publication of books would be increasingly associated to other genres like, for instance, textbooks or syntheses. Following the thread of textbooks, say, from Minding's treatise on, rather convincingly reveals the constitution of a discipline based on the D.A. Two works, conceived at the end of the 1850s, would incarnate it in the following decades, serving as standard references for the numerous textbooks on number theory from the end of the century:[186] Henry John Smith's *Report on the Theory of Numbers* and Dedekind's first edition of Dirichlet's *Vorlesungen über Zahlentheorie*.

While Peacock in 1834 had integrated the D.A. in a report on analysis, twenty-five years later number theory received an extensive and detailed report by itself at the British Association for the Advancement of Science. Smith had begun to tackle number-theoretical questions in the mid-1850s essentially in Legendre's style. However, his report, published between 1857 and 1865, covered very thoroughly the research from Gauss to Kummer. Smith classified the material under two main headings, following more or less Gauss's table of contents:

> There are two principal branches of the higher arithmetic:– the Theory of Congruences, and the Theory of Homogeneous Forms. … It might, at first sight, appear as if there was not sufficient foundation for the distinction. But in the present state of our knowledge, the methods applicable to, and the researches suggested by these two problems, are sufficiently distinct to justify their separation from one another. … Those miscellaneous investigations, which do not properly come under either of them, we shall place in a third division by themselves.[187]

Richard Dedekind adopted roughly the same presentation when in 1863 he edited Dirichlet's Göttingen lectures of 1857–58 for the first time:[188] after a first chapter on general properties of integers, as in Minding, two chapters are devoted to congruences and two others to binary quadratic forms, while miscellanea, including results on the cyclotomic equation, are exiled into several supplements.

Delicatessen lieber gar nicht ein. On the idea of proof analysis and Dirichlet's importance for this issue, see [Haubrich 1992], p. 14.

186. In his 1890 survey on number theory, for instance, Thomas Stieltjes would recognize his debts to both of them and two years later George Mathews would write in his own book that "he derived continual assistance" from them; for these examples and others, see the following chap. I.2.

187. [Smith 1859–1865], pp. 39–40.

188. On the *Vorlesungen über Zahlentheorie*, see [Goldstein 2005].

In both cases, novel developments were forced into this bipartite scheme, the D.A. giving the lead to the whole text. Smith is explicit about it:

> Instead of confining our attention exclusively to the most recent researches in the Theory of Quadratic Forms, we propose … to give a brief but systematic *résumé* of the theory itself, as it appears in the Disq. Arith., introducing in their proper places, notices, … of the results obtained by later mathematicians. We adopt this method, partly to render the later researches themselves more easily intelligible, by showing their connexion with the whole theory; but partly also in the hope of facilitating to some persons the study of the Fifth Section of the Disq. Arith.[189]

Similarly, Smith presented Kummer's ideal numbers in the section on higher congruences, in connection with reciprocity laws. Dirichlet's chapters on quadratic forms integrate his simplifications of the D.A. on this topic, as well as his class number formula. But several analytic lemmas and his theorem on primes in arithmetical progressions are relegated to the supplements, just as Smith relegated Jacobi's theta functions and their applications to quadratic forms to the end of his report.[190]

Gauss's strong impact on shaping survey publications can be measured *a contrario* by two other books on number theory published around 1850. That of Eugène Desmarest, a pharmacist and amateur number-theorist, [Desmarest 1852], is mostly devoted to exhibiting effective, "practical" solutions to quadratic Diophantine equations. Despite its Legendre-like appearance, it was only with respect to Gauss's D.A. that Desmarest felt the need in 1852 to justify the "foolhardiness," as he put it, of his encompassing title *Théorie des nombres*. And although he rejected Gauss's notation for congruences, he did organize his treatise into a first part on the resolution of binary quadratic equations modulo a prime number, a second part on the representation of numbers by binary quadratic forms (including proper and improper equivalence), finally applying them in a third part to his main problem.

As for Pafnuti L. Čebyšev, who published in 1849 a treatise on the theory of

189. [Smith 1859–1865], p. 169.

190. It is interesting to contrast the situation of analytic methods here with what Kummer said about them at the same time, in Dirichlet's obituary of 1860: "that in those applications, analysis would be made to serve number theory in such a way that it does not just yield coincidentally some isolated results, but is bound to yield with necessity the solutions to certain general types of problems of arithmetic as yet inaccessible in other ways," and thus that "these methods of Dirichlet's … would have to be recognized as the creation of a new mathematical discipline, if they extended not just to certain types of problems but uniformly to all problems of number theory" ([Dirichlet 1889–1897], vol. 2, p. 327: *in ihnen die Analysis der Zahlentheorie in der Art dienstbar gemacht wird, dass sie nicht mehr nur zufällig manche vereinzelte Resultate für dieselbe abwirft, sondern dass sie die Lösungen gewisser allgemeiner Gattungen, auf anderen Wegen noch ganz unzugänglicher Probleme der Arithmetik mit Notwendigkeit ergeben muss. … sie würden auch … als Schöpfung einer neuen mathematischen Disciplin anerkannt werden müssen, wenn sie sich nicht bloss auf gewisse Gattungen, sondern auf alle Probleme der Zahlentheorie gleichmässig erstreckten.*) A proper research discipline, centered around Dirichlet series, but encompassing neither all number theory, nor arithmetic algebraic analysis, would flourish in the second half of the century.

congruences based on his Saint-Petersburg dissertation, he was of course no marginal figure like Desmarest, but his keen regard for practical applications of mathematics put his own papers mostly outside of arithmetic algebraic analysis. In his number-theoretical papers from the 1850s,[191] he refers to Dirichlet, but not to the *Disquisitiones Arithmeticae*. At the same time, he was coediting Leonhard Euler's number theoretical memoirs with Viktor Yakovlevič Bunyakovski. In his own book, he elected neither Diophantus, nor Fermat, nor Gauss, but Euler as the father of number theory. He claimed in the preface not to follow Legendre or Gauss – implicitly, however, he used the *Disquisitiones Arithmeticae* to restructure Euler's results:

> Among Euler's numerous investigations in the domain of number theory, the memoirs which had the most important influence on the success of this science are those on the two following topics: (1) On the powers of numbers considered with respect to their residues when divided by a given number. (2) On numbers that are represented as a sum of two numbers, of which one is a square and the other is the product of a square by a given number. The memoirs on the first topic provided the basis of the theory of indices, of the theory of binomial congruences in general and of quadratic residues in particular; the memoirs on the second topic built the beginning of the theory of quadratic forms.[192]

His book focuses on the first part, congruences, quadratic forms appearing as a subsection in the chapter on quadratic congruences, while his own analytic results were put at the end and were eventually omitted from the German translation.

To summarize, the role of the *Disquisitiones Arithmeticae* in the constitution of number theory fifty years after its publication was two-fold. On the one hand, the D.A. provided number theory with the features of its self-organization[193] as an academic discipline. It shaped what number theory[194] was and ought to be: congruences

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191. Where he proved Bertrand's postulate, a result on the mean values of arithmetical functions, and the fact that if $\pi(x) \cdot \frac{\log x}{x}$, where $\pi(x)$ denotes the number of primes less than x, has a limit as x tends to ∞, then this limit has to be 1.

192. From the preface of [Čebyšev 1849]: Между многими изысканиями Эйлера в теории чисел наиболее имели влияния на успех этой науки изыскания его по следующим двум предметам: 1) о степенях чисел в отношении остатков, получаемых при делении их на данное число, и 2) о числах, представляющих сумму двух чисел, из которых одно есть квадрат, а другое произведение квадрата на данное число. Первые положили основание теории указателей, сравнений двучленных вообще и в особенности теории квадратичных вычетов; вторые были началом теории квадратичных форм. Hearty thanks to Ilia Itenberg, Strasbourg, for finding this original citation, and advising us on the translation.

193. Cf. Rudolf Stichweh, *Zur Entstehung des modernen Systems wissenschaftlicher Disziplinen. Physik in Deutschland 1740–1890.* Frankfurt: Suhrkamp, 1984, chap. I: "The differentiation of disciplines … is a mechanism of self-organization of the system."

194. Ralf Haubrich has coined the expression "Gaussian Number Theory" to designate this core. We would like to stress that in the middle of the century, as the titles of books witness, it was intended to be seen as "Number Theory" *per se*. Even Diophantine analysis bore its marks. "Gaussian" thus is meant to allude to the role of the D.A. to

and forms with integer coefficients (with their possible generalizations). This image informed advanced textbooks and helped to structure them, and it would, for an even longer time, structure classifications of mathematics. On the other hand, the D.A. had launched an active research field, with a firm grasp on number theory, algebra and analysis, supported by close and varied readings of the book. It provided the field with technical tools, and a stock of proofs to scrutinize and adapt. It also provided concrete examples of the very links between different branches of mathematics that created the field, often articulated around richly textured objects and formulae, such as the cyclotomic equation or Gauss sums. The (meta)stability of the field was not guaranteed by any unicity of purpose or concept (individual mathematicians might have their own priorities, mix differently the resources available or disregard some of them), nor by a merging into a larger domain,[195] but by a constant circulation from one branch to another, a recycling of results and innovations. How certain branches emancipated themselves, and with which consequences for number theory and for the role of the D.A., will be the subject of the next chapter.

References

ABEL, Niels Henrik. 1881. *Œuvres complètes*, ed. L. Sylow, S. Lie. 2 vols. Christiania: Grøndahl & Søn.

———. 1902. *Mémorial publié à l'occasion du centenaire de sa naissance*. Kristiania: Jacob Dybwad; Paris: Gauthier-Villars; Leipzig: Teubner; London: Williams & Norgate.

BABBAGE, Charles. 1813. Preface. *Memoirs of the Analytical Society* 1, i-xxii. Repr. in *Science and Reform. Selected Works of Charles Babbage*, ed. A. Hyman, pp. 11–34. Cambridge: Cambridge University Press, 1989.

BACHMANN, Paul. 1911. *Über Gauß' zahlentheoretische Arbeiten*. Leipzig: Teubner. Repr. slightly revised in [Gauss 1922–1933], Abhandlung 1.

BARLOW, Peter. 1811. *An Elementary Investigation of the Theory of Numbers, with its application to the indeterminate and diophantine analysis, the analytical and geometrical division of the circle and several other curious algebraical and arithmetical problems*. London: Johnson.

BELHOSTE, Bruno. 1991. *Augustin-Louis Cauchy. A Biography*. New York, etc.: Springer-Verlag.

———. 1996. Autour d'un mémoire inédit: la contribution d'Hermite au développement de la théorie des fonctions elliptiques. *Revue d'Histoire des mathématiques* 2, 1–66.

BETTI, Enrico. 1903. *Opere matematiche,* vol. I, ed. Reale Accademia dei Lincei. Milano: Ulrico Hoepfli.

BIERMANN, Kurt-R. 1977. Aus unveröffentlichten Aufzeichnungen des jungen Gauß (zum 200. Geburtstag von C.F. Gauß). *Wissenschaftliche Zeitschrift der Technischen Hochschule Ilmenau* 4, 7–24.

———. 1988. *Die Mathematik und ihre Dozenten an der Berliner Universität 1810–1933*. Berlin: Akademie-Verlag.

shape, but not to qualify or restrict its domain.

195. The situation is thus markedly different from the earlier period, see § 2 above, when some of the contents of the D.A. merged into algebra.

Boncompagni, Baldassare. 1882. Intorno alla vita ed ai lavori di Antonio Carlo Marcellino Poullet-Delisle. *Bullettino di Bibliografia e Storia delle Scienze Matematiche e Fisiche* 15, 670–678.

Bos, Henk J. M., Kers, Cees, Oort, Franz, Raven, Diederik W. 1987. Poncelet's Closure Theorem, its history, its modern formulation, a comparison of its modern proof with those by Poncelet and Jacobi, and some mathematical remarks inspired by these early proofs. *Expositiones Mathematicae* 5, 289–364.

Bourdieu, Pierre. 1976. Quelques propriétés des champs. Exposé à l'ENS, novembre 1976. In *Questions de sociologie*, pp. 113–120. Paris: Les Editions de Minuit, 2002.

Bühler, Walter Kaufmann. 1981. *Gauss. A Biographical Study.* Berlin, Heidelberg, etc.: Springer.

Bullynck, Maarten. 2006a. A Note on Article 36 in Gauss's *Disquisitiones Arithmeticae*. A Ramified Story in the Margin of the Re-Writing of Section II. *Bulletin of the Belgian Mathematical Society. Simon Stevin*, to appear.

———. 2006b. *Vom Zeitalter der Formalen Wissenschaften. Anleitung zur Verarbeitung von Erkenntnissen anno 1800, vermittelst einer parallelen Geschichte*. Thesis, Universiteit Ghent. Ghent.

Cajori, Florian. 1928–1929. *A History of Mathematical Notations*. 2 vols. Chicago: Open Court.

Cauchy, Augustin-Louis. 1813–1815. Démonstration du théorème général de Fermat sur les nombres polygones. *Mémoires des sciences mathématiques et physiques de l'Institut de France* 14, 1st ser., 177–220. Repr. in *Œuvres complètes*, ed. Académie des sciences, 2nd ser., vol. 6, pp. 320–353. Paris: Gauthier-Villars, 1887.

———. 1815. Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment. *Journal de l'Ecole polytechnique* 17e cahier, 10, 29–97. Repr. in *Œuvres complètes*, ed. Académie des sciences, 2nd ser., vol. 1, pp. 91–169. Paris: Gauthier-Villars, 1905.

———. 1840. Mémoire sur la théorie des nombres. *Mémoires de l'Académie des Sciences* 17, 249–768. Repr. in *Œuvres complètes*, ed. Académie des sciences, Ist ser., vol. 3, pp. 5–449. Paris: Gauthier-Villars, 1911.

Čebyšev, Pafnuti Lvovič. 1849. *Teoria sravnenii.* Sanktpetersburg: Tipographia Akademii Nauk. German transl. by H. Schapira: *Theorie der Congruenzen. Elemente der Zahlentheorie*. Berlin: Mayer & Müller, 1889; repr. 1902.

Décaillot, Anne-Marie. 1999. *Edouard Lucas (1842–1891): le parcours original d'un scientifique français dans la deuxième moitié du XIXe siècle*. Thèse de l'université René Descartes. Paris.

Dedekind, Richard. 1930–1932. *Gesammelte mathematische Werke*, ed. E. Noether, R. Fricke, O. Ore. 3 vols. Braunschweig: Vieweg.

Delambre, Jean-Baptiste Joseph. 1810. *Rapport historique sur les progrès des sciences mathématiques depuis 1789, et sur leur état actuel*. Paris: Imprimerie impériale.

Desmarest, Eugène. 1852. Théorie des nombres, Traité de l'analyse indéterminée du second degré à deux inconnues, suivi de l'application de cette analyse à la recherche des racines primitives, avec une table de ces racines pour tous les nombres premiers compris entre 1 et 10000. Paris: Hachette.

Dickson, Leonard Eugene. 1919–1923. *History of the Theory of Numbers.* 3 vols. Washington: The Carnegie Institute. Repr. New York: Chelsea, 1956.

Dirichlet, Johann Peter Gustav Lejeune-. 1863. *Vorlesungen über Zahlentheorie*, ed. R. Dedekind. Braunschweig: Vieweg. English transl. J. Stillwell. History of Mathematics Sources 16. Providence: AMS, London: LMS, 1999.

———. 1889–1897. *Werke*, ed. L. Kronecker, L. Fuchs. 2 vols. Berlin: Reimer.

Dirksen, Enno Heeren. 1845. *Organon der gesammten transcendenten Analysis. Erster Theil. Transcendente Elementarlehre.* Berlin: Reimer.

Dunnington, Guy Waldo. 1955. *Carl Friedrich Gauss, Titan of Science. A Study of his Life and Work.* New York: Exposition Press. Repr. (with additional material by J. Gray and F.-E. Dohse). The Mathematical Association of America, 2004.

Edwards, Harold M. 1975–1977. The Background of Kummer's Proof of Fermat's Last Theorem for Regular Primes. *Archive for History of Exact Sciences* 14, 219–236. Postscript to "The Background of Kummer's Proof …." *Archive for History of Exact Sciences* 17, 381–394

———. 1977. *Fermat's Last Theorem. A Genetic Introduction to Algebraic Number Theory*. New York: Springer.

———. 1980. The Genesis of Ideal Theory. *Archive for History of Exact Sciences* 23, 321–378.

Eisenstein, Gotthold. 1975. *Mathematische Werke*. 2 vols. New York: Chelsea.

Ewald, William Bragg. 1996. *From Kant to Hilbert, A Source Book in the Foundations of Mathematics*. 2vols. Oxford: Clarendon Press.

Folkerts, Menso. 1983–1984. Der Mathematiker E.H. Dirksen und C.F. Gauß. *Mitteilungen der Gauß-Gesellschaft Göttingen* 20–21, 66–76.

Galois, Évariste. 1962. *Écrits et mémoires mathématiques*, ed. R. Bourgne, J.-P. Azra. Paris: Gauthier-Villars.

Gauss, Carl Friedrich. 1796–1814. [Mathematical Diary.] Original manuscript in Latin: Handschriftenabteilung Niedersächsische Staats- und Universitätsbibliothek Göttingen, Cod. Ms. Gauß Math. 48 Cim. 1st comm. ed. F. Klein: Wissenschaftliches Tagebuch 1796–1914. *Festschrift zur Feier des hundertfünfzigjährigen Bestehens der Königlichen Gesellschaft der Wissenschaften zu Göttingen*. Berlin: Weidmann, 1901. Repr. *Mathematische Annalen* 57 (1903), 1–34. 2nd comm. ed.: Abdruck des Tagebuchs (Notizenjournals) (with Fac-simile). In [Gauss 1917], pp. 483–575. Ed. with German transl. by E. Schuhmann, a historical introduction by K.-R. Biermann, and new annotations by H. Wußing und O. Neumann: *Mathematisches Tagebuch 1796–1814*. 5th ed. Ostwalds Klassiker der exakten Wissenschaften 256. Leipzig: Akademische Verlagsgesellschaft Geest & Portig; Frankfurt a.M., Thun: Harry Deutsch, 2005. French comm. transl. P. Eymard, J.-P. Lafon: Le journal mathématique de Gauss. *Revue d'histoire des sciences et de leurs applications* 9 (1956), 21–51.English comm. transl. J. Gray: A commentary on Gauss's mathematical diary, 1796-1814, with an English translation. *Expositiones Mathematicae* 2 (1984), 97–130. Repr. in [Dunnington 1855/2004], pp. 409-505.

———. 1863. *Werke*, vol. II, *Höhere Arithmetik*, ed. Königliche Gesellschaft der Wissenschaften zu Göttingen. Göttingen: Universitäts-Druckerei. 2nd augm. ed., 1876.

———. 1866. *Werke*, vol. III, *Analysis*, ed. Königliche Gesellschaft der Wissenschaften zu Göttingen. Göttingen: Universitäts-Druckerei.

———. 1900. *Werke*, vol. VIII, *Arithmetik und Algebra: Nachträge zu Band 1–3*, ed. Königliche Gesellschaft der Wissenschaften zu Göttingen. Leipzig: Teubner.

———. 1917. *Werke*, vol. X.1, *Nachtraege zur reinen Mathematik*, ed. Königliche Gesellschaft der Wissenschaften zu Göttingen. Leipzig: Teubner.

———. 1922–1933. *Werke*, vol. X.2, *Abhandlungen über Gauß' wissenschaftliche Tätigkeit auf den Gebieten der reinen Mathematik und Mechanik*, ed. Königliche Gesellschaft der Wissenschaften zu Göttingen. Berlin: Springer.

GAUSS, Carl Friedrich & GERLING, Christian Ludwig. 1927. *Briefwechsel zwischen Carl Friedrich Gauß und Christian Ludwig Gerling*, ed. C. Schäfer. Berlin: Elsner. Repr. in C. F. Gauss, *Werke. Ergänzungsreihe* 3. Hildesheim: Olms, 1975.

GAUSS, Carl Friedrich & OLBERS, Wilhelm. 1900–1909. *Briefwechsel zwischen Olbers und Gauß*, ed. C. Schilling and I. Kramer. *Wilhelm Olbers, sein Leben und seine Werke*, ed. C. Schilling, vol. 2. Berlin: J. Springer. Repr. in C. F. Gauss, *Werke. Ergänzungsreihe* 4. 2 parts. Hildesheim: Olms, 1976.

GAUSS, Carl Friedrich & SCHUMACHER, Heinrich C. 1860–1865. *Briefwechsel.* 6 vols, ed. C.A.F. Peters. Altona: Esch. Repr. Hildesheim: Olms, 1975.

GEERTZ, Clifford. 1973. Thick Description: Toward an Interpretative Theory of Culture. In *The Interpretation of Cultures: Selected Essays*, pp. 3–30. New York: Basic.

GOLDSTEIN, Catherine. 1989. Le métier des nombres aux 17e et 19e siècles. In *Éléments d'Histoire des Sciences*, ed. M. Serres, pp. 274–295. Bordas: Paris. English transl. in *A History of Scientific Thought*, pp. 344–371. Oxford: Blackwell, 1995.

———. 2003. Les *Disquisitiones Arithmeticae* en France: un parcours de recherche. *Revue de la Bibliothèque nationale de France* 14, 48–55.

———. 2005. The *Vorlesungen über die Zahlentheorie* by P. G. Dirichlet. In *Landmarks of the History of Mathematics*, ed. I. Grattan-Guinness, pp. 480–490. Amsterdam: Elsevier.

GUNTAU, Martin, LAITKO, Hubert. 1987. Entstehung und Wesen wissenschaftlicher Disziplinen. In *Der Urspung der modernen Wissenschaften. Studien zur Entstehung wissenschaftlicher Disziplinen*, ed. M. Guntau, H. Laitko, pp. 17-89. Berlin: Akademie-Verlag.

HARTMANN, Klaus. 1972. Hegel: A non-metaphysical view. In *Hegel. A collection of criticial essays*, ed. A. MacIntyre, pp. 101–124. Garden City, NY: Anchor Books, Doubleday.

HAUBRICH, Ralf. 1992. *Zur Entstehung der algebraischen Zahlentheorie Richard Dedekinds*. Dissertation, Georg-August-Universität Göttingen. Göttingen.

HEATH-BROWN, David Rodney, PATTERSON, Samuel James. 1979. The distribution of Kummer sums at prime arguments, *Journal für die Reine und Angewandte Mathematik* 310, 111–130.

HEGEL, Georg Wilhelm Friedrich. 1801. *Differenz des Fichte'schen und Schelling'schen Systems der Philosophie.* Jena: Seidler. Repr. Philosophische Bibliothek 62a. Hamburg: Meiner, 1962.

———. 1812–1816. *Wissenschaft der Logik.* 2 vols. in 3 books. Nürnberg: J.L. Schrag. Repr. Philosophische Bibliothek 377. Hamburg: Meiner, 1986–1994.

HERMITE. 1893. *1822–1892. Jubilé de M. Hermite*. Paris: Gauthier-Villars.

HOUZEL, Christian. 1978. Fonctions elliptiques et intégrales abéliennes. In *Abrégé d'histoire des mathématiques 1700–1900*, J. Dieudonné (ed.), vol. II, pp. 1–113. Paris: Hermann.

Jacobi, Carl Gustav Jacob. 1829. *Fundamenta nova theoriae functionum ellipticarum.* Königsberg: Gebrüder Borntraeger. Repr. in [Jacobi 1881–1891], vol. 1, pp. 49–239.

———. 1836–1837. *Theorie der Zahlen.* Lecture notes by J.G. Rosenhain, copied by hand.[196]

———. 1839. Über die complexen Prinzahlen, welche in der Theorie der Reste der 5$^{\text{ten}}$, 8$^{\text{ten}}$ und 12$^{\text{ten}}$ Potenzen zu betrachten sind. *Monatsbericht der Akademie der Wissenschaften zu Berlin* Mai 1839, pp. 86–91. Repr. in *Journal für die reine und angewandte Mathematik* 19 (1839), 314–318. Repr. in [Jacobi 1881–1891], vol. 6, pp. 275–280. French transl. *Journal de mathématiques pures et appliquées* 8 (1843), 268–272.

———. 1881–1891. *Gesammelte Werke,* ed. C.W. Borchardt, K. Weierstrass. 7 vols. Berlin: Reimer.

Jacobi & Humboldt. 1987. *Briefwechsel zwischen Alexander von Humboldt und Carl Gustav Jacob Jacobi*, ed. H. Pieper. Berlin: Akademie-Verlag.

Jahnke, Hans Niels. 1990. *Mathematik und Bildung in der Humboldtschen Reform.* Studien zur Wissenschafts-, Sozial- und Bildungsgeschichte der Mathematik 8. Göttingen: Vandenhoeck & Ruprecht.

Jordan, Camille. 1870. *Traité des substitutions et des équations algébriques.* Paris: Gauthier-Villars.

Kiernan, B. Melvin. 1971. The Development of Galois Theory from Lagrange to Artin. *Archive for History of Exact Sciences* 8, 40–154.

Königsberger, Leo. 1904. *Carl Gustav Jacob Jacobi. Festschrift zur Feier der hundertsten Wiederkehr seines Geburtstags.* Leipzig: Teubner.

Kronecker, Leopold. 1891. Vorlesungen Sommersemester 1891. Manuscript: Strasbourg: Bibliothèque de l'IRMA. Ed. in J. Boniface & N. Schappacher, "Sur le concept de nombre dans la mathématique". Cours inédit de Leopold Kronecker à Berlin (1891). *Revue d'Histoire des mathématiques* 7 (2001), 207–275.

———. 1901. *Vorlesungen über Zahlentheorie*, vol. 1, ed. K. Hensel. Leipzig: Teubner. Repr. Berlin, Heidelberg, New York: Springer, 1978.

———. 1895–1930. *Werke*, ed. K. Hensel. 5 vols. Leipzig: Teubner. Repr. New York: Chelsea, 1968.

Kummer, Ernst Eduard. 1975. *Collected Papers*, ed. A. Weil. 2 vols. Berlin, Heidelberg, etc.: Springer.

Lacroix, Sylvestre François. 1804. *Complément des Élemens d'algèbre, à l'usage de l'École centrale des quatre-nations.* 3$^{\text{rd}}$ ed. Paris: Courcier.

Lagrange, Joseph Louis. 1770 (an vi). *Traité de la résolution des équations numériques de tous les degrés.* Paris: Duprat. 2$^{\text{nd}}$ ed., Paris: Courcier, 1808. Repr. in [Lagrange 1867–1892], vol. 8, pp. 1-370. 3$^{\text{rd}}$ ed., with [Poinsot 1808], Paris: Bachelier, 1826.

———. 1867–1892. *Œuvres*, ed. J.-A Serret. 14 vols. Paris: Gauthier-Villars, 1879. Repr. Hildesheim, New York: Olms, 1973.

Lambert, Johann Heinrich. 1764. *Neues Organon oder Gedanken über die Erforschung und Bezeichnung des Wahren und dessen Unterscheidung vom Irrthum und Schein.* Leipzig: Johann Wendler.

196. We have used the 2 copies kept at the library of IRMA, Strasbourg: n$^{\text{o}}$ L 2244 (former property of Hesse, Weierstrass) and n$^{\text{o}}$ L 3052 (former property of Kronecker, Hensel). Quotes follow the spelling and the page-by-page numbering of the former.

———. 1770. *Zusätze zu den logarithmischen und trigonometrischen Tabellen zur Erleichterung und Abkürzung der bey Anwendung der Mathematik vorfallenden Berechnungen.* Berlin: Haude und Spener. Repr. in *Opera mathematica*, ed. A. Speiser, vol. 2, pp. 1–111. Zürich: Orell Füssli, 1948.

———. 1771. *Anlage zur Architectonic, oder Theorie des Ersten und des Einfachen in der philosophischen und mathematischen Erkenntniß.* Riga: Hartknoch.

LAUBENBACHER, Reinhard, PENGELLEY, David. 1998. *Mathematical Expeditions: Chronicles by the Explorers.* New York: Springer.

LEGENDRE, Adrien-Marie. 1788. Recherches d'analyse indéterminée. *Histoire de l'Académie royale des sciences de Paris. Année 1785*, Mémoires, 465–559.

———. 1798. *Essai sur la théorie des nombres.* Paris: Duprat. Repr. with suppl., 1816. 2nd augm. ed. Paris: Courcier, 1808; supplements, 1825.

———. 1811. *Exercices de calcul intégral sur divers ordres de transcendantes et sur les quadratures.* Paris: Courcier.

———. 1825–1828. *Traité des fonctions elliptiques et des intégrales Euleriennes avec des tables pour en faciliter le calcul numérique.* 3 vols. Paris: Huzard-Courcier.

———. 1830. *Théorie des nombres.* 2 vols. Paris: Didot.

LEBESGUE, Victor-Amédée. 1837. Recherches sur les nombres. *Journal de mathématiques pures et appliquées* 2, 253-292.

LEMMERMEYER, Franz. 2000. *Reciprocity Laws from Euler to Eisenstein.* Berlin, etc.: Springer.

LUCAS, Edouard. 1891. *Théorie des nombres.* Paris: Gauthier-Villars.

MAENNCHEN, Philipp. 1930. *Gauß als Zahlenrechner.* In [Gauss 1922–1933], Abhandlung 6.

MERZ, John Theodor. 1896–1914. *A History of European Thought in the Nineteenth Century.* 4 vols. Edinburgh, London: Blackwood and sons.

MERZBACH, Uta C. 1981. An Early Version of *Disquisitiones Arithmeticae.* In *Mathematical Perspectives, Essays on mathematics and its historical development presented to Prof. Dr. Kurt-R. Biermann on the occasion of his 60th birthday*, ed. J. Dauben, pp. 167–177. New York etc.: Academic Press.

MILNOR, John. 1971. *Introduction to Algebraic K-Theory.* Annals of Mathematics Studies 72. Princeton: Princeton University Press.

MINDING, Ferdinand. 1832. *Anfangsgründe der höheren Arithmetik.* Berlin: Reimer.

NEUKIRCH, Jürgen. 1999. *Algebraic Number Theory.* Grundlehren der mathematischen Wissenschaften 322. Berlin, etc.: Springer.

NEUMANN, Olaf. 1979–1980. Bemerkungen aus heutiger Sicht über Gauss' Beiträge zu Zahlentheorie, Algebra und Funktionentheorie. *NTM-Schriftenreihe* 16:2, 22–39; 17:1, 32–48; 17:2, 38–58.

———. 1981. Über die Anstöße zu Kummers Schöpfung der "Idealen Complexen Zahlen." In *Mathematical Perspectives, Essays on mathematics and its historical development presented to Prof. Dr. Kurt-R. Biermann on the occasion of his 60th birthday*, ed. J. Dauben, pp. 179–199. New York etc.: Academic Press.

———. 2005. Carl Friedrich Gauss's *Disquisitiones Arithmeticae* (1801). In *Landmark Writings in Western Mathematics, 1640–1940*, ed. I. Grattan-Guinness, pp. 303–315. Amsterdam: Elsevier.

ORE, Oystein. 1957. *Niels Henrik Abel. Mathematician Extraordinary*. Minneapolis: University of Minnesota Press. Repr. New York: Chelsea 1974.

PEACOCK, George. 1834. Report on the Recent Progress and Present State of Certain Branches of Analysis. In *Report on the Third Meeting of the British Association for the Advancement of Science held at Cambridge in 1833*, pp. 185–352. London: Murray.

PETRI, Birgit, SCHAPPACHER, Norbert. 2004. From Abel to Kronecker. Episodes from 19th Century Algebra. In *The Legacy of Niels Henrik Abel*, ed. O.A. Laudal & R. Piene, pp. 227–266. Berlin, etc.: Springer.

POINSOT, Louis. 1808. [Review of] Traité de la résolution des équations numériques de tous les degrés … par J. L. Lagrange. *Magazin encyclopédique, ou Journal des sciences, des lettres et des arts* 4, 343–375. Repr. in [Lagrange 1770/1826], pp. v–xx.

———. 1819–1820. Mémoire sur l'application de l'algèbre à la théorie des nombres. *Mémoires de l'Académie royale des sciences. Années 1829 et 1820* (1824), 99–183.

REICH, Karin. 1996. Frankreich und Gauss, Gauss und Frankreich. Ein Beitrag zu den deutsch-französischen Wissenschaftsbeziehungen in den ersten Jahrzehnten des 19. Jahrhunderts. *Berichte zur Wissenschaftsgeschichte* 19, 19–34.

———. 2000. Die Entdeckung und frühe Rezeption der Konstruierbarkeit des regelmäßigen 17-Ecks und dessen geometrische Konstruktion durch Johannes Erchinger (1825). In *Mathesis. Festschrift zum siebzigsten Geburtstag von Matthias Schramm*, ed. R. Thiele, pp. 101–118. Berlin: Diepholz.

RIEGER, Georg Johann. 1957. Die Zahlentheorie bei C.F. Gauss. In *C.F. Gauss 1777–1855. Gedenkband anlässlich des 100. Todestages am 23. Februar 1955*, ed. H. Reichardt, pp. 37–77. Leipzig: Teubner.

RITTER, Joachim, GRÜNDER, Karlfried (eds.). 1998. *Historisches Wörterbuch der Philosophie*, vol. 10, St–T. Basel: Schwabe & Co.

SCHAPPACHER, Norbert. 1997. Some Milestones of Lemniscatomy. In *Algebraic Geometry. Proceedings Bilkent Summer School, Ankara 1995*, ed. S. Sertöz, pp. 257–290. Lecture Notes in Pure and Applied Mathematics Series 193. New York: Marcel Dekker.

———. 1998. On the History of Hilbert's Twelfth Problem. A comedy of errors. In *Matériaux pour l'histoire des mathématiques au XX^e^ siècle. Actes du colloque à la mémoire de Jean Dieudonné (Nice, 1996)*, pp. 243–273. Séminaires et Congrès 3. Paris: Société Mathématique de France.

SCHLESINGER, Ludwig. 1922. *Über Gauß' Arbeiten zur Funktionentheorie*. In [Gauss 1922–1933], Abhandlung 2.

SCHÖNEMANN, Theodor. 1845. Grundzüge einer allgemeinen Theorie der höheren Congruenzen, deren Modul eine reelle Primzahl ist. *Journal für die reine und angewandte Mathematik* 31, 269–325.

———. 1846. Von denjenigen Moduln, welche Potenzen von Primzahlen sind. *Journal für die reine und angewandte Mathematik* 32, 93–105.

———. 1850. Notiz. *Journal für die reine und angewandte Mathematik* 40, 188.

SERRE, Jean-Pierre. 1967. Complex Multiplication. In *Algebraic Number Theory*, ed. J.W.S. Cassels, A. Fröhlich, pp. 292–296. London, New York: Academic Press.

SERRET, Joseph-Alfred. 1849. *Cours d'algèbre supérieure*. Paris: Bachelier. 2nd ed. Paris: Mallet-Bachelier, 1854. 3rd ed. 2 vols. Paris: Gauthier-Villars, 1866.

SHIMURA, Goro, TANIYAMA, Yutaka. 1961. *Complex Multiplication of Abelian Varieties ad its Applications to Number Theory.* Publications of the Mathematical Society of Japan 6. Tokyo: Mathematical Society of Japan.

SMITH, Henry John Stephen. 1859–1865. Report on the Theory of Numbers. *Report of the British Association for the Advancement of Science* 1859, 228-267; 1860, 120–169; 1861, 292–340; 1862, 503–526; 1863, 768–786; 1865, 322–375. Repr. in *The Collected Mathematical Papers*, ed. J.W.L. Glaisher, vol. 1. Oxford: Clarendon Press, 1894.

SOMMER, Julius. 1907. *Vorlesungen über Zahlentheorie. Einführung in die Theorie der algebraischen Zahlkörper.* Leipzig: Teubner.

TATE, John. 1971. Symbols in Arithmetic. In *Actes du Congrès international des Mathématiciens. 1–10 septembre 1970, Nice*, vol. 1, pp. 201–211. Paris: Gauthier-Villars.

ULLRICH, Peter. 2001. Gotthold Eisensteins Exemplar der Gauß'schen *Disquisitiones Arithmeticae*, erneut betrachtet. In *Neue Welten. Wilhelm Olbers und die Naturwissenschaften um 1800*, ed. G. Biegel, G. Oestmann, K. Reich, pp. 202–221. Braunschweig: Braunschweigisches Landesmuseum.

VIERHAUS, Rudolf. 1987. Wilhelm von Humboldt. In *Berlinische Lebensbilder. Wissenschaftspolitik in Berlin,* pp. 63–76 Einzelveröffentlichungen der Historischen Kommission zu Berlin 60. Berlin: Colloquium Verlag.

VLĂDUŢ, Sergeï G. 1991. *Kronecker's Jugendtraum and Modular Functions*. New York, Philadelphia, etc.: Gordon & Breach.

WALTERSHAUSEN, Wolfgang SARTORIUS VON. 1856. *Gauss zum Gedächtniss.* Leipzig: Hirzel.

WEIL, André. 1976. *Elliptic Functions According to Eisenstein and Kronecker*. Ergebnisse der Mathematik und ihrer Grenzgebiete 88. Berlin, etc.: Springer.

———. 1979. *Œuvres scientifiques. Collected Papers.* 3 vols. New York, Heidelberg, Berlin: Springer.

———. 1984. *Number Theory. An Approach Through History from Hammurapi to Legendre.* Boston, etc.: Birkhäuser.

———. 1986. Gauss et la composition des formes quadratiques binaires. In *Aspects of Mathematics and its Applications*, pp. 895–912. North-Holland Mathematical Library 34. Amsterdam: North-Holland.

WUSSING, Hans. 1969. *Die Genesis des abtrakten Gruppenbegriffs. Ein Beitrag zur Entstehungsgeschichte der abtrakten Gruppentheorie.* Berlin: VEB Deutscher Verlag der Wissenschaften. English transl. A. Shenitzer. Cambridge: MIT Press, 1984.

———. 2001. Implicit Group Theory in the Domain of Number Theory, Especially Gauß and the Group Theory in his "Disquisitiones Arithmeticae" (1801). *Revista Brasileira de História da Matemática* 1-1, 57–65.

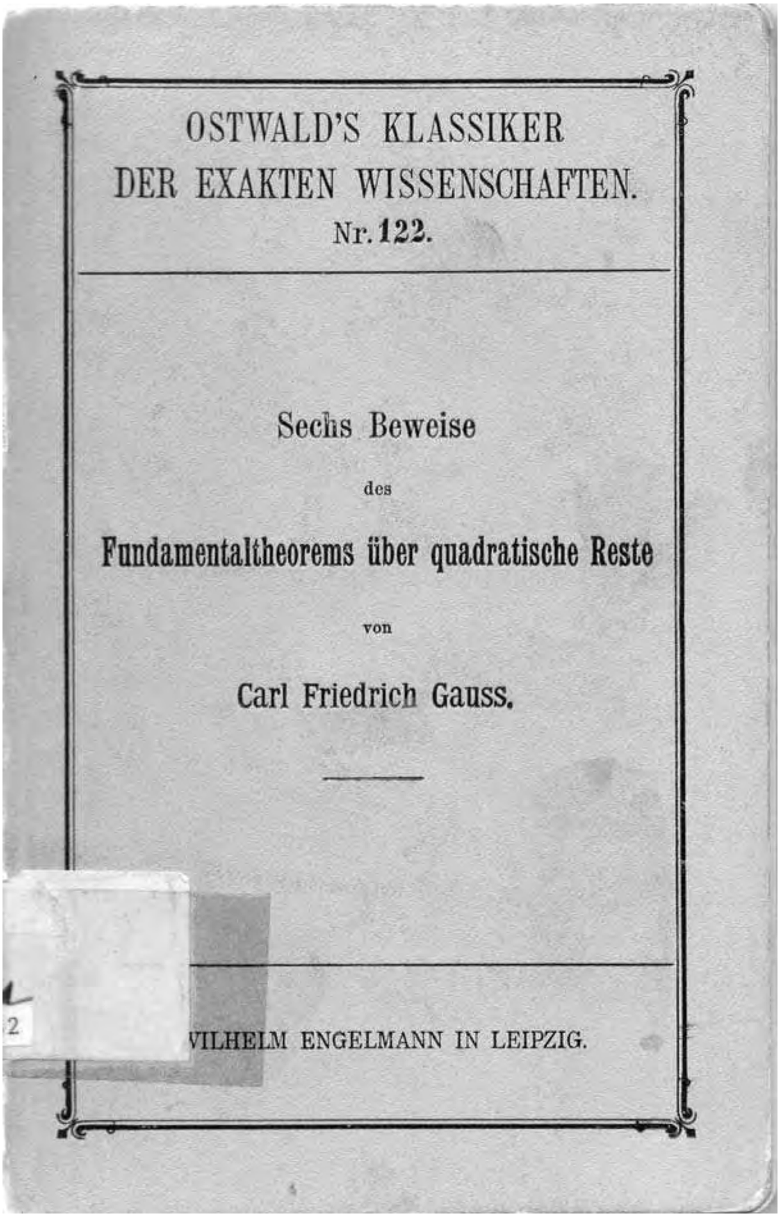

Fig. I.2A. Gauss's *theorema fundamentale* of the D.A. becomes a popular German classic: volume 122 of Ostwalds Klassiker der Exakten Wissenschaften (Courtesy of the Bibliothèque de l'IRMA, Strasbourg)

I.2

Several Disciplines and a Book (1860–1901)

Catherine Goldstein and Norbert Schappacher

Carl Friedrich Gauss died on February 2, 1855. Jacobi had died almost exactly four years earlier, and Eisenstein in 1852. Cauchy died in 1857. Dirichlet became Gauss's successor in Göttingen, and died in 1859.[1] Kummer, for his part, was then turning to geometry, publishing only an occasional number-theoretical paper. In France, Hermite's research was shifting to invariant theory and differential equations. Thus, the erstwhile proud and active leading group of European researchers in the domain opened up by the *Disquisitiones Arithmeticae* was decimated dramatically by the 1860s. Only Leopold Kronecker in Berlin represented a strong element of continuity across these years.

After Gauss's and Dirichlet's deaths, the Göttingen scene was dominated by non-arithmetical occupations with Gauss's legacy. Among the brains of deceased Göttingen colleagues that the physiologist Rudolf Wagner managed to collect and examine were Gauss's and Dirichlet's. In an attempt to go beyond weighing and superficial descriptions, he used in particular Gauss's brain and that of the professor of pathology Conrad Heinrich Fuchs to develop and calibrate new parameters. They provided him with rankings: Göttingen university professors arrived on top and orang-outangs at the bottom.[2]

1. Literary Activities

A more traditional approach to Gauss's remains was the grand publication of his collected works (*Werke*), including pieces from his *Nachlass*, his unpublished notes

1. Dirichlet's successor was Bernhard Riemann who would die in 1866, the year that Göttingen became a Prussian town. The chair was then offered to Kronecker, but he preferred to stay in Berlin, with Kummer and Weierstrass. The position was finally given to Alfred Clebsch.
2. [Hagner 2004], pp. 142–149.

and correspondence. This major undertaking of the *Königliche Gesellschaft der Wissenschaften zu Göttingen* was finally realized in two ventures, separated by some twenty years. The first phase was directed by the mathematician-astronomer Ernst Schering[3] who actually did most of the work himself, editing in particular single-handedly the first volume, a newly corrected printing of the *Disquisitiones Arithmeticae*, which appeared in 1861. For Gauss's other arithmetic publications and some related *Nachlass* material in vol. II (1863) – in particular, the early manuscript of what may have been the planned and never published sec. 8 of the D.A. – Schering enlisted the help of Richard Dedekind,[4] who was also editing Dirichlet's lectures on number theory. Schering brought out six volumes by 1874;[5] he died in 1897, at a time when a renewed interest in Gauss's *Nachlass* was growing, as Paul Stäckel's success in locating Gauss's mathematical diary in 1898 shows. The subsequent six volumes were published between 1900 and 1929, under Felix Klein's supervision[6] with the astronomer Martin Brendel as managing editor. Here the parts having to do with arithmetic, as well as with elliptic and modular functions, were commented by Paul Bachmann, Paul Stäckel, Ludwig Schlesinger, and Robert Fricke.[7] We shall see that this second phase of the Gauss edition coincided with a new boom in number theory in which Göttingen played a central role.

In the middle of these editorial achievements, concerns arose about making Gauss more readily accessible, in particular to students of mathematics.

> The handsome complete editions which have been realized of the works of almost all great mathematicians since *Lagrange* have a double significance. On the one hand, they are to be a dignified monument for those illustrious minds; on the other hand, they are to render their creations more accessible to a studious posterity than the originals, of which some are scattered in periodicals, and others have become rare. But there can be no doubt that this latter goal can only be imperfectly achieved. For the volume and the layout of the complete editions imply that – most literally – the means to ascend to these sources are rarely at the disposal of those who would like to quench their thirst there.[8]

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3. He became in a way Gauss's successor when he was appointed head of the newly created section of theoretical astronomy at the Göttingen observatory in 1867.
4. A personal conflict between the two resulted from this collaboration, see [Lipschitz 1986], p. 90. Riemann died before delivering anything useful about Gauss's *Nachlass* on elliptic functions: see Schering's grudging remark about him in [Gauss 1866], p. 492.
5. He also published (Gotha: F.A. Perthes, 1871) a new printing of Gauss's 1809 *Theoria motus corporum coelestium* in a format which looked like vol. 7, except that it was edited by himself, "member of the Göttingen Society," instead of "by the Society." Furthermore, Schering oversaw some "second printings" (*Zweiter Abdruck*) which for vol. 2, reedited in 1876, contains additional *Nachlass* material with Schering's comments.
6. Klein had obtained a chair in Göttingen in 1886.
7. Bachmann who had attended Dirichlet's lectures in Göttingen obtained his doctorate under Kummer in 1862, see § 3.1 below. The three others obtained theirs between 1885 and 1887, Stäckel and Schlesinger in Berlin, partly with Kronecker, Fricke with Klein.
8. From Heinrich Simon's preface to his German edition of Gauss's "Disquisitiones generales circa seriem infinitam …" (Berlin: Springer, 1888): *Die stattlichen Gesammt-Ausgaben,*

The problem with the D.A. for German students, besides its rarity or price, was the Latin. A selection of Gauss's texts in German (translated where applicable) was in fact published in these decades, including several issues of the popular low-budget series *Ostwald's Klassiker der exakten Naturwissenschaften*.[9] There was also a project to convince Schering[10] to translate the D.A. into German, as the following letter shows – incidentally reflecting the antagonism between Dedekind and Kronecker to which we shall return below:

> If you write once more to Professor Schering, would you please tell him roughly this: There can be no doubt about the importance of Gauss's *Disquisitiones Arithmeticae* for the development of mathematics. It is a work which holds in mathematics approximately the same position as Kant's *Critique of Pure Reason* holds in philosophy. … In spite of this eminent importance, the work is hardly read. I am convinced that a statistical investigation would show that not even 3% of all mathematicians have read the work. Professor Schering is completely right: the majority of the students get by on surrogates. It is in particular the Dirichlet-Dedekind lectures which help satisfy the desire for arithmetical knowledge. But this is precisely the trouble. Professor Kronecker, probably one of Dirichlet's best students, thinks … that it is not an advantage to see through the spectacles of commentators. The students have to study such a work themselves. … However, … not all mathematicians have the necessary linguistic faculties.[11]

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die von den Werken fast aller grossen Mathematiker seit Lagrange *veranstaltet worden sind, haben eine zwiefache Bedeutung. Sie sollen einerseits ein würdiges Denkmal jener erlauchten Geister sein, andererseits aber die Schöpfungen derselben der lernenden Nachwelt zugänglicher machen, als dies bei den Originalen der Fall ist, die teils in periodischen Schriften zerstreut, teils selten geworden sind. Nun ist nicht zu verkennen, dass der letzere Zweck nur in beschränktem Massse erreicht werden kann. Denn Umfang und Ausstattung der Gesammt-Ausgaben bringen es mit sich, dass – im prosaischsten Sinne – die Mittel, durch die man zu diesen Quellen steigt, denen, die daselbst zu schöpfen begehren, nur selten zu Gebote stehen.*

9. In this series, the following numbers reproduced works by Gauss : 2, 5, 14, 19, 53, 55, 122, 153, 167, 177, 225, 256. The last is Gauss's mathematical diary; the only strictly arithmetical little volume is no. 122, published by Eugen Netto in 1901, which contains Gauss's six proofs of the quadratic reciprocity law.

10. Springer, who at that time was just starting to get into mathematics publishing, would indeed publish such a translation four years later; but it was Hermann Maser who translated, not just the D.A., but also related texts by Gauss from vol. 2 of Gauss's *Werke*, and included Dedekind's comments with the latter's permission, [Gauss 1889]. One may note that Maser dropped Gauss's dedication of the D.A. to the archduke. Maser had published a German translation of Adrien-Marie Legendre's *Théorie des nombres* in 1886.

11. Letter of Carl Itzigsohn to Julius Springer, March 23, 1885 (Korrespondenzarchiv, Springer-Verlag Heidelberg, Abteilung A, Weierstraß 23; Weierstraß/Itzigsohn): *Wenn Sie nochmals an Herrn Professor Scheering* [sic] *schreiben, so bitte ich dem Herrn etwa Folgendes zu sagen: Welche Wichtigkeit Gauß:* Disquisitiones Arithmeticae *für die Entwicklung der Mathematik gehabt haben, darüber existirt wohl kein Zweifel. Es ist ein Werk, das ungefähr in der Mathematik dieselbe Stellung einnimmt, wie die* Kritik der reinen Vernunft *von Kant in der Philosophie. … Trotz dieser eminenten Wichtigkeit wird*

The "surrogates" (in Itzigsohn's words) included of course Dedekind's edition of Dirichlet's *Vorlesungen über Zahlentheorie*, which, as explained in chap. I.1, simplified and popularized the D.A., but also the various syntheses or textbooks which integrated parts of the D.A. in the second half of the century; an example is Bachmann's textbook on cyclotomy, [Bachmann 1872], which would finally become the third of six volumes of his "attempt at a comprehensive presentation" of number theory.[12] They provided a uniform basic training in number theory,[13] and their mediating role is testified to from many quarters by the generation born in the 1860s. The young Hermann Minkowski, for example, would read Dirichlet's *Vorlesungen über Zahlentheorie* first and then the *Disquisitiones Arithmeticae*.[14] Edmond Maillet, an engineer trained at the *Ecole Polytechnique*, who obtained the *Grand Prix* of the French Academy of Sciences in 1896 for a memoir on finite groups and was one of the rare French mathematicians to use Kummer's ideal factors, characteristically answered, when asked about his formative readings:

> On M. Jordan's advice, I read on the one hand Serret's *Algèbre supérieure*, Dirichlet-Dedekind's *Zahlentheorie*, Bachmann's *Kreistheilung*, probably some Gauss…[15]

But one may wonder if Carl Itzigsohn's complaint about the very small readership of the D.A. does not correspond to a deeper and larger phenomenon: a low point, or at least a decline[16] of the ebullient research field opened at the middle of the century, as its most important contributors left it. This at any rate is the impression conveyed by the contemporary mathematicians themselves.

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das Werk fast gar nicht gelesen. Ich bin fest überzeugt, daß eine statistische Untersuchung ergeben würde, daß nicht 3% aller Mathematiker das Werk gelesen haben. Herr Prof. Sch.[ering] hat vollkommen recht, ein großer Theil der Studirenden behilft sich mit Surrogaten. Namentlich sind es die Dirichlet-Dedekindschen Vorlesungen, welche das Bedürfnis nach arithmetischen Kenntnissen befriedigen helfen. Indeß hierin liegt eben gerade der Fehler. Herr Prof. Kronecker, wohl einer der besten Schüler Dirichlet's, ist der Ansicht …, daß es kein Vorteil ist, durch die Brille von Commentatoren zu sehen. Die Studenten müssen ein derartiges Orgina[le]s [sic] *Werk selbst studiren. … Indeß, … stehen nicht allen Mathematikern diejenigen philologischen Hilfsmittel zu Gebote.* We heartily thank Reinhard Siegmund-Schultze for having shared this letter and his transcription of it with us. About Itzigsohn, see [Bölling 1994], pp. 11–20.

12. The whole series, mostly written after Bachmann's early retirement from his Münster chair in 1890, comprises 5 parts in 6 physical volumes. [Bachmann 1872] was reedited posthumously in 1921 by Robert Haussner, as vol. 3 of the series. Bachmann also published other number-theoretical books, for instance on Fermat's Last Theorem.
13. Gaston Darboux, reviewing the third edition of Dirichlet's *Vorlesungen* in the *Bulletin des sciences mathématiques et astronomiques* 3 (1872), p. 168, stated that "the order followed in the book is that adopted by all the professors."
14. See [Strobl 1985], p. 144, and J. Schwermer's chap. VIII.1 below.
15. This appears in a survey organized by the journal *L'Enseignement mathématique* 8 (1906), p. 222: *Sur le conseil de M. Jordan, je lus* l'Algèbre supérieure *de Serret, la* Zahlentheorie *de Dirichlet-Dedekind, la* Kreistheilung *de Bachmann, du Gauss probablement…*
16. Ralf Haubrich speaks of *Niedergang* as far as higher reciprocity laws were concerned, [Haubrich 1992], p. 35.

Dedekind published his own development of Kummer's work on algebraic numbers – in particular his theory of ideals, to which we shall return – within the supplements to his successive editions of Dirichlet's *Vorlesungen*, as "the safest means to win a larger circle of mathematicians to work in this field."[17] But his expectations were disappointed. On March 11, 1876, Rudolf Lipschitz proposed a French translation of the relevant supplement, regretting at the same time that the theory it contained was "not appreciated at its true value, even in Germany." In his grateful reply of April 29, Dedekind wrote that until now only one person, Heinrich Weber, had expressed an active interest in his work.[18]

On the more classical topic of quadratic forms, Hermite wrote to Henry Smith in 1882 about a prize posted by the Paris Academy on the representation of integers as sums of five squares:

> Until now, I do not know of any paper submitted. This is explained by the direction of the mathematical trend which does not go now toward arithmetic. You are the only one in England to follow the path opened by Eisenstein. M. Kronecker is the only one in Germany; among us, M. Poincaré, after putting forward some good ideas on what he calls arithmetical invariants, now seems to think only about Fuchsian functions and differential equations.[19]

Dedekind's and Hermite's declarations seem to agree. But the fact that the names mentioned in both are different points to another process: the dissociation of several components of what we have called arithmetic algebraic analysis in chap. I.1. The research work on the theory of forms, for instance, was more and more oriented toward algebra, in particular invariant theory, with no concern for the nature of the coefficients of the forms, as is witnessed by the works of Francesco Faà di Bruno, Arthur Cayley, Alfred Clebsch, and others.[20] Meanwhile, complex analytic functions were developing into an independent topic. And the track from them to congruences and reciprocity laws seemed progressively washed out.

2. Citation Networks

To go beyond such local testimonies and get a global view of the situation, historians have at their disposal a new tool in the 1870s, the *Jahrbuch über die Fortschritte*

17. See [Dedekind 1930–1932], vol. 3, p. 464. Dedekind added his theory inside a new supplement (the tenth) on the composition of forms, in the second edition in 1871; he would later expand and rewrite it, rendering it autonomous as an eleventh supplement to the editions of 1879 and 1894.
18. [Lipschitz 1986], pp. 47–49. Quote on p. 47: *dabei selbst in Deutschland nicht nach ihrer vollen Gebühr allgemein gewürdigt werden.*
19. [Smith 1894], vol. 1, p. lxvii: *Jusqu'ici je n'ai pas eu connaissance qu'aucune pièce ait été envoyée, ce qui s'explique par la direction du courant mathématique qui ne se porte plus maintenant vers l'arithmétique. Vous êtes seul en Angleterre à marcher dans la voie ouverte par Eisenstein. M. Kronecker est seul en Allemagne, et chez nous, M. Poincaré, qui a jeté en avant quelques idées heureuses sur ce qu'il appelle les invariants arithmétiques semble maintenant ne plus songer qu'aux fonctions fuchsiennes et aux équations différentielles.* On the prize, see J. Schwermer's chap. VIII.1.
20. At least the latter two also integrated projective geometry into their agenda.

der Mathematik. These yearly volumes edited in Berlin with the support of Borchardt, Kronecker and Weierstrass, provided reviews of all mathematical publications. Number theory occupied the 3rd section, after a section on pedagogical and historical questions, and one on algebra.[21] It included elementary arithmetic (*Niedere Zahlentheorie*, that is, here, school arithmetic); higher arithmetic, divided into generalities and the theory of forms; continued fractions.[22] A quantitative analysis of the *Jahrbuch* reveals that between 1870 and World War I, higher arithmetic filled up 3.5% to 4% of the pages of the *Jahrbuch*, which gives some weight and some perspective to Itzigsohn's evaluation above, but still represents some 3500 papers of all kinds, from short notes on a Diophantine question in the *Educational Times* to book-length memoirs in the *Journal für die reine und angewandte Mathematik*.[23]

A finer structuring of this corpus of publications is of course necessary. It can be obtained by pursuing the web of mutual references, implicit and explicit.[24] This allows us to verify that higher arithmetic was parcelled out and to distinguish several clusters of articles; inside each cluster, references to each other's articles and results are frequent, the papers are often close in terms of methods, points of view, or objectives; at least an awareness of the works of others is indicated, while the *mathematical* exchanges between two different clusters of articles are limited, in some cases even tainted by technical misunderstanding or disapproval.[25] For reasons of space, we only delineate briefly here the (numerically) most important clusters

21. This place may seem natural to us. However, the *Répertoire bibliographique des sciences mathématiques*, a parallel enterprise begun in the 1880s, would classify number theory almost at the end of the analysis section (under the letter "I," i.e., in the ninth place, as an *application* of analysis). On the *Répertoire*, see [Rollet, Nabonnand 2002]. The history of the *Jahrbuch* is discussed in [Siegmund-Schultze 1993].

22. During the first decade of the *Jahrbuch*, generalities, theory of forms, and continued fractions were the three subsections of number theory. The separate section on continued fractions reminds us of the numerical importance of the topic during the XIXth century, where it was linked in particular to approximation; see [Brezynski 1991].

23. For more details, in particular on the distribution according to countries and on the evolution through time, see [Goldstein 1994] and [Goldstein 1999].

24. Such an approach was advocated decades ago by several authors, in particular Eugène Garfield, Derek Price, and Thomas Kuhn. To identify research groups, Kuhn suggested "the recourse … above all to formal and informal communication networks including those discovered in correspondence and in the linkage among citations," [Kuhn 1970], p. 178. See also [MacKenzie 1986]. However, applying this idea to XIXth century mathematical texts like ours to study content-related issues is not straightforward, and automatic quotation indexation would not be adequate; see [Goldstein 1999].

25. Examples involving Ernest de Jonquières and Rudolf Lipschitz, or James Sylvester and Dedekind, are given in [Goldstein 1994]. We would like to stress here that our groupings of texts *do not* usually coincide with the personal links between mathematicians, that is with the "communication networks," and thus of course have no bearing on their works in other domains. For instance, Hermite entertained a friendly correspondence with Sylvester and had many mathematical exchanges on invariant theory, but disagreed on his number-theoretical choices for his students at Johns Hopkins, see [Parshall 1998], p. 221 and chap. VI.1 below.

and their respective relations to the D.A.

The largest group by far in the last decades of the XIX$^{\text{th}}$ century – we shall designate it as "cluster L-G" (for Legendre and Gauss) – attests to the interest in number theory inside and outside the academic milieu, some contributors being engineers or highschool teachers, others university professors. However, it also counts among the authors of its papers renowned figures like James Sylvester, Angelo Genocchi, and Edouard Lucas.[26] As these names suggest, the group is quite international, though German authors are underrepresented in it. These articles witness the vitality of research themes dating back to the first decades of the XIX$^{\text{th}}$ century: primitive roots, Legendre symbol and quadratic reciprocity, prime numbers, as well as cyclotomic and Diophantine equations.[27] Accordingly, they will find their sources in both Legendre's *Théorie des nombres* and the D.A., sometimes quoted through a "surrogate," in particular the first chapters of Dirichlet's *Vorlesungen*. The most distinctive common feature is negative: these papers avoid recourse to complex functions, sometimes also to complex numbers, and some of the authors make their opposition to the use of analysis in number theory quite explicit. Edouard Lucas for instance presented his book project in these terms to Ernesto Cesàro:

> My treatise is based on a scheme totally different from anything that exists; there is no notion of continuity, exponential, logarithm, not even a $\sqrt{2}$.[28]

Several authors, like Genocchi or Lucas, were also involved in historical work, retrieving and editing texts of medieval and early-modern algebraists, from Fibonacci to Fermat, sometimes even promoting them as a source for mathematical inspiration. From the D.A., they borrowed of course research topics, but also a taste for thorough and precise explorations of their subject, with no external techniques. The lack of advanced tools did not prevent these articles from being innovative and fruitful, like Sylvester's on cubic ternary equations or Lucas's on primality.[29] Indeed, this approach was not only in favour among outsiders from the academic milieu, it was also a privileged entrance door to number theory in places where no advanced academic tradition existed in this domain and thus is important to take into account in an international perspective.[30] However, most of the authors of this group did not train students in number theory[31] and with time and the development of multi-country

26. On Sylvester, see [Parshall 2006]; on Genocchi, see [Conte, Giacardi 1991] and A. Brigaglia's chap. VII.1 below; on Lucas, see [Décaillot 1999] and chap. VI.2.

27. See [Dickson 1919–1923], vol. 1 and vol. 2.

28. Letter of October 4, 1890, quoted in [Décaillot 1999], vol. 1, p. 64 and p. 154: *Mon ouvrage repose sur un plan absolument différent de tout ce qui existe; on n'y trouve aucune notion de continuité, d'exponentielle, de logarithme, pas même* $\sqrt{2}$. We avoid using the tempting word "elementary" to describe these papers, because this word had a different use in the 1870s.

29. On Sylvester's algebraico-geometric perspective, cf. [Schappacher 1991] and [Lavrinenko 2002]. On Lucas and his relation to the D.A., see A.-M. Décaillot's chap. VI.2 below.

30. See on this question the issues raised concerning Leonard Dickson's *History of the Theory of Numbers* in D. Fenster's chap. VII.3 below.

31. One of the rare exceptions is Sylvester, who launched a small cohort of mathematicians

education,[32] most of these papers tended to drift away from the research journals.

A second group of articles, say the cluster D (for Dirichlet), shares not only interreferences but also citations to Dirichlet's analytic work. Well represented until the 1890s, it then first declined, but came back to the forefront in the first decade of the XX[th] century with a more sophisticated, complex-analytic approach, inherited from Riemann and centering around Dirichlet series, i.e., series of the type $\sum_n \frac{a_n}{n^s}$, for a complex variable s and (real or complex) coefficients a_n. In the 1870s and 1880s, a small industry also developed around arithmetical functions, i.e., functions defined only on integers, such as the function giving the sum of divisors of an integer or the function counting the number of prime factors. This topic is illustrated by names such as Pafnuti Čebyšev, Ernesto Cesàro, Nicolai Bugaiev, but also Joseph Liouville, James Glaisher, and Leopold Gegenbauer.[33] Linked on the one hand to the evaluation of mean properties and asymptotic values (and thus referring to Dirichlet's seminal articles on this topic), it connects on the other hand to the cluster L-G since certain constructions allow one to dispense with advanced complex analysis. Direct references to the relevant articles of the D.A. (arts. 302, 304, etc.) are rare, serving merely to indicate the historical origin of some of the questions.

The papers of the third group – which we shall call H-K (for Hermite and Kronecker) – became sporadic during the period considered, which is compatible with Hermite's remark to Smith quoted above; they were, however, often seminal papers of otherwise well-known mathematicians, with central academic positions: from Emile Picard via Leo Königsberger to Aleksandr Nikolaevič Korkin, and Luigi Bianchi. Thematically, they dealt with modular equations and above all with the arithmetic theory of forms[34] developing in particular the explicit theory of ternary, and then quaternary, quadratic forms, as well as the concepts needed to adapt Gauss's classification to general n-ary forms.[35] The main common references here are Hermite's articles on continuous reduction and the work of Hermite and Kronecker which is situated at the junction of the arithmetic of forms and elliptic functions. References to the D.A. again read more and more like historical notes rather than mathematical reference points, even though we have many testimonies that their authors read and pondered Gauss's book. This cluster of articles appears to continue the tradition of arithmetic algebraic analysis for a few decades,[36] but explicit connections to con-

working on partitions. Of course, again, we are taking into account here only Sylvester's articles which are reviewed under the label "number theory," not his articles on algebraic invariant theory for instance.

32. One can compare in this respect the two cases, separated by 40 years, of Angelo Genocchi and Luigi Bianchi, in A. Brigaglia's chap. VII.1 below.

33. For the work of Čebyšev and Bugaiev, see [Ozhigova, Yuškevič 1992], pp. 171–201.

34. But the papers belonging to this group do *not* coincide with those reviewed in the section on forms of the *Jahrbuch*; for instance, some papers like those of Théophile Pépin reviewed in this section refer only to the D.A. and clearly belong to the cluster L-G.

35. On these topics, see C. Houzel's chap. IV.2, C. Goldstein's chap. VI.1, A. Brigaglia's chap. VII.1, J. Schwermer's chap. VIII.1.

36. Geometrical interpretations in them, in particular via lattices, follow Gauss's 1831 com-

gruences and reciprocity laws were generally abandoned.[37] During the summer of 1844, when Kummer began his investigations, Eisenstein wrote to Moritz Stern:

> The difficulty [of higher reciprocity laws] depends on the first elements of the complex numbers about which not much is known now. … Also Jacobi agrees completely with me that the theory of general complex numbers can only be accomplished by a complete theory of *higher forms*.[38]

The study of decomposable forms – forms of higher degree which factor over a given domain of algebraic (complex) numbers, and are expressible in terms of norms of those numbers – appeared to the cluster H-K, as it did in Hermite's programme, to be a viable alternative to Kummers's theory of ideal factors.[39]

3. Beyond Kummer and Back to Gauss

The reader familiar with the development of number theory during the XIX$^{\text{th}}$ century as it is usually presented may be surprised by our survey and wonder what happened to ideal factors and Kummer's achievements after the 1860s. An early announcement by Kummer himself in 1859, promoting a generalization of his theory to all algebraic numbers about to be published by Kronecker, hinted again at decomposable forms:

> Regarding … the general propositions which are common to all theories of complex numbers, I may also refer to a work by *Herr Kronecker* which will appear soon, and in which the theory of the most general complex numbers, in its connection with the theory of decomposable forms of all degrees, is developed completely and in magnificent simplicity.[40]

But Kronecker did not publish anything along these lines until the 1880s, and when he did, his theory had to compete with Dedekind's theory of ideals already alluded to above. Together, these works are usually seen as paving the way from the D.A. to the domain of number theory called " algebraic number theory," which blossomed in the first decades of the XX$^{\text{th}}$ century. Algebraic number theory is often perceived as a natural extension of the D.A., via Kummer's work and that of Dedekind and Kronecker. The global description of number-theoretical publications presented above

ments on August Ludwig Seeber's thesis. On this development towards Minkowski's geometry of numbers and beyond, see J. Schwermer's chap. VIII.1 below.

37. One of the exceptions is the work of Kronecker discussed in chap. I.1, § 4.3. See also [Dickson 1919–1923], vol. 3, chap. XIX.

38. [Eisenstein 1975], vol. 2, p. 793: *aber die Schwierigkeit hängt hier von den ersten Elementen der complexen Zahlen ab, über welche man noch gar nichts weiß. … Auch Jacobi ist ganz meiner Ansicht, dass die Theorie der allgemeinen complexen Zahlen erst durch eine vollständige Theorie der* höheren Formen *ihre Vollendung erhalten kann.*

39. See [Haubrich 1992], p. 36–37, and the literature cited there which proves the continuing activity in the field of decomposable forms all through the XIX$^{\text{th}}$ century.

40. [Kummer 1975], vol. 1, p. 737: *Ich kann in Betreff … der allgemeinen Sätze, welche allen Theorien complexer Zahlen gemein sind auch auf eine Arbeit von Hrn.* Kronecker *verweisen, welche nächstens erscheinen wird, in welcher die Theorie der allgemeinsten complexen Zahlen, in ihrer Verbindung mit der Theorie der zerlegbaren Formen aller Grade, vollständig und in großartiger Einfachheit entwickelt wird.*

shows that there were alternative paths of development. We shall now reevaluate the relation to the D.A. of these articles taking up and generalizing Kummer's theory, and more generally of algebraic number theory.

In the 1870s, a mere handful of papers[41] – the small number confirms Dedekind's disillusion and the decline of interest in these questions at the time – were devoted to generalizing Kummer's theory of ideal numbers from the domain of complex numbers generated by the λ^{th} roots of unity, to domains generated by the roots of other equations.[42] The difficulties of such a generalization included the right choice of domain; e.g., if a and b vary over all integers, the domain of numbers of the form $a+b\sqrt{-3}$ does not admit unique prime factorization, while the slightly larger domain of numbers $a+b\big(\frac{-1+\sqrt{-3}}{2}\big)$, which is generated by a 3^{rd} root of 1, does. Another, related, problem was to find a correct invariant to play the role of the discriminant of the cyclotomic equation in Kummer's setting, and to characterize the primes which divide it.[43]

3.1. Early Attempts

Three mathematicians took up this task in the 1860s: Richard Dedekind (1831–1916), Eduard Selling (1834–1920), and Paul Bachmann (1837–1920). They had all attended Dirichlet's lectures in Göttingen at the end of the 1850s, and had read the D.A. early on. Kronecker played for them a role which bears some analogy to Gauss's mighty shadow hanging over Abel, Jacobi, and others some thirty years before. Selling, for instance, admitted in his paper:

> Both my timidity to touch the big questions connected with the theory of these numbers, and my knowing that *Herr Kronecker* has been pursuing these investigations for a long time and has already overcome all the difficulties which had stopped me at first, and the hope that the mathematical public would soon be gratified by an extensive publication of his results, has so far kept me from publishing this study

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41. Besides those named below, one might add Arnold Meyer's dissertation. Born in 1844, Meyer studied with Karl Weierstrass and Kummer in Berlin, then with Hermite in Paris, before joining Ludwig Schäfli in Zürich. His dissertation written in 1870–1871 used ideal factorization in domains generated by roots of a cubic equation as a means to study certain decomposable ternary forms; however, his subsequent work on ternary forms belongs strictly to our cluster H-K of papers and the dissertation itself was published only in 1897.
42. While establishing his higher reciprocity laws, Kummer himself began the task for domains generated by all λ^{th} roots of unity and a root of $X^\lambda - a$, where a itself is a λ-cyclotomic number.
43. The two problems we are alluding to here are that of hitting on the notion of algebraic integer, and that of correctly defining the discriminant of a number field (which in general is a proper divisor of the greatest common divisor of the discriminants of all algebraic integers of the field) and to relate it to ramification; see [Edwards 1980], pp. 330–337, and [Haubrich 1992], pp. 57–59. Dedekind gave $\mathbf{Q}(\alpha)$, where $\alpha^3 - \alpha^2 - 2\alpha - 8 = 0$, as an example of an extension where taking the g.c.d. of the discriminants of the minimal polynomials of the elements leads to a parasitic factor, namely 2.

which I had elaborated in less generality already in the autumn of 1859.[44]

Selling's goal was to handle the domain generated by all the roots of an arbitrary irreducible polynomial f with integer coefficients. His work is most interesting as an early courageous attempt[45] to generalize Kummer directly, determining the ideal prime decomposition of a rational prime p in this domain by reading f modulo (powers of) p. Mathematically, his convoluted style makes difficult reading.[46] Kurt Hensel would later toil to understand it and, even though he did not like the experience, it is very likely that he received key insights from it for his introduction of p-adic numbers.[47]

But after this attempt, Selling turned to the more convivial topic of ternary forms in the Hermitian tradition and his few number-theoretical papers then belong to the cluster H-K mentioned above.

As for Paul Bachmann, he returned from Göttingen to Berlin where he obtained his dissertation on group theory under Kummer's supervision in 1862 and prepared for his *Habilitation* on complex units which he obtained at Breslau University in 1864. In 1867, he published in the *Journal für die reine und angewandte Mathematik* a short article on the arithmetic of complex numbers generated by two independent square roots: he studied the prime ideal factors, the units, the class number. But then, like Selling, he devoted his subsequent number-theoretical papers to ternary forms, reproving and completing some of Hermite's results.[48]

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44. [Selling 1865], p. 17: *Sowohl die Scheu, die grossen mit der Theorie dieser Zahlen zusammenhängenden Fragen zu berühren, als die Kenntniss davon, dass Herr* Kronecker *dieselben Untersuchungen seit lange* [sic] *pflege und bereits alle Schwierigkeiten, die mir zunächst ein Ziel gesetzt hatten, überwunden habe und die Hoffnung, dass das mathematische Publikum sich bald einer ausgedehnten Veröffentlichung seiner Resultate zu erfreuen habe, hatte mich bisher abgehalten, diese in geringerer Allgemeinheit schon im Herbste 1859 ... ausgearbeitete Untersuchung zu veröffentlichen.* Dedekind would also cautiously refer to Kronecker's announced, und still unpublished, theory in the prefaces to his second and third editions of Dirichlet's *Vorlesungen*, in 1871 and 1879, where he explained his own theory. To be fair, one has to add that the Bavarian Ministry appointed Selling *Extraordinarius* at Würzburg in 1860, against the vote of the faculty, on the strength of a letter from Kronecker.
45. Partly anticipated by unpublished work of Dedekind; see [Haubrich 1992], p. 164–165.
46. Indeed, it seems to have fooled most commentators, even Bourbaki (Weil) who wrote that Selling's boldness could only lead to nonsense; see the historical note on commutative algebra and algebraic number theory in their *Eléments d'histoire des mathématiques*. One crucial problem for a modern reader is Selling's misleading notation – for instance [Selling 1865], p. 23: "$f(j) \equiv 0 \pmod{p}$" – which suggests that he worked over a finite field $\mathbf{F}_{p^r}$, while he meant a higher congruence modulo p which he could later refine modulo powers of p.
47. We are indebted to Hans-Joachim Vollrath and Birgit Petri for all our information about Selling; see B. Petri's forthcoming thesis on Kurt Hensel for more details.
48. A good testimony of the links expected inside our cluster H-K in the 1870s (here centred around ternary quadratic forms) is provided by a letter of Georg Cantor to Richard Dedekind of November 1873, see [Dugac 1976], p. 226. Having received a copy of

3.2. Dedekind's Ideals

Richard Dedekind had been a kind of mentor for Selling and Bachmann – they attended his Göttingen lectures – and he was to be the most influential of the three.[49]

According to his friend, Hans Zincke, Dedekind had studied the D.A. already when he was a pupil at the Braunschweig *Collegium Carolinum* in 1849–1850.[50] In Göttingen, however, Dedekind was above all influenced by Dirichlet and Riemann with whom he shared a predilection for conceptual definitions. He would explain to Rudolf Lipschitz in 1876 how, in his number-theoretical work, he had tried to "build research not on accidental forms of presentation but on simple basic concepts."[51] Twenty years later, he would still put forward the same point, this time in his memories of the D.A.:

> I first recall a beautiful passage from the *Disquisitiones Arithmeticae* which had made the deepest impression on me already in my youth: "But in our judgment, such truths ought to be extracted from notions rather than notations." These last words, when taken in the most general sense, express a great scientific idea, a preference for the intrinsic against the extrinsic. This antithesis repeats itself in mathematics in almost all fields; for instance Riemann's definition of functions via intrinsic characteristic properties from which the extrinsic forms of representations necessarily flow.[52]

Dedekind started his own investigations with an algebraic project, linked to the

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Bachmann's work on ternary forms and learning that Selling would publish on this theme, Cantor reminded Dedekind of his own habilitation of 1869 on the same subject, and wondered if they would quote Smith's work of 1867.

49. Dedekind's work has been extensively studied. We shall be very brief here, stressing above all his relation to the D.A. See [Dugac 1976], [Edwards 1980], [Edwards 1983], [Edwards 1992b], [Edwards, Neumann, Purkert 1981], [Neumann 1979–1980], and the very thorough thesis [Haubrich 1992] which regrettably remains unpublished.

50. [Dugac 1976], p. 14, citing Zincke's 1916 "Reminiscences of Richard Dedekind." Hans (Zincke) Sommer was to become both a musicologist and a specialist of optics at Braunschweig. Dedekind had also studied, before 1855, H. Seeger's notes of Dirichlet's lectures on the applications of calculus to number theory and on number theory.

51. Letter of June 10, 1876, in [Lipschitz 1986], p. 60, quoted in [Haubrich 1992, p. 12]: *die Forschung nicht auf zufällige Darstellungsformen der Ausdrücke sondern auf einfache Grundbegriffe zu stützen.* In [Dugac 1976], p. 71, it is recalled that substituting concepts for computations was for Dirichlet a hallmark of modern analysis, as he wrote in his obituary of Jacobi.

52. [Dedekind 1930–1932], vol. 2, pp. 54–55, with a reference to D.A., art. 76: *Ich erinnere zunächst an eine schöne Stelle der Disquisitiones Arithmeticae, die schon in meiner Jugend den tiefsten Eindruck auf mich gemacht hat:* At nostro quidem judicio huiusmodi veritates ex notionibus potius quam ex notationibus hauriri debebant. *In diesen letzten Worten liegt, wenn sie im allgemeinsten Sinne genommen werden, der Ausspruch eines großen wissenschaftlichen Gedankens, die Entscheidung für das Innerliche im Gegensatz zu dem Äußerlichen. Dieser Gegensatz wiederholt sich auch in der Mathematik auf fast allen Gebieten; man denke nur an die Funktionentheorie, an Riemanns Definition der Funktionen durch innerliche charakteristiche Eigenschaften, aus welchen die äußerlichen Darstellungsformen mit Notwendigkeit entspringen.*

elaboration of a theory of higher congruences.[53] He studied Gauss's cyclotomy in depth, read Abel and Galois, and taught this topic at Göttingen. This led him first to reshape Gauss's presentation which failed to meet his main criterion:

> In Gauss's synthetic presentation the aim to be brief won out over the principle of deducing everything from a unified algebraic idea. When I first thoroughly studied cyclotomy over the Whitsun holidays in 1855, I had to fight for a long time, even though I understood all the details, to recognize irreducibility as the principle which would drive me with necessity to all the details once I asked simple, natural questions about it.[54]

In this perspective, the auxiliary (irreducible) equation, whose roots are the e so-called periods of f terms ($ef = \lambda - 1$, λ a prime number) associated to the cyclotomic equation in Gauss's sec. 7, corresponds to the subgroup of order f of the Galois group $(\mathbf{Z}/\lambda\mathbf{Z})^*$. From there Dedekind turned to Kummer's theory in December 1855 trying to find an algebraic core in Kummer's computational arguments.[55] This first stage of Dedekind's theory, which he never published, suffered according to his own account from an unsatisfactory concept of the discriminant and from a number of exceptions requiring specific treatments.

The second stage, delayed by his editorial activities for the collected works of Gauss, Dirichlet, and later Riemann, was finally written up, as mentioned above, in Supplement X of the 2nd edition of Dirichlet's *Vorlesungen* in 1871, and completed and reformulated in Supplement XI of the subsequent editions (1879 and 1894).[56] Although the supplement as a whole is devoted to Gauss's theory of composition of binary quadratic forms, Dedekind passed quickly from Gauss and Dirichlet to Kummer's theory of ideal numbers, and then to his own "higher standpoint," introducing a concept

53. See [Scharlau 1982]. Dedekind's paper was surely written before he saw Gauss's manuscript of the *Caput octavum* which he would edit in vol. 2 of Gauss's *Werke*; see [Haubrich 1992], pp. 139–145, and G. Frei's chap. II.4, § 3.2.1 below.

54. [Dedekind 1930–1932], vol. 3, pp. 414–415, quoted in [Haubrich 1992], p. 94: ... *daß in der synthetischen Darstellung von Gauß das Streben nach Kürze den Sieg über die Forderung davongetragen hat, alles aus einem einheitlichen algebraischen Gedanken abzuleiten. Bei meinem ersten gründlichen Studium der Kreisteilung in den Pfingstferien 1855 hatte ich, obgleich ich das Einzelne wohl verstand, doch lange zu kämpfen, bis ich in der Irreduktibilität das Prinzip erkannte, an welches ich nur einfache, naturgemäße Fragen zu richten brauchte, um zu allen Einzelheiten mit Notwendigkeit getrieben zu werden.* Cf. O. Neumann's chap. II.1 below.

55. He actually found an error in it in 1857, as did Cauchy and other French mathematicians who were scrutinizing Kummer's articles in the context of Fermat's Last Theorem, the topic of the 1857 prize of the Paris Academy; see [Edwards 1975]. Dedekind's notes on the gap were communicated by Dirichlet to Liouville; they are reproduced in [Haubrich 1992], appendix 2, pp. 192–194.

56. Our topic being the history of the D.A., we do not go into the differences between the successive versions of Dedekind's theory. We refer instead to [Dedekind 1930–1932], vol. 1, pp. 202–203, and to the two studies [Edwards 1980] and [Haubrich 1992], chap. IX, which look at this evolution from opposite angles.

> that seems well adapted to serve as a foundation for higher algebra and the parts of number theory linked with it,[57]

that of a field (*Körper*), i.e., for him, a system of real or complex numbers stable under the four usual operations. He mostly concentrated on those generated over the rationals by a finite number of algebraic numbers. An algebraic integer in a given field is *defined* as a number of the field satisfying a polynomial equation with leading coefficient 1 and (ordinary) integer coefficients. Dedekind also introduced "ideals," i.e., systems of numbers stable under addition, subtraction and multiplication by the integers of the field; the name alludes of course to Kummer's ideal numbers. As is well known, he defined a concept of primality for these ideals and obtained a complete theory of unique prime factorization, at the level of ideals.[58] A few years later, in collaboration with Heinrich Weber, he successfully transposed the same ideas to the theory of algebraic functions. The fully fledged theory of ideals would in turn serve as a model for so-called *modern algebra* after World War I, in particular in the works of Emmy Noether.

While most authors before the 1860s had been happy to explore families of explicitly given complex numbers, in order to prove new cases of the reciprocity laws, Dedekind on the contrary, wanted to

> deflect attention from the integers and thus to let the concept of a finite field [i.e. here, a field of finite degree over the rationals] Ω come to the fore more clearly.[59]

This emphasis should be appreciated with respect to Dedekind's global view of the structure of mathematics. His brand of logicism, developed in parallel with his theory of ideals and fields, would base pure mathematics on numbers, and thus ultimately on set-theoretical concepts inherited from them, such as the notion of field. According to Dedekind's view, they alone provided the theory with both unity and rigour.[60] For Dedekind, pure mathematics *was arithmetic*, but an arithmetic which amalgamated number theory and algebra. He would thus describe the theory of equations itself as the "science of the relationship between fields" (*die Wissenschaft von der Verwandtschaft der Körper*); in turn, he would perceive the concept of field arithmetically, describing the inclusion of fields, for instance, as a division.[61] The

57. [Dedekind 1930–1932], vol. 3, p. 224: ... *welcher wohl geeignet scheint, als Grundlage für die höhere Algebra und die mit ihr zusammenhängenden Teile der Zahlentheorie zu dienen.*

58. He also saw in 1873 how Gauss's theory of cyclotomy could be presented in terms of fields and "substitutions," i.e., field homomorphisms; see [Dedekind 1930–1932], vol. 3, pp. 414–416.

59. Letter to Lipschitz on June 10, 1876, [Lipschitz 1986], p. 61: ... *um zunächst die Aufmerksamkeit von den ganzen Zahlen abzulenken und dadurch den Begriff eines endlichen Körpers* Ω *deutlicher hervortreten zu lassen.*

60. See Dedekind's *Was sind und was sollen die Zahlen?*, [Dedekind 1930–1932], vol. III, pp. 335–391. On Dedekind and arithmetization, see [Dugac 1976], [Ferreirós 1999], as well as B. Petri's and N. Schappacher's chap. V.2 below.

61. [Dedekind 1930–1932], vol. 3, p. 409. Cf. [Haubrich 1992], pp. 106–107, and [Corry 1996/2004], Part One, chap. 2.

theory of number fields and their ideals incarnates Dedekind's whole programme.

3.3. Zolotarev's Theory of Algebraic Numbers

Another successful generalization of Kummer's theory was proposed in the same decade by Egor Ivanovič Zolotarev. A student of Čebyšev's at Saint-Petersburg, Zolotarev was led to this question as an auxiliary problem for treating certain integrals; see for instance [Zolotarev 1880], p. 51. Using a mixture of elementary arithmetic and Jacobi's theory of transformations, he had proved in [Zolotarev 1872] the validity of an algorithm proposed by Čebyšev to determine the reducibility of integrals of the form $\int \frac{(x+A)dx}{\sqrt{x^4+\alpha x^3+\beta x^2+\gamma x+\delta}}$ for rational coefficients $\alpha, \beta, \gamma, \delta$; he now wanted to generalize these results to arbitrary algebraic coefficients.

Just as Selling and Dedekind had initially set out, Zolotarev followed Kummer and tried to describe the (ideal) factorization of the prime number p in a domain generated by a root α of the algebraic equation $F(x) = 0$ by factoring F modulo p and then defining the divisibility of an element of the domain by a corresponding "ideal factor" of p via higher congruences, modulo powers of p and $F(x)$. Treating one p at a time and looking at what we would describe now as the semi-local situation above p, he finally arrived at a complete theory which was, however, only published after his untimely death in 1878.[62]

Zolotarev also collaborated with Korkin on the theory of quadratic forms, in Hermite's lineage. For $n = 2, 3, 4, 5$, they gave precise bounds for the minima of n-ary quadratic forms evaluated on integers.[63] Connecting approximations, elliptic integrals, quadratic forms, higher congruences and algebraic numbers, Zolotarev's is one of the very rare works to incarnate the survival of arithmetic algebraic analysis. However, while Kronecker and Eisenstein used Kummer's theory of ideal numbers to give arithmetic interpretations of results in the theory of elliptic functions, Zolotarev emphasized *the other direction*; he developed ideal factorization in order to get new results in integral calculus. This order of things was probably better received around Čebyšev, and it also found a positive echo in Smith's Presidential Address to the London Mathematical Society in 1876, where Smith relied on such applications of number theory to advocate its development in Great-Britain; see [Smith 1894], vol. 2, pp. 175–176.

3.4. General Arithmetic According to Kronecker

Leopold Kronecker considered himself a faithful follower of Gauss in general, and of the *Disquisitiones Arithmeticae* in particular. He liked to present findings of his as observations that brought out the true, general essence of an idea which occured in

62. Zolotarev's theory is described in P. Piazza's chap. VII.2 below. Further references and developments are indicated in [Haubrich 1992], pp. 163–164. This case shows the problems of communities, in particular linguistic ones, as late as 1880: Dedekind and his followers would criticize Zolotarev's work on the basis of an incomplete version of his theories, the only part available to them.

63. These papers, as opposed to those on algebraic numbers, fully belong to our cluster H-K. See also J. Schwermer's chap. VIII.1 below.

the D.A. in a specific context. For instance, before the Berlin Academy in December 1870, he took the final articles 305, 306 of D.A., sec. 5 – where Gauss discussed the cyclicity of the classes of quadratic forms in the principal genus – as the starting point for a showcase application of his approach:

> The exceedingly simple principles on which the Gaussian method rests can be applied not only at the indicated place, but also in the most elementary parts of number theory. This indicates, as it is easy to convince oneself, that those principles belong to a more general and abstract sphere of ideas. It therefore seems adequate to rid their presentation of all inessential restrictions, so that one no longer has to repeat the same deductions in the various cases where they are used.[64]

The abstract result that Kronecker thus developed amounts to what is called today the structure theorem of finitely generated abelian groups.[65]

We saw above that Kummer announced Kronecker's theory of algebraic numbers as forthcoming already in 1859. It seems impossible to say what this original theory would have looked like had it been published in a timely manner and how it would have then been tied to the D.A.[66]

Kronecker offered the most important publication on his theory, the *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* (Foundations of an arithmetical theory of algebraic magnitudes), to his teacher and friend Kummer in 1881, for the 50th anniversary of Kummer's doctorate.[67] Over the domain of rational numbers, a genus domain (*Gattungsbereich*) is obtained by adjoining a root λ of an

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64. [Kronecker 1895–1931], vol. 1, pp. 274–275: *Die überaus einfachen Principien, auf denen die* Gauss*'sche Methode beruht, finden nicht blos an der bezeichneten Stelle, sondern auch sonst vielfach und zwar schon in den elementarsten Theilen der Zahlentheorie Anwendung. Dieser Umstand deutet darauf hin, und es ist leicht sich davon zu überzeugen, dass die erwähnten Principien einer allgemeineren, abstrakteren Ideensphäre angehören. Deshalb erscheint es angemessen die Entwickelung derselben von allen unwesentlichen Beschränkungen zu befreien, sodass man alsdann einer Wiederholung derselben Schlussweise in den verschiedenen Fällen des Gebrauchs überhoben wird.*
65. Cf. [Wussing 1969], pp. 44–48. In art. 306, Gauss himself had stressed the analogy between various structures encountered in different parts of the D.A.; see chap. I.1, § 1.4 above. Kronecker first applied his observations to a system of ideal numbers.
66. In his publications of the 1880s, Kronecker would take pains to convince his readers of the early conception of his work, emphasizing its connection with Dirichlet's and Eisenstein's work, and even with his own 1845 dissertation. But he also mentioned Weierstrass's influence which made him extend his programme to include algebraic functions with complex coefficients along with algebraic numbers. See [Kronecker 1895–1931], vol. 2, pp. 324–326; also pp. 195–200. In the preface of this latter article, which was published in 1881, but presented as an 1862 communication to the Berlin Academy, Kronecker, apparently to fix priorities, mentioned his Berlin courses on related topics from the 1850s and 1860s, listing colleagues who had attended them, as well as his exchanges with Weierstrass, Dedekind, and Weber. But we are not aware of any reliable source for Kronecker's theory of algebraic numbers from before the 1870s.
67. [Kronecker 1895–1931], vol. 2, pp. 239–387. For a very readable first introduction to this theory in the case of algebraic numbers, see [Edwards 1980], pp. 353–368.

irreducible polynomial f.[68] A crucial difference with Selling and Zolotarev is that Kronecker, instead of reading the (minimal) equation f itself modulo primes p, passed to the "fundamental form" of the domain, $\omega = \omega_1 u_1 + \ldots + \omega_n u_n$, where u_i are indeterminates and ω_i a system of elements of the domain such that any algebraic integer of the domain is a linear combination of the ω_i with *integral* coefficients. This ω satisfies an algebraic equation $F(\omega) = 0$ whose coefficients are functions of the indeterminates u_i (with integral coefficients). Taking this "fundamental equation" F modulo a prime p yields the ideal decomposition of p in the domain.[69]

Forms are a pillar of Kronecker's late theory; in the *Grundzüge*, they are in general just polynomials in arbitrarily many variables, with no homogeneity required. Another pillar is the concept of "module systems" (*Modulsysteme*): an element m of the genus domain is said to be divisible by a system of elements $m_1, \ldots, m_k$ of the domain if one has $m = a_1 m_1 + a_2 m_2 + \ldots a_k m_k$, where the a_i belong to the domain. This is also expressed as a (generalized) congruence $m \equiv 0\,(\text{modd.}\, m_1, \ldots, m_k)$. The system $m_1, \ldots, m_k$ is called a module system; studying it comes down to studying the form $u_1 m_1 + u_2 m_2 + \ldots u_k m_k$, with indeterminates u_i. Module systems can be composed, and thus decomposed into products of other, eventually indecomposable, module systems. They are used by Kronecker to give a concrete definition of the greatest common divisor of several algebraic integers, and more generally to provide a theory of divisibility in genus domains.[70]

There are frequent allusions and references to Gauss in Kronecker's late expositions of his theory. The algebraic basis Kronecker relied upon owed much to Gauss's work on the fundamental theorem of algebra. When introducing the term "rationality domain" (*Rationalitätsbereich*), Kronecker recalls "Gauss's classical model of borrowing terminology from the classification of the descriptive natural sciences."[71] The introduction of indeterminates into the theory of numbers is attributed to Gauss

68. That is, for us, a number field. We translate "Gattung" by "genus" since Kronecker himself indicated *genus* as the Latin equivalent of his term; see [Kronecker 1895–1931], vol. 2, p. 251. Kronecker developed his theory not only over the "absolute rationality domain," i.e., **Q**, but also over any rationality domain (*Rationalitätsbereich*) obtained from **Q** by the adjunction of finitely many indeterminates. Thus algebraic functions and algebraico-geometric applications are included; see [Edwards 1992b] for more details and a comparison with the theory of Dedekind and Weber.

69. Using such ω_i to represent all algebraic integers of the domain solves the first problem mentioned before in § 3.1 above. As for the second, the discriminant of F is equal to $dU^2(u_1, \ldots, u_n)$, where U is a form in the indeterminates with coprime coefficients and the integer d is the correct discriminant of the domain. It may happen that for all integral values of the u_i, the values of U have a common divisor. This accounts for the parasitic factors of the discriminant occurring in theories which deal directly with the discriminants of the (minimal) equations of algebraic integers of the domain.

70. Module systems correspond more or less to Dedekind's ideals, but Kronecker's main notions are independent of the ambient domain. Moreover, the operations available on their respective objects are not completely equivalent. See in particular [Edwards 1980], pp. 355–364, [Edwards 1992a], pp. 9–17, and his chap. II.2 below, and [Neumann 2002].

71. [Kronecker 1895–1931], vol. 2, p. 249: *durchweg nach* Gauss' *klassischem Muster der Systematik der beschreibenden Naturwissenschaften entlehnten Bezeichnungen.* Other

and module systems are presented as a simultaneous generalization of Gauss's congruences and of the equivalence of quadratic forms.[72] Like their Gaussian analogues, module systems allow one to avoid any recourse to infinite sets and they are adapted to explicit computations; the principles of their composition, as well as that of forms, find their obvious sources in the D.A.[73] This selection from the resources offered by the D.A. fits well Kronecker's wish to take "refuge in the safe haven of actual mathematics":

> I recognize a true scientific value – in the field of *mathematics* – only in concrete mathematical truths, or, to put it more pointedly, only in mathematical formulas.[74]

Kronecker also mentioned a particular reason for not publishing his theory in the 1860s, which indicates a difference of scope between him and Dedekind; Kronecker had failed to obtain a general theory for what he called the "association of genera" (*Assoziation der Gattungen*).[75] This alludes to his work on complex multiplication where he had seen how to use modular functions to realize with actual algebraic numbers (in a bigger genus domain) the properties of the module systems belonging to a given imaginary quadratic domain:

> In fact, all deeper properties of the genus $\sqrt{-n}$ pertaining to composition and partition into classes [of the module systems in this genus] have, so to say, their image in the elementary properties of the associated genus Γ.[76]

This is a strong reminder of the Kronecker we encountered in chap. I.1 above, through his work on complex multiplication, as a champion of the integrated field which we called arithmetic algebraic analysis. That aspect did continue to be very much present in the minds of some of his contemporaries and in his own work.[77] In his obituary of

parts of his terminology are closer to Dirichlet, but Gauss is still mentioned; see [Kronecker 1895–1931], p. 262–263. On mathematics as a science in Kronecker's writings, see J. Boniface's chap. V.1 below.

72. [Kronecker 1895–1931], vol. 3.1, pp. 147–154, 211–213, 249–275.
73. [Kronecker 1895–1931], vol. 2, pp. 237–388, § 21, V, and § 22, V. Also seen in the Gaussian vein was Kronecker's theory of composition of abelian equations; see [Kronecker 1895–1931], vol. 4, pp. 115–121, which refers to D.A., art. 358, at the beginning.
74. Letter to Cantor from 1884, quoted (with English translation) from [Edwards 1995], pp. 45–46: *Ich [habe] mich in den sicheren Hafen der wirklichen Mathematik geflüchtet.... Einen wahren wissenschaftlichen Werth erkenne ich – auf dem Felde der* Mathematik *– nur in concreten mathematischen Wahrheiten, oder schärfer ausgedrückt, 'nur in mathematischen Formeln.'* For a comparison with Gauss's foundational views, see H. Edwards's chap. II.2 and J. Boniface's chap. V.1 below.
75. [Kronecker 1895–1931], vol. 2, p. 321–324.
76. [Kronecker 1895–1931], vol. 2, p. 323: *Es haben überhaupt alle tieferen, auf die Composition und Classenteintheilung bezüglichen Eigenschaften der Gattung* $\sqrt{-n}$ *in den elementaren Eigenschaften der associirten Gattung* Γ, *so zu sagen, ihr Abbild.* In modern terms, this means that Kronecker hesitated to publish because he lacked an explicit construction of what we call the Hilbert class field of an algebraic number field. See [Edwards 1980], pp. 328–330.
77. Kronecker enthusiastically endorsed Dirichlet's analytical methods in number theory,

Kronecker, in 1892, Hermite did not even mention the *Grundzüge*, but commented:

> M. Kronecker completely showed that the theory of quadratic forms of negative determinant was an anticipation of the theory of elliptic functions, in such a way that the concepts of classes and genus … could have been obtained by the analytical study and the examination of the properties of the transcendental function.[78]

In Jules Tannery's 1895 lectures on number theory, on the other hand, Kronecker's is the only theory of algebraic numbers mentioned, and it is presented as belonging to the algebraic, not to the arithmetic part of the book, which allows the display of links from arithmetic to algebra to analysis and back:

> I have hardly taken up the question of the nature of *algebraic numbers* in these lectures, … and I have scarcely indicated how Kronecker was able, using congruences and module systems, to build his exposition of algebra on a purely arithmetical basis, and how, from a completely different viewpoint, one could consider algebra not as an extension of arithmetic but as part of analysis, an algebraic number being just a special, well defined case of a general irrational number. … This study leads to some of the results owed to Kronecker.[79]

However, during the last decade of his life, Kronecker's arithmetization programme increasingly determined his outlook on the theory of algebraic numbers and functions. The ultimate goal then was to unify number theory, algebra, and analysis on a new basis, with the rational integers as the only fundamental objects, and such that the encompassing status of the theory was obtained by freely adjoining indeterminates and working modulo congruences according to module systems:[80]

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practiced elliptic and modular functions himself, and devoted numerous papers to the analysis of the existing proofs of reciprocity laws. He likened the difference between purely arithmetical and analytical methods to that between manual and machine work; see A. Hurwitz, notes for a scientific biography of L. Kronecker, Archives of ETH Zürich, Hs 582 : 142, sheet 3v: *Der rein arithmetische Weg zur Lösung einer arithmetischen Aufgabe, sagt Kronecker, verhält sich zu dem Wege, der die Analysis zu Hilfe nimmt, wie Handarbeit zur Maschinenarbeit. Während jeder einzelne Schritt bei der Handarbeit auch dem Nichtkenner plausibel ist, geschieht bei der Maschinenarbeit Vieles innerhalb der Maschine, was dem Auge verborgen bleibt.* See also C. Houzel's chap. IV.2 below.

78. [Hermite 1905–1917], vol. 4, p. 341: *M. Kronecker a mis en complète evidence que la théorie des formes quadratiques, de déterminant négatif, a été une anticipation de la théorie des fonctions elliptiques, de telle sorte que les notions de classes et de genres … auraient pu s'obtenir par l'étude analytique et l'examen des propriétés de la transcendante.* See C. Goldstein's chap. VI.1 below.

79. [Tannery 1895], pp. iii–iv: *C'est à peine si dans les conférences, j'avais soulevé la question de la nature des* nombres algébriques, *… c'est à peine si j'avais indiqué comment, par l'emploi des congruences et des systèmes de modules, Kronecker avait pu fonder l'exposition de l'Algèbre sur une base purement arithmétique, et comment à un point de vue tout autre, on pouvait regarder l'Algèbre, non comme un prolongement de l'Arithmétique, mais comme une partie de l'analyse, le nombre algébrique n'étant qu'un cas particulier, nettement défini d'ailleurs, de l'irrationnelle générale. … Cette étude conduit à quelques-uns des résultats que l'on doit à Kronecker.*

80. See J. Boniface's chap. V.1, and § 2 of B. Petri's and N. Schappacher's chap. V.2 below.

> With the *systematic* introduction of "indeterminates" which goes back to *Gauss*, the special theory of integers has expanded into the general arithmetic theory of polynomials in the indeterminates with integer coefficients. This general theory allows one to avoid all concepts foreign to arithmetic proper, that of negative, of fractional, of real and imaginary algebraic numbers. The concept of negative number can be avoided when one replaces in the formulas the factor -1 by the indeterminate x and the sign of equality by the Gaussian sign of congruence. Thus the equation $7-9=3-5$ will be transformed into the congruence $7+9x \equiv 3+5x \pmod{x+1}$.[81]

This programme to restructure pure mathematics thus also took its cue from the D.A. It has its historical counterpart in a specific reinterpretation of the lines of development opened by Gauss in the D.A. Kronecker's student Kurt Hensel wrote in the introduction of his edition of Kronecker's number-theoretical lectures:

> In the introduction of his "Disquisitiones arithmeticae" Gauss fixes the domain of the natural integers as the field of arithmetic, but he himself was forced to extend this domain, as he added in the fifth section of this same work the realm of quadratic forms with two variables, in the seventh the functions of x which, once set equal to zero, produce the cyclotomic equation. Kronecker characterizes then the investigation of rational numbers and rational functions of one and several variables to be the task of general arithmetic a priori.[82]

3.5. Newcomers

Early in 1857, Dedekind had written to his sister that Kronecker and he were the only readers of each other's work, because of its great abstraction. However, they were

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Kronecker's presentations grew increasingly explicit in this respect; compare for instance the *Grundzüge*, [Kronecker 1895–1931], vol. 2, pp. 237–388, § 13, where the existence of a real zero of polynomials of odd degree is assumed, to the 1897 article on the concept of number [Kronecker 1895–1931], vol. 3.1, pp. 271–272.

81. [Kronecker 1895–1931], vol. 3.1, p. 260: *mit der* principiellen *Einführung von "Unbestimmten"* (indeterminatae)*, welche von* Gauss *herrührt, hat sich die specielle Theorie der ganzen Zahlen zu der allgemeinen arithmetischen Theorie der ganzen ganzzahligen Functionen von Unbestimmten eweitert. Diese allgemeine Theorie gestattet alle der eigentlichen Arithmetik fremden Begriffe, den der negativen, der gebrochenen, der reellen und der imaginären algebraischen Zahlen, auszuscheiden. Der Begriff der* negativen *Zahlen kann vermieden werden, indem in den Formeln der Factor* -1 *durch eine Unbestimmte x und das Gleichheitszeichen durch das* Gauss'*sche Congruenzzeichen* modulo $x+1$ *ersetzt wird. So wird die Gleichung* $7-9=3-5$ *in die Congruenz* $7+9x \equiv 3+5x \pmod{x+1}$ *transformirt.* Cf. also [Kronecker 1895–1931], vol. 2, p. 355.

82. [Kronecker 1901], p. VI: *Gauss bestimmt in der Einleitung zu seinen "Disquisitiones arithmeticae" das Gebiet der natürlichen ganzen Zahlen als das Feld der Arithmetik, aber er selbst war gezwungen, dieses Gebiet dadurch zu erweitern, daß er in der fünften Sektion desselben Werkes das Reich der quadratischen Formen von zwei Variablen, in der siebenten die Funktionen von x hinzunahm, welche gleich null gesetzt die Kreisteilungsgleichungen ergeben. Kronecker bezeichnet nun von vorn herein die Untersuchung der rationalen Zahlen und der rationalen Funktionen von einer und von mehreren Variablen als die Aufgabe der allgemeinen Arithmetik.*

not satisfied with each other's final approach. Kronecker would criticize Dedekind's theory of ideals for its use of completed infinites, and its invention of unnecessary new concepts, and Dedekind would find holes in the *Grundzüge* and consider polynomials a device foreign to the matter in question.[83]

In the last decades of the century, several younger people – born between 1859 and 1864 – entered this field, announcing the blossoming of the early XX$^{\text{th}}$ century; Kronecker's student Kurt Hensel in Berlin whose 1884 dissertation concerned the delicate issue of the divisors of the discriminant,[84] Adolf Hurwitz in Leipzig who had just written a dissertation on elliptic modular functions under Felix Klein's supervision, as well as Hermann Minkowski and David Hilbert in Königsberg. Whereas Hensel was closely associated with Kronecker's programme, the latter three, who became acquainted when Hurwitz obtained a position in Königsberg,[85] cared little about the preferences of either Dedekind or Kronecker. An anecdote describes how Hilbert and Hurwitz went for a walk during which one of them presented Dedekind's proof for the unique decomposition into prime ideals, the other Kronecker's analogue, and they found both awful.[86] More positively, Minkowski – whose prize-winning 1882 memoir on quadratic forms referred only to Gauss, Dirichlet, Eisenstein, and (for a small technical point) Weber, who had discovered his talent[87] – would devote his Bonn *Habilitationschrift* to a question raised by Kronecker's *Grundzüge*, about the concept of equivalence of forms, but would later express some of his results in terms of Dedekind's number fields. In several papers of the mid 1890s, while also using Dedekind's notions of (number) field and ideal, Hurwitz defined ideals via finite sets

83. See [Kronecker 1895–1931], vol. 3.1, p. 156, footnote; [Dedekind 1930–1932], vol. 2, p. 53; [Edwards 1980], [Edwards, Neumann, Purkert 1982], [Haubrich 1992], pp. 128–130. On February 17, 1882, Dedekind wrote to Cantor that Kronecker's "way would, I believe, please me more if he had completely separated the theory of numbers from that of functions;" see [Dugac 1976], p. 254. Of course, function fields and number fields would both be subsumed later into the concept of global field, but for Kronecker and Dedekind, and for their own historians, two different projects for algebraic numbers (and functions), and eventually for pure mathematics, were available to students in Germany in the 1880s.

84. His reflections about the analogy between algebraic numbers and functions eventually led him to the creation of a new arithmetic-algebraic-analytic object, the p-adic numbers, which in turn would drive the abstract algebraic study of general fields; see [Purkert 1971–1973], [Ullrich 1998] and the forthcoming thesis by B. Petri. Doctoral students of Kronecker in the 1880s include Adolf Kneser, who turned to other topics, Mathias Lerch, whose number-theoretical articles pursued Dirichlet's analytical tradition, and Paul Stäckel, the discoverer of Gauss's mathematical diary and collaborator in the edition of the Gauss's *Werke*.

85. D. Fenster's and J. Schwermer's chap. II.3 below throws light on Hurwitz's further work on quadratic forms which we do not touch upon here.

86. [Blumenthal 1935], p. 397. Cf. Hilbert's letter to Hurwitz criticizing the fourth edition of Dirichlet-Dedekind, [Dugac 1976], p. 270.

87. This memoir, the laudatory letter sent by Weber to Dedekind about Minkowski, and Minkowski's relation to Gauss's D.A. are discussed in J. Schwermer's chap. VIII.1.

of generators, and used a basically Kroneckerian approach based on polynomials in several unknowns to derive the unique decomposition of ideals into prime ideals.[88]

3.6. Hilbert's Zahlbericht

In 1897, Hilbert published for the recently created *Deutsche Mathematiker-Vereinigung* a report on algebraic numbers, *Die Theorie der algebraischen Zahlkörper*, which would become known as the *Zahlbericht*.[89] Hilbert thoroughly read the literature in the preparation of it, and although he complained about Kummer's computations (*Rechnereien*), he quoted his papers abundantly. The *Zahlbericht* presents the arithmetic theory of a general (number) field K: integers, discriminant, units, ideals, ideal classes; it studies in detail the decomposition of the prime ideals of K in a Galois extension of K,[90] as well as particular cases, such as quadratic or cyclotomic fields. Moreover, Hilbert again gave a central position to higher reciprocity laws, dear to Gauss and Kummer, albeit in a new framework of number fields.[91]

Throughout the report, Hilbert placed number fields at the centre of attention and also defined ideals in Dedekind's style as sets of algebraic integers which are closed under linear combinations with algebraic integer coefficients. But for several proofs, in particular for the uniqueness of decomposition into prime ideals in arbitrary number fields and for the divisors of the discriminant, he adopted essentially the Kronecker-Hurwitz method. However, this methodological syncretism progressively fell into oblivion as Dedekind's conceptual heritage increasingly won the day; in Hilbert's own summary of his report for the *Enyclopädie der mathematischen Wissenschaften mit Einschluß ihrer Anwendungen*, published in 1900, proofs are mostly omitted so that Dedekind's concepts come out prominently, a tendency which would be amplified in the following century with the success of modern algebra.

Gauss and the D.A. have almost vanished from the main text of Hilbert's *Zahlbericht*,[92] but they do figure prominently in Hilbert's imposing preface. This preface is a manifesto for the incipient discipline that Hilbert presented in the main text, and whose position within pure mathematics is defined here. First, Hilbert appealed to "our master Gauss" to celebrate through well-chosen (and oft-repeated since) quotes "the charms of the investigations" in this "divine science" of number theory. Like Dedekind and Kronecker, Hilbert reserved for arithmetic, "the Queen

88. Much to Dedekind's chagrin, who criticized this approach as lacking methodological purity and conceptual unity; see [Dedekind 1930–1932], vol. 2, pp. 50–58, and [Hurwitz 1895], p. 198: … *in der Meinung, dass meine Arbeit keine Verteidigung verdient, wenn sie nicht für sich selbst spricht.* See also [Dugac 1976], p. 266.

89. See [Schappacher 2005] for an introduction to the writing and content of this report.

90. That is an extension of K generated by all the roots of an irreducible polynomial. Thus, Hilbert considered relative situations where the base field was no longer the field of rationals but an arbitrary number field K.

91. He shifted the emphasis from ℓ^{th} powers to norms of elements in an extension generated by the root of a polynomial of degree ℓ.

92. There are 7 references to Gauss among which 5 are to the D.A., against 52 references to Kummer. Hilbert made it clear that he offered a new synthesis, not a historical report; all references are technical.

of mathematics" (another allusion to Gauss), a key position inside pure mathematics. But with Hilbert this "royalty claim"[93] is no longer restricted to the unification of number theory and algebra, or even analysis:

> Thus we see how arithmetic, the "Queen" of the mathematical sciences, conquers large areas in algebra and function theory, and takes the leading role in them ... Finally, there is the additional fact that, if I am not mistaken, the modern development of pure mathematics takes place chiefly *under the sign of number*: Dedekind's and Weierstrass's definitions of fundamental concepts of arithmetic and Cantor's general construction of the concept of number lead to an *arithmetization of function theory*. ... The *arithmetization of geometry* is accomplished by the modern investigations of non-Euclidean geometry.[94]

Thus Hilbert endowed number theory with two different roles. On the one hand, the *Zahlbericht* itself presents a model for what should be a fully developed discipline, here number theory:[95] it defines its proper subject matter,[96] algebraic number fields; it has its key problems, that will be made explicit in Hilbert's subsequent papers, like prime decomposition in a field extension, relative reciprocity laws and the construction of class fields; it has a rigorous system of proofs. Moreover, as a closed, mature system, number theory, as described by Hilbert, integrates its own development:

> Instead of that erratic progress characteristic of the youngest age of a science, a sure and continuous development occurs now thanks to the systematic construction of the theory of number fields.[97]

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93. Erhard Scholz coined this expression at our 1999 Oberwolfach workshop.

94. [Hilbert 1932–1935], vol. 1, pp. 65–66: *So sehen wir, wie die Arithmetik, die "Königin" der mathematischen Wissenschaft, weite algebraische und funktionentheoretische Gebiete erobert und in ihnen die Führerrolle übernimmt ... Es kommt endlich hinzu, daß, wenn ich nicht irre, überhaupt die moderne Entwicklung der reinen Mathematik vornehmlich* unter dem Zeichen der Zahl *geschieht: Dedekinds und Weierstrass' Definitionen der arithmetischen Grundbegriffe und Cantors allgemeine Zahlgebilde führen zu einer* Arithmetisierung der Funktionentheorie ... *Die* Arithmetisierung der Geometrie *vollzieht sich durch die modernen Untersuchungen über Nicht-Euklidische Geometrie.* On arithmetization at the beginning and at the end of the XIX$^{\text{th}}$ century, cf. J. Ferreiròs's chap. III.2, and B. Petri's and N. Schappacher's chap. V.2 below.

95. See chap. I.1, § 5, in particular footnote 184.

96. Ralf Haubrich pointed out to us that the very concept of "the subject matter of a discipline" changed during the century: while it was a thing "already there" at the beginning and had the status of a natural object, like a star, the subject matter of a discipline at the end of the century was *defined* within the discipline. On the related simultaneous change in the ontology of mathematical objects, see [Gray 1992].

97. [Hilbert 1932–1935], vol. 1, pp. 65–66: *An Stelle eines solchen für das früheste Alter einer Wissenschaft charakteristischen, sprunghaften Fortschrittes ist heute durch den systematischen Aufbau der Theorie der Zahlkörper eine sichere und stetige Entwicklung getreten.*The allusion is to Gauss's difficult determination of the sign of Gauss sums. The allusion is all the more fitting in that Gauss's presentation of this episode, on the contrary, reveals the Romantic values of the early XIX$^{\text{th}}$ century, see S. Patterson's chap. VIII.2. Similarly, one can contrast the idea of system present in Hilbert's preface to that in the

This development includes a representation of the past as well as one of the future, and the D.A. again is used here both as a prestigious origin and as a foil:

> As to the *position of number theory* inside the whole of mathematical science, Gauss presents number theory in the preface of the *Disquisitiones Arithmeticae* still only as a theory of the natural integers with the explicit exclusion of all imaginary numbers. … [Now] the theory of number fields … has become the most essential part of modern number theory.[98] The service, of laying down the first germ of the theory of number fields is again due to Gauss. Gauss recognized the natural source for the laws of biquadratic residues in an "extension of the domain of arithmetic." [99]

On the other hand, this internal reorganization is completed by an external one, positing number theory in the foundational enterprise of the turn of the century to which Hilbert would soon contribute his new *Grundlagen der Geometrie* (1899). Comforted by the fact that both in Dedekind's (algebraico-set-theoretical) and in Kronecker's (arithmetized) programmes, reflections on numbers in general and number-theoretical results are elaborated side by side, what may almost appear as a play on the expression "theory of numbers"[100] allows Hilbert to link number theory to the recent movement of arithmetization of mathematics – in its Göttingen form, that is, including geometry. Thus the idea of number theory as a royal discipline that was taking the lead in all fields of mathematics, and the purposeful reference to the D.A. and its author – the edition of whose *Werke* was just starting then afresh in Göttingen under Felix Klein's supervision – fit well with Hilbert's – and Göttingen's – most far-reaching projects at the time.[101]

4. Long Shadows

At the turn of the century, both Gauss's D.A. and number theory in general had thus found quite a stable and prestigious place inside mathematics. Or, more precisely,

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D.A., see chap. I.1, § 2.4.

98. In view of the papers really published in number theory at that moment, such a statement is a *coup de force*. It serves in fact as a normative *definition* of what ought to be "modern" number theory.

99. [Hilbert 1932–1935], vol. 1, p. 64: *Was die* Stellung der Zahlentheorie *innerhalb der gesamten mathematischen Wissenschaft betrifft, so faßt Gauss in der Vorrede zu den Disquisitiones Arithmeticae die Zahlentheorie noch lediglich als eine Theorie der ganzen natürlichen Zahlen auf mit ausdrücklicher Ausschließung aller imaginären Zahlen. … Die Theorie der Zahlkörper insbesondere ist … der wesentlichste Bestandteil der modernen Zahlentheorie geworden. Das Verdienst, den ersten Keim für die Theorie der Zahlkörper gelegt zu haben, gebührt wiederum Gauss. Gauss erkannte die natürliche Quelle für die Gesetze der biquadratischen Reste in einer "Erweiterung des Feldes der Arithmetik."* Here, Hilbert follows Dedekind's point of view on the extension of arithmetic, see the end of § 3.4.

100. This expression was used for number theory, but also for various theories of *real* numbers, such as those of Dedekind and Cantor, mentioned in Hilbert's preface.

101. See in particular [Rowe 1989].

places, as neither the position of number theory, nor the uses of the D.A. were uniform from country to country, or even within each country.[102]

In Germany, the Neo-Gaussian movement created by the commentaries on Gauss's *Werke* and the reedition of Gauss's articles[103] expanded after the *Zahlbericht*. Around 1910, Bachmann was happy to announce that number theory was being taught in twelve German universities, see [Bachmann 1921], p. viii. Theses and articles on number-theoretical and related topics multiplied. The discipline initiated by the *Zahlbericht*, which would soon be called "algebraic number theory,"[104] blossomed especially, although not exclusively,[105] within the international area of Göttingen's influence:[106] Rudolf Fueter, Philipp Furtwängler, Alexander Ostrowski,[107] Legh Wilber Reid, Luigi Bianchi, Andreas Speiser, Teiji Takagi figured among its young authors. A harvest of textbooks in several languages framed and reinforced this development, all the way from Julius Sommer's 1907 *Vorlesungen über Zahlentheorie. Einführung in die Theorie der algebraischen Zahlkörper*, or Reid's 1910 *The Elements of the Theory of Algebraic Numbers* which were both supported by Hilbert and focused on quadratic number fields, to the *Lezioni sulla teoria dei numeri algebrici* by Bianchi in 1924.[108]

But the situation was not uniform. We have seen above that Smith in 1876 had

102. Detailed examples illustrating this variety of research groups, inside a country or on the contrary beyond frontiers, are given in parts VI and VII below, centered on France, Italy, Russia, and the United States. For global comparative data between the number-theoretical production of members of the Deutsche Mathematiker-Vereinigung and of the Société mathématique de France between 1897 and WW1, see [Gispert, Tobies 1996], in particular p. 430.
103. To those should be added the various celebrations organized in Germany around Gauss's hundredth anniversary in 1877: for instance, Dedekind's contribution to the Gauss Festschrift organized in Braunschweig is a display of his familiarity with the D.A., see [Dedekind 1930–1932], vol. 1, pp. 105–158.
104. Edmund Landau used *Algebraische Zahlentheorie* as the heading of the corresponding part of his *Vorlesungen* in 1927.
105. Heinrich Weber also continued to play a key role in this development, as did some young Berliners like Robert Remak and Edmund Landau (who then joined the Göttingen staff), also Hensel's students, most notably Adolf Fraenkel and Helmut Hasse. Gauss's editors included the older generation (Bachmann), and people who had defended their theses in the middle of the 1880s, either with Klein (Fricke), or with Kronecker (Stäckel, Schlesinger).
106. Although the name of Klein is rarely associated with number theory, he not only took charge of the Gauss edition and related projects (editing in particular Gauss's diary in 1901), fostering what we call the Neo-Gaussian movement, but himself taught number theory at Göttingen. Geometrical insights, like those provided by Minkowski's geometry of numbers, particularly appealed to him, both because they connected number theory to other fields and because they were supposed to mitigate its overly abstruse aspects; see [Klein 1894] and [Klein 1926/1967], pp. 26–27, in perfect harmony with the preface of the *Zahlbericht*, and presumably comforting his opposition to Berlin trends.
107. Ostrowski worked first with Hensel in Marburg before joining Göttingen during WW I.
108. See A. Brigaglia's chap. VII.1.

tried to promote the study of number theory in Great Britain on the strength of its applications to more indigenous topics:

> It is worthy of remembrance that some of the most fruitful conceptions of modern algebra had their origin in arithmetic, and not in geometry or even in the theory of equations. The characteristic properties of an invariant, and of a contravariant, appear with distinctness for the first time in the *Disquisitiones Arithmeticae*.[109]

Besides Zolotarev's applications of his theory of algebraic numbers to integral calculus, Smith also emphasized asymptotic results[110] and the possible bridges to analysis provided by the approximation processes used by Jacobi and Hermite.

Analysis, indeed, and not algebraic structures nor the general arithmetic of polynomials, appeared to many as the greatest unifying force. In his well-known 1951 "The Queen of Mathematics," Eric Temple Bell – after first declaring that "[Gauss's] work is as vital as it was in 1801, when he published his *Disquisitiones Arithmeticae*," stated:

> There was remarkable progress since the time of Gauss, and especially since 1914, when modern analysis was applied to problems in the theory of numbers that had withstood the strongest efforts of Gauss's successors for over a century.[111]

"Modern analysis" referred in particular to the achievements of Godfrey H. Hardy, John E. Littlewood, and Edmund Landau (all belonging to our cluster D above). The royalty claim for arithmetic was challenged by no less a figure than Henri Poincaré in his lecture at the First International Congress of Mathematicians in 1897:

> Analysis unfolds for us infinite perspectives that arithmetic has not dreamed of; it shows us at a glance a grandiose composition, whose arrangement is simple and symmetric. Against this, in number theory, where the unforeseen reigns, the view is so to speak blocked at every step.[112]

Poincaré's own number-theoretical contributions mostly belong to our group H-K and this commentary, when compared to Hilbert's preface, underlines the nature

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109. [Smith 1894], vol. 2, pp. 166–190. Smith mentions specifically arts. 157, 267, 268.

110. Again referring to the D.A. as their origin: "the first asymptotic results that were obtained are due to Gauss and are given without demonstration in the Disquisitiones arithmeticae," (arts. 302, 304, *additamenta* to 306.X).

111. [Bell 1951/1956], pp. 498–499 for the quotes, and p. 508 for details along the same lines as Smith.

112. [Poincaré 1897/1991], p. 26: *L'analyse nous déroule des perspectives infinies que l'arithmétique ne soupçonne pas: elle nous montre d'un coup d'œil un ensemble grandiose dont l'ordonnance est simple et symétrique; au contraire, dans la théorie des nombres où règne l'imprévu, la vue est pour ainsi dire arrêtée à chaque pas.* The quote and its context are discussed in C. Goldstein's chap. VI.1 below. The increasing international tension of the time could render statements more political: in the *Revue du Mois*, a cultural journal created by Emile Borel, a comment on a result of Hilbert and Landau concerning the expression of a definite polynomial as a sum of squares ends with the remark that there is no need to regret that "the young French mathematical school has forsaken these arithmetical studies, favoured in Germany, to attach themselves to the immense field of investigations linked with differential and integral calculus," see the motto, part VI below.

of the dissociation already mentioned of the research field built on the D.A. in the 1850s. Although both Hilbert and Poincaré shared the same diagnosis on past number theory, the domain of the "erratic," "where the unforeseen reigns," and although both extolled a majestic layout, they put forward opposed remedies.

It would be misleading to summarize this dissociation by opposing algebraic trends to analytic ones, or France to Germany, for instance. The promotors of an "algebraic theory of numbers" at the end of the XIX$^{\text{th}}$ century included projects as different as those of Lucas and Hilbert, mathematicians for whom algebra meant a classical theory of equations or, on the contrary, group theory, those for whom analysis should be avoided at all costs, or assimilated, or even used *faute de mieux.*

As our global picture of number theory has stressed, contrasted views coexisted inside a single country. We have seen that, when Hilbert and Hensel affirmed the necessity of extending number theory beyond ordinary integers, they meant different extensions, they referred to different parts of Gauss's corpus to sustain their views, and they even interpreted differently that part of the *Disquisitiones Arithmeticae* (sec. 7 on cyclotomy) which they both claimed. On a larger scale, Genocchi, Cesàro, and Bianchi in Italy, Lucas, Hadamard, and Picard in France, provide representatives of our three main groups in each country. *Contra* Reid, Leonard Dickson, a prominent proponent of number theory in the United States, devoted his textbooks on this topic to forms, not to algebraic numbers.[113] A few years after Poincaré's statement, Albert Châtelet proposed his own syncretism, in his *Leçons sur la théorie des nombres*, based on Hermite's approach, but integrating Minkowski's work on the geometry of numbers, as well as "the first elements of the theory of the complex integers of a field and of their arithmetic."[114] Distinct networks of mathematical solidarities cut across national boundaries.[115]

While the chapter entitled "Forms" of the *Jahrbuch über die Fortschritte der Mathematik* regularly displayed a few articles at the beginning of the XX$^{\text{th}}$ century, its chapter "Generalities," hosting papers on the distribution of primes or laws of reciprocity in number fields, besides those on congruences or Diophantine problems, tended to explose. It was a reclassification that was used to register the new views on number theory at the beginning of the XX$^{\text{th}}$ century. The *Jahrbuch* first gathered number theory and algebra under the same section, then identified algebraic numbers and analytic methods as incipient topics, gathering them together in a separate subsection, and finally, in the 1930s, recognized both as autonomous branches in this section, under specific headings, *Idealtheorie* and *Analytische Zahlentheorie*.

113. See resp. chap. VII.1, part VI, and chap. VII.3.

114. [Châtelet 1913], p. viii: *Les Chapitres IV, V et VII, plus spécialement consacrés aux nombres algébriques, constituent les premiers éléments de la théorie des entiers complexes d'un corps et de son arithmétique*. Châtelet mentioned that he did not want to tie himself to a specific school and he borrowed procedures indifferently from Dedekind and Kronecker, as well as from Minkowski and Hurwitz.

115. We have underlined several times the role of lectures and students to provide mathematical continuities. It has been a crucial element in German countries, see *a contrario* the case of Italy in chap. VII.1.

This *Jahrbuch* classification was compatible with that adopted by Bachmann for his presentation of number theory in different branches (each discussed in a specific volume), and also with the one used in the *Encyclopädie der mathematischen Wissenschaften*. This new order of topics defined a different place for the main content of the D.A.: congruences, quadratic reciprocity, and also binary quadratic forms – equivalence, classes, reduction, but usually not genus theory nor the composition of forms – were now seen as *elementary* number theory.[116] The other parts[117] of number theory at that time – namely the theory of forms, algebraic number fields, and analytic number theory – entertained varied relations with the D.A.: all of them could be seen in some respects as stemming from Gauss's book, while none of them could claim Gauss's disciplinary heritage for itself alone.[118]

However, the structure of the *Disquisitiones Arithmeticae* and its history did inform other classifications of number theory.[119] The Subject Index of the *Catalogue of Scientific Papers 1800–1900* classifies number theory following an order which, after some introductory sections on divisibility and Diophantine equations, reminds one of Smith's *Report on the Theory of Numbers*, see chap. I.1, § 5.2: congruences and forms are handled consecutively and the list concludes with a number of miscellanea on cyclotomy, presented as the application of circular functions to number theory, and on other analytical aspects. Of particular interest is the fact that the *Catalogue* distinguishes between "forms of higher degree which cannot be considered as products of linear factors," and "forms of higher degree which can be considered as products of linear factors." The last are classified with "algebraic numbers" and "ideals" – thus joining together the various programmes proposed for the arithmetical study of algebraic numbers, an amusing testimony of the historical crossroads we have already met.

The shadow of the table of contents of the D.A. does not haunt only such retrospective catalogues surveying the mathematical literature of the past. In his famous Paris lecture, in 1900, Hilbert dedicated to number theory 6 out of 23 problems (nos. 7–12).[120]

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116. What Gauss called "elementary" in the preface of the D.A. was the part of arithmetic dealing with the writing of integers and the usual operations. On the other hand, books on the *elements* of the theory of number fields, like Reid's, would soon restructure the arithmetic study of **Z** on the model of the *Zahlbericht* or propose a detailed presentation of the simplest case, that of quadratic fields.
117. Probably because of the expansion of advanced teaching, the links between the academic disciplines and the research fields seem at this time closer than what we described in chap. I.1 for the 1850s.
118. At the 2001 Oberwolfach meeting, Ralf Haubrich proposed seven criteria to characterize a mathematical discipline (subject matter, key concepts and results, systematization, proofs,...) and showed that algebraic number theory and Gaussian number theory (see chap. I.1, § 5.2) differ on all accounts. Alain Herreman has semiotically opposed Gauss's sec. 5 with set-theoretical formulations in "Vers une analyse sémiotique de la théorie des ensembles: hiérarchies et réflexivité," http://perso.univ-rennes1.fr/alain.herreman/.
119. See also Dickson's *History* discussed in D. Fenster's chap. VII.3 below.
120. We can compare this to the mere 4% of publications that this field represented at the time.

Schedule of Classification

		PAGE
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	Permutations ...	87–88
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2030	Characteristic properties of linear substitutions; types of linear substitutions. (*See also* 1210.)	140
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2400	General	156
2410	Elements of the theory of algebraic equations; existence of roots, symmetric functions; rational fractions; partial fractions	158
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2420	Reality, multiplicity, separation, of roots	163
2430	Equations of the second, third and fourth orders: other particular equations ...	169
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2440	Numerical solution of equations (*including graphical methods*)	174
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2450	General resolution of equations; theory of Galois. (*See also* 1210.)	179
	Equations, 5th order	180–183
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2810	Divisibility; linear congruences	192
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2815	Continued fractions and indeterminate equations	197
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2820	Quadratic residues	204
2830	Quadratic binary forms ...	205
2840	Quadratic forms of three or more variables; bilinear forms. (*See also* 2060, 2070.)	209
	Decomposition of numbers into sums of squares	209–210
	Bilinear forms ...	213–214
2850	Congruences other than linear; cubic and higher residues	214
2860	Forms of higher degree which cannot be considered as products of linear factors. (*See also* 2060, 2070.) ...	218
	Fermat's theorem on $x^n+y^n=z^n$ ...	218–219
2870	Forms of higher degree which can be considered as products of linear factors; algebraic numbers; ideals. (*See also* 0820.)	220
2880	Application of trigonometrical functions to arithmetic; cyclotomy	222
	Binomial equations ...	222
2890	Application of other transcendental functions to arithmetic	224
2900	Distribution of prime numbers	225
2910	Special numbers and numerical functions	228
	Bernoulli's numbers	228–229

ii

Fig. I.2B. The Theory of Numbers in the *Catalogue of Scientific Papers* (Courtesy of the Bibliothèque Mathématiques-Recherche Jussieu, Paris)

Of these, the first two lie outside the scope of the D.A.: nos. 7, 8, which concern irrationality and the distribution of primes. But the three[121] problems 9, 11, 12 correspond neatly, and in the order of Hilbert's list, to the *Disquisitiones*, although presented from Hilbert's point of view, i.e., centered around the notion of number field: no. 9 asks for the generalization to number fields of the general reciprocity law; no. 11 for a theory of forms over such fields; and no. 12 concerns complex multiplication and is thus at the junction of number theory, algebra, and function theory, just like Gauss's famous sec. 7. In problem 11, for instance, Hilbert wrote:

> Our present knowledge of the theory of quadratic number fields puts us in a position *to successfully attack the theory of quadratic forms with any number of variables and with any algebraic numerical coefficients.* This leads in particular to the interesting problem: to solve a given quadratic equation with algebraic numerical coefficients in any number of variables by integral or fractional numbers belonging to the algebraic realm of rationality determined by the coefficients.[122]

The *Disquisitiones Arithmeticae* survived at the beginning of the XX$^{\text{th}}$ century as a cultural icon, as a historical source, as a frame for number theory. But this did not mean that its role as a mathematical resource had come to an end. On various occasions in this book, the authors allude to the way in which relatively recent results by André Weil, Kurt Heegner, John Tate, and – even more recently – Manjul Bhargava, and others, connect to specific articles or techniques of the D.A.[123]

We find it fitting to mention as a final example an article which puts us precisely one century after the publication of the *Disquisitiones Arithmeticae*, and links it, unexpectedly, to perhaps the single important branch of number theory which has only fully blossomed in the XX$^{\text{th}}$ century, Diophantine geometry,[124] and which, moreover, is related to precisely that part of number theory which Gauss essentially excluded from the D.A.: Diophantine analysis. In his 1901 article on the arithmetic of algebraic curves, Henri Poincaré proposed to interpret Diophantine questions geometrically[125] and to classify them up to birational equivalence: he not only

121. The remaining problem, the tenth, is about Diophantine equations, and, here as in Dickson's *History of the Theory of Numbers*, is squeezed between divisibility and the theory of forms.

122. [Hilbert 1932–1935], vol. 3, pp. 310–311: *Unsere jetzige Kenntnis der Theorie der quadratischen Zahlkörper setzt uns in den Stand,* die Theorie der quadratischen Formen mit beliebig vielen Variablen und beliebigen algebraischen Zahlkoeffizienten erfolgreich in Angriff zu nehmen. *Damit gelangen wir insbesondere zu der interessanten Aufgabe, eine vorgelegte quadratische Gleichung beliebig vieler Variablen mit algebraischen Zahlkoeffizienten in solchen ganzen oder gebrochenen Zahlen zu lösen, die in dem durch die Koeffizienten bestimmten algebraischen Rationalitätsbereiche gelegen sind.*

123. See chap. I.1, footnotes 21, 37, 79; part II and part VIII below.

124. Cf. the volumes on number theory in the *Encyclopaedia of Mathematical Sciences*, published in the 1990s.

125. Interesting new links between geometry and number theory have been created during the last decades of the XX$^{\text{th}}$ century. Only a few hints have been given here, see J. Schwermer's chap. VIII.1 on Minkowski and [Schappacher 1991].

alluded to Gauss's equivalence of quadratic forms in the introduction, as a model for this birational equivalence of curves, but he also proposed as a point of departure of his work a fresh look at a specific article of the D.A.:

> To recognize when a conic has a rational point is a problem that Gauss has taught us how to solve in the chapter of his *Disquisitiones Arithmeticae* entitled *Repraesentatio ciffrae*. The conics without a rational point are arranged in several classes and the conditions of this arrangement follow immediately from the principles of the same chapter in Gauss.[126]

5. Paradises Lost

That the *Disquisitiones Arithmeticae* is both a mathematical work and a historical event does not mean that it is an obvious object for the history of mathematics. Immediately after its publication, concepts, results, themes, were detached, reintegrated in other points of views, and grew to have each their own story to tell.[127] The way they could be isolated or amalgamated depends, as we have seen, on many factors, some of which stem from the D.A., some from other publications; some are the result of generational shifts among the readers or of the ambient scientific priorities. Different parts of the D.A. have been activated at different times, then left aside for a while and reactivated again, with completely different research horizons in mind.

Even if we could follow each of these fragments, we would still miss our target. The *Disquisitiones Arithmeticae* had several functions: a mythical model for number-theoretical activities, a technical reference, a familiar companion of everyday mathematical experience. The content of the book has at times defined number theory, and at others has been cut and reshaped to fit new disciplinary views. All these different scales matter for the history of the book. They require the reconstitution of relevant strata of textual organization and of the contexts which have allowed these strata to be interpreted and used efficiently for doing mathematics. Our choice here to stick to explicit mentions of the D.A. has already led us to historical evidence of all kinds, from personal correspondence to lectures, from treatises of cultural history to political gazettes, from catalogues to research papers. And these mentions have been of all sorts: one article of the D.A., a notation, a whole section, a way of thinking.

Our main goal in this part has been to reshape the global representation of the history of the D.A. – this will help, we hope, to situate more accurately the results of the following chapters. We were not satisfied with the usual summary – a period of latency and awe, then a succession of a small number of brilliant contributors, one or two per generation,who were deeply involved with the book, and finally the blossoming of algebraic number theory at the turn of the twentieth century. What we

126. [Poincaré 1901], p. 485: *Reconnaître si une conique admet un point rationnel, c'est un problème que Gauss nous a enseigné à résoudre, dans son chapitre des* Disquisitiones*, intitulé* Repraesentatio ciffrae*. Les coniques qui n'ont pas de point rationnel se répartissent en plusieurs classes et les conditions de cette répartition se déduisent immédiatement des principes de ce même chapitre de Gauss.* The reference is to D.A., art. 299, on ternary forms; a plane conic is the zero set of a ternary quadratic form.

127. Specific examples are traced in part II, part IV, and part VIII below.

wanted was to pay more attention to the relations among mathematicians (and among their results), and to the actual mechanisms of knowledge transfer, in particular from one generation to another,[128] that is, to understand some part of the dynamics in the changing role of the D.A. and of number theory.

What we have seen here is, first, that a serious study of a limited part of the book (all that touching the theory of equations) took place quite quickly, and that it nourished a vast expansion of algebra.[129] For a while, it tended to relegate congruences to a supporting role – the topic which, in what survives of Gauss's original plans, secured the coherence of the whole – and it pointed toward an assimilation of arithmetic and algebra.

Then, from the mid 1820s on, new types of readers of the D.A. appeared: they immersed themselves in the D.A., often on their own, early in their mathematical life; and they made it the arithmetical seed of a large international area of research, this time mixing number theory, algebra, and analysis. The mixture operated in many ways: using analytical techniques to prove statements left open in the D.A.; encapsulating links between properties of integers and continuous functions in an algebraic formula; and creating algebraic analytic concepts by means of those introduced in the D.A. Here we find the well-known names of Jacobi, Dirichlet, Kummer, Eisenstein, Hermite, Kronecker, and others. But we want to stress that they intervened in an essentially unique network, precipitated out by intense exchanges among these mathematicians, accompanied by a characteristic discourse concerning the value of unity, even when their work displayed different mathematical agendas.

The mechanisms at work in the 1860s are still unclear for us.[130] During the following decades, textbooks in various languages gave ready access to large parts of the D.A. The edition of Gauss's *Werke* and related publications also set up, specially in Germany, a "usable" Gauss,[131] a classic, of whom set pieces were made accessible to a larger audience. But in the last quarter century, components of research that were closely linked in the preceding period seem to drift apart: number theory now encompasses several rather well-established disciplines, each with its own privileged problems and relations to the D.A.

Because of its status in the standard history, we have revisited the development of one component, that of Kummer's ideal numbers. It presents several intriguing features: first of all, unlike other components, such as forms or modular equations,

128. We think that this issue, which ought to take into account teaching, available techniques and problems, and cultural agendas, is important for the understanding of scientific development and has not yet been sufficiently studied in the history of mathematics.

129. This stage has already been identified in [Neumann 1979–1980].

130. This puzzlement is due partly to the increased size of the mathematical scene, but especially to the discontinuity in our sources, due in turn to the almost simultaneous disappearance, for various reasons, of the main contributors of the preceding period. The *Jahrbuch*, on the other hand, only appeared at the end of the 1860s.

131. We mean by this something of public utility, contributing to the self-understanding and to the action of a community, here mathematical; see William J. Bouwsma, *A Usable Past. Essays in European Cultural History*. Berkeley: University of California Press, 1990.

it concerns a mere handful of articles and appears to be mainly a German affair.[132] Then, it makes evident the paradoxical nature of Dedekind's activities. The most important influences on him were those of Dirichlet and Riemann; unlike most of his German colleagues, he had virtually no research students.[133] But he mediated in three different and important ways the number-theoretical work of the first half century for the generation coming to (mathematical) age in the 1880s. He edited Dirichlet's *Vorlesungen über Zahlentheorie*, one of the most influential "surrogates" for the D.A.; he annotated and wrote commentaries on Gauss's writings for the *Werke*, thus imprinting his mark on the interpretation of Gauss for decades to come; and finally, in his own mathematical work, he synthetized various threads coming from the D.A.; in particular Galois (group) theory, higher congruences, and Kummer's ideal numbers. However the continuity running from his work to the new generation is really a product of the latter.

Indeed a decisive event at the end of the century was the meeting of Hilbert, Minkowski, and Hurwitz,[134] soon connected to Klein's Göttingen. Besides their syncretic approach – which nonetheless privileged Dedekind's ideals and number fields as key objects – their skill in evocative presentation and their numerous students around the world were instrumental in establishing within a few years a new *subdiscipline* within number theory, algebraic number theory – one which put reciprocity laws back at center stage.[135]

This group produced not only influential mathematics but equally influential views of the development of mathematics, such as Klein's *Vorlesungen über die Entwicklung der Mathematik in XIX*$^{\text{ten}}$ *Jahrhundert* and the preface to Hilbert's *Zahlbericht*. Indeed historical representations are often created by mathematicians themselves; they seem all the more natural because they stem from mathematical practice and are coherent with it.[136] However, as seen above, a multitude of practices stem from the D.A. (with, perhaps, still more to come). As opposed to a person, a book like the D.A. has several identities.

Some of these identities, some of the works and memories linked to them, have been transmitted to the present. But only some. Those which have been lost for a while may well serve to give a more concrete meaning to the present:

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132. It is still difficult to properly evaluate the impact of Zolotarev's theory.

133. An evocative, if incomplete, information is contained in the Mathematics Genealogy Project, which lists no descendent at all for Dedekind, but, for instance, 3867 for his Berlin contemporary Lazarus Fuchs.

134. It is remarkable that Minkowski began with the theory of forms, Hilbert with invariant algebra, and Hurwitz with modular elliptic functions, each incarnating key components of arithmetic algebraic analysis originating from the D.A.

135. See the interesting list of books given in [Lemmermeyer 2000], p. xii, note 6, which are all linked to this milieu.

136. In this respect, it is interesting to contrast the various types of historical activities in some of the number theorists we have looked at: Lucas's involvement with Fermat's manuscripts, Hilbert's rational reconstruction of the evolution of number theory, Dickson's topical *History of the Theory of Numbers*.

Yes, if a memory, thanks to forgetfulness, has been unable to contract any tie, to forge any link between itself and the present, if it has remained in its own place, of its own date, if it has kept its distance, its isolation in the hollow of a valley or on the peak of a mountain, it makes us suddenly breathe an air new to us just because it is an air we have formerly breathed, an air purer than that the poets have vainly called Paradisiacal, which offers that deep sense of renewal only because it has been breathed before, inasmuch as the true paradises are paradises we have lost.[137]

References

BACHMANN, Paul. 1872. *Die Lehre von der Kreistheilung und ihre Beziehungen zur Zahlentheorie*. Leipzig: Teubner.

———. 1892. *Zahlentheorie. Versuch einer Gesamtdarstellung dieser Wissenschaft in ihren Hauptteilen.* Part 1: *Die Elemente der Zahlentheorie*. Leipzig: Teubner.

———. 1921. *Grundlehre der neueren Zahlentheorie*. 2nd ed. Göschens Lehrbücherei I.3. Berlin, Leipzig: de Gruyter.

BELL, Eric Temple. 1951. *Mathematics—Queen and Servant of Science*. New York: McGraw-Hill. Part. repr. The Queen of Mathematics. In *The World of Mathematics*, ed. J. Newman, vol. 1, pp. 498–518. New York: Simon and Schuster, 1956.

BLUMENTHAL, Otto. 1935. Lebensgeschichte. In [Hilbert 1932–1935], vol. 3, pp. 388–429.

BÖLLING, Reinhard. 1994. *Das Fotoalbum für Weierstraß – A Photo Album for Weierstrass*. Braunschweig, Wiesbaden: Vieweg.

BREZYNSKI, Claude. 1991. *History of Continued Fractions and Padé Approximants*. Berlin: Springer.

CHÂTELET, Albert. 1913. *Leçons sur la théorie des nombres*. Paris: Gauthier-Villars.

CONTE, Alberto, GIACARDI, Livia (eds.). 1991. *Angelo Genocchi e i suoi interlocutori scientifici. Contributi dall'epistolario.* Studi e Fonti per la Storia della Università di Torino 4. Torino: Deputazione subalpina di storia patria.

CORRY, Leo. 1996. *Modern Algebra and the Rise of Mathematical Structures*. Science Networks 17. Basel, Boston, Berlin: Birkhäuser. 2nd ed., 2004.

DÉCAILLOT, Anne-Marie. 1999. *Edouard Lucas (1842–1891): le parcours original d'un scientifique français dans la deuxième moitié du XIXe siècle*. Thèse de l'université René Descartes. Paris.

DEDEKIND, Richard. 1930–1932. *Gesammelte mathematische Werke*, ed. R. Fricke, E. Noether, O. Ore. 3 vols. Braunschweig: Vieweg.

DICKSON, Leonard Eugene. 1919–1923. *History of the Theory of Numbers.* 3 vols. Washington: The Carnegie Institute. Repr. New York: Chelsea, 1956.

137. Stephen Hudson's transl. from Marcel Proust, *Le temps retrouvé*: *Oui, si le souvenir, grâce à l'oubli, n'a pu contracter aucun lien, jeter aucun chaînon entre lui et la minute présente, s'il est resté à sa place, à sa date, s'il a gardé ses distances, son isolement dans le creux d'une vallée ou à la pointe d'un sommet; il nous fait tout à coup respirer un air nouveau, précisément parce que c'est un air qu'on a respiré autrefois, cet air plus pur que les poètes ont vainement essayé de faire régner dans le Paradis et qui ne pourrait donner cette sensation profonde de renouvellement que s'il avait été respiré déjà, car les vrais paradis sont les paradis qu'on a perdus.*

DUGAC, Pierre. 1976. *Richard Dedekind et les fondements des mathématiques*. Paris: Vrin.

EDWARDS, Harold M. 1975. The Background of Kummer's Proof of Fermat's Last Theorem for Regular Primes. *Archive for History of Exact Sciences* 14, 219–236.

———. 1980. The Genesis of Ideal Theory. *Archive for History of Exact Sciences* 23, 321–378.

———. 1983. Dedekind's Invention of Ideals. *Bulletin of the London Mathematical Society* 15, 8–17.

———. 1992a. Kronecker's Arithmetical Theory of Algebraic Quantities. *Jahresberichte der Deutschen Mathematiker-Vereinigung* 94, 130–139.

———. 1992b. Mathematical Ideas, Ideals, and Ideology. *The Mathematical Intelligencer* 14, 6–19.

———. 1995. Kronecker on the Foundations of Mathematics. In *From Dedekind to Gödel. Essays on the Development of the Foundations of Mathematics*, ed. J. Hintikka, pp. 45–52. Synthese Library, Studies in Epistemology, Logic, Methodology, and Philosophy of Science 251. Dordrecht, Kluwer.

EDWARDS, Harold M., NEUMANN, Olaf, PURKERT, Walter. 1982. Dedekinds "Bunte Bemerkungen" zu Kroneckers "Grundzüge." *Archive for History of Exact Sciences* 27, 49–85.

EISENSTEIN, Gotthold. 1975. *Mathematische Werke*. 2 vols. New York: Chelsea.

FERREIRÓS, Jose. 1999. *Labyrinth of Thought. A History of Set Theory and its Role in Modern Mathematics*. Basel, Boston: Birkhäuser.

GAUSS, Carl Friedrich. 1866. *Werke*, vol. III, *Analysis*, ed. Königliche Gesellschaft der Wissenschaften zu Göttingen. Göttingen: Universitäts-Druckerei.

———. 1889. *Untersuchungen über höhere Arithmetik*, ed. and German transl. H. Maser. Berlin: Springer. Repr. New York: Chelsea, 1965; 2nd ed., 1981.

GISPERT, Hélène, TOBIES, Renate. 1996. A Comparative Study of the French and German Mathematical Societies before 1914. In *L'Europe mathématique. Mathematical Europe*, ed. C. Goldstein, J. Gray, J. Ritter, pp. 407–430. Paris: Maison des sciences de l'homme.

GOLDSTEIN, Catherine. 1994. La théorie des nombres dans les *Comptes rendus de l'Académie des sciences* (1870–1914) : un premier examen. *Rivista di Storia della Scienza* 2nd ser. 2, 137–160.

———. 1999. Sur la question des méthodes quantitatives en histoire des mathématiques : le cas de la théorie des nombres en France (1870–1914). *Acta historiæ rerum naturalium necnon technicarum* New ser. 3, 187–214.

———. 2005. Johann Peter Gustav Lejeune-Dirichlet, *Vorlesungen über Zahlentheorie*, first edition (1863). In *Landmark Writings in Western Mathematics*, ed. I. Grattan-Guinness, pp. 480–490. Amsterdam, Boston, etc.: Elsevier.

GRAY, Jeremy. 1996. The Nineteenth-Century Revolution in Mathematical Ontology. In *Revolutions in Mathematics*, ed. D. Gillies, pp. 226-248. Oxford: Oxford University Press.

HAGNER, Michael. 2004. *Geniale Gehirne. Zur Geschichte der Elitegehirnforschung.* Göttingen: Wallstein.

HAUBRICH, Ralf. 1992. *Zur Entstehung der algebraischen Zahlentheorie Richard Dedekinds*. Dissertation, Georg-August-Universität Göttingen. Göttingen.

HERMITE, Charles. 1905–1917. *Œuvres*, ed. E. Picard. 4 vols. Paris: Gauthier-Villars.

HILBERT, David. 1932–1935. *Gesammelte Abhandlungen*. 3 vols. Berlin: Julius Springer.

HURWITZ, Adolf. 1985. Über einen Fundamentalsatz der arithmetischen Theorie der algebraischen Grössen. *Nachrichten von der königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse* 1985, 230–240. Repr. in *Mathematische Werke*, vol. 2, pp. 198–207. Basel: Birkhäuser, 1963.

KLEIN, Felix. 1894. *Lectures on Mathematics. The Evanston Colloquium*. New York: Macmillan. Repr. Providence (R.I): AMS, 1911; 2000.

———. 1926. *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, ed. R. Courant, O. Neugebauer, vol. 1. Berlin: Springer; repr. (with vol. 2) New York: Chelsea, 1967; repr. Berlin, Heidelberg, New York: Springer, 1979.

KRONECKER, Leopold. 1895–1931. *Werke*, ed. K. Hensel. 5 vols. Leipzig: Teubner. Rep. New York: Chelsea, 1968.

———. 1901. *Vorlesungen über Zahlentheorie*, vol. 1, ed. K. Hensel. Leipzig: Teubner. Repr. Berlin, Heidelberg, New York: Springer, 1978.

KUHN, Thomas S. 1970. *The Structure of Scientific Revolutions*. International Encyclopedia of Unified Science 2, II. 2nd enlarged ed. Chicago: Chicago University Press.

KUMMER, Ernst Eduard. 1975. *Collected Papers*, ed. A. Weil. 2 vols. Berlin, Heidelberg, etc.: Springer.

LAVRINENKO, Tatiana. 2002. Solving an indeterminate third degree equation in rational numbers. Sylvester and Lucas. *Revue d'histoire des mathématiques* 8-1, 67–111.

LEMMERMEYER, Franz. 2000. *Reciprocity Laws from Euler to Eisenstein*. Berlin, etc.: Springer.

LIPSCHITZ, Rudolf. 1986. *Briefwechsel mit Cantor, Dedekind, Helmholtz, Kronecker, Weierstrass*, ed. W. Scharlau. Dokumente zur Geschichte der Mathematik 2. Braunschweig, Wiesbaden: Vieweg.

MCKENZIE, Donald F. 1986. *Bibliography and the Sociology of Texts*. London: British Library. 2e ed. Cambridge: Cambridge University Press, 1999.

NEUMANN, Olaf. 1979–1980. Bemerkungen aus heutiger Sicht über Gauss' Beiträge zu Zahlentheorie, Algebra und Funktionentheorie. *NTM-Schriftenreihe* 16:2, 22–39; 17:1, 32–48; 17:2, 38–58.

———. 2002. Was sollen und was sind Divisoren?, *Mathematische Semesterberichte* 48, 139–192.

OZHIGOVA, Elena Petrovna, YUŠKEVIČ, Adolph Andrei Pavlovič. 1992. Problems of Number Theory. In *Mathematics of the 19th Century. Mathematical Logic. Algebra. Number Theory. Probability Theory*, ed. A. N. Kolmogorov and A. P. Yuškevič, chap. 3. Basel, Boston, Berlin: Birkhäuser.

PARSHALL, Karen. 1998. *James Joseph Sylvester. Life and Work in Letters*. Oxford: Clarendon Press.

———. 2006. *James Joseph Sylvester: Jewish Mathematician in a Victorian World*. Baltimore: The Johns Hopkins University Press.

POINCARÉ, Henri. 1897. Les rapports de l'analyse et de la physique mathématique. *Acta Mathematica* 21, 331–341. Repr. *Revue générale des sciences pures et appliquées* 8, 857–861. Repr. in *L'analyse et la recherche*, ed. G. Ramunni. Paris: Hermann, 1991.

———. 1901. Sur les propriétés arithmétiques des courbes algébriques. *Journal de mathématiques pures et appliquées* 5e ser., 7-3, 161–233. Repr. in *Œuvres*, vol. 5, ed. A. Châtelet, pp. 483–550. Paris, Gauthier-Villars, 1950.

PURKERT, Walter. 1971–1973. Zur Genesis des abstrakten Körperbegriffs. *NTM-Schriftenreihe* 8:1, 23–37; 10:2, 8–20.

———. 1981. Richard Dedekind – zum 150. Geburtstag. *Mitteilungen der Mathematischen Gesellschaft der Deutschen Demokratischen Republik* 2/4, 84–110.

ROLLET, Laurent, NABONNAND, Philippe. 2002. Une bibliographie mathématique idéale ? Le *Répertoire bibliographique des sciences mathématiques*. *Gazette des mathématiciens* 92, 11–26.

ROWE, David. 1989. Klein, Hilbert, and the Göttingen Mathematical Tradition. *Osiris* 5-2, 186–213.

SCHAPPACHER, Norbert. 1991. Développement de la loi de groupe sur une cubique. In *Séminaire de Théorie des Nombres de Paris 1988–1989*, ed. C. Goldstein, pp. 159–184. Progress in Mathematics 91. Boston, Basel, etc.: Birkhäuser.

———. 2005. David Hilbert, Report on Algebraic Number Fields (*Zahlbericht*) (1897). In *Landmark Writings in Western Mathematics*, ed. I. Grattan-Guinness, pp. 700–709. Amsterdam, Boston, etc.: Elsevier.

SCHARLAU, Winfried. 1982. Unveröffentliche algebraische Arbeiten Richard Dedekinds aus seiner Göttinger Zeit 1855–1858. *Archive for History of Exact Sciences* 36, 63–74.

SELLING, Eduard. 1865. Über die idealen Primfactoren der complexen Zahlen, welche aus den Wurzeln einer beliebigen irreductiblen Gleichung rational gebildet sind. *Zeitschrift für Mathematik und Physik* 10, 17–47.

SIEGMUND-SCHULTZE, Reinhard. 1993. *Mathematische Berichterstattung in Hitlerdeutschland. Der Niedergang des "Jahrbuchs über die Fortschritte der Mathematik."* Göttingen: Vandenhoeck & Ruprecht.

SMITH, Henry John Stephen. 1894. *The Collected Mathematical Papers*, ed. J.W.L. Glaisher. 2vols. Oxford: Clarendon Press.

STROBL, Walter. 1985. Aus den wissenschaftlichen Anfängen Hermann Minkowskis. *Historia Mathematica* 12, 142–156.

TANNERY, Jules. 1895. *Introduction à l'étude de la théorie des nombres et de l'algèbre supérieure*. Paris: Nony.

ULLRICH, Peter. 1998. The Genesis of Hensel's *p*-adic Numbers. In *Karl der Grossen und sein Nachwirken. 1200 Jahre Kultur und Wissenschaft in Europa*, ed. P.L Butzer, H. Th. Jongen, and W. Oberschelp, vol. 2, pp. 163-178. Turnhout: Brepols.

WUSSING, Hans. 1969. *Die Genesis des abtrakten Gruppenbegriffs. Ein Beitrag zur Entstehungsgeschichte der abtrakten Gruppentheorie.* Berlin: VEB Deutscher Verlag der Wissenschaften. English transl. A. Shenitzer. Cambridge: MIT Press, 1984.

ZOLOTAREV, Egor Ivanovich. 1872. Sur la méthode d'intégration de M. Tchebychef. *Mathematische Annalen* 5, 560–580. Repr. *Journal de mathématiques pures et appliquées* 2nd ser. 19 (1874), 161–188.

———. 1880. Sur la théorie des nombres complexes. *Journal de mathématiques pures et appliquées* 3e ser. 6, 51–94, 129–166. Repr. in *Polnoe Sobranie Sochineny Egora Ivanovicha Zolotareva* (Collected Papers of Egor lvanovich Zolotarev), vol. 1, pp. 72–179. Leningrad: V.A. Steklov Institute of Physics and Mathematics.