Remarks about Intuition in Italian Algebraic Geometry

Norbert Schappacher

Italian Algebraic Geometry has repeatedly been criticised for its alleged lack of rigour. Accordingly, the first rewriting of Algebraic Geometry, which was realized in the 1930ies and 1940ies principally by Bartel L. van der Waerden, Oscar Zariski and André Weil, has been portrayed as a restoration of rigour in this domain. Furthermore, in most accounts of these events the criticism for lack of rigour is linked to the alleged *intuitive* character of the Italian research on Algebraic Geometry—see the three clippings below. The aim of the talk was to show that this association of intuition with lack of rigour—a topos which incidentally is in itself interesting from a historical or philosophical point of view—is misleading and ought to be discarded in order to clear the way for an adequate historiographical appraisal of the Italian production.

Just two quotes to illustrate what I am alluding to: Commenting on his 1941 paper "On the Riemann hypothesis for function fields", André Weil recalled in 1979 discussions about Algebraic Geometry from the 1930ies and 1940ies (my translation from [11], p. 555):

> There was still quite a bit of confusion as to Algebraic Geometry. A growing number of mathematicians, among them the followers of Bourbaki, had convinced themselves of the necessity to ground all of mathematics on set theory; but others had doubts whether this would be possible. As counterexamples they pointed to probability calculus, differential geometry, and algebraic geometry. They claimed that these needed autonomous foundations, or even (thus confusing the needs of invention with those of logic) that they required the constant intervention of some mysterious intuition. But it had become increasingly difficult to sustain an unlimited confidence in Algebraic Geometry. Too many fractures appeared which made one fear that the whole edifice would collapse at the next blow. This is what Zariski experienced when he wrote his famous volume *Algebraic Surfaces* whose explicit goal was above all the critical evaluation of the main discoveries of Italian geometers in their favourite area of research.

And in 2009, we read on the first page of [1] (and we wonder which other 'schools' the authors may be thinking of):

> There were, of course, other important schools of algebraic geometry in other countries, but the Italian school stood out because of its unique mathematical style, especially its strong appeal to geometric intuition.

Since intuition, taken in a broad sense, accompanies any scientific activity, we have to make our question more precise: To which extent did the geometric visualisation of the researched objects constitute validating elements of mathematical proof in Italian Algebraic Geometry, say, between the 1880ies and 1920ies? To answer this question we will look at the way in which figures relate to the surrounding discourse in various sorts of texts produced by Italian Algebraic Geometers. Four cases were presented in the talk:

(1) The *Encyklopädie* overseen by Felix Klein. It contains well illustrated chapters, for instance the one on (systems of) conic sections [7], where figures are linked to the text almost as closely as in Euclid's *Elements.* Also Kohn's and Loria's exposition of special algebraic curves [10], which introduces basic objects of Algebraic Geometry, contains at least occasional illustrations. The most spectacular drawings of surfaces can be admired in the chapter on topology [6], see for instance p. 197. All these images show that there were no production constraints on inserting figures into the text in the *Encyklopädie*. Yet the two famous chapters on algebraic surfaces, [3] and [4], by Guido Castelnuovo and Federigo Enriques carry not a single illustration. By the way, the same is true of Enriques's textbook [9].

(2) But to be sure, our Italian Algebraic Geometers *were* in the habit of using illustrations in other sorts of texts they produced. Looking for instance at Enriques's lectures on Projective Geometry [8], we find numerous lettered diagrams which are clearly meant to be read as part of the proofs given. This means that substantial basic knowledge required of any researcher preparing to work in Algebraic Geometry was invested with an essential illustrative component. More generally, there can be no doubt that basic objects of algebraic geometry—such as individual algebraic curves, for example—were naturally pictured (with or without actually drawing them) by all those working with them.

(3) This basic reflex of visualising given objects and constellations is nicely documented in the recently edited notes, by an unidentified notetaker, of Castelnuovo's last lecture course (1922–23) on plane curves and space curves [5]. Studying these notes, one begins to understand why figures tend to disappear from research publications in Algebraic Geometry. In fact, as they were shaping algebraic geometry, the Italian geometers were led to analyzing constellations of objects which are increasingly difficult to visualise adequately: Their work takes place in *iperspazio*—i.e., in higher dimensions, often needed to conveniently project down from, even when the basic objects are initially given in the plane or in 3-space. Furthermore, adjoint objects, linear series, and other families of geometric objects are studied which cut out things on underlying geometric objects. These families are often defined, and always ultimately controlled in terms of polynomial algebra. The corresponding arguments can typically not be drawn, nor can subtle genericity assumptions—another hallmark of classical algebro-geometric reasoning—be visually controlled in a figure.

Thus turning pages in [5], we find many results accompanied by skilful illustrations of the situation addressed. However, these do not visualise the core arguments of the proof which is then developed. In the talk, this was explained for Theorem 3.11 ([5], p. 45, where the adjoint curve whose existence is finally established as an application of polynomial algebra via Noether's theorem is not shown in the figure), Lemma 3.26 ([5], p. 48, where the existence claim is established by counting constants), and Theorem 4.28 ([5], p. 67–68, where the higher dimensional situation is beautifully sketched, but the proof is quintessentially algebraic in that it rests on the independence of various intersection conditions).

(4) We have also inspected the occasional jottings in Enriques's letters to Castelnuovo [2] (in the talk we just commented the drawings on p. 244 and p. 477). They confirm the impression that such spontaneous sketches of geometrical constellations are not meant to carry the weight of a geometric construction or a concluding argument.

To sum up, since objects whose existence is finally established in an algebraic way are typically absent from the drawings, it is plausible to interpret the drawings as a spontaneous reflex when setting up an investigation, rather than viewing them as a key element of the argument. Even choices made in the course of a proof—often of objects in general position—are typically not recorded in the diagrams. If lack of rigour there was, it thus has to be studied in the discourse and the algebraic reasoning of the Italian Algebraic Geometers.

References

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