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Holomorphic L^p-type for sub-Laplacians on connected Lie groups ☆

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Abstract

We study the problem of determining all connected Lie groups G which have the following property (hlp): every sub-Laplacian L on G is of holomorphic L^p-type for $1 \leqslant p < \infty$, $p \neq 2$. First we show that semi-simple non-compact Lie groups with finite center have this property, by using holomorphic families of representations in the class one principal series of G and the Kunze–Stein phenomenon. We then apply an L^p-transference principle, essentially due to Anker, to show that every connected Lie group G whose semi-simple quotient by its radical is non-compact has property (hlp). For the convenience of the reader, we give a self-contained proof of this transference principle, which generalizes the well-known Coifman–Weiss principle. One is thus reduced to studying those groups for which the semi-simple quotient is compact, i.e. to compact extensions of solvable Lie groups. In this article, we consider semi-direct extensions of exponential solvable Lie groups by connected compact Lie groups. It had been proved in joint work by W. Hebisch and the two first named authors that every exponential solvable Lie group S, which has a non-$*$ regular co-adjoint orbit whose restriction to the nilradical is closed, has property (hlp), and we show here that (hlp) remains valid for compact extensions of these groups.

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Contents

1. Introduction

A comprehensive discussion of the problem studied in this article, background information and references to further literature can be found in [14]. We shall therefore content ourselves in this introduction by recalling some notation and results from [14].

Let $(X, d\mu)$ be a σ-finite measure space. If A is a self-adjoint linear operator on the L^2-space $L^2(X, d\mu)$, with spectral resolution $A = \int_{\mathbb{R}} \lambda\, dE_\lambda$, and if F is a bounded Borel function on $\mathbb{R}$, then we call F an *L^p-multiplier for* A ($1 \leqslant p < \infty$), if $F(A) := \int_{\mathbb{R}} F(\lambda)\, dE_\lambda$ extends from $L^p \cap L^2(X, d\mu)$ to a bounded linear operator on $L^p(X, d\mu)$. We shall denote by $\mathcal{M}_p(A)$ the space of all L^p-multipliers for A, and by $\sigma_p(A)$ the L^p-spectrum of A. We say that A is of *holomorphic L^p-type*, if there exist some non-isolated point λ_0 in the L^2-spectrum $\sigma_2(A)$ and an open complex neighborhood $\mathcal{U}$ of λ_0 in $\mathbb{C}$, such that every $F \in \mathcal{M}_p(A) \cap C_\infty(\mathbb{R})$ extends holomorphically to $\mathcal{U}$. Here, $C_\infty(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$ vanishing at infinity.

Assume in addition that $-A$ generates a symmetric diffusion semigroup (as in the case of the sub-Laplacians that we shall examine later). Then, if A is of holomorphic L^p-type, one can show that

$$\sigma_2(A) \neq \sigma_p(A).$$

Moreover, if this semigroup is hypercontractive (as, for instance, in the case where A is a sub-Laplacian on a unimodular Lie group), then the set $\mathcal{U}$ belongs to the L^p-spectrum of A, i.e.

$$\bar{\mathcal{U}} \subset \sigma_p(A), \tag{1.1}$$

and

$$\sigma_2(A) \subsetneq \sigma_p(A) \tag{1.2}$$

(see [8]).

Throughout this article, G will denote a connected Lie group.

Let dg be a *left-invariant* Haar measure on G. If $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G on the Hilbert space $\mathcal{H} = \mathcal{H}_\pi$, then we denote the integrated representation of $L^1(G) = L^1(G, dg)$ again by π, i.e. $\pi(f)\xi := \int_G f(g)\pi(g)\xi\, dg$ for every $f \in L^1(G)$, $\xi \in \mathcal{H}$. For $X \in \mathfrak{g}$, we denote by $d\pi(X)$ the infinitesimal generator of the one-parameter group of unitary operators $t \mapsto \pi(\exp tX)$. Moreover, we shall often identify X with the corresponding *right-invariant* vector field $X^r f(g) := \lim_{t\to 0} \frac{1}{t}[f((\exp tX)g) - f(g)]$ on G and write $X = X^r$.

Let $X_1, \dots, X_k$ be elements of $\mathfrak{g}$ which generate $\mathfrak{g}$ as a Lie algebra, which just means that the corresponding right-invariant vector fields satisfy Hörmander's condition. The corresponding sub-Laplacian $L := -\sum_{j=1}^k X_j^2$ is then essentially self-adjoint on $\mathcal{D}(G) \subset L^2(G)$ and hypoelliptic.

As has been pointed out to us by W. Hebisch, if such a right-invariant sub-Laplacian L is of holomorphic L^p-type, then properties (1.1) and (1.2) hold true for $A := L$, even if the group G is non-unimodular.

This is due to the existence of approximations to the identity by convolution from the right by smooth functions with compact support (cf. Lemma A.2 in Appendix A).

Denote by $\{e^{-tL}\}_{t>0}$ the heat semigroup generated by L. Since L is right G-invariant, for every $t > 0$, e^{-tL} admits a convolution kernel h_t such that

$$e^{-tL} f = h_t * f,$$

where $*$ denotes the usual convolution product in $L^1(G)$. The function $(t, g) \mapsto h_t(g)$ is smooth on $\mathbb{R}_{>0} \times G$, since the differential operator $\frac{\partial}{\partial t} + L$ is hypoelliptic. Moreover, by [19, Theorems VIII.4.3 and V.4.2], the heat kernel h_t as well as its right-invariant derivatives admit Gaussian type estimates in terms of the Carnot–Carathéodory distance δ associated to the Hörmander system $X_1, \dots, X_k$.

In particular, for every right-invariant differential operator D on G, there exist constants $c_{D,t}, C_{D,t} > 0$, such that, for all $g \in G$, $t > 0$,

$$\left|Dh_t(g)\right| \leqslant C_{D,t} e^{-c_{D,t}\delta(g,e)^2}. \tag{1.3}$$

Let now $F_0 \in \mathcal{M}_p(L)$. By duality, we may assume that $1 \leqslant p \leqslant 2$. With F_0, also the function $\lambda \mapsto F(\lambda) := e^{-\lambda} F_0(\lambda)$ lies in $\mathcal{M}_p(L)$, since $F(L) = e^{-L} F_0(L)$, where the heat operator e^{-L} is bounded on every $L^p(G)$ $(1 \leqslant p < \infty)$. Now by [14, Lemma 6.1], the operator $F_0(L)$ is bounded also on all the spaces $L^q(G)$, $p \leqslant q \leqslant p'$. Hence for every test function f on G,

$$\begin{aligned} F(L)(f) &= F_0(L)\big(e^{-L}(f)\big) = F_0(L)(h_1 * f) \\ &= F_0(L)(h_{1/2} * h_{1/2} * f) = \big(F_0(L)h_{1/2}\big) * h_{1/2} * f, \end{aligned}$$

by the right-invariance of the operator $F_0(L)$. Since $h_{1/2}$ is contained in every $L^q(G)$, $1 \leqslant q \leqslant \infty$, in particular in $L^1(G)$, we see that the operator $F(L)$ acts by convolution from the left with the function $(F_0(L)h_{1/2}) * h_{1/2}$ which is contained in every $L^q(G)$, $p \leqslant q \leqslant p'$, and so are all its derivatives from the right. We can thus identify the operator $F(L)$ with the C^∞-function $F(L)\delta := (F_0(L)h_{1/2}) * h_{1/2}$, i.e.

$$F(L)(f) = \bigl(F(L)\delta\bigr) * f, \quad f \in \bigcup_{p \leqslant q \leqslant p'} L^q(G).$$

Recall that the *modular function* Δ_G on G is defined by the equation

$$\int_G f(xg)\,dx = \Delta_G(g)^{-1} \int_G f(x)\,dx, \quad g \in G.$$

Put, for $g \in G$:

$$\check{f}(g) := f\bigl(g^{-1}\bigr),$$
$$f^*(g) := \Delta_G^{-1}(g)\overline{f\bigl(g^{-1}\bigr)}.$$

Then $f \mapsto f^*$ is an isometric involution on $L^1(G)$, and for any unitary representation π of G, we have

$$\pi(f)^* = \pi(f^*). \tag{1.4}$$

The group G is said to be *symmetric*, if the associated group algebra $L^1(G)$ is symmetric, i.e. if every element $f \in L^1(G)$ with $f^* = f$ has a real spectrum with respect to the involutive Banach algebra $L^1(G)$.

In this paper we consider connected Lie groups for which every sub-Laplacian is of holomorphic L^p-type. First, in Section 2, we consider connected semi-simple Lie groups G with finite center. We construct a holomorphic family of representations $\pi_{(z)}$ of G on mixed L^p-spaces (see Section 2.2). Applying these representations to h_1, we obtain a holomorphic family of compact operators on these spaces (see Section 2.3). Using the Kunze–Stein phenomenon on semi-simple Lie groups (see Section 2.4), the eigenvectors of the operators $\pi_{(z)}(h_1)$ allow us to construct a holomorphic family of L^p-functions on G which are eigenvectors for $F(L)$, if $F \in \mathcal{M}_p(T) \cap C_\infty(\mathbb{R})$. From the corresponding holomorphic family of eigenvalues we can read off that F admits a holomorphic extension in a neighborhood of some element in the spectrum of L (see Section 2.5). This gives us:

Theorem 1.1. *Let G be a non-compact connected semi-simple Lie group with finite center. Then every sub-Laplacian on G is of holomorphic L^p-type, for $1 \leqslant p < \infty$, $p \neq 2$.*

Remark 1. Even if at the end of the proof, we consider only ordinary L^p-spaces, we need representations on mixed L^p-spaces. They are used to get some isometry property and then to apply the Kunze–Stein phenomenon.

In Sections 3.1 and 3.2, we discuss respectively p-induced representations and a generalization of the Coifman–Weiss transference principle [5]. We consider a separable locally compact group G, and an isometric representation ρ of a closed subgroup S of G on spaces of L^p-type, e.g. L^p-spaces $L^p(\Omega)$. Denote by $\pi_p := \mathrm{ind}_{p,S}^G \rho$ the p-induced representation of ρ. We prove, among other results, that, for any function $f \in L^1(G)$, the operator norm of $\pi_p(f)$ is bounded by the norm of the convolution operator $\lambda_G(f)$ on $L^p(G)$, provided the group S is amenable. Here, λ_G denotes the left-regular representation. It should be noted that we do not require the group G to be amenable. As an application we obtain the L^p-transference of a convolution operator on G to a convolution operator on the quotient group G/S, in the case where S is an amenable closed, normal subgroup.

When preparing this article, we were not aware of J.-Ph. Anker's article [1] which, to a large extent, contains these transference results, and which we also recommend for further references to this topic. We are indebted to N. Lohoué for informing us on Anker's work [1] as well as on the influence of C. Herz on the development of this field (compare [9]). For the convenience of the reader, we have nevertheless decided to include our approach to these transference results, since it differs from Anker's by the use of a suitable cross section for G/S, which we feel makes the arguments a bit easier.

Applying this transference principle, we obtain the following generalization of Theorem 1.1 in Section 4.

Theorem 1.2. *Let $G = \exp \mathfrak{g}$ be a connected Lie group, and denote by $S = \exp \mathfrak{s}$ its radical. If G/S is not compact, then every sub-Laplacian on G is of holomorphic L^p-type, for any $1 \leqslant p < \infty$, $p \neq 2$.*

It then suffices to study connected Lie groups for which G/S is compact. In Section 5, we shall consider groups G which are the semi-direct product of a compact group K with a non-symmetric exponential solvable group S from a certain class. The exponential solvable non-symmetric Lie groups have been completely classified by Poguntke [17] (with previous contributions by Leptin, Ludwig and Boidol) in terms of a purely Lie-algebraic condition (B). Let us describe this condition, which had been first introduced by Boidol in a different context [3].

Recall that the unitary dual of S is in one to one correspondence with the space of coadjoint orbits in the dual space $\mathfrak{s}^*$ of $\mathfrak{s}$ via the Kirillov map, which associates with a given point $\ell \in \mathfrak{s}^*$ an irreducible unitary representation π_ℓ (see e.g. [8, Section 1]).

If ℓ is an element of $\mathfrak{s}^*$, denote by

$$\mathfrak{s}(\ell) := \bigl\{X \in \mathfrak{s} \mid \ell\bigl([X,Y]\bigr) = 0, \text{ for all } Y \in \mathfrak{s}\bigr\}$$

the *stabilizer* of ℓ under the coadjoint action ad^*. Moreover, if $\mathfrak{m}$ is any Lie algebra, denote by

$$\mathfrak{m} = \mathfrak{m}^1 \supset \mathfrak{m}^2 \supset \cdots$$

the descending central series of $\mathfrak{m}$, i.e. $\mathfrak{m}^2 = [\mathfrak{m},\mathfrak{m}]$, and $\mathfrak{m}^{k+1} = [\mathfrak{m},\mathfrak{m}^k]$. Put

$$\mathfrak{m}^\infty = \bigcap_k \mathfrak{m}^k.$$

Then $\mathfrak{m}^\infty$ is the smallest ideal $\mathfrak{k}$ in $\mathfrak{m}$ such that $\mathfrak{m}/\mathfrak{k}$ is nilpotent. Put

$$\mathfrak{m}(\ell) := \mathfrak{s}(\ell) + [\mathfrak{s}, \mathfrak{s}].$$

Then we say that ℓ, respectively the associated coadjoint orbit $\Omega(\ell) := \mathrm{Ad}^*(G)\ell$, satisfies *Boidol's condition* (B), if

$$\ell\,|_{\mathfrak{m}(\ell)^\infty} \neq 0. \tag{B}$$

According to [17], the group S is non-symmetric if and only if there exists a coadjoint orbit satisfying Boidol's condition.

If Ω is a coadjoint orbit, and if $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$, then

$$\Omega|_{\mathfrak{n}} := \{\ell|_{\mathfrak{n}} \mid \ell \in \Omega\} \subset \mathfrak{n}^*$$

will denote the restriction of Ω to $\mathfrak{n}$.

We show that the methods developed in [8] can also be applied to the case of a compact extension of an exponential solvable group and thus obtain:

Theorem 1.3. *Let $G = K \ltimes S$ be a semi-direct product of a compact Lie group K with an exponential solvable Lie group S, and assume that there exists a coadjoint orbit $\Omega(\ell) \subset \mathfrak{s}^*$ satisfying Boidol's condition, whose restriction to the nilradical $\mathfrak{n}$ is closed in $\mathfrak{n}^*$. Then every sub-Laplacian on G is of holomorphic L^p-type, for $1 \leqslant p < \infty$, $p \neq 2$.*

Remarks. (a) A sub-Laplacian L on G is of holomorphic L^p-type if and only if every continuous bounded multiplier $F \in \mathcal{M}_p(L)$ extends holomorphically to an open neighborhood of a non-isolated point in $\sigma_2(L)$.

(b) If the restriction of a coadjoint orbit to the nilradical is closed, then the orbit itself is closed (see [8, Theorem 2.2]).

(c) What we really use in the proof is the following property of the orbit Ω:

Ω is closed, and for every real character ν of $\mathfrak{s}$ which does not vanish on $\mathfrak{s}(\ell)$, there exists a sequence $\{\tau_n\}_n$ of real numbers such that $\lim_{n\to\infty}(\Omega + \tau_n\nu) = \infty$ in the orbit space.

This property is a consequence of the closedness of $\Omega|_{\mathfrak{n}}$. There are, however, many examples where the condition above is satisfied, so that the conclusion of the theorem still holds, even though the restriction of Ω to the nilradical is not closed (see e.g. [8, Section 7]). We do not know whether the condition above automatically holds whenever the orbit Ω is closed.

Observe that, contrary to the semisimple case, we need to consider representations on mixed L^p-spaces till the end of the proof.

2. The semi-simple case

2.1. Preliminaries

In the following, if M is a topological space, $C_0(M)$ will mean the space of compactly supported continuous functions on M.

As usual, if S is a Lie group, $\mathfrak{s}$ will denote its Lie algebra.

If E is a vector space, denote by E^* its algebraic dual. If it is real, $E_{\mathbb{C}}$ denotes its complexification. Let F be a vector subspace of E. We identify the restriction $\lambda_{|F}$ of $\lambda \in E^*$ or $E_{\mathbb{C}}^*$ with an element of respectively F^* or $F_{\mathbb{C}}^*$.

Let G be a connected semisimple real Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Fix a Cartan involution θ of G and denote by K the fixed point group for θ. The Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to θ is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ the -1-eigenspace in $\mathfrak{g}$ for the differential of θ, denoted again by θ. We fix a subspace $\mathfrak{a}$ of $\mathfrak{p}$ which is maximal with respect to the condition that it is an abelian subalgebra of $\mathfrak{g}$. It is endowed with the scalar product $(\cdot,\cdot)$ given by the Killing form B, which is positive definite on $\mathfrak{p}$. By duality, we endow $\mathfrak{a}^*$ with the corresponding, induced scalar product, which we also denote by $(\cdot,\cdot)$. Let $|\cdot|$ be the associated norm on $\mathfrak{p}$ and $\mathfrak{a}^*$.

For any root $\alpha \in \mathfrak{a}^*$, we denote by $\mathfrak{g}_\alpha$ the corresponding root space, i.e. $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H,X] = \alpha(X),\ H \in \mathfrak{a}\}$. We fix a set R^+ of positive roots of $\mathfrak{a}$ in $\mathfrak{g}$. Let P denote the corresponding minimal parabolic subgroup of G, containing $A := \exp \mathfrak{a}$, and $P = MAN$ its Langlands decomposition.

Denote by ρ the linear form on $\mathfrak{a}$ given by

$$\rho(X) := \frac{1}{2} \operatorname{tr}(\operatorname{ad} X_{|\mathfrak{n}}), \qquad X \in \mathfrak{a},$$

where $\mathfrak{n}$ is the Lie algebra of N.

Let $\|\cdot\|$ denote the "norm" on G defined in [2, §2]. Recall that, for $g \in G$, $\|g\|$ is the operator norm of $\operatorname{Ad} g$ considered as an operator on $\mathfrak{g}$, endowed with the real Hilbert structure, $(X,Y) \mapsto -B(X,\theta Y)$ as scalar product. This norm is K-biinvariant and, according to [2, Lemma 2.1], and satisfies the following properties:

$$\begin{aligned}
&\|\cdot\| \text{ is a continuous and proper function on } G,\\
&\|g\| = \big\|\theta(g)\big\| = \big\|g^{-1}\big\| \geqslant 1;\\
&\|xy\| \leqslant \|x\|\|y\|;\\
&\text{there exists } c_1, c_2 > 0 \text{ such that,}\\
&\quad \text{for } Y \in \mathfrak{p},\ \text{then } e^{c_1|Y|} \leqslant \|\exp Y\| \leqslant e^{c_2|Y|};\\
&\text{for all } a \in A,\ n \in N,\ \|a\| \leqslant \|an\|.
\end{aligned} \tag{2.1}$$

We choose a basis for $\mathfrak{a}^*$, following, for example, [6, p. 220].

Let $\alpha_1, \ldots, \alpha_r$ denote the simple roots in R^+. By the Gram–Schmidt process, one constructs from the basis $\{\alpha_1, \ldots, \alpha_r\}$ of $\mathfrak{a}^*$ an orthonormal basis $\{\beta_1, \ldots, \beta_r\}$ of $\mathfrak{a}^*$ in a such way that, for every $j = 1, \ldots, r$, the vector space $\operatorname{Vect}\{\beta_1, \ldots, \beta_j\}$ spanned by $\{\beta_1, \ldots, \beta_j\}$ agrees with $\operatorname{Vect}\{\alpha_1, \ldots, \alpha_j\}$, and, for every $1 \leqslant k < j \leqslant r$, $(\beta_j, \alpha_k) = 0$. Define H_j $(j = 1, \ldots, r)$ as the element of $\mathfrak{a}$ given by $\beta_k(H_j) = \delta_{jk}$ $(k = 1, \ldots, r)$, and put:

$$\mathfrak{a}_j := \mathbb{R}H_j, \qquad \mathfrak{a}^j := \sum_{k=1}^{j} \mathfrak{a}_k;$$
$$R_j := R^j \setminus R^{j-1}, \quad \text{with } R^0 := \emptyset, \qquad R^j := R^+ \cap \operatorname{Vect}\{\alpha_1, \dots, \alpha_j\};$$
$$\mathfrak{n}^j := \sum_{\alpha \in R^j} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_j := \sum_{\alpha \in R_j} \mathfrak{g}_\alpha.$$

We define, for $j = 1, \dots, r$, the reductive Lie subalgebra $\mathfrak{m}^j$ of $\mathfrak{g}$ by setting:

$$\mathfrak{m}^j := \theta(\mathfrak{n}^j) + \mathfrak{m} + \mathfrak{a}^j + \mathfrak{n}^j.$$

In this way, we obtain a finite sequence of reductive Lie subalgebras of $\mathfrak{g}$,

$$\mathfrak{m} =: \mathfrak{m}^0 \subset \mathfrak{m}^1 \subset \cdots \subset \mathfrak{m}^r = \mathfrak{g},$$

such that

$$\mathfrak{m}^j = \theta(\mathfrak{n}_j) + \mathfrak{m}^{j-1} + \mathfrak{a}_j + \mathfrak{n}_j \quad (j = 1, \dots, r).$$

Then $\mathfrak{m}^{j-1} \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j$ is a parabolic subalgebra of $\mathfrak{m}^j$.

Observe that G is a real reductive Lie group in the Harish-Chandra class (see e.g. [7, p. 58], for the definition of this class of reductive Lie groups). We can then inductively define a decreasing sequence of reductive real Lie groups M^j in the Harish-Chandra class, starting from $M^r = G$, in the following way.

Let P_j denote the parabolic subgroup of M^j corresponding to the parabolic subalgebra $\mathfrak{m}^{j-1} \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j$, and $P_j = M^{j-1} A_j N_j$ its Langlands decomposition. Here A_j (respectively N_j) is the analytic subgroup of M^j with Lie algebra $\mathfrak{a}_j$ (respectively $\mathfrak{n}_j$), $M^{j-1}A_j$ is the centralizer in M^j of A_j, and

$$M^{j-1} := \bigcap_{\chi \in \operatorname{Hom}(M^{j-1}A_j, \mathbb{R}_+^\times)} \operatorname{Ker} \chi$$

(see e.g. [7, Theorem 2.3.1]).

Moreover, $M^{j-1}A_j$ normalizes N_j and $\theta(N_j)$, and M^{j-1} is a reductive Lie subgroup of M^j, in the Harish-Chandra class, with Lie algebra $\mathfrak{m}^{j-1}$ (see [7, Proposition 2.1.5]).

Put $K^j := M^j \cap K$ $(j = 1, \dots, r)$. Then K^j is the maximal compact subgroup of M^j related to the Cartan involution $\theta_{|M^j}$ of M^j (see e.g. [7, Theorem 2.3.2, p. 68]). Hence, M^j is the product

$$K^j P_j = K^j M^{j-1} A_j N_j.$$

Fix invariant measures dk, dm, da, dn, dm_j, dk_j, da_j, dn_j for respectively K, M, A, N, M^j, K^j, A_j, N_j.

Choose an invariant measure dx on G such that

$$\int_G \varphi(x)\,dx = \int_{K \times A \times N} a^{2\rho} \varphi(kan)\,dk\,da\,dn, \quad \text{for all } \varphi \in C_0(G) \tag{2.2}$$

(see e.g. [7, Proposition 2.4.2]).

We shall next recall an integral formula. Let S be a reductive Lie group in the Harish-Chandra class, and let $S = K \exp \mathfrak{p}$ be its Cartan decomposition, where K is a maximal compact subgroup of S. Let Q be a parabolic subgroup of S related to the above Cartan decomposition, and let $Q = M_Q A_Q N_Q$ be its Langlands decomposition.

Let $K_Q := K \cap Q = K \cap M_Q$, and put, for $k \in K$, $[k] := kK_Q$ in K/K_Q. We extend this notation to S by putting, for $s = kman$, $(k, m, a, n) \in K \times M_Q \times A_Q \times N_Q$, $[s] := [k]$. This is still well defined even though the representation of s in $KM_Q A_Q N_Q$ is not unique. In fact,

$$s = kman = k'm'a'n'$$

if and only if $a' = a$, $n' = n$, and $k' = kk_Q$, $m' = k_Q^{-1} m$ for some $k_Q \in K_Q$ (see e.g. [7, Theorem 2.3.3]). From this we see that the decomposition above becomes unique, if we require m to be in $\exp(\mathfrak{m}_Q \cap \mathfrak{p})$.

Every $s \in S$ thus admits a unique decomposition $s = kman$, with $(k, m, a, n) \in K \times \exp(\mathfrak{m}_Q \cap \mathfrak{p}) \times A_Q \times N_Q$. We then write $k_Q(s) := k$, $m_Q(s) := m$, $a_Q(s) := a$ and $n_Q(s) := n$, i.e.

$$s = k_Q(s) m_Q(s) a_Q(s) n_Q(s).$$

In particular, $[s] = k_Q(s) K_Q$.

For $y \in S$ and $k \in K$, we define $y[k] \in K/K_Q$ as follows:

$$y[k] := [yk].$$

Moreover, for any $\gamma \in \mathfrak{a}_{\mathbb{C}}^*$ and $Y \in \mathfrak{a}$, we put $(\exp Y)^{\gamma} := e^{\gamma(Y)}$.

One can deduce, for example from [20, Lemma 2.4.1], the following lemma.

Lemma 2.1. *Fix an invariant measure dk on K and let $d[k]$ denote the corresponding left-invariant measure on K/K_Q. For any $y \in S$, we then have*

$$d\big(y[k]\big) = a_Q(yk)^{-2\rho_Q} d[k],$$

where $\rho_Q \in \mathfrak{a}_Q^*$ is given by $\rho_Q(X) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} X_{|\mathfrak{n}_Q})$ $(X \in \mathfrak{a}_Q)$; that is, for any $f \in C(K/K_Q)$,

$$\int_{K/K_Q} f\big([k]\big)\, d[k] = \int_{K/K_Q} a_Q(yk)^{-2\rho_Q} f\big(y[k]\big)\, d[k].$$

We return now to our semisimple Lie group G. In the following, we shall use another basis of $\mathfrak{a}^*$, given as follows. For $j = 1, \ldots, r$, let ρ_j denote the element of $\mathfrak{a}_j^*$ defined by

$$\rho_j(X) := \frac{1}{2}\operatorname{tr}(\operatorname{ad} X_{|\mathfrak{n}_j}) \quad \text{for all } X \in \mathfrak{a}_j.$$

Notice that we can identify ρ_j with the restriction $\rho_{|\mathfrak{a}_j}$ of ρ to $\mathfrak{a}_j$.

By [6, Lemma 4.1], ρ_j and β_j are scalar multiples of each other. In particular, the family $\{\rho_j\}$ is an orthogonal basis of $\mathfrak{a}^*$, and therefore of $\mathfrak{a}^*_{\mathbb{C}}$. For every $\nu \in \mathfrak{a}^*$ (respectively $\nu \in \mathfrak{a}^*_{\mathbb{C}}$) and $j = 1, \ldots, r$, we define $\nu_j \in \mathbb{R}$ (respectively $\mathbb{C}$) by the following:

$$\nu = \sum_{j=1}^{r} \nu_j \rho_j.$$

Recall that, for $j = 1, \ldots, r$, P_j is a parabolic subgroup of the real reductive Lie group in the Harish-Chandra class, M_j. Therefore, by taking $(S, K, Q) := (M^j, K^j, P_j)$ in the discussion above, put $k_j := k_{P_j}$, $m_{j-1} := m_{P_j}$, $a_j := a_{P_j}$ and $n_j := n_{P_j}$. Thus, any $g \in M^j$ has a unique decomposition $g = k_j(g)m_{j-1}(g)a_j(g)n_j(g)$, with $k_j(g) \in K^j$, $m_{j-1}(g) \in \exp(\mathfrak{m}^{j-1} \cap \mathfrak{p})$, $a_j(g) \in A_j$ and $n_j(g) \in N_j$. Notice that $\mathfrak{m}_0 \cap \mathfrak{p} = \{0\}$, i.e. $m_0(g) = e$.

Lemma 2.2. *Denote by r_y the right multiplication with $y \in G$. Let $j \in \{1, \ldots, r\}$, $g \in M^j$ and $k_l \in K^l$ $(l = 1, \ldots, j)$.*

We define recursively the element g_l of M^l, $l = 1, \ldots, j$, starting from $l = j$, by putting $g_j := g$ and $g_{l-1} := m_{l-1}(g_l k_l)$, i.e.

$$g_l = m_l \circ (r_{k_{l+1}} \circ m_{l+1}) \circ \cdots \circ (r_{k_j} \circ m_j)(g), \quad 1 \leqslant l \leqslant j-1.$$

Then, the following estimate holds:

$$\left\| \prod_{l=j}^{1} a_l\big(g_l k^l\big) \right\| \leqslant \|g\|.$$

Proof. We first show that, for $1 \leqslant p \leqslant j$,

$$\|g\| = \left\| \prod_{l=j}^{p} a_l(g_l k_l) \cdot m_{p-1}(g_p k_p) \cdot \prod_{l=p}^{j} n_l(g_l k_l) k_l^{-1} \right\|. \tag{2.3}$$

(Here the products are non-commutative products, in which the order of the factors is indicated by the order of indices.) We use an induction, starting from $p = j$. If $p = j$ and $g \in M^j$, then

$$\|g\| = \big\|g k_j k_j^{-1}\big\| = \big\|k_j(g k_j) m_{j-1}(g k_j) a_j(g k_j) n_j(g k_j) k_j^{-1}\big\|.$$

Using the left K-invariance of the norm and the fact that $a_j(gk_j) \in A_j$ and $m_{j-1}(gk_j) \in M^{j-1}$ commute, we find that

$$\|g\| = \big\|a_j(g k_j) m_{j-1}(g k_j) n_j(g k_j) k_j^{-1}\big\|,$$

so that (2.3) holds for $p = j$. Assume now, by induction, that (2.3) is true for $p + 1$ in place of p, i.e.

$$\|g\| = \left\| \prod_{l=j}^{p+1} a_l(g_l k_l) \cdot m_p(g_{p+1} k_{p+1}) \cdot \prod_{l=p+1}^{j} n_l(g_l k_l) k_l^{-1} \right\|.$$

We then decompose:

$$m_p(g_{p+1}k_{p+1})k_p = g_p k_p = k_p(g_p k_p)m_{p-1}(g_p k_p)a_p(g_p k_p)n_p(g_p k_p).$$

Since $k_p(g_p k_p) \in K^p \subset M^l$, for $p \leqslant l \leqslant j$, it commutes with $a_l(g_l k_l)$, for $l = p+1, \ldots, j$, and therefore, because of the K-invariance of $\|\cdot\|$, we have:

$$\|g\| = \left\| \prod_{l=j}^{p+1} a_l(g_l k_l) \cdot m_{p-1}(g_p k_p)a_p(g_p k_p)n_p(g_p k_p)k_p^{-1} \cdot \prod_{l=p+1}^{j} n_l(g_l k_l)k_l^{-1} \right\|.$$

Moreover, $a_p(g_p k_p)$ commutes with $m_{p-1}(g_p k_p)$, and so (2.3) follows.

Applying now (2.3) for $p = 1$, we obtain

$$\left\| \prod_{l=j}^{1} a_l(g_l k_l) \prod_{l=1}^{j} n_l(g_l k_l)k_l^{-1} \right\| = \|g\|. \tag{2.4}$$

By right K-invariance of the norm, the left-hand side of this equation is equal to

$$\left\| \prod_{l=j}^{1} a_l(g_l k_l) \prod_{l=1}^{j} n_l(g_l k_l)k_l^{-1} \prod_{l'=j}^{1} k_{l'} \right\|.$$

Notice that we can write $\prod_{l=1}^{j} n_l(g_l k_l)k_l^{-1} \prod_{l'=j}^{1} k_{l'}$ as follows:

$$n_1(g_1k_1)\big(k_1^{-1}n_2(g_2k_2)k_1\big)\big((k_2k_1)^{-1}n_3(g_3k_3)k_2k_1\big)$$
$$\cdots\left(\left(\prod_{l=j-1}^{1} k_l\right)^{-1} n_j(g_j k_j) \prod_{l'=j-1}^{1} k_{l'}\right).$$

For $2 \leqslant p \leqslant j$, $(\prod_{l'=p-1}^{1} k_{l'})^{-1}$ lies in $K^{p-1} \subset M^{p-1}$ and thus normalizes N_p. Hence, we get that

$$\prod_{l=1}^{j} n_l(g_l k_l)k_l^{-1} \prod_{l'=j}^{1} k_{l'} \in N.$$

Using the last property of the norm given in (2.1), the left-hand side of (2.4) is then greater or equal to $\|\prod_{l=j}^{1} a_l(g_l k_l)\|$, which proves the lemma. □

2.2. *A holomorphic family of representations of G on mixed L^p-spaces*

For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, let $\mathcal{M}(G, P, \nu)$ denote the space of complex-valued measurable functions f on G satisfying the following covariance property:

$$f(gman) = a^{-(\nu+\rho)} f(g) \quad \text{for all } g \in G,\ m \in M,\ a \in A,\ n \in N.$$

The space $\mathcal{M}(G, P, \nu)$ is endowed with the left regular action of G, denoted by $\tilde{\pi}_\nu$, i.e. $[\tilde{\pi}_\nu(g)f](g') = f(g^{-1}g')$. The representations $\tilde{\pi}_\nu$ form the class-one principal series.

Let $\mathcal{M}(K/M)$ be the space of right M-invariant measurable functions on K.

The restriction to K of functions on G gives us a linear isomorphism from $\mathcal{M}(G, P, \nu)$ onto $\mathcal{M}(K/M)$. Denote by $I_\nu : f \mapsto f_\nu$ the inverse mapping. Then $f_\nu \in \mathcal{M}(G, P, \nu)$ is given by

$$f_\nu(kan) := a^{-(\nu+\rho)} f(k) \quad \text{for all } k \in K,\ a \in A,\ n \in N,$$

if $G = KAN$ denotes the Iwasawa decomposition of G.

If we intertwine the representation $\tilde{\pi}_\nu$ with I_ν, we obtain a representation π_ν of G on $\mathcal{M}(K/M)$, given by

$$\big(\pi_\nu(g)f\big)_\nu = \tilde{\pi}_\nu(g) f_\nu, \quad \text{if } f \in \mathcal{M}(K/M),\ g \in G.$$

For $j = 1, \dots, r$, denote by $d\dot{k}_j$ the quotient measure on K^j/K^{j-1} coming from dk_j. It is invariant by left translations. Notice that $K^{j-1} = K^j \cap M^{j-1}$.

We choose a left-invariant measure $d\dot{k}$ on K/M such that, for any $f \in C(K/M)$,

$$\int\limits_{K/M} f(k)\, d\dot{k} = \int\limits_{K^r/K^{r-1}} \cdots \int\limits_{K^1/M} f(k_r \cdots k_1)\, d\dot{k}_1 \dots d\dot{k}_r. \tag{2.5}$$

Let $\underline{p} = (p_1, \dots, p_r) \in [1, +\infty[^r$. One can easily see that, for every $f \in \mathcal{M}(K/M)$, $k' \in K$, the function on K^j given by

$$k \mapsto \Bigg(\int\limits_{K^{j-1}/K^{j-2}} \cdots \Bigg(\int\limits_{K^1/M} \big|f(k'kk_{j-1} \dots k_1)\big|^{p_1}\, d\dot{k}_1 \Bigg)^{p_2/p_1} \dots d\dot{k}_{j-1} \Bigg)^{1/p_{j-1}},$$

is right K^{j-1}-invariant.

We can thus define the mixed L^p-space, $L^{\underline{p}}(K/M)$, as the space of all (equivalent classes of) functions f in $\mathcal{M}(K/M)$ whose mixed L^p-norm:

$$\|f\|_{\underline{p}} := \Bigg(\int\limits_{K^r/K^{r-1}} \cdots \Bigg(\int\limits_{K^1/M} \big|f(k_r \dots k_1)\big|^{p_1}\, d\dot{k}_1 \Bigg)^{p_2/p_1} \dots d\dot{k}_r \Bigg)^{1/p_r}$$

is finite, endowed with this norm. This definition extends to the case where some of the p_j are infinite, by the usual modifications.

Let d denote the right G- and left K-invariant metric on G, given by

$$d(g, g') := \frac{1}{c_1} \log\big\|g'g^{-1}\big\| \quad (g, g' \in G),$$

where c_1 is the positive constant appearing in (2.1). Notice that $d(g, e) = 0$ if and only if g lies in the center of G. In particular, d is not separating.

Then, for $a = \exp Y$, with $Y \in \mathfrak{a} \subset \mathfrak{p}$, and $\gamma \in \mathfrak{a}^*$, we have, in view of the fourth property of $\|\cdot\|$ in (2.1), that

$$a^\gamma = \left|e^{\gamma(Y)}\right| \leqslant e^{|\gamma||Y|} = \left(e^{c_1|Y|}\right)^{|\gamma|/c_1} \leqslant \|a\|^{\frac{|\gamma|}{c_1}} = e^{|\gamma| d(a,e)}. \tag{2.6}$$

Proposition 2.1. *For every $f \in L^{\underline{p}}(K/M)$ and $g \in G$, we have*:

$$\left\|\pi_\nu(g) f\right\|_{\underline{p}} \leqslant e^{|\sum_j (\frac{2}{p_j} - \operatorname{Re} \nu_j - 1)\rho_j| d(g,e)} \|f\|_{\underline{p}}.$$

Thus π_ν defines a representation $\pi_{\nu,\underline{p}}$ of G on $L^{\underline{p}}(K/M)$. Furthermore, this gives an analytic family $(\pi_{\nu,\underline{p}})_{\nu \in \mathfrak{a}^*_{\mathbb{C}}}$ of representations of G on $L^{\underline{p}}(K/M)$.

Before giving the proof, we show the following statement. We keep the same notations as in Lemma 2.2.

Lemma 2.3. *Let $g \in M^j, k \in K$ and $f_\nu \in \mathcal{M}(G, P, \nu)$. Then*:

$$\begin{aligned}
&\left(\int\limits_{K^j/K^{j-1}} \cdots \left(\int\limits_{K^1/M} \left|f_\nu(kgk_j \dots k_1)\right|^{p_1} d\dot{k}_1\right)^{p_2/p_1} \dots d\dot{k}_j\right)^{1/p_j} \\
&= \left(\int\limits_{K^j/K^{j-1}} \cdots \left(\int\limits_{K^1/M} \left|\prod_{l=j}^{1} a_l(g_l k_l)^{-(\operatorname{Re} \nu_l + 1)\rho_l} f_\nu\left(k \prod_{l=j}^{1} k_l(g_l k_l)\right)\right|^{p_1} d\dot{k}_1\right)^{p_2/p_1} \dots d\dot{k}_j\right)^{1/p_j}.
\end{aligned}$$

Proof. We use induction on j. For $j = 0$, one has, by right M-invariance of f and since $g \in M^0 = M$, that

$$\left|f_\nu(kg)\right| = \left|f_\nu(k)\right|.$$

Assume that the statement is true for $j - 1$. Observe that $a_j(gk_j)$ commutes with $k_{j-1} \dots k_1 \in M^{j-1}$, and that $(k_{j-1} \dots k_1)^{-1} n_j(gk_j) k_{j-1} \dots k_1 \in N$. Therefore, the covariance property of f_ν applied to the integration over K^j/K^{j-1}, implies:

$$\begin{aligned}
&\left(\int\limits_{K^j/K^{j-1}} \cdots \left(\int\limits_{K^1/M} \left|f_\nu(kgk_j \dots k_1)\right|^{p_1} d\dot{k}_1\right)^{p_2/p_1} \dots d\dot{k}_j\right)^{1/p_j} \\
&= \left(\int\limits_{K^j/K^{j-1}} \cdots \left(\int\limits_{K^1/M} \left|a_j(gk_j)^{-(\operatorname{Re} \nu_j + 1)\rho_j} f_\nu\left(kk_j(gk_j) m_{j-1}(gk_j) k_{j-1} \dots k_1\right)\right|^{p_1} d\dot{k}_1\right)^{p_2/p_1} \dots d\dot{k}_j\right)^{1/p_j}.
\end{aligned}$$

Since $g = g_j$ and $m_{j-1}(gk_j) = g_{j-1} \in M^{j-1}$, the statement holds, using the induction hypothesis. $\square$

Proof of Proposition 2.1. If we apply (2.6) to $\gamma := \sum_{j=1}^{r}(\frac{2}{p_j} - \operatorname{Re}\nu_j - 1)\rho_j$ and notice that the $\mathfrak{a}_l$'s are pairwise orthogonal with respect to $(\cdot,\cdot)$, we get, in view of Lemma 2.2:

$$\sup_{k_j \in K^j,\, j=1,\dots,r} \prod_{j=1}^{r} a_j(g_j k_j)^{(\frac{2}{p_j} - \operatorname{Re}\nu_j - 1)\rho_j} \leqslant \|g\|^{\frac{|\gamma|}{c_1}} = e^{|\gamma| d(g,e)}.$$

On the other hand, according to the above lemma and Lemma 2.1, applied successively to the integrations over K^j/K^{j-1}, $j = 1,\dots,r$, we have:

$$\left\|\pi_\nu\left(g^{-1}\right) f\right\|_{\underline{p}} \leqslant \sup_{k_j \in K^j,\ j=1,\dots,r} \left(\prod_{j=r}^{1} a_j(g_j k_j)^{(\frac{2}{p_j} - \operatorname{Re}\nu_j - 1)\rho_j} \right) \|f\|_{\underline{p}}.$$

Since $d(g^{-1}, e) = d(g, e)$, the first assertion of the proposition follows.

In order to prove the analyticity of the family of representations $\pi_{\nu,\underline{p}}$, choose $\underline{p} = (p_1,\dots,p_r) \in [1,\infty[^r$ and denote by $\underline{p}' = (p_1',\dots,p_r') \in\,]1,\infty]^r$ the tuple of conjugate exponents, i.e., $1/p_j + 1/p_j' = 1$. Then, for $f \in L^{\underline{p}}(K/M)$, $u \in L^{\underline{p}'}(K/M) = (L^{\underline{p}}(K/M))'$ and $g \in G$, we have:

$$\begin{aligned}
\left\langle \pi_{\nu,\underline{p}}(g) f, u \right\rangle &= \int_{K/M} \left(\pi_{\nu,\underline{p}}(g) f\right)(k)\overline{u(k)}\, d\dot{k} \\
&= \int_{K/M} a\left(g^{-1}k\right)^{-(\nu+\rho)} f\left(\kappa\left(g^{-1}k\right)\right)\overline{u(k)}\, d\dot{k}.
\end{aligned}$$

Here, the functions $\kappa(\cdot), a(\cdot), n(\cdot)$ on G are given by the unique factorization $g = \kappa(g)a(g)n(g)$ of g, according to the Iwasawa decomposition $G = KAN$.

Obviously, the expression above is analytic in $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, which finishes the proof. $\square$

For $t = (t_1,\dots,t_r) \in\,]0,+\infty[^r$, let

$$\Omega_t := \left\{\nu \in \mathfrak{a}_{\mathbb{C}}^* \;\middle|\; |\operatorname{Re}\nu_j| < t_j,\ j = 1,\dots,r\right\}.$$

Moreover, for $p \geqslant 0$, let

$$\overline{p} := (p,\dots,p) \in \mathbb{R}^r.$$

According to our choice of measure on K/M (cf. (2.5)), notice that

$$L^{\overline{p}}(K/M) = L^p(K/M), \qquad \|\cdot\|_{\overline{p}} = \|\cdot\|_{L^p(K/M)}.$$

Proposition 2.2.

(i) *For all $\underline{p} \in [1,+\infty[^r$, $f \in L^{\underline{p}}(K/M)$, $\nu \in \Omega_t$, $g \in G$, we have*

$$\left\|\pi_{\nu,\underline{p}}(g) f\right\|_{\underline{p}} \leqslant e^{\sum_j (t_j+1)|\rho_j| d(g,e)} \|f\|_{\underline{p}}.$$

(ii) *Let* $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, *and let* $\underline{q}$ *be an element of* $[1, +\infty[^r$ *satisfying*

$$\operatorname{Re}\nu_j = \frac{2}{q_j} - 1, \quad j = 1, \dots, r.$$

Then, for all $g \in G$, $f \in L^{\underline{q}}(K/M)$,

$$\left\| \pi_{\nu,\underline{q}}(g) f \right\|_{\underline{q}} = \| f \|_{\underline{q}}.$$

Furthermore, for $\nu \in i\mathfrak{a}^*$, $\pi_{\nu,\bar{2}}$ *is a unitary representation of* G.

Proof. (i) results immediately from the estimate given in Proposition 2.1 and (ii) from Lemmas 2.3 and 2.1, since, for such $\underline{q}$, we have $a^{-(\operatorname{Re}\nu_l+1)\rho_l} = a^{-2\rho_l/q_l}$, if $a \in A_l$. □

2.3. A holomorphic family of compact operators

Let $L = -\sum_1^k X_j^2$ be a fixed sub-Laplacian on G. The estimate (1.3), in combination with the estimate in Proposition 2.2 (i), easily implies that the operator

$$\pi_{\nu,\underline{p}}(h_1) f := \int_G h_1(x) \pi_{\nu,\underline{p}}(x) f \, dx, \quad f \in L^{\underline{p}}(K/M),$$

is well defined and bounded on $L^{\underline{p}}(K/M)$. In fact, these operators are even compact. To see this, let us put, for $\nu \in \Omega_1$, $k_1, k_2 \in K$,

$$K_\nu(k_1, k_2) := c_G \int_{M \times A \times N} a^{-\nu+\rho} h_1\big(k_1 (man)^{-1} k_2^{-1}\big) \, dm \, da \, dn, \tag{2.7}$$

where c_G is the positive constant given by $d(x^{-1}) = c_G \, dx$ (which exists, since G is unimodular).

Lemma 2.4. *The integral in* (2.7) *is absolutely convergent and defines a continuous, right* M*-invariant kernel function on* $K \times K$, *in the sense that* $K_\nu(k_1 m', k_2 m') = K_\nu(k_1, k_2)$ *for every* $m' \in M$.

Proof. In order to prove that the integral in (2.7) is absolutely convergent, we put

$$I := \int_{M \times A \times N} \left| a^{-\nu+\rho} h_1\big(k_1 (man)^{-1} k_2^{-1}\big) \right| dm \, da \, dn.$$

Then, in view of (1.3), we have

$$I \leqslant C \int_{M \times A \times N} a^{-\operatorname{Re}\nu+\rho} e^{-c d(k_1 (man)^{-1} k_2^{-1}, e)^2} \, dm \, da \, dn.$$

Using the K-bi-invariance of the norm $\|\cdot\|$ on G and the inclusion $M \subset K$, we get, for any $k \in K$, that

$$d\big(k_1(man)^{-1}k_2^{-1}, e\big) = d(kan, e).$$

Moreover, by (2.6) and (2.1),

$$a^{-2\rho}a^{-\operatorname{Re}\nu+\rho} = a^{-\operatorname{Re}\nu-\rho} \leqslant e^{|\operatorname{Re}\nu+\rho|\,d(kan,e)}.$$

Since $|\operatorname{Re}\nu + \rho| \leqslant 2\sum_j |\rho_j|$ for $\nu \in \Omega_1$, we obtain:

$$I \leqslant C \int\limits_{K\times A\times N} a^{2\rho} e^{2\sum_j |\rho_j|\,d(kan,e)} e^{-cd(kan,e)^2}\, dk\, da\, dn,$$

which is in fact equal to

$$C\int\limits_G e^{2\sum_j |\rho_j|\,d(x,e)} e^{-cd(x,e)^2}\, dx.$$

Since G is unimodular and has exponential volume growth, it is easy to see that this integral is finite. Moreover, since the integrand in (2.7) depends continuously on k_1 and k_2, we see that K_ν is continuous.

In order to prove the right M-invariance of K_ν, let $m' \in M$. One has, for any $(m, a, n) \in M \times A \times N$:

$$(man)^{m'} = m^{m'} a n^{m'}.$$

According to the invariance of dm, we then have, for any k_1, $k_2 \in K$:

$$K_\nu(k_1 m', k_2 m') = c_G \int\limits_{M\times A\times N} a^{-\nu+\rho} h_1\big(k_1 (man^{m'})^{-1} k_2^{-1}\big)\, dm\, da\, dn.$$

Furthermore, it is easy to check that, for any $\varphi \in C_0(N)$,

$$\int\limits_N \varphi\big(n^{m'}\big)\, dn = \int\limits_N \varphi(n)\, dn.$$

Indeed, since $G = KAN$, there exists $\phi \in C_0(G)$ such that

$$\varphi(n) = \int\limits_{K\times A} a^{2\rho}\phi(kan)\, dk\, da.$$

According to our choice of the Haar measure dx on G (cf. (2.2)), we have:

$$\int\limits_G \phi(x)\, dx = \int\limits_{K\times A\times N} a^{2\rho}\phi(kan)\, dk\, da\, dn = \int\limits_N \varphi(n)\, dn.$$

Using the invariance of dx and dk, in combination with the commutation and normalization properties of $m' \in M$, we see that

$$\int_N \varphi(n)\,dn = \int_G \phi\big(x^{m'}\big)\,dx = \int_{K\times A\times N} a^{2\rho}\phi\big(kan^{m'}\big)\,dk\,da\,dn = \int_N \varphi\big(n^{m'}\big)\,dn.$$

We thus conclude that K_ν is right M-invariant. $\square$

Put, for $\nu \in \Omega_1 := \Omega_{\bar{1}}$,

$$T(\nu) := \pi_\nu(h_1).$$

Proposition 2.3. *For any $\nu \in \Omega_1$, $T(\nu)$ is a kernel operator with kernel K_ν, and induces, for any $\underline{p} \in [1, +\infty[^r$, a compact operator $T_{\underline{p}}(\nu)$ on $L^{\underline{p}}(K/M)$, given by $\pi_{\nu,\underline{p}}(h_1)$.*

The family $\nu \mapsto T_{\underline{p}}(\nu)$ of compact operators is analytic (in the sense of Kato [10]*) on Ω_1.*

Furthermore, for $\nu \in i\mathfrak{a}^*$, $T_{\bar{2}}(\nu)$ is a self-adjoint operator on $L^2(K/M)$.

Proof. Let $\nu \in \Omega_1$, $f \in L^1(K/M)$ and $k_1 \in K$. By definition,

$$\big(T(\nu)f\big)(k_1) = \int_G h_1(x)\big(\pi_\nu(x)f\big)(k_1)\,dx.$$

By invariance of dx, this is equal to

$$c_G \int_G h_1\big(k_1x^{-1}\big) f_\nu(x)\,dx.$$

According to our choice of dx and using the covariance property of f_ν, we obtain

$$\big(T(\nu)f\big)(k_1) = c_G \int_{K\times A\times N} a^{2\rho}a^{-(\nu+\rho)}h_1\big(k_1(an)^{-1}k^{-1}\big) f_\nu(k)\,dk\,da\,dn.$$

Using the right M-invariance of f_ν, and since dk is invariant and $M \subset K$, this can be written as follows:

$$\big(T(\nu)f\big)(k_1) = c_G \int_{K\times M\times A\times N} a^{-\nu+\rho}h_1\big(k_1(man)^{-1}k^{-1}\big) f_\nu(k)\,dk\,dm\,da\,dn.$$

But, $f_\nu = f$ on K. According to Fubini's theorem, this shows that $T(\nu)$ is a kernel operator with kernel K_ν.

Since K_ν is continuous on the compact space $K \times K$, it follows from Lemma 2.4 that $T_{\underline{p}}(\nu)$ defines a compact operator on $L^{\underline{p}}(K/M)$, and the analytic dependence of K_ν, which is evident by (2.7), implies that, for any $\underline{p} \in [1, +\infty[^r$, the family of operators $T_{\underline{p}}(\nu)$ is analytic on Ω_1.

Finally, if $\nu \in i\mathfrak{a}^*$, then $\pi_{\nu,\bar{2}}$ is unitary, and since $h_1(x) = \overline{h_1(x^{-1})}$, we see (by (1.4)) that the operator $\pi_{\nu,\bar{2}}(h_1)$ is self-adjoint. $\square$

2.4. Some consequences of the Kunze–Stein phenomenon

Observe that, by Hölder's inequality, for any $\underline{p} \in [1,2]^r$ and any $\underline{q} \in [\underline{p}, \underline{p}']$, we have

$$\|f\|_{\underline{q}} \leqslant \|f\|_{\underline{p}'}, \quad \text{for all } f \in L^{\underline{p}'}(K/M), \tag{2.8}$$

since the compact space K/M has normalized measure 1. Therefore, $L^{\underline{p}'}(K/M)$ is a subspace of $L^{\underline{q}}(K/M)$.

As a consequence of the Kunze–Stein phenomenon (see [12] and [6]), we shall prove:

Proposition 2.4. *Let* $1 < p_0 < 2$ *and* $\nu_0 \in \mathfrak{a}^* \setminus \{0\}$. *There exist* $\varepsilon > 0$ *and* $C > 0$, *such that, for any* $\xi, \eta \in L^{p_0}(K/M)$ *and* $z \in \mathbb{C}$ *with* $|\operatorname{Re} z| < \varepsilon$,

$$\big\|\langle \pi_{z\nu_0}(\cdot)\xi, \eta\rangle\big\|_{L^{p_0'}(G)} \leqslant C\|\xi\|_{p_0'}\|\eta\|_{p_0'}. \tag{2.9}$$

Proof. Observe that, for every $\nu \in i\mathfrak{a}^*$, the representation π_ν is unitarily equivalent to $\tilde{\pi}_\nu$. Therefore, given $\delta > 0$, we obtain from [6], that there is a constant $C_\delta > 0$, such that, for any $2 + \delta \leqslant r' \leqslant \infty$ and $\xi, \eta \in L^2(K/M)$, we have

$$\big\|\langle \pi_\nu(\cdot)\xi, \eta\rangle\big\|_{L^{r'}(G)} \leqslant C_\delta\|\xi\|_2\|\eta\|_2, \quad \text{provided } \operatorname{Re}\nu = 0. \tag{2.10}$$

Indeed, in [6], this is only stated for $\nu = 0$, but the proof easily extends to arbitrary $\nu \in i\mathfrak{a}^*$.

On the other hand, since $\pi_{\nu,\underline{q}}$ is isometric, we have, as an immediate consequence of Proposition 2.2(ii), the estimate

$$\big\|\langle \pi_\nu(\cdot)\xi, \eta\rangle\big\|_{L^\infty(G)} \leqslant \|\xi\|_{\underline{q}}\|\eta\|_{\underline{q}'}, \quad \underline{q} \in [1, +\infty[^r, \tag{2.11}$$

for any $\xi \in L^{\underline{q}}(K/M)$, $\eta \in L^{\underline{q}'}(K/M)$, provided that

$$\operatorname{Re}\nu_j = \frac{2}{q_j} - 1, \quad j = 1, \ldots, r. \tag{2.12}$$

Let $\theta_0 \in]0,1[$ be given by $\frac{2}{p_0} = 1 + \theta_0$. If $\underline{q}$ satisfies (2.12) and $|\operatorname{Re}\nu_j| \leqslant \theta_0$ for any $j = 1, \ldots, r$, $\underline{q} \in [\overline{p}_0, \overline{p}_0']$. Thus, since $2 \in [p_0, p_0']$, we can unify (2.10) and (2.11), using (2.8), as follows.

Given $\delta > 0$, there exists a constant $C_\delta \geqslant 1$ such that, for any $\xi \in L^{p_0}(K/M)$, $\eta \in L^{p_0'}(K/M)$:

if $\operatorname{Re}\nu = 0$, for all $r' \in [2+\delta, +\infty]$, then $\|\langle \pi_\nu(\cdot)\xi, \eta\rangle\|_{L^{r'}(G)} \leqslant C_\delta\|\xi\|_{p_0'}\|\eta\|_{p_0'}$,

and

if $|\operatorname{Re}\nu_j| \leqslant \theta_0$, $j = 1, \ldots, r$, then $\|\langle \pi_\nu(\cdot)\xi, \eta\rangle\|_{L^\infty(G)} \leqslant \|\xi\|_{p_0'}\|\eta\|_{p_0'}$.

If we choose $\nu = z\nu_0$, and put, for $\xi, \eta \in L^{p_0'}(K/M)$ fixed, $\Psi_z := \langle \pi_{z\nu_0}(\cdot)\xi, \eta\rangle$, we obtain that

$$\|\Psi_{iy}\|_{L^{r'}(G)} \leqslant C_\delta\|\xi\|_{p_0'}\|\eta\|_{p_0'}, \quad \text{for all } r' \in [2+\delta, +\infty] \text{ and } y \in \mathbb{R},$$

and

$$\|\Psi_{\pm\theta_1+iy}\|_{L^\infty(G)} \leqslant C_\delta \|\xi\|_{p_0'}\|\eta\|_{p_0'}, \quad \text{for all } y \in \mathbb{R},$$

with $\theta_1 := \theta_0/\max_{j=1,\dots,r}|\mathrm{Re}\,\nu_{0,j}|$.

Since Ψ_z depends analytically on z, we can apply Stein's interpolation theorem (cf. [18, Theorem 4.1]), and obtain, for every $r' \geqslant 2+\delta$, that

$$\text{if } |\mathrm{Re}\,z| \leqslant \theta_1 \text{ and } q' := \frac{r'}{1-|\mathrm{Re}\,z|/\theta_1}, \text{ then } \|\Psi_z\|_{L^{q'}(G)} \leqslant C_\delta\|\xi\|_{p_0'}\|\eta\|_{p_0'}. \tag{2.13}$$

But $p_0' > 2$. Hence we can choose $\delta > 0$ and $\varepsilon > 0$ so small that $(1-\frac{\varepsilon}{\theta_1})p_0' \geqslant 2+\delta$. Then, for $|\mathrm{Re}\,z| \leqslant \varepsilon$, if we take $r' = p_0'(1-\frac{|\mathrm{Re}\,z|}{\theta_1})$ in (2.13), we have $r' \geqslant 2+\delta$, and hence

$$\|\Psi_z\|_{L^{p_0'}(G)} \leqslant C_\delta\|\xi\|_{p_0'}\|\eta\|_{p_0'}. \qquad \square$$

2.5. *Proof of Theorem 1.1*

Let $p \in [1,\infty[$, $p \neq 2$. The aim is to find a non-isolated point λ_0 in the L^2-spectrum $\sigma_2(L)$ of L and an open neighborhood $\mathcal{U}$ of λ_0 in $\mathbb{C}$ such that, if $F_0 \in C_\infty(\mathbb{R})$ is an L^p-multiplier for L, then F_0 extends holomorphically to $\mathcal{U}$. Recall that $C_\infty(\mathbb{R})$ denotes the space of continuous functions on $\mathbb{R}$ vanishing at infinity.

Since the L^2-spectrum of L is contained in $[0,+\infty[$, we may assume that $F_0 \in C_\infty([0,+\infty[)$. Moreover, according to [8, Lemma 6.1], it suffices to consider the case where $2 < p' < \infty$.

As in Section 1, we can replace F_0 by the function $F = F_0 e^{-\cdot}$, so that $F(L)$ acts on the spaces $L^q(G)$, $q \in [p,p']$, by convolution with the function $F(L)\delta \in \bigcap_{q=p}^{p'} L^q(G)$. The Kunze–Stein phenomenon implies now that every L^p function defines a bounded operator on $L^2(G)$ and also on every Hilbert space $\mathcal{H}$ of any unitary representation π of G, which is weakly contained in the left regular representation. Indeed, we know that for any coefficient $x \mapsto c^\pi_{\xi,\eta}(x) := \langle\pi(x)\xi,\eta\rangle$ of π, we have, for some constant $C_p > 0$, that

$$\left\|c^\pi_{\xi,\eta}\right\|_{p'} \leqslant C_p\|\xi\|\|\eta\|, \quad \xi,\eta \in \mathcal{H}.$$

Hence for $f \in L^p(G)$,

$$\left|\int_G f(x)c^\pi_{\xi,\eta}(x)\,dx\right| \leqslant \|f\|_p\left\|c^\pi_{\xi,\eta}\right\|_{p'} \leqslant C_p\|f\|_p\|\xi\|\|\eta\|.$$

Hence there exists a unique bounded operator $\pi(f)$ on $\mathcal{H}$, such that $\|\pi(f)\|_{\mathrm{op}} \leqslant C_p\|f\|_p$ and

$$\langle\pi(f)\xi,\eta\rangle = \int_G f(x)c^\pi_{\xi,\eta}(x)\,dx, \quad \xi,\eta \in \mathcal{H}.$$

Choosing now a sequence $(f_\nu)_\nu$ of continuous functions with compact support, which converges in the L^p-norm to $F(L)\delta$, we see that the operators $\lambda(f_\nu)$ converge in the operator norm to

$\lambda(F(L)) = F(\lambda(L))$, and thus for any unitary representation $(\pi, \mathcal{H})$ of G which is weakly contained in the left regular representation λ, we have that

$$\int_G \big(F(L)\delta\big)(x) c^\pi_{\xi,\eta}(x)\, dx = \lim_{\nu\to\infty} \int_G f_\nu(x) c^\pi_{\xi,\eta}(x)\, dx = \lim_{\nu\to\infty} \big\langle \pi(f_\nu)\xi, \eta\big\rangle$$
$$= \big\langle \pi\big(F(L)\big)\xi, \eta\big\rangle = \big\langle F\big(\pi(L)\big)\xi, \eta\big\rangle, \quad \xi, \eta \in \mathcal{H}.$$

In particular,

$$\big(F(L)\delta\big) * \big(c^\pi_{\xi,\eta}\big)\check{}\,(x) = \int_G \big(F(L)\delta\big)(y)\big\langle \pi(y)\xi, \pi(x)\eta\big\rangle\, dy$$
$$= \big\langle F\big(\pi(L)\big)\xi, \pi(x)\eta\big\rangle, \quad x \in G,\ \xi, \eta \in \mathcal{H}. \tag{2.14}$$

In a first step, in order to find $\lambda_0 \in \mathbb{R}$ and its neighborhood $\mathcal{U}$, we choose a suitable direction ν_0 in $\mathfrak{a}^*$. To this end, let ω be the Casimir operator of G, and $\nu \in i\mathfrak{a}^*$. The representation π_ν is then unitary, and we can define the operator $d\pi_\nu(\omega)$ on the space of smooth vectors in $L^2(K/M)$ with respect to π_ν. Moreover, π_ν is irreducible (see [11, Theorem 1]), and therefore

$$d\pi_\nu(\omega) = \chi(\nu)\,\mathrm{Id},$$

where χ is a polynomial function on $\mathfrak{a}^*$, given by the Harish-Chandra isomorphism. Thus, χ is in fact a quadratic form.

Choose $\nu_0 \in \mathfrak{a}^*$, $\nu_0 \neq 0$, such that $\chi(\nu_0) \neq 0$. Then, clearly,

$$\big|\chi(iy\nu_0)\big| \to +\infty \quad \text{as } y \to +\infty \text{ in } \mathbb{R}. \tag{2.15}$$

Put $p_0 := p'$. According to Proposition 2.4, there exist $\varepsilon > 0$ and $C > 0$ such that (2.9) holds for every $z \in U_1 := \{z \in \mathbb{C} \mid |\mathrm{Re}\, z| < \varepsilon\}$. Put:

$$\pi_{(z)} := \pi_{z\nu_0} \quad \text{and} \quad \widetilde{T}(z) := T(z\nu_0).$$

From Proposition 2.3, $(\widetilde{T}(z))_{z\in U_1}$ is an analytic family of compact operators on $L^{p_0}(K/M)$. And, by an obvious analogue to [8, Proposition 5.4], there exist an open connected neighborhood U_{y_0} of some point iy_0 in U_1, with $y_0 \in \mathbb{R}$, and two holomorphic mappings

$$\lambda : U_{y_0} \to \mathbb{C} \quad \text{and} \quad \xi : U_{y_0} \to L^{p_0'}(K/M)$$

such that, for some constant $C > 0$, we have, for all $z \in U_{y_0}$:

$$\widetilde{T}(z)\xi(z) = \lambda(z)\xi(z);$$
$$\xi(z) \neq 0 \quad \text{and} \quad \big\|\xi(z)\big\|_{p_0'} \leqslant C. \tag{2.16}$$

Since $\pi_{(iy)}$ is unitary for every $y \in \mathbb{R}$, λ is real-valued on $U_{y_0} \cap i\mathbb{R}$.

Fix a non-trivial function η in $C^\infty(K/M)$. Let $z \in U_{y_0}$ and Φ_z denote the function on G given by

$$\Phi_z(g) := \langle \pi_{(z)}(g^{-1})\xi(z), \eta \rangle.$$

We have that $\Phi_z(g)$ depends continuously on z and g. Moreover, by (2.9) and (2.16), there exists a constant $C_0 > 0$, such that

$$\text{for all } z \in U_{y_0}, \quad \|\Phi_z\|_{L^{p_0'}(G)} \leqslant C_0. \tag{2.17}$$

Thus, for any $z \in U_{y_0}$, $\Phi_z \in L^{p_0'}(G)$, and consequently, since F is an $L^{p_0'}$-multiplier for L, $F(L)\Phi_z \in L^{p_0'}(G)$ is well defined.

Put, for $z \in U_{y_0}$, $\mu(z) := -\log \lambda(z)$, where log denotes the principal branch of the logarithm. For $z \in U_{y_0}$, $\xi(z)$ is an eigenvector of $\widetilde{T}(z) = \pi_{(z)}(h_1)$ associated to the eigenvalue $\lambda(z)$, where h_1 is the convolution kernel of e^{-L}. Thus, one has by (2.14), for all $z \in U_{y_0} \cap i\mathbb{R}$, $g \in G$,

$$\begin{aligned}(F(L)\Phi_z)(g) &= \langle F(\pi_{(z)}(L))\xi(z), \pi_{(z)}\eta \rangle \\ &= F(\mu(z))\langle \pi_{(z)}(g^{-1})\xi(z), \eta \rangle. \end{aligned} \tag{2.18}$$

Let ψ be a fixed element of $C_0(G)$ such that

$$\int_G \Phi_{iy_0}(x)\psi(x)\,dx \neq 0.$$

By shrinking U_{y_0}, if necessary, we may assume that $\int_G \Phi_z(x)\psi(x)\,dx \neq 0$ for all $z \in U_{y_0}$.

Then, (2.18) implies that

$$(F \circ \mu)(z) = \frac{\int_G (F(L)\Phi_z)(x)\psi(x)\,dx}{\int_G \Phi_z(x)\psi(x)\,dx}, \quad \text{for } z \in U_{y_0} \cap i\mathbb{R}. \tag{2.19}$$

Observe that the numerator and the denominator in the right-hand side of (2.19) are holomorphic functions in $z \in U_{y_0}$. Indeed, $F(L)^*\overline{\psi} \in L^{p_0}$, and, by (2.17), $\|\Phi_z\|_{L^{p_0'}} \leqslant C$. Moreover

$$\langle F(L)\Phi_z, \overline{\psi} \rangle = \langle \Phi_z, F(L)^*\overline{\psi} \rangle.$$

This implies that the mapping $z \mapsto \langle F(L)\Phi_z, \overline{\psi} \rangle$ is continuous, and the holomorphy of this mapping then follows easily from Fubini's and Morera's theorems.

Therefore, $F \circ \mu$ has a holomorphic extension to U_{y_0}.

Moreover, since $\omega h_1 \in L^1(G)$, in view of Proposition 2.2, for $y \in \mathbb{R}$, the norm

$$\|\pi_{(iy)}(\omega h_1)\|_{\mathrm{op}} \leqslant \|\omega h_1\|_{L^1(G)}$$

is uniformly bounded. On the other hand,

$$\pi_{(iy)}(\omega h_1) = d\pi_{(iy)}(\omega)\pi_{(iy)}(h_1) = \chi(iy\nu_0)\pi_{(iy)}(h_1),$$

and so (2.15) implies that

$$\lim_{y\to+\infty}\|\widetilde{T}(iy)\| = \lim_{y\to+\infty}\|\pi_{(iy)}(h_1)\| = 0.$$

This shows that λ is not constant, and hence, varying y_0 slightly, if necessary, we may assume that $\mu'(iy_0) \neq 0$. It follows that μ is a local bi-holomorphism near iy_0. In combination with (2.19), this implies that F has a holomorphic extension to a complex neighborhood of $\lambda_0 := \mu(iy_0) \in \mathbb{R}$.

3. Transference for p-induced representations

3.1. *p-Induced representations*

Let G be a separable locally compact group and $S < G$ a closed subgroup. By [15], there exists a Borel measurable cross-section $\sigma : G/S \to G$ for the homogeneous space $H := G/S$ (i.e. $\sigma(x) \in x$ for every $x \in G/S$) such that $\sigma(K)$ is relatively compact for any compact subset K of H. Then, every $g \in G$ can be uniquely decomposed as

$$g = \sigma(x)s, \quad \text{with } x \in H,\ s \in S.$$

We put $\Phi : H \times S \to G$, $\Phi(x, s) := \sigma(x)s$. Then Φ is a Borel isomorphism, and we write:

$$\Phi^{-1}(g) =: \big(\eta(g), \tau(g)\big).$$

Thus

$$g = \sigma \circ \eta(g)\tau(g), \quad g \in G.$$

For later use, we also define:

$$\tau(g, x) := \tau\big(g^{-1}\sigma(x)\big), \qquad \eta(g, x) := \eta\big(g^{-1}\sigma(x)\big),$$
$$g \in G,\ x \in H.$$

Let dg denote the left-invariant Haar measure on G, and Δ_G the modular function on G, i.e.

$$\int_G f(gh)\,dg = \Delta_G(h)^{-1}\int_G f(g)\,dg, \quad h \in G.$$

Similarly, ds denotes the left-invariant Haar measure on S, and Δ_S its modular function.

On a locally compact measure space Z, we denote by $\mathcal{M}_b(Z)$ the space of all essentially bounded measurable functions from M to $\mathbb{C}$, and by $\mathcal{M}_0(Z)$ the subspace of all functions which have compact support, in the sense that they vanish a.e. outside a compact subset of Z. For $f \in \mathcal{M}_0(G)$, let $\tilde{f}$ be the function on G given by

$$\tilde{f}(g) := \int_S f(gs)\Delta_{G,S}(s)\,ds, \quad g \in G,$$

where we have put $\Delta_{G,S}(s) := \Delta_G(s)/\Delta_S(s)$, $s \in S$. Then $\tilde{f}$ lies in the space

$$\mathcal{E}(G,S) := \bigl\{\tilde{h} \in \mathcal{M}_{\mathrm{b}}(G) \mid \tilde{h} \text{ has compact support modulo } S, \text{ and} \\ \tilde{h}(gs) = \bigl(\Delta_{G,S}(s)\bigr)^{-1}\tilde{h}(g) \text{ for all } g \in G,\ s \in S\bigr\}.$$

In fact, one can show that $\mathcal{E}(G,S) = \{\tilde{f} \mid f \in \mathcal{M}_0(G)\}$. Moreover, one checks easily, by means of the use of a Bruhat function, that $\tilde{f} = 0$ implies $\int_G f(g)\,dg = 0$.

From here it follows that there exists a unique positive linear functional, denoted by $\int_{G/S} d\dot{g}$, on the space $\mathcal{E}(G,S)$, which is left-invariant under G, such that

$$\int\limits_G f(g)\,dg = \int\limits_{G/S} \tilde{f}(g)\,d\dot{g} = \int\limits_{G/S}\int\limits_S f(gs)\Delta_{G,S}(s)\,ds\,d\dot{g}. \tag{3.1}$$

By means of the cross-section σ, we can next identify the function $\tilde{h} \in \mathcal{E}(G,S)$ with the measurable function $h \in \mathcal{M}_0(H)$, given by

$$h(x) := R\tilde{h}(x) := \tilde{h}\bigl(\sigma(x)\bigr), \quad x \in H.$$

Notice that, given $h \in \mathcal{M}_0(H)$, the corresponding function $\tilde{h} =: R^{-1}h \in \mathcal{E}(G,S)$ is given by

$$\tilde{h}\bigl(\sigma(x)s\bigr) = h(x)\Delta_{G,S}(s)^{-1}.$$

The mapping $h \mapsto \int_{G/S} \tilde{h}(g)\,d\dot{g}$ is then a positive Radon measure on $C_0(H)$, so that there exists a unique regular Borel measure dx on $H = G/S$, such that

$$\int\limits_{G/S} \tilde{h}(g)\,d\dot{g} = \int\limits_H h(x)\,dx, \quad h \in C_0(H). \tag{3.2}$$

Formula (3.1) can then be re-written as

$$\int\limits_G f(g)\,dg = \iint\limits_{H\,S} f\bigl(\sigma(x)s\bigr)\Delta_{G,S}(s)\,ds\,dx. \tag{3.3}$$

Notice that the left-invariance of $\int_{G/S} d\dot{g}$ then translates into the following quasi-invariance property of the measure dx on H:

$$\int\limits_H h\bigl(\eta(g,x)\bigr)\Delta_{G,S}\bigl(\tau(g,x)\bigr)^{-1}\,dx = \int\limits_H h(x)\,dx \quad \text{for every } g \in G. \tag{3.4}$$

Formula (3.3) remains valid for all $f \in L^1(G)$.

Next, let ρ be a strongly continuous isometric representation of S on a complex Banach space $(X, \|\cdot\|_X)$, so that in particular,

$$\bigl\|\rho(s)v\bigr\|_X = \|v\|_X \quad \text{for every } s \in S,\ v \in X.$$

Fix $1 \leqslant p < \infty$, and let $L^p(G, X; \rho)$ denote the Banach space of all Borel measurable functions $\tilde{\xi} : G \to X$, which satisfy the covariance condition

$$\tilde{\xi}(gs) = \Delta_{G,S}(s)^{-1/p} \rho\big(s^{-1}\big)\big[\tilde{\xi}(g)\big] \quad \text{for all } g \in G,\ s \in S,$$

and have finite L^p-norm $\|\tilde{\xi}\|_p := (\int_{G/S} \|\tilde{\xi}(g)\|_X^p \, d\dot{g})^{1/p}$.

Notice that the function $g \mapsto \|\tilde{\xi}(g)\|_X^p$ satisfies the covariance property of functions in $\mathcal{E}(G, S)$, so that the integral $\int_{G/S} \|\tilde{\xi}(g)\|_X^p \, d\dot{g}$ is well defined.

The *p-induced representation* $\pi_p = \operatorname{ind}_{p,S}^G \rho$ is then the left-regular representation $\lambda_G = \lambda$ of G acting on $L^p(G, X; \rho)$, i.e.,

$$\big[\pi_p(g)\tilde{\xi}\big](g') := \tilde{\xi}\big(g^{-1}g'\big), \quad g, g' \in G,\ \tilde{\xi} \in L^p(G, X; \rho).$$

By means of the cross-section σ, one can realize π_p on the L^p-space $L^p(H, X)$.

To this end, given $\tilde{\xi} \in L^p(G, X; \rho)$, we define $\xi \in L^p(H, X)$ by

$$\xi(x) := \mathcal{T}\tilde{\xi}(x) := \tilde{\xi}\big(\sigma(x)\big), \quad x \in H.$$

Because of (3.2), $\mathcal{T} : L^p(G, X; \rho) \to L^p(H, X)$ is a linear isometry, with inverse

$$\mathcal{T}^{-1}\xi\big(\sigma(x)s\big) := \tilde{\xi}\big(\sigma(x)s\big) = \Delta_{G,S}(s)^{-1/p} \rho\big(s^{-1}\big)\big[\xi(x)\big].$$

Since, for $g \in G$, $y \in H$ and $\tilde{\xi} \in L^p(G, X; \rho)$,

$$\begin{aligned}
\tilde{\xi}\big(g^{-1}\sigma(y)\big) &= \tilde{\xi}\big(\sigma \circ \eta\big(g^{-1}\sigma(y)\big)\tau\big(g^{-1}\sigma(y)\big)\big) \\
&= \tilde{\xi}\big(\sigma\big(\eta(g, y)\big)\tau(g, y)\big) \\
&= \Delta_{G,S}\big(\tau(g, y)\big)^{-1/p} \rho\big(\tau(g, y)^{-1}\big)\big[\tilde{\xi}\big(\sigma\big(\eta(g, y)\big)\big)\big],
\end{aligned}$$

we see that the induced representation π_p can also be realized on $L^p(H, X)$, by

$$\big[\pi_p(g)\xi\big](y) = \Delta_{G,S}\big(\tau(g, y)\big)^{-1/p} \rho\big(\tau(g, y)^{-1}\big)\big[\xi\big(\eta(g, y)\big)\big], \tag{3.5}$$

for $g \in G$, $y \in H$, $\xi \in L^p(H, X)$.

Observe that $\pi_p(g)$ acts isometrically on $L^p(H, X)$, for every $g \in G$. This is immediate from the original realization of π_p on $L^p(G, X; \rho)$, but follows also from (3.4), in the second realization given by (3.5).

Examples 3.1. (a) If $S \lhd G$ is a closed, normal subgroup, then $H = G/S$ is again a group, and one finds that, for a suitable normalization of the left-invariant Haar measure dx on H, we have

$$\int_G f(g)\, dg = \int_H \int_S f\big(\sigma(x)s\big)\, ds\, dx, \quad f \in L^1(G).$$

In particular, $\Delta_G|_S = \Delta_S$, so that $\Delta_{G,S} = 1$ and dx in (3.3) agrees with the left-invariant Haar measure on H.

Furthermore, there exists a measurable mapping $q : H \times H \to S$, such that

$$\sigma(x)^{-1}\sigma(y) = \sigma\big(x^{-1}y\big)q(x, y), \quad x, y \in H,$$

since $\sigma(x)^{-1}\sigma(y) \equiv \sigma(x^{-1}y)$ modulo S. Thus, if $g = \sigma(x)s$, then

$$\begin{aligned} g^{-1}\sigma(y) &= s^{-1}\sigma(x)^{-1}\sigma(y) = s^{-1}\sigma\big(x^{-1}y\big)q(x, y) \\ &= \sigma\big(x^{-1}y\big)\big(\big(s^{-1}\big)^{\sigma(x^{-1}y)^{-1}}q(x, y)\big). \end{aligned}$$

(Here we use the notation $s^g := gsg^{-1}$, $s \in S$, $g \in G$.)

This shows that $\tau(g, y) = (s^{-1})^{\sigma(x^{-1}y)^{-1}}q(x, y)$ and $\eta(g, y) = x^{-1}y$. Hence π_p is given as follows:

$$\big[\pi_p\big(\sigma(x)s\big)\xi\big](y) = \rho\big(q(x, y)^{-1}s^{\sigma(x^{-1}y)^{-1}}\big)\big[\xi\big(x^{-1}y\big)\big], \tag{3.6}$$

for $(x, s) \in H \times S,\ y \in H,\ \xi \in L^p(H, X)$.

We remark that it is easy to check that

$$q(x, y)^{-1}s^{\sigma(x^{-1}y)^{-1}} = s^{\sigma(y)^{-1}\sigma(x)}q(x, y)^{-1}.$$

Notice that (3.6) does not depend on p.

(b) In the special case where $\rho = 1$ and S is normal, the induced representation $\iota = \mathrm{ind}_S^G 1$ is given by

$$\big[\iota\big(\sigma(x)s\big)\xi\big](y) = \xi\big(x^{-1}y\big).$$

For the integrated representation, we then have:

$$\begin{aligned} \big[\iota(f)\xi\big](y) &= \int_H\int_S f\big(\sigma(x)s\big)\xi\big(x^{-1}y\big)\,ds\,dx \\ &= \int_H \tilde{f}(x)\xi\big(x^{-1}y\big)\,dx \\ &= \big[\lambda_H(\tilde{f})\xi\big](y), \end{aligned}$$

i.e.

$$\iota(f) = \lambda_H(\tilde{f}), \quad \text{where } \tilde{f}(x) := \int_S f\big(\sigma(x)s\big)\,ds,$$

i.e. $\tilde{f}$ is the image of f under the quotient map from G onto G/S.

3.2. A transference principle

If $\xi \in L^p(H, X)$, and if $\phi : S \to \mathbb{C}$, we define the *ρ-twisted tensor product*

$$\xi \otimes_\rho^p \phi : G \to X \quad \text{by}$$
$$\left[\xi \otimes_\rho^p \phi\right]\left(\sigma(x)s\right) := \phi(s)\Delta_{G,S}(s)^{-1/p}\rho\left(s^{-1}\right)\left[\xi(x)\right], \quad (x, s) \in H \times S.$$

Let us denote by X^* the dual space of X. For any complex vector space Y, we denote by $\overline{Y}$ its complex conjugate, which, as an additive group, is the space Y, but with scalar multiplication given by $\bar{\lambda}y$, for $\lambda \in \mathbb{C}$ and $y \in Y$. In the following, we assume that X contains a dense, ρ-invariant subspace X_0, which embeds via an anti-linear mapping $i : X_0 \hookrightarrow \overline{X^*}$ into the complex conjugate of the dual space of X, in such a way that, for every $x \in X$,

$$\|x\| = \sup_{\{v \in X_0:\ \|i(v)\|_{X^*}=1\}} \left|\langle x, v\rangle\right|. \tag{3.7}$$

Here, we have put

$$\langle x, v\rangle := i(v)(x), \quad v \in X_0,\ x \in X.$$

Moreover, we assume that

$$\left\|i\left(\rho(s)v\right)\right\|_{X^*} = \left\|i(v)\right\|_{X^*} \quad \text{for every } v \in X_0,\ s \in S, \tag{3.8}$$

and

$$\left\langle\rho(s)x, \rho(s)v\right\rangle = \langle x, v\rangle \quad \text{for every } x \in X,\ v \in X_0,\ s \in S. \tag{3.9}$$

The most important example for us will be an L^p-space $X = L^p(\Omega)$, $1 \leqslant p < \infty$, on a measure space $(\Omega, d\omega)$, and a representation ρ of G which acts isometrically on $L^p(\Omega)$ as well as on its dual space $L^{p'}(\Omega)$ (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$). In this case, by interpolation, we have $\|\rho(g)\xi\|_r \leqslant \|\xi\|_r$, for $|\frac{1}{r} - \frac{1}{2}| \leqslant |\frac{1}{p} - \frac{1}{2}|$, $g \in G$, which implies that indeed $\rho(g)$ acts isometrically on $L^r(\Omega)$, for $|\frac{1}{r} - \frac{1}{2}| \leqslant |\frac{1}{p} - \frac{1}{2}|$. In particular, ρ is a unitary representation on $L^2(\Omega)$. We can then choose $X_0 := L^{p'}(\Omega) \cap L^p(\Omega) \subset L^2(\Omega)$, and put

$$i(\eta)(\xi) := \int_\Omega \xi(\omega)\overline{\eta(\omega)}\, d\omega, \quad \eta \in L^{p'}(\Omega) \cap L^p(\Omega),\ \xi \in L^p(\Omega).$$

Notice that, if ρ is a unitary character, (3.8) and (3.9) are always satisfied.

Lemma 3.1. *Let $\phi \in L^p(S)$, $\psi \in L^{p'}(S)$, $\xi \in L^p(H, X_0)$ and $\eta \in L^{p'}(H, X_0)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for every $g \in G$,*

$$\left\langle\lambda_G(g)\left(\xi \otimes_\rho^p \phi\right), \eta \otimes_\rho^{p'} \psi\right\rangle = \int_H \phi * \overset{\vee}{\bar{\psi}}\left(\tau(g, x)\right)\left\langle\left[\pi_p(g)\xi\right](x), \eta(x)\right\rangle dx. \tag{3.10}$$

Proof. By (3.3), we have:

$$\begin{aligned}
&\big\langle \lambda_G(g)\big(\xi \otimes_\rho^p \phi\big), \eta \otimes_\rho^{p'} \psi\big\rangle \\
&\quad= \int_H\int_S \big\langle \xi \otimes_\rho^p \phi\big(g^{-1}\sigma(x)s\big), \eta \otimes_\rho^{p'} \psi\big(\sigma(x)s\big)\big\rangle \Delta_{G,S}(s)\, ds\, dx \\
&\quad= \int_H\int_S \big\langle \xi \otimes_\rho^p \phi\big(\sigma\big(\eta(g,x)\big)\tau(g,x)s\big), \eta \otimes_\rho^{p'} \psi\big(\sigma(x)s\big)\big\rangle \Delta_{G,S}(s)\, ds\, dx \\
&\quad= \int_H\int_S \Delta_{G,S}\big(\tau(g,x)s\big)^{-\frac{1}{p}} \Delta_{G,S}(s)^{-\frac{1}{p'}} \phi\big(\tau(g,x)s\big)\overline{\psi}(s) \\
&\qquad \big\langle \rho\big(s^{-1}\tau(g,x)^{-1}\big)\big[\xi\big(\eta(g,x)\big)\big], \rho\big(s^{-1}\big)\big[\eta(x)\big]\big\rangle \Delta_{G,S}(s)\, ds\, dx \\
&\quad= \int_H\int_S \Delta_{G,S}\big(\tau(g,x)\big)^{-\frac{1}{p}} \phi\big(\tau(g,x)s\big)\overline{\psi}(s)\, ds \big\langle \rho\big(\tau(g,x)^{-1}\big)\big[\xi\big(\eta(g,x)\big)\big], \eta(x)\big\rangle dx.
\end{aligned}$$

Here, we have used that, by (3.9), $\langle \rho(s^{-1})v_1, \rho(s^{-1})v_2\rangle = \langle v_1, v_2\rangle$ for all $v_1, v_2 \in X_0$.

But,

$$\int_S \phi\big(\tau(g,x)s\big)\overline{\psi}(s)\, ds = \int_S \phi(s)\psi\big(\tau(g,x)^{-1}s\big)\, ds = \phi * \check{\psi}\big(\tau(g,x)\big),$$

and

$$\Delta_{G,H}\big(\tau(g,x)\big)^{-\frac{1}{p}} \big\langle \rho\big(\tau(g,x)^{-1}\big)\big[\xi\big(\eta(g,x)\big)\big], \eta(x)\big\rangle = \big\langle \big[\pi_p(g)\xi\big](x), \eta(x)\big\rangle,$$

and thus (3.10) follows. □

From now on, we shall assume that the group S is *amenable*.

Since G is separable, we can then choose an increasing sequence $(A_j)_j$ of compacta in S such that $A_j^{-1} = A_j$ and $S = \bigcup_j A_j$, and put

$$\phi_j = \phi_j^p := \frac{\chi_{A_j}}{|A_j|^{1/p}}, \qquad \psi_j = \psi_j^{p'} := \frac{\chi_{A_j}}{|A_j|^{1/p'}},$$

where χ_A denotes the characteristic function of the subset A. Then $\check{\psi}_j = \psi_j$, $\|\phi_j\|_p = \|\psi_j\|_{p'} = 1$, and, because of the amenability of S (see [16]), we have:

$$\chi_j := \phi_j * \psi_j \text{ tends to 1, uniformly on compacta in } S. \tag{3.11}$$

Proposition 3.1. *Let $\pi_p = \mathrm{ind}_{p,S}^G \rho$ be as before, where S is amenable, and let $\xi, \eta \in C_0(H, X_0)$. Then*

$$\big\langle \pi_p(g)\xi, \eta\big\rangle = \lim_{j\to\infty} \big\langle \lambda_G(g)\big(\xi \otimes_\rho^p \phi_j\big), \eta \otimes_\rho^{p'} \psi_j\big\rangle,$$

uniformly on compacta in G.

Proof. By Lemma 3.1,

$$\big\langle \lambda_G(g)\big(\xi \otimes_\rho^p \phi_j\big), \eta \otimes_\rho^{p'} \psi_j \big\rangle = \int_H \chi_j\big(\tau(g,x)\big)\big\langle \big[\pi_p(g)\xi\big](x), \eta(x)\big\rangle\, dx.$$

Fix a compact set $K = K^{-1} \subset H$ containing the supports of ξ and η, and let $Q \subset G$ be any compact set. We want to prove that $\{\tau(g,x) \mid g \in Q,\ x \in K\}$ is relatively compact, for then, by (3.11), we immediately see that

$$\lim_{j\to\infty} \big\langle \lambda_G(g)\big(\xi \otimes_\rho^p \phi_j\big), \eta \otimes_\rho^{p'} \psi_j \big\rangle = \int_H \big\langle \big[\pi_p(g)\xi\big](x), \eta(x)\big\rangle\, dx = \big\langle \pi_p(g)\xi, \eta\big\rangle,$$

uniformly for $g \in Q$.

Recall that $\tau(g,x) = \tau(g^{-1}\sigma(x))$. Therefore, since $\sigma(K)$ is relatively compact, it suffices to prove that τ maps compact subsets of G into relatively compact sets in S. So, let again Q denote a compact subset of G, and put $M := Q$ mod $S \subset H = G/S$. Then M is compact, so that $\overline{\sigma(M)}$ is compact in S. And, since $\tau(\sigma(x)s) = s$ for every $x \in H$, $s \in S$, we have:

$$\tau(Q) = \big\{ s \in S \mid \sigma(x)s \in Q \text{ for some } x \in M \big\} = \sigma(M)^{-1}Q,$$

which shows that $\tau(Q)$ is indeed relatively compact. □

Theorem 3.1. *For every bounded measure $\mu \in M^1(G)$, we have*:

$$\big\| \pi_p(\mu) \big\|_{L^p(H,X)\to L^p(H,X)} \leqslant \big\| \lambda_G(\mu) \big\|_{L^p(G,X)\to L^p(G,X)}.$$

Proof. Let $\xi, \eta \in C_0(H, X_0)$. Observe first that, for $g \in G$,

$$\begin{aligned}
&\big|\big\langle \lambda_G(g)\big(\xi \otimes_\rho^p \phi_j\big), \eta \otimes_\rho^{p'} \psi_j \big\rangle\big| \\
&\quad \leqslant \big\| \lambda_G(g)\big(\xi \otimes_\rho^p \phi_j\big)\big\|_{L^p(G,X)} \big\| i \circ \big(\eta \otimes_\rho^{p'} \psi_j\big)\big\|_{L^{p'}(G,X^*)} \\
&\quad = \big\| \xi \otimes_\rho^p \phi_j \big\|_{L^p(G,X)} \big\| i \circ \big(\eta \otimes_\rho^{p'} \psi_j\big)\big\|_{L^{p'}(G,X^*)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\big\| \xi \otimes_\rho^p \phi_j \big\|_{L^p(G,X)}^p &= \iint_{H\,S} \big|\phi_j(s)\big|^p \Delta_{G,S}(s)^{-1} \big\| \rho\big(s^{-1}\big)\big[\xi(x)\big]\big\|_X^p \Delta_{G,S}(s)\, ds\, dx \\
&= \int_S \big|\phi_j(s)\big|^p\, ds \int_H \big\|\xi(x)\big\|_X^p\, dx \\
&= \|\xi\|_{L^p(H,X)}^p,
\end{aligned}$$

since $\rho(s^{-1})$ is isometric on X, so that

$$\big\| \xi \otimes_\rho^p \phi_j \big\|_{L^p(G,X)} = \|\xi\|_{L^p(H,X)}. \tag{3.12}$$

Similarly, because of (3.8),

$$\left\|i \circ \left(\eta \otimes_\rho^{p'} \psi_j\right)\right\|_{L^{p'}(G,X^*)} = \|i \circ \eta\|_{L^{p'}(H,X^*)}. \tag{3.13}$$

This implies

$$\left|\left\langle \lambda_G(g)\left(\xi \otimes_\rho^p \phi_j\right), \eta \otimes_\rho^{p'} \psi_j\right\rangle\right| \leqslant \|\xi\|_{L^p(H,X)} \|i \circ \eta\|_{L^{p'}(H,X^*)}.$$

Therefore, if $\mu \in M^1(G)$, Proposition 3.1 implies, by the dominated convergence theorem, that

$$\left\langle \pi_p(\mu)\xi, \eta\right\rangle = \lim_{j\to\infty} \left\langle \lambda_G(\mu)\left(\xi \otimes_\rho^p \phi_j\right), \eta \otimes_\rho^{p'} \psi_j\right\rangle. \tag{3.14}$$

Moreover, by (3.12) and (3.13),

$$\begin{aligned}
&\left|\left\langle \lambda_G(\mu)\left(\xi \otimes_\rho^p \phi_j\right), \eta \otimes_\rho^{p'} \psi_j\right\rangle\right| \\
&\quad = \left|\int_G \left\langle \lambda_G(\mu)\left(\xi \otimes_\rho^p \phi_j\right), \eta \otimes_\rho^{p'} \psi_j\right\rangle dg\right| \\
&\quad \leqslant \left\|\lambda_G(\mu)\right\|_{L^p(G,X)\to L^p(G,X)} \|\xi\|_{L^p(H,X)} \|i \circ \eta\|_{L^{p'}(H,X^*)}.
\end{aligned}$$

By (3.14), we therefore obtain:

$$\left|\left\langle \pi_p(\mu)\xi, \eta\right\rangle\right| \leqslant \left\|\lambda_G(\mu)\right\|_{L^p(G,X)\to L^p(G,X)} \|\xi\|_{L^p(H,X)} \|i \circ \eta\|_{L^{p'}(H,X^*)}. \tag{3.15}$$

In view of (3.7) and since $C_0(H, X_0)$ lies dense in $L^p(H, X)$, this implies the theorem. $\square$

Corollary 1 *(Transference). Let $X = L^p(\Omega)$. Then, for every $\mu \in M^1(G)$, we have*

$$\left\|\pi_p(\mu)\right\|_{L^p(H,L^p(\Omega))\to L^p(H,L^p(\Omega)))} \leqslant \left\|\lambda_G(\mu)\right\|_{L^p(G)\to L^p(G)}.$$

Proof. If $X = L^p(\Omega)$ and $h \in L^p(G, X)$, then, by Fubini's theorem,

$$\left\|\lambda_G(\mu)h\right\|^p_{L^p(G,X)} = \int_\Omega \left\|\mu * h(\cdot, \omega)\right\|^p_{L^p(G)} d\omega \leqslant \left\|\lambda_G(\mu)\right\|^p_{L^p(G)\to L^p(G)} \|h\|_{L^p(G,X)}.$$

Hence,

$$\left\|\lambda_G(\mu)\right\|_{L^p(G,X)\to L^p(G,X)} \leqslant \left\|\lambda_G(\mu)\right\|_{L^p(G)\to L^p(G)}.$$

In combination with (3.15), we obtain the desired estimate. $\square$

Remark 2. We call a Banach space X to be of L^p*-type*, $1 \leqslant p < \infty$, if there exists an embedding $\iota : X \hookrightarrow L^p(\Omega)$ into an L^p-space and a constant $C \geqslant 1$ such that, for every $x \in X$,

$$\frac{1}{C}\|x\|_X \leqslant \left\|\iota(x)\right\|_{L^p(\Omega)} \leqslant C\|x\|_X.$$

For instance, any separable Hilbert space $\mathcal{H}$ is of L^p-type, for $1 \leqslant p < \infty$, or, more generally, any space $L^p(Y, \mathcal{H})$. This follows easily from Khintchin's inequality. By an obvious modification of the proof, Corollary 1 remains valid for spaces X of L^p-type.

Denote by $C^*_{\mathrm{r}}(G)$ the reduced C^*-algebra of G. If $p = 2$, we can extend (3.14) to $C^*_{\mathrm{r}}(G)$.

Proposition 3.2. *If $p = 2$ and $X = L^2(\Omega)$, then the unitary representation π_2 is weakly contained in the left-regular representation λ_G. In particular, for any $K \in C^*_{\mathrm{r}}(G)$, the operator $\pi_2(K) \in \mathcal{B}(L^2(H, L^2(\Omega)))$ is well defined.*

Moreover, for all $\xi, \eta \in C_0(H, L^2(\Omega))$, we have

$$\langle \pi_2(K)\xi, \eta \rangle = \lim_{j \to \infty} \langle \lambda_G(K)\big(\xi \otimes^2_\rho \phi_j\big), \eta \otimes^2_\rho \phi_j, \rangle. \tag{3.16}$$

Proof. If $K \in C^*_{\mathrm{r}}(G)$, then we can find a sequence $(f_k)_k$ in $L^1(G)$, such that $\lambda_G(K) = \lim_{k\to\infty} \lambda_G(f_k)$ in the operator norm $\|\cdot\|$ on $L^2(G)$. But, (3.15) implies, for all $f \in L^1(G)$ that

$$\big\|\pi_2(f)\big\| \leqslant \big\|\lambda_G(f)\big\|. \tag{3.17}$$

Here $\|\cdot\|$ denotes the operator norm on $\mathcal{B}(L^2(H, L^2(\Omega)))$ and $\mathcal{B}(L^2(G))$, respectively. Therefore, the $(\pi_2(f_k))_k$ form a Cauchy sequence in $\mathcal{B}(L^2(H, L^2(\Omega)))$, whose limit we denote by $\pi_2(K)$.

It does not depend on the approximating sequence $\{f_k\}_k$. Moreover, from (3.17) we then deduce that, for all $K \in C^*_{\mathrm{r}}(G)$,

$$\big\|\pi_2(K)\big\| \leqslant \big\|\lambda_G(K)\big\| = \|K\|_{C^*_{\mathrm{r}}(G)}. \tag{3.18}$$

In particular, we see that π_2 is weakly contained in λ_G. It remains to show (3.16).

Given $\varepsilon > 0$, we choose $f \in C_0(G)$ such that $\|K - f\|_{C^*_{\mathrm{r}}(G)} < \varepsilon/4$. Next, by (3.15), we can find j_0 such that, for all $j \geqslant j_0$,

$$\big|\langle \pi_2(f)\xi, \eta \rangle - \langle \lambda_G(f)\big(\xi \otimes^2_\rho \phi_j\big), \eta \otimes^2_\rho \phi_j \rangle\big| < \varepsilon/4.$$

Assume without loss of generality that $\|\xi\|_2 = \|\eta\|_2 = 1$. Then, by (3.18),

$$\big|\langle \pi_2(K)\xi, \eta \rangle - \langle \pi_2(f)\xi, \eta \rangle\big| \leqslant \|K - f\|_{C^*_{\mathrm{r}}(G)} \|\xi\|_2 \|\eta\|_2 < \varepsilon/4,$$

and furthermore

$$\begin{aligned}
&\big|\langle \lambda_G(K)\big(\xi \otimes^2_\rho \phi_j\big), \eta \otimes^2_\rho \phi_j \rangle - \langle \lambda_G(f)\big(\xi \otimes^2_\rho \phi_j\big), \eta \otimes^2_\rho \phi_j \rangle\big| \\
&\quad \leqslant \|K - f\|_{C^*_{\mathrm{r}}(G)} \big\|\xi \otimes^2_\rho \phi_j\big\|_2 \big\|\eta \otimes^2_\rho \phi_j\big\|_2 \\
&\quad < \frac{\varepsilon}{4} \|\xi\|_2 \|\eta\|_2 = \varepsilon/4.
\end{aligned}$$

Combining these estimates, we find that, for all $j \geqslant j_0$,

$$\big|\langle \pi_2(K)\xi, \eta \rangle - \langle \lambda_G(K)\big(\xi \otimes^2_\rho \phi_j\big), \eta \otimes^2_\rho \phi_j \rangle\big| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon. \qquad \square$$

Corollary 3. *Assume that ρ is a unitary representation on a separable Hilbert space X, for instance a unitary character of S, and that $\Delta_{G,S} = 1$. Let $K \in C_{\mathrm{r}}^*(G)$, and assume that $\lambda_G(K)$ extends from $L^2(G) \cap L^p(G)$ to a bounded linear operator on $L^p(G)$, where $1 \leqslant p < \infty$.*

Then $\pi_2(K)$ extends from $L^2(G/S, X) \cap L^p(G/S, X)$ to a bounded linear operator on $L^p(G/S, X)$, and

$$\left\|\pi_2(K)\right\|_{L^p(G/S,X)\to L^p(G/S,X)} \leqslant \left\|\lambda_G(K)\right\|_{L^p(G)\to L^p(G)}. \tag{3.19}$$

Moreover, for $f \in L^1(G)$, we have $\pi_p(f) = \pi_2(f)$ on $C_0(G/S, X)$.

Proof. If $\xi, \eta \in C_0(H, X)$, then, since $\Delta_{G,S} = 1$,

$$\left\langle \pi_2(K)\xi, \eta \right\rangle = \lim_{j\to\infty} \left\langle \lambda_G(K)\left(\xi \otimes_\rho^2 \phi_j^2\right), \eta \otimes_\rho^2 \psi_j^2 \right\rangle = \lim_{j\to\infty} \left\langle \lambda_G(K)\left(\xi \otimes_\rho^p \phi_j^p\right), \eta \otimes_\rho^{p'} \psi_j^{p'} \right\rangle$$

and,

$$\begin{aligned} &\left|\left\langle \lambda_G(K)\left(\xi \otimes_\rho^p \phi_j^p\right), \eta \otimes_\rho^{p'} \psi_j^{p'} \right\rangle\right| \\ &\quad \leqslant \left\|\lambda_G(K)\right\|_{L^p(G)\to L^p(G)} \left\|\xi \otimes_\rho^p \phi_j^p\right\|_{L^p(G)} \left\|\eta \otimes_\rho^{p'} \psi_j^{p'}\right\|_{L^{p'}(G)} \\ &\quad \leqslant \left\|\lambda_G(K)\right\|_{L^p(G)\to L^p(G)} \|\xi\|_{L^p(H,X)} \|\eta\|_{L^{p'}(H,X)}. \end{aligned}$$

Estimate (3.19) follows.

That $\pi_p(f) = \pi_2(f)$ on $C_0(H, X)$, if $f \in L^1(G)$, is evident, since $\Delta_{G,S} = 1$. $\square$

4. The case of a non-compact semi-simple factor

In this section, we shall give our proof of Theorem 1.2. Let us first notice the following consequence of Corollary 3.

Assume that S is a closed, normal and amenable subgroup of G, and let $L = -\sum_j X_j^2$ be a sub-Laplacian on G. Denote by $\iota_2 := \mathrm{ind}_S^G 1$ the representation of G induced by the trivial character of S (compare Example 3.1), and let $\tilde{L} = -\sum_j (X_j \bmod \mathfrak{s})^2 = d\iota_2(L)$ be the corresponding sub-Laplacian on the quotient group $H := G/S$. Then

$$\mathcal{M}_p(L) \cap C_\infty(\mathbb{R}) \subset \mathcal{M}_p(\tilde{L}) \cap C_\infty(\mathbb{R}). \tag{4.1}$$

In particular, if $\tilde{L}$ is of holomorphic L^p-type, then so is L.

In order to prove (4.1), assume that F is an L^p-multiplier for L contained in $C_\infty(\mathbb{R})$. Then $F(L)$ lies in $C_{\mathrm{r}}^*(G)$, and by Corollary 3 the operator $\iota_2(F(L)) = F(d\iota_2(L)) = F(\tilde{L})$ extends from $L^2(H) \cap L^p(H)$ to a bounded operator on $L^p(H)$, so that $F \in \mathcal{M}_p(\tilde{L}) \cap C_\infty(\mathbb{R})$.

Let now G be a connected Lie group, with radical $S = \exp \mathfrak{s}$. Then there exists a connected, simply connected semi-simple Lie group H such that G is the semi-direct product of H and S, and this Levi factor H has a discrete center Z (see [4]). Let L be a sub-Laplacian on G, and denote by $\tilde{L}$ the corresponding sub-Laplacian on $G/S \simeq H$ and by $\tilde{\tilde{L}}$ the sub-Laplacian on H/Z corresponding to $\tilde{L}$ on H. We have that Z and S are amenable groups, and H/Z has finite center. From Theorem 1.1, we thus find, if we assume H to be non-compact, that $\tilde{\tilde{L}}$ is of holomorphic

L^p-type for every $p \neq 2$, and (4.1) then allows us to conclude that the same is true of $\tilde{L}$, and then also of L.

5. Compact extensions of exponential solvable Lie groups

5.1. Compact operators arising in induced representations

Let now $K = \exp \mathfrak{k}$ be a connected compact Lie group acting continuously on an exponential solvable Lie group $S = \exp \mathfrak{s}$ by automorphisms $\sigma(k) \in \mathrm{Aut}(S)$, $k \in K$. We form the semi-direct product $G = K \ltimes S$ with the multiplication given by

$$(k, s) \cdot (k', s') = \left(kk', \sigma\left(k'^{-1}\right)ss'\right), \quad k, k' \in K,\ s, s' \in S.$$

The left Haar measure dg is the product of the Haar measure of K and the left Haar measure of S. Let us choose a K-invariant scalar product $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{s}$ of S. Denote by $\mathfrak{n}$ the nil-radical of $\mathfrak{s}$. Since every derivation d of $\mathfrak{s}$ maps the vector space $\mathfrak{s}$ into the nil-radical, it follows that the orthogonal complement $\mathfrak{b}$ of $\mathfrak{n}$ in $\mathfrak{s}$ is in the kernel of $d\sigma(X)$ for every $X \in \mathfrak{k}$. The following decomposition of the solvable Lie algebra $\mathfrak{s}$ has been given in [3]. Choose an element $X \in \mathfrak{b}$, which is in general position for the roots of $\mathfrak{s}$, i.e., for which $\lambda(X) \neq \mu(X)$ for all roots $\mu \neq \lambda$ of $\mathfrak{s}$. Let $\mathfrak{s}_0 = \{Y \in \mathfrak{s};\ \mathrm{ad}^l(X)Y = 0 \text{ for some } l \in \mathbb{N}^*\}$. Then $\mathfrak{s}_0$ is a nilpotent subalgebra of $\mathfrak{s}$, which is K-invariant (since $[X, \mathfrak{k}] = \{0\}$) and $\mathfrak{s} = \mathfrak{s}_0 + \mathfrak{n}$. Let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{n} \cap \mathfrak{s}_0$ in $\mathfrak{s}_0$. Then $\mathfrak{a}$ is also a K-invariant subspace of $\mathfrak{s}$ (but not in general a subalgebra) and $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$. Let $N = \exp \mathfrak{n} \subset S$ be the nil-radical of the group S. Then S is the topological product of $A = \exp \mathfrak{a}$ and N. Finally our group G is the topological product of K, A and N. Hence every element g of G has the unique decomposition

$$g = k_g \cdot a_g \cdot n_g, \quad \text{where } k_g \in K,\ a_g \in A \text{ and } n_g \in N.$$

We shall use the notations and constructions of [8] in the following but we have to replace there the symbol G with the letter S.

Let $h : G \to \mathbb{C}$ be a function. For every $x \in G$, we denote by $\tilde{h}(x)$ the function on S defined by

$$\tilde{h}(x)(s) = h(xs), \quad s \in S.$$

Also, for a function $r : S \to \mathbb{C}$ and for $x \in G$, let ${}^x r : S \to \mathbb{C}$ be defined by

$${}^x r(s) := r\left(xsx^{-1}\right).$$

We say that a Borel measurable function $\omega : G \to \mathbb{R}_+$ is a weight, if $1 \leqslant \omega(x) = \omega(x^{-1})$ and $\omega(xy) \leqslant \omega(x)\omega(y)$, for every $x, y \in G$. Then the space

$$L^p(G, \omega) = \left\{f \in L^p(G) \;\middle|\; \|f\|_{\omega,p} := \left\|f\omega^{1/p}\right\|_p < \infty\right\},$$

for $1 \leqslant p \leqslant \infty$, is a subspace of $L^p(G)$. For $p = 1$, it is even a Banach algebra for the norm $\|\cdot\|_{\omega,1}$.

Proposition 5.1. *Let G be a locally compact group and let S be a closed normal subgroup of G. Let ω be a continuous weight on G such that the inverse of its restriction to S is integrable with respect to the Haar measure on S. Let $f, g : G \to \mathbb{C}$ be two continuous functions on G, such that $\omega \cdot g$ is uniformly bounded and such that $f \in L^1(G,\omega)$. Let $h := f * g \in L^1(G,\omega)$. Then for every $t \in G$, the function $\tilde{h}(t)$ is in $L^1(S)$ and the mapping $(t,u) \mapsto {}^u\tilde{h}(t)$ from $G \times G$ to $L^1(S)$ is continuous.*

Proof. Since ω is a weight, we have that $\omega(s) \leqslant \omega(u)\omega(u^{-1}s)$, i.e. $\frac{1}{\omega(u^{-1}s)} \leqslant \frac{\omega(u)}{\omega(s)},\ s,u \in G$. Hence, for $t \in G,\ s \in S$,

$$\begin{aligned}\left|\tilde{h}(t)(s)\right| &= \left|\int_G f(u)g\left(u^{-1}ts\right)du\right| = \left|\int_G f(tu)g\left(u^{-1}s\right)du\right| \\ &\leqslant \int_G \left|f(tu)\right|\left|g\left(u^{-1}s\right)\right|\frac{\omega(u^{-1}s)}{\omega(u^{-1}s)}\,du \leqslant \int_G \left|f(tu)\right|\omega(u)\left|g\left(u^{-1}s\right)\right|\frac{\omega(u^{-1}s)}{\omega(s)}\,du,\end{aligned}$$

and so

$$\begin{aligned}\left\|\tilde{h}(t)\right\|_1 &\leqslant \int_S\int_G \left|f(tu)\right|\omega(u)\left|g\left(u^{-1}s\right)\right|\frac{\omega(u^{-1}s)}{\omega(s)}\,du\,ds \\ &\leqslant \int_S\int_G \left|f(tu)\right|\omega(u)\frac{\|g\omega\|_\infty}{\omega(s)}\,du\,ds \\ &\leqslant \int_S\int_G \omega\left(t^{-1}\right)\left|f(tu)\right|\omega(tu)\frac{\|g\omega\|_\infty}{\omega(s)}\,du\,ds \\ &= \omega(t)\|f\|_{\omega,1}\|g\omega\|_\infty\left\|\left(\frac{1}{\omega}\right)_{|S}\right\|_1. \end{aligned} \tag{5.1}$$

Thus, for every $t \in G$, the function $\tilde{h}(t)$ is in $L^1(S)$. Furthermore, by (5.1), for $t, t' \in G$,

$$\begin{aligned}\left\|h(t) - h(t')\right\|_1 &\leqslant \int_S\int_G \left|f(tu) - f(t'u)\right|\omega(u)\frac{\|g\omega\|_\infty}{\omega(s)}\,du\,ds \\ &\leqslant \left\|\left(\lambda\left(t^{-1}\right)f - \lambda\left(t'^{-1}\right)f\right)\right\|_{\omega,1}\|g\omega\|_\infty\left\|\left(\frac{1}{\omega}\right)_{|S}\right\|_1,\end{aligned}$$

where λ denotes left translation by elements of G. Since left translation in $L^1(G,\omega)$ and conjugation in $L^1(S)$ are continuous, it follows that the mapping $(t,u) \mapsto {}^u\tilde{h}(t)$ from $G \times G$ to $L^1(S)$ is continuous too. $\square$

Let, as in (1.3), δ denote the Carathéodory distance associated to our sub-Laplacian L on G and $(h_t)_{t>0}$ its heat kernel. Then the function $\omega_d(g) := e^{d\delta(x,e)}$, $g \in G$, $d \in \mathbb{R}_+$, defines a weight on G. Since we have the Gaussian estimate

$$\left|h_t(g)\right| \leqslant C_t e^{-c_t \delta(g,e)^2}, \quad \text{for all } g \in G,\ t > 0,$$

it follows that

$$h_t \in L^1(G, \omega_d) \cap L^\infty(G, \omega_d) \quad \text{for every } t > 0 \text{ and } d > 0. \tag{5.2}$$

Proposition 5.2. *Let G be the semidirect product of a connected compact Lie group K acting on an exponential solvable Lie group S. Then there exists a constant $d > 0$, such that $\frac{1}{\omega_d}|_S$ is in $L^1(S)$.*

Proof. Let U be a compact symmetric neighborhood of e in G containing K. Since S is connected, we know that $G = \bigcup_{k\in\mathbb{N}} U^k$. This allows us to define $\tau_U = \tau : G \to \mathbb{N}$ by

$$\tau(x) = \min\left\{k \in \mathbb{N} \mid x \in U^k\right\}.$$

Then τ is sub-additive and thus defines a distance on G, which is bounded on compact sets. Since τ is clearly connected in the sense of [19], it follows that τ and the Carathéodory distance δ are equivalent at infinity, i.e.

$$1 + \tau(x) \leqslant D\bigl(1 + \delta(x)\bigr) \leqslant D'\bigl(1 + \tau(x)\bigr), \quad x \in G.$$

We choose now a special compact neighborhood of e in the following way. We take our K-invariant scalar-product on $\mathfrak{s}$, the unit-ball $B_\mathfrak{a}$ in $\mathfrak{a}$ and the unit-ball $B_\mathfrak{n} \in \mathfrak{n}$. Both balls are K-invariant. Let $U_\mathfrak{a} = \exp B_\mathfrak{a}$ and $U_\mathfrak{n} = \exp B_\mathfrak{n}$. Then $U = KU_\mathfrak{a}U_\mathfrak{n} \cap U_\mathfrak{n}U_\mathfrak{a}K$ is a compact symmetric neighborhood of e. Let us give a rough estimate of the radii of the "balls" U^l, $l \in \mathbb{N}$. For simplicity of notation, we shall denote all the positive constants which will appear in the following arguments (and which will be assumed to be integers, if necessary) by C.

Let $k_i a_i n_i \in KU_\mathfrak{a}U_\mathfrak{n}$, $i = 1, \dots, l$, and $g := \prod_{i=1}^l k_i a_i n_i$. We have:

$$g = \prod_{i=1}^l k_i a_i n_i = \left(\prod_{i=1}^l k_i a_i\right)\bigl((k_2a_2\cdots k_la_l)^{-1} n_1 (k_2a_2\cdots k_la_l)\cdots (k_la_l)n_{l-1}(k_la_l)n_l\bigr).$$

Since $U_\mathfrak{a}$ is K-invariant, it follows that

$$g = \prod_{i=1}^l k_i a_i n_i = k'a' \prod_{i=1}^l \bigl(a_i''k_i''\bigr)n_i\bigl(a_i''k_i''\bigr)^{-1},$$

where $k', k_1'', \dots, k_l'' \in K$, $a' \in U_\mathfrak{a}^l$, $a_1'' \in U_\mathfrak{a}^{l-1}, \dots, a_{l-1}'' \in U_a$. Hence there exists $X_1, \dots, X_l \in B_\mathfrak{a}$, such that

$$a' = \exp X_1 \cdots \exp X_l = \exp\left(X_1 + \cdots + X_l\right) \exp q_l(X_1, \dots, X_l)$$

for some element $q_l(X_1, \ldots, X_l) \in \mathfrak{n} \cap \mathfrak{s}_0$. Since $\mathfrak{s}_0$ is a nilpotent Lie algebra we have that $\|q_l(X_1, \cdots, X_l)\| \leqslant C(1+l)^C$, $l \in \mathbb{N}$. Hence

$$a' \in \exp(lB_a) \exp\bigl[C(1+l)^C B_{\mathfrak{n}}\bigr] \subset \exp(lB_{\mathfrak{a}}) U_{\mathfrak{n}}^{C(1+l)^C}.$$

Furthermore, because $U_{\mathfrak{a}}$ is compact, $\sup_{a \in U_{\mathfrak{a}}} \|\operatorname{Ad}(a)\|_{\mathrm{op}} \leqslant C < \infty$ and so, for $i = 1, \ldots, l$, $(a_i'' k_i'') n_i (a_i'' k_i'')^{-1} \in \exp C^{(l-i)} B_{\mathfrak{n}} \subset U_{\mathfrak{n}}^{C^{l-i}}$. Finally, for some integer constants C,

$$\begin{aligned} g = k'a' \prod_{i=1}^{l} \bigl(a_i'' k_i''\bigr) n_i \bigl(a_i'' k_i''\bigr)^{-1} &\in K \exp l U_a U_{\mathfrak{n}}^{C(1+l)^C} \left(\prod_{i=1}^{l-1} U_{\mathfrak{n}}^{C^{l-i}} \right) U_{\mathfrak{n}} \\ &\subset K \exp l U_a U_{\mathfrak{n}}^{C(1+l)^C + \sum_{i=1}^{l-1} C^{l-i} + 1} \\ &\subset K \exp l U_a U_{\mathfrak{n}}^{C^l} \\ &\subset K \exp l U_a \exp C^l B_{\mathfrak{n}}. \end{aligned} \tag{5.3}$$

Hence, for any $g \in G$, for $\tau_U(g) = l$, we have that $g \in (K U_{\mathfrak{a}} U_{\mathfrak{n}})^l$. Thus, denoting by $\operatorname{Log}: S \to \mathfrak{s}$ the inverse map of $\exp: \mathfrak{s} \to S$, we get that $g = k_g a_g n_g$, with $k_g \in K$, $a_g \in \exp \mathfrak{a}$, $\|\operatorname{Log}(a_g)\| \leqslant l = \tau_U(g)$ and $n_g \in N$ with $\|\operatorname{Log}(n_g)\| \leqslant C^l$, i.e. $\log(1 + \|\operatorname{Log}(n_g)\|) \leqslant Cl = C\tau_G(g)$. Whence for our weight ω_d, ($d \in \mathbb{R}_+$), we have that

$$\begin{aligned} \omega_d(g) = e^{d\delta(g)} \geqslant C e^{dC\tau_U(g)} &\geqslant C e^{dC(\|\operatorname{Log}(a_g)\| + \log(1 + \|\operatorname{Log}(n_g)\|))} \\ &= C e^{dC\|\operatorname{Log}(a_g)\|} \bigl(1 + \bigl\|\operatorname{Log}(n_g)\bigr\|\bigr)^{dC}. \end{aligned}$$

Therefore, for d big enough,

$$\begin{aligned} \int_S \frac{1}{\omega_d(s)}\, ds &= \iint_{\mathfrak{a}\ \mathfrak{s}} \frac{1}{\omega_d(\exp X \exp Y)}\, dY\, dX \\ &\leqslant C \iint_{\mathfrak{a}\ \mathfrak{s}} e^{-dC\|X\|} \frac{1}{(1 + \|Y\|)^{dC}}\, dY\, dX < \infty. \qquad \square \end{aligned}$$

Proposition 5.3. *Let T be a compact topological space and let $k: T \times T \to \mathcal{K}(\mathcal{H})$ be a continuous mapping into the space of compact operators on a Hilbert space $\mathcal{H}$. Let μ be a Borel probability measure on T. Then the linear mapping K from $L^2(T, \mathcal{H})$ to $L^2(T, \mathcal{H})$ given by*

$$K\xi(t) := \int_T k(t,u)\xi(u)\, du, \quad t \in T,\ \xi \in L^2(T, \mathcal{H}),$$

is compact too.

Proof. We show that K is the norm-limit of a sequence of operators of finite rank. Let $\varepsilon > 0$. Since T is compact and k is continuous, there exists a finite partition of unity of $T \times T$ consisting of continuous non-negative functions $(\varphi_i)_{i=1}^N$, such that $\|k(t,t') - k(u,u')\|_{\mathrm{op}} < \frac{\varepsilon}{2}$ for every (t,t'), (u,u') contained in the support φ_i. Choose, for $i = 1, \ldots, N$ an element (t_i, t_i') in $\operatorname{supp} \varphi_i$.

Since $k(t_i, t_i')$ is a compact operator, we can find a bounded endomorphism F_i of $\mathcal{H}$ of finite rank, such that $\|k(t_i, t_i') - F_i\|_{\mathrm{op}} < \frac{\varepsilon}{2}$, hence $\|k(t, t') - F_i\|_{\mathrm{op}} < \varepsilon$ for every $(t, t') \in \operatorname{supp} \varphi_i$, $i = 1, \dots, N$. The finite rank operator F_i has the expression $F_i = \sum_{k=1}^{N_i} P_{\eta_{i,k}, \eta'_{i,k}}$, where, for $\eta, \eta' \in \mathcal{H}$, $P_{\eta,\eta'}$ denotes the rank one operator $P_{\eta,\eta'}(\eta'') = \langle \eta'', \eta' \rangle \eta$, $\eta'' \in \mathcal{H}$.

We approximate, by tensors $\psi_i = \sum_{j=1}^{M_i} \varphi_{i,j} \otimes \varphi'_{i,j} \in C(T, \mathbb{R}_+) \otimes C(T, \mathbb{R}_+)$, the continuous functions φ_i uniformly on $T \times T$ up to an error of at most $\frac{\varepsilon}{R}$ for some $R > 0$ to be determined later on. Let K_ε be the finite rank operator

$$K_\varepsilon = \sum_{i=1}^{N} \sum_{j=1}^{M_i} \sum_{k=1}^{N_i} P_{\varphi_{i,j} \otimes \eta_{i,k}, \varphi'_{i,j} \otimes \eta'_{i,k}}.$$

In order to estimate the difference $K - K_\varepsilon$, we let first $K_{\varepsilon,1}$ be the kernel operator with kernel $k_{\varepsilon,1}(s, t) = \sum_{i=1}^{N} \varphi_i(s, t) F_i$. Then, for $\xi \in L^2(T, \mathcal{H})$,

$$\begin{aligned} \|K_{\varepsilon,1}\xi - K\xi\|_2^2 &= \int_T \left\| \sum_{i=1}^{N} \int_T \varphi_i(s, t)\big(k(s, t) - F_i\big)\xi(t)\, dt \right\|^2 ds \\ &\leqslant \int_T \left(\sum_{i=1}^{N} \int_T \varphi_i(s, t)\varepsilon \big\|\xi(t)\big\|\, dt \right)^2 ds = \int_T \left(\int_T \varepsilon \big\|\xi(t)\big\|\, dt \right)^2 ds \leqslant \varepsilon^2 \|\xi\|^2. \end{aligned}$$

Hence $\|K - K_{\varepsilon,1}\|_{\mathrm{op}} \leqslant \varepsilon$. Moreover,

$$\begin{aligned} \big\|(K_{\varepsilon,1} - K_\varepsilon)\xi\big\|^2 &= \int_T \left\| \int_T \sum_{i=1}^{N} \left(\varphi_i(s, t) - \sum_{j=1}^{M_i} \varphi_{i,j}(s)\varphi'_{i,k}(t) \right) F_i \xi(t)\, dt \right\|^2 ds \\ &\leqslant \int_T \left(\int_T \sum_{i=1}^{N} \frac{\varepsilon}{R} \|F_i\|_{\mathrm{op}} \big\|\xi(t)\big\|\, dt \right)^2 ds \leqslant \frac{\varepsilon^2}{R^2} \left(\sum_{i=1}^{N} \|F_i\|_{\mathrm{op}} \right)^2 \|\xi\|^2. \end{aligned}$$

So, if we let $R = \frac{1}{1 + \sum_{i=1}^{N} \|F_i\|_{\mathrm{op}}}$, then

$$\|K - K_\varepsilon\|_{\mathrm{op}} \leqslant \|K - K_{\varepsilon,1}\|_{\mathrm{op}} + \|K_{\varepsilon,1} - K_\varepsilon\|_{\mathrm{op}} \leqslant 2\varepsilon. \qquad \square$$

Let now π be an isometric representation of the group S on a Banach space X and denote by $\rho^p := \operatorname{ind}_S^G \pi$ be the corresponding induced representation of G on $L^p(G, X; \pi)$. Here we follow the notation of Section 3.1.

Let h be in $L^1(G)$, and assume furthermore that $\tilde{h}(g) \in L^1(S)$ for all $g \in G$ and that the mapping $\tilde{h} : G \to L^1(S)$ is continuous. Then the operator $\rho^p(h)$ is a kernel operator, whose kernel $k(t, u)$, $t, u \in G$, is given by

$$k(t, u) = \Delta_G\big(u^{-1}\big)\pi\big({}^u\tilde{h}\big(tu^{-1}\big)\big) \tag{5.4}$$

(in the notations of Proposition 5.1). Indeed, for $\xi \in L^p(G, X; \pi)$, $t \in G$,

$$\begin{aligned}
\big[\rho^p(h)\xi\big](t) &= \int_G h(g)\xi\big(g^{-1}t\big)\,dg = \int_G \Delta_G\big(g^{-1}\big)h\big(tg^{-1}\big)\xi(g)\,dg \\
&= \int_{G/S}\int_S \Delta_G\big(s^{-1}g^{-1}\big)h\big(ts^{-1}g^{-1}\big)\xi(gs)\,ds\,dg \\
&= \int_{G/S}\int_S \Delta_G\big(g^{-1}\big)\Delta_S\big(s^{-1}\big)h\big(ts^{-1}g^{-1}\big)\xi(gs)\,ds\,dg \\
&= \int_{G/S}\int_S \Delta_G\big(g^{-1}\big)h\big(tg^{-1}\big(gsg^{-1}\big)\big)\pi(s)\xi(g)\,ds\,dg \\
&= \int_{G/S} \Delta_G\big(g^{-1}\big)\pi\big({}^g\tilde{h}\big(tg^{-1}\big)\big)\xi(g)\,dg.
\end{aligned}$$

Moreover the kernel k satisfies the following covariance property under S:

$$k(ts, us') = \pi\big(s^{-1}\big)k(t,u)\pi(s'), \quad t,u \in G,\ s,s' \in S. \tag{5.5}$$

Proposition 5.4. *Let G be the semidirect product of a connected compact Lie group K acting on an exponential solvable Lie group S. Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of the normal closed subgroup S of G whose Kirillov orbit $\Omega_\pi = \Omega \subset \mathfrak{s}^*$ is closed. Let $\rho = \mathrm{ind}_S^G\,\pi$. Then the operator $\rho(h_t)$ is compact for every $t > 0$.*

Proof. Let $t > 0$. By the relation (5.2), for $d > 0$, the function h_t is in $L^1(G, \omega_d) \cap L^\infty(G, \omega_d)$. Furthermore, we have that $h_t = h_{t/2} * h_{t/2}$. Hence by Propositions 5.2 and 5.1 the mapping $G \times G \to L^1(S)$, $(s,u) \mapsto {}^u\tilde{h}_t(su^{-1})$, is continuous. Therefore, the operator valued kernel function $k(s,u) := \Delta_G(u^{-1})\pi({}^u\tilde{h}_t(su^{-1}))$ is continuous too. It follows from the preceding discussion that k is just the integral kernel of the operator $\rho(h_t)$. The fact that the Kirillov orbit of $\pi \in \widehat{S}$ is closed in $\mathfrak{s}^*$ implies that, for any $\varphi \in L^1(S)$, the operator $\pi(\varphi) = \int_S f(s)\pi(s)\,ds$ is compact (see [13] and [8]). Hence $k(s,u)$ is compact for every $(s,u) \in G \times G$ and in particular for every $(s,u) \in K \times K$. We apply Proposition 5.3 to the restriction of k to $K \times K$. The related kernel operator on $L^2(K, \mathcal{H})$ is then compact. Now, since π is unitary, the restriction map to K is an isometric isomorphism from $L^2(G, \mathcal{H}; \pi)$ onto $L^2(K, \mathcal{H})$. Thus we see that $\rho(h_t)$ is compact. □

5.2. *Proof of Theorem 1.3*

We now turn to the proof of Theorem 1.3, which follows closely the notation and argumentation in [8]. In the following, we always make the

Assumption. $\ell \in \mathfrak{s}^*$ satisfies Boidol's condition (B) and $\Omega(\ell)|_{\mathfrak{n}}$ is closed.

Since ℓ satisfies (B), the stabilizer $\mathfrak{s}(l)$ is not contained in $\mathfrak{n}$. Let ν be the real character of $\mathfrak{s}$, which has been defined in [8, Section 5], trivial on $\mathfrak{n}$ and different from 0 on $\mathfrak{s}(\ell)$. We denote by $\pi_\ell = \mathrm{ind}_P^S\,\chi_\ell$ the irreducible unitary representation of S associated to ℓ by the Kirillov map. Here

$P = P(\ell)$ denotes a suitable polarizing subgroup for ℓ, and χ_ℓ the character $\chi_\ell(p) := e^{i\ell(\operatorname{Log} p)}$ of P.

For any complex number z in the strip

$$\Sigma := \{\zeta \in \mathbb{C} \mid |\operatorname{Im}\zeta| < 1/2\},$$

let Δ_z be the complex character of S given by

$$\Delta_z(\exp X) := e^{-iz\nu(X)}, \quad X \in \mathfrak{s},$$

and χ_z the unitary character

$$\chi_z(\exp X) := e^{-i\operatorname{Re} z\nu(X)}, \quad X \in \mathfrak{s}.$$

If we define $p(z) \in]1, \infty[$ by the equation

$$\operatorname{Im} z = 1/2 - 1/p(z), \tag{5.6}$$

it is shown in [8] that the representation π_ℓ^z, given by

$$\pi_\ell^z(x) := \Delta_z(x)\pi_\ell(x) = \chi_z(x)\pi_\ell^{\overline{p(z)}}(x), \quad x \in G, \tag{5.7}$$

is an isometric representation on the mixed L^p-space $L^{\overline{p(z)}}(S/P, \ell)$. Here, $\pi_\ell^{\overline{p(z)}}$ denotes the $\overline{p(z)}$-induced representation of S on $L^{\overline{p(z)}}(S/P, \ell)$ defined in [8]. Here $\bar{q}$ is a multi-index of the form $(q, \dots, q, 2, \dots, 2)$.

Observe that, for $\tau \in \mathbb{R}$, we have $p(\tau) = 2$, and $\pi_\ell^\tau = \chi_\tau \otimes \pi_\ell$ is a unitary representation on $L^{\bar{2}}(S/P, \ell)$. Moreover, since the mapping $f \mapsto \overline{\chi}_\tau f$ intertwines the representations $\chi_\tau \otimes \pi_\ell$ and $\pi_{\ell-\tau\nu}$:

$$\pi_\ell^\tau \simeq \pi_{\ell-\tau\nu}. \tag{5.8}$$

We take now, for $z \in \Sigma$, the $p(z)$-induced representation $\rho_\ell^z := \operatorname{ind}_{p(z),S}^G \pi_\ell^z$ of G which acts on the space:

$$L^{\overline{p(z)}}(G/P, \ell) := L^{p(z)}\big(G, L^{\overline{p(z)}}(S/P, \ell); \pi_\ell^z\big).$$

Let us shortly write:

$$L^{\bar{q}} := L^{\bar{q}}(G/P, \ell), \quad 1 \leqslant q < \infty,$$

for the space of ρ_ℓ^z.

We can extend the character Δ_z, $z \in \Sigma$, of S to a function on G by letting

$$\Delta_z(kan) := \Delta_z(an) = e^{-iz\nu(\operatorname{Log}(a))}, \quad k \in K,\ a \in A,\ n \in N.$$

Since ν is trivial on $\mathfrak{n}$ and since, for all $k \in K, a \in A$, $kak^{-1} \in aN$, we have that

$$\Delta_z(kank') = \Delta_z(an), \quad k, k' \in K,\ a \in A,\ n \in N,$$

and in particular Δ_z is a character of G.

Define the operator $T(z),\ z \in \Sigma$, by

$$T(z) := \rho_\ell^z(h_1).$$

Then, by the relations (5.4) and (5.7), for $z \in \Sigma$ and $\xi \in L^{\overline{p(z)}}$ (since Δ_z is K-invariant)

$$\begin{aligned} T(z)\xi(k) &= \int_K \pi_\ell^z\big({}^{k'}\tilde{h}_1\big(kk'^{-1}\big)\big)\xi(k')\,dk' \\ &= \int_K \pi_\ell\big((\Delta_{z|S})^{k'}\tilde{h}_1\big(kk'^{-1}\big)\big)\xi(k')\,dk' \\ &= \int_K \pi_\ell\big({}^{k'}\widetilde{(\Delta_z h_1)}\big(kk'^{-1}\big)\big)\xi(k')\,dk' \\ &= \big[\rho_\ell(\Delta_z h_1)\big]\big(\xi(k)\big). \end{aligned}$$

Hence

$$T(z) = \rho_\ell^z(h_1) = \rho_\ell(\Delta_z h_1), \quad z \in \Sigma. \tag{5.9}$$

By (5.2), for every continuous character χ of G which is trivial on N, the function χh_1 is in $L^1(G)$. Then, it follows from [8, Corollary 5.2 and Proposition 3.1] that the operator $T(z)$ leaves $L^{\overline{q}}$ invariant for every $1 \leqslant q < \infty$, and is bounded on all these spaces. Moreover, by Proposition 5.4, $T(\tau)$ is compact for $\tau \in \mathbb{R}$. From here on we can proceed exactly as in the proof of [8, Theorem 1], provided that we can prove a "Riemann–Lebesgue" type lemma like [8, Theorem 2.2] in our present setting, since $G = K \ltimes S$ is amenable.

We must show that $T(\tau)$ tends to 0 in the operator norm if τ tends to ∞ in $\mathbb{R}$. The condition we have imposed on the coadjoint orbit Ω of ℓ, namely that the restriction of Ω to $\mathfrak{n}$ is closed, tells us that $\lim_{\tau\to\infty} \Omega + \tau\nu = \infty$ in the orbit space, which means that $\lim_{\tau\to\infty} \|\pi_{\ell+\tau\nu}(f)\|_{\mathrm{op}} = 0$ for every $f \in L^1(S)$. Now, by (5.4), the operator $T(\tau) = \rho_\ell^\tau(h_1)$ is a kernel operator whose kernel K_τ has values in the bounded operators on $\mathcal{H}_\ell$. The kernel K_τ is given by

$$K_\tau(k,k') = \int_S \Delta_\tau(s) h_1\big(k^{-1}sk'^{-1}\big)\pi_\ell(s)\,ds = \pi_\ell^\tau\big(h_1(k,k')\big),$$

where $h_1(k,k')$ is the function on S defined by $h_1(k,k')(s) := h_1(ksk'^{-1})$. Hence

$$\lim_{\tau\to\infty} \big\|\pi_\ell^\tau\big(h_1(k,k')\big)\big\|_{\mathrm{op}} = 0$$

for every $k, k' \in K$. Moreover for $k, k' \in K$,

$$\big\|\pi_\ell^\tau\big(h_1(k,k')\big)\big\|_{\mathrm{op}} \leqslant \big\|h_1(k,k')\big\|_1 \leqslant \sup_{k''\in K} \big\|\tilde{h}_1(k'')\big\|_1.$$

We know from Proposition 5.1 that, for every $k'' \in K$,

$$\big\|h_1(k'')\big\|_1 \leqslant \|\omega_d|_K\|_\infty \|h_{1/2}\|_{\omega_d,1}\|h_{1/2}\|_{\omega_d,\infty}\left\|\left(\frac{1}{\omega_d}\right)\Big|_S\right\|_1,$$

which is finite by Proposition 5.2 and relation (5.2) (if d is big enough). Hence, by Lebesgue's dominated convergence theorem, we see that

$$\lim_{\tau\to\infty}\int_K\int_K \big\|\pi_\ell^\tau\big(h_1(k,k')\big)\big\|_{\mathrm{op}}^2\,dk\,dk' = 0.$$

This shows that

$$\lim_{\tau\to\infty}\big\|\rho_\ell^\tau(h_1)\big\|_{\mathrm{op}} = 0.$$

Acknowledgments

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Appendix A. On the spectra of sub-Laplacians of holomorphic L^p-type

In this section, we shall give a proof of Hebisch's observation that (1.1) and (1.2) hold true for sub-Laplacians on connected Lie groups. The key idea is the use of an approximation to the identity by right convolution with smooth functions with compact support. We shall also make use of the following lemma, which is of independent interest and which simplifies at the same time the proof suggested originally to us by Hebisch.

Lemma A.1. *Let $(X, d\mu)$ be a σ-finite measure space, and let $1 \leqslant p < \infty$. Let A be a self-adjoint operator generating a C_0-semigroup $\{T_2(t)\colon t > 0\}$ on $L^2(X)$. Assume further*:

(i) *There exist consistent C_0-semigroups $\{T_q(t)\colon t > 0\}$, $q = p, p'$ (if $q' = \infty$ we only require the weak*-continuity of the semigroup), i.e., $T_{q_1}(t) = T_{q_2}(t)$ on $L^{q_1}(X) \cap L^{q_2}(X)$ for all $q_1, q_2 \in \{p, 2, p'\}$ and every $t > 0$.*

(ii) $$\big\langle T_p(t)f, g\big\rangle = \big\langle f, T_{p'}(t)g\big\rangle, \quad \forall t > 0,\ f \in L^p(X),\ g \in L^{p'}(X)$$
(*which holds automatically if* $1 < p < \infty$).

(iii) *$T_q(t)$ maps real-valued functions to real-valued functions for every q and $t > 0$.*

We denote by A_q the generator of $\{T_q(t)\colon t > 0\}$, and by $\sigma(A_q)$ and $\rho(A_q)$ the spectrum and the resolvent set of A_q. Then

$$\sigma(A_p) = \sigma(A_{p'}). \tag{A.1}$$

By A_{p+q} we denote the operator

$$A_{p+q}(f+g) := A_p f + A_q g, \quad f \in \mathcal{D}(A_p),\ g \in \mathcal{D}(A_q),$$

defined on the sum $\mathcal{D}(A_p)+\mathcal{D}(A_q)$ of the domains of A_p and A_q and taking values in $L^p(X)+L^q(X)$. Notice that A_{p+q} is well defined, since A_p and A_q agree on the intersections of their domains.

Assume that $z\in\rho(A_p)\cap\rho(A_q)$, and that $z-A_{p+q}$ is injective on $\mathcal{D}(A_p)+\mathcal{D}(A_q)$. Then the resolvents $R(z,A_p)$ and $R(z,A_q)$ are consistent, i.e.,

$$R(z,A_p)=R(z,A_q)\quad \text{on } L^p(X)\cap L^q(X). \tag{A.2}$$

Proof. By Proposition 8.1 in [8], we know that (A.1) holds true. Assume next that $h\in L^p(X)\cap L^q(X)$, and put $f:=R(z,A_p)h\in\mathcal{D}(A_p)$, $g:=R(z,A_q)h\in\mathcal{D}(A_q)$. Then

$$(z-A_{p+q})(f-g)=(z-A_p)f-(z-A_q)g=h-h=0,$$

so that by our assumption $f=g$. This proves (A.2). □

Lemma A.2. *Let L be a right-invariant sub-Laplacian on the connected Lie group G, and let $1\leqslant p<\infty$. Then for $z\in\rho(L_p)=\rho(L_{p'})$, the resolvents $R(z,L_p)$ and $R(z,L_{p'})$ are consistent.*

As a consequence, if L is of holomorphic L^p-type, then (1.1) *and* (1.2) *hold true for $A=L$.*

Proof. Fix an approximation to the identity $\{\varphi_\nu\}_\nu$ in $\mathcal{D}(G)$ such that the supports of the φ_ν shrink to the identity element as $\nu\to\infty$, and let us write $q:=p'$. Let $f\in\mathcal{D}(L_p)$, $g\in\mathcal{D}(L_q)$, and assume that $z\in\rho(L_p)\cap\rho(L_q)$ and

$$(z-L_{p+q})(f+g)=0.$$

Then, since convolution from the right commutes with L,

$$(z-L_p)(f*\varphi_\nu)+(z-L_q)(g*\varphi_\nu)=0$$

for every ν. But, observe that $f*\varphi_\nu\in L^p(X)\cap L^\infty(X)\subset L^{p'}(X)$, so that $(z-L_p)(f*\varphi_\nu)=(z-L_q)(f*\varphi_\nu)$. Since $z\in\rho(L_q)$, we thus see that $f*\varphi_\nu+g*\varphi_\nu=0$ a.e., and passing to the limit as $\nu\to\infty$, we obtain $f+g=0$ a.e.

By Lemma A.1, the resolvents $R(z,L_p)$ and $R(z,L_{p'})$ are thus consistent, and we can from here on follow the proof of Proposition 8.1 in [8] in order to conclude the proof of Lemma A.2. □

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