Paley-Wiener theorem(s) for real reductive Lie groups

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Outline

- Introduction
- 2 The Paley-Wiener theorem(s) for G
- 4 Holomorphic families of representations and their successive derivatives
- 4 The generalized Hecke algebra for (G, K)
- 5 The Paley-Wiener theorem(s) for G and some reformulation
- 6 Application

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Paley-Wiener theorem?

Describe the image of the Fourier transform

Let G be a real reductive Lie group of the Harish-Chandra class, e.g. G semisimple, connected, with finite center.

Let $K = G^{\theta}$, θ the associated Cartan involution.

Let $\mathfrak{g} := Lie(G)$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, its Cartan decomposition w.r.t. θ .

Let $\mathfrak a$ be a fixed maximal abelian subspace of $\mathfrak p$, $A=\exp{(\mathfrak a)}$.

For instance, if $G = SL(n, \mathbb{R})$, then one can take K = SO(n), $\mathfrak{a} \simeq \mathbb{R}^{n-1}$.

The classical Euclidean case

Consider the Euclidean space \mathfrak{a} , $\mathfrak{a}_{\mathbb{C}} := \mathfrak{a} \otimes \mathbb{C}$, $\mathfrak{a}_{\mathbb{C}}^*$ dual of $\mathfrak{a}_{\mathbb{C}}$.

Paley-Wiener thm

$$(C_c^{\infty}(\mathfrak{a}))^{\wedge} = PW(\mathfrak{a}).$$

with

$$\begin{split} \mathrm{PW}(\mathfrak{a}) := \{ \phi \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*) : \; \exists_{\mathrm{R} > 0} \forall_{n > 0} \exists_{C_n > 0}, \\ |\phi(\lambda)| \le C_n (1 + |\lambda|)^{-n} e^{\mathrm{R}|\mathrm{Re}\lambda|} \} \end{split}$$

Fourier transform of $f \in C_c^{\infty}(\mathfrak{a})$: $\widehat{f}(\lambda) := \int_{\mathfrak{a}} f(X) \pi_{1,\lambda}(X) dX$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ where $\pi_{1,\lambda}$ 1-dim. rep-n of \mathfrak{a} (as an additive group): $\pi_{1,\lambda}(X) = e^{-\lambda(X)}$

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Paley-Wiener thm

$$(C_c^{\infty}(\mathfrak{a})_{\mathbb{R}})^{\wedge} = PW(\mathfrak{a})_{\mathbb{R}}.$$

with

$$\begin{split} \mathbf{C}_c^{\infty}(\mathfrak{a})_{\mathbf{R}} &:= \{ f \in \mathbf{C}_c^{\infty}(\mathfrak{a}) : \| \mathbf{supp}(f) \| \leq \mathbf{R} \}, \text{ and} \\ \mathbf{PW}(\mathfrak{a})_{\mathbf{R}} &:= \{ \phi \in \mathcal{O}(\mathfrak{a}_{\varepsilon}^*) : \ \forall_{n > 0} \exists_{C_n > 0}, \\ & |\phi(\lambda)| \leq C_n (1 + |\lambda|)^{-n} e^{\mathbf{R}|\mathrm{Re}\lambda|} \} \end{split}$$

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The (operator valued) Fourier transform

Let P be a fixed minimal parabolic subgroup, with P = MAN its Langlands decomposition,

e.g., if $G = SL(n, \mathbb{R})$, P is the upper triangular group with A block-diagonal, N strictly upper triangular.

Let M^{\wedge} be the unitary dual of M.

Definition (Minimal principal series of G)

For $(\xi,\lambda)\in M^{\wedge}\times \mathfrak{a}_{\mathbb{C}}^{*}$, $\overline{\pi}_{\xi,\lambda}$ is the right regular rep-n of G on $C^{\infty}(G\colon \xi\otimes\lambda)$ where:

$$C^{\infty}(G: \xi \otimes \lambda) := \{ \psi \in C^{\infty}(G, V_{\xi}) : \psi(manx) = a^{\lambda + \rho_{P}} \xi(m) \psi(x) \}$$

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Definition (Minimal principal series of G)

For $(\xi,\lambda)\in M^\wedge\times\mathfrak{a}_{\mathbb{C}}^*$, $\pi_{\xi,\lambda}$ is the rep-n of the compact realization of the smooth minimal p-s of G on $C^\infty(K\,:\,\xi)$, given by transport of structure from $\overline{\pi}_{\xi,\lambda}$ under res-n to K:

$$C^{\infty}(G:\xi\otimes\lambda)\stackrel{\simeq}{\longrightarrow} C^{\infty}(K:\xi)$$

Example

Consider $G = SL(2,\mathbb{R})$. Here $\widehat{M} = \{+,-\}$, $N = \{\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\}$ and $\mathfrak{a}_{\mathbb{C}}^* \simeq \mathbb{C}$. Under restriction of functions on G to $\theta(N)$, the rep-n $\pi_{\pm,\lambda}$ is equiv-t to $p_{\pm,\lambda}$

$$p_{\pm,\lambda}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(y) = \begin{cases} |by+d|^{1+\lambda} f\left(\frac{ay+c}{by+d}\right) & \text{if } + \\ \operatorname{sgn}(by+d)|by+d|^{1+\lambda} f\left(\frac{ay+c}{by+d}\right) & \text{if } -. \end{cases}$$

Example

Consider $G = SL(2,\mathbb{R})$. Here $\widehat{M} = \{+,-\}$, $N = \{\begin{pmatrix} 1 & \mathcal{Y} \\ 0 & 1 \end{pmatrix}\}$ and $\mathfrak{a}_{\mathbb{C}}^* \simeq \mathbb{C}$. Under restriction of functions on G to $\theta(N)$, the rep-n $\pi_{\pm,\lambda}$ is equiv-t to $p_{\pm,\lambda}$

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Definition (Fourier transform)

For $f \in C_c^{\infty}(G)$, $\xi \in M^{\wedge}$,

$$\widehat{f}(\xi,\lambda) := \int_C f(x) \pi_{\xi,\lambda}(x) \, dx, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

$$f \mapsto \widehat{f} \text{ maps } C_c^{\infty}(G) \text{ into } \oplus_{\xi \in M^{\wedge}} \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*) \otimes \operatorname{End}(C^{\infty}(K : \xi)).$$

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The Paley-Wiener theorems

Let $C_c^{\infty}(K\backslash G/K)$ denote the subspace of bi-K-invariant elements in $C_c^{\infty}(G)$.

Let W the Weyl group associated to $(\mathfrak{g}, \mathfrak{a})$.

Theorem (Helgason '66, estimates by Gangolli '71)

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$$C_c^{\infty}(K\backslash G/K)^{\wedge} = PW(\mathfrak{a})^W$$

Let $C_c^{\infty}(G/K)_K$ denote the subspace of right K-invariant and left K-finite elements in $C_c^{\infty}(G)$.

Theorem (Helgason '73)

$$(C_c^{\infty}(G/K)_K)^{\wedge} = PW(G/K)$$

with
$$PW(G/K) = {\varphi \in PW(\mathfrak{a}) \otimes L^2(K/M) : \varphi(w\lambda) = A_w(\lambda)\varphi(\lambda)}$$

where $A_w(\lambda) \in End(L^2(K/M)_K)$ normalized standard intertwining operator (with rational coeff-s in λ).

Let $C_c^\infty(G,K)$ denote the space of bi-K-finite elements in $C_c^\infty(G)$. For $\xi\in M^\wedge$, let

$$\mathscr{S}(\xi) := \operatorname{End}(\operatorname{C}^{\infty}(K : \xi)_{K \times K}).$$

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Theorem (Arthur '82, Campoli '80 for real rank one case)

$$C_c^{\infty}(G, K)^{\wedge} = PW_{Arth}(G, K)$$

with $PW_{Arth}(G, K) = \{ \phi \in PW(\mathfrak{a}) \otimes (\bigoplus_{\xi \in M^{\wedge}} \mathscr{S}(\xi)) : \phi \text{ satisfies } A\text{-}C \}.$

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Theorem (Delorme '06)

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Without using the Paley-Wiener theorems, we want to show

$$PW_{Arth}(G,K) = PW_{Del}(G,K)$$

Some notation

For any $\eta \in \mathfrak{a}_{\mathbb{C}}^*$, $\Phi \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*)$, let define

$$\Phi(\lambda;\eta) := \frac{d}{dz} \big(\Phi(\lambda + z\eta) \big)_{|z=0}, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

The map

$$\eta \mapsto [\Phi \mapsto \Phi(\cdot;\eta)]$$

extends uniquely to an algebra hom of $S(\mathfrak{a}_{\mathbb{C}}^*)$.

For any $\eta \in \mathfrak{a}_{\mathbb{C}}^*$ and $\Phi \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*, End(V))$ (V a F-space), let define $\Phi^{(\eta)} \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*, End(V \oplus V))$ by

$$\Phi^{(\eta)}(\lambda) := \left(\begin{smallmatrix} \Phi(\lambda) & \Phi(\lambda;\eta) \\ 0 & \Phi(\lambda) \end{smallmatrix} \right).$$

By iterating the process, one can generalize this definition to any finite sequence in $\mathfrak{a}_{\mathbb{C}}^*$.

Arthur-Campoli relations

An Arthur-Campoli sequence is a finite sequence $(\xi_i, \psi_i, \lambda_i, u_i)_i$ with $\xi_i \in M^{\wedge}$, $\psi \in \mathscr{S}(\xi_i)_{K \times K}^*$, $\lambda_i \in \mathfrak{a}_{\mathbb{C}}^*$ and $u_i \in S(\mathfrak{a}_{\mathbb{C}}^*)$ s.t

$$\sum_i \langle \pi_{\xi_i, \lambda_i; u_i}(x), \psi_i \rangle = 0, \quad x \in G.$$

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$$\sum_{i} \langle \pi_{\xi_i, \lambda_i; u_i}(x), \psi_i \rangle = 0, \quad x \in G.$$

Let

$$\mathscr{F} := \oplus_{\xi \in \mathcal{M}^{\wedge}} \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^{*}) \otimes \mathscr{S}(\xi).$$

Definition

 $\phi \in \mathscr{F}$ satisfies the Arthur-Campoli relations if

$$\sum_i \langle \phi(\xi_i, \lambda_i; u_i), \psi_i \rangle = 0, \quad \text{for any } (\xi_i, \psi_i, \lambda_i, u_i)_i \text{ A-C sequence.}$$

Intertwining relations

Let $\delta=(\xi,\lambda,\eta)\in \mathscr{D}$ with $\xi\in M^{\wedge},\,\lambda\in\mathfrak{a}_{\mathbb{C}}^{*}$ and η a finite sequence in $\mathfrak{a}_{\mathbb{C}}^{*}.$

Define the rep-n π_{δ} of G in $V_{\pi_{\delta}} := C^{\infty}(K : \xi)^{(\eta)}$ by $\pi_{\delta} := \pi_{\xi, \lambda}^{(\eta)}$.

Similarly, define for $\phi\in\mathscr{F},\,\phi_{\delta}\in End(C^{\infty}(K\,;\xi)^{(\eta)})$ by $\phi_{\delta}:=\phi^{(\eta)}(\xi,\lambda).$

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Similarly, define for $\phi \in \mathscr{F}$, $\phi_\delta \in End(C^\infty(K \, ; \, \xi)^{(\eta)})$ by $\phi_\delta := \phi^{(\eta)}(\xi, \lambda)$.

Definition (Intertwining relations)

A function $\phi \in \mathscr{F}$ satisfies Delorme's intertwining relations if

- for every $N \in \mathbb{Z}_+$ and each $\delta \in \mathcal{D}^N$, ϕ_δ preserves all closed inv-nt s-spaces of π_δ ;
- for all $N_1, N_2 \in \mathbb{Z}_+$, all $\delta_1 \in \mathcal{D}^{N_1}$ and $\delta_2 \in \mathcal{D}^{N_2}$, and any two sequences of closed invariant subspaces $U_j \subset V_j$ for π_{δ_j} ,

$$\begin{array}{ccc} V_1/U_1 & & \overline{\phi}_{\delta_1} & V_2/U_2 \\ T & & & \downarrow T \\ V_1/U_1 & & \overline{\phi}_{\delta_2} & V_2/U_2 \end{array}$$

The space of functions $\phi \in \mathscr{F}$ satisfying (a) and (b) is denoted by $\mathscr{F}(\mathscr{D})$.

Some examples of rank one

where "non-classical" conditions occur already.

Example (PW for G = SU(2,1) on 2-dim-l K-types)

Besides growth and symmetric cond-ns, one has extra cond-ns related to holomorphic families of intertwining op-rs between p. s. rep-ns.

Some examples of rank one

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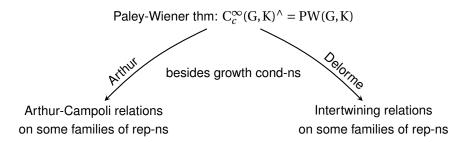
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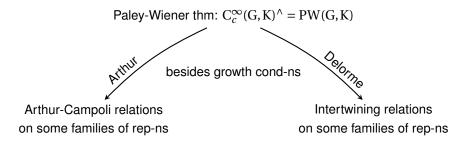
Besides growth and symmetric cond-ns, one has extra cond-ns related to holomorphic families of intertwining op-rs between p. s. rep-ns.

Example (PW for G = SU(2, 1) on 4-dim-l K-types)

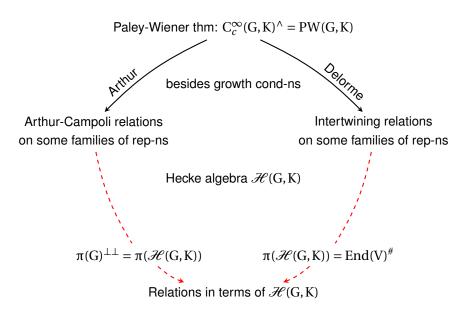
Besides growth and symmetric cond-ns, one has extra cond-ns related to derivatives of holomorphic families of intertwining op-rs between p. s. rep-ns.

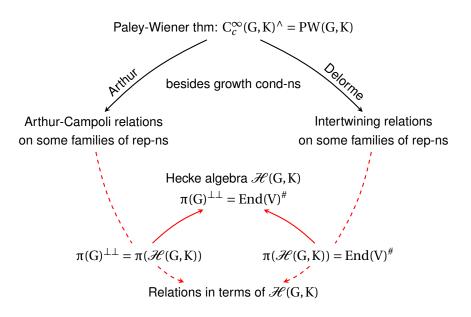
Paley-Wiener thm: $C_c^{\infty}(G, K)^{\wedge} = PW(G, K)$





Hecke algebra $\mathcal{H}(G, K)$





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Holomorphic families of representations

Definition (Holomorphic families of representations)

The pair (π, V) is a holomorphic family of smooth rep-ns of G over $\mathfrak{a}_{\mathbb{C}}^*$ if

- V is a Fréchet space
- $\pi: G \to \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*, End(V))$ is a group homomorphism, continuous, with rest-n to K constant on $\mathfrak{a}_{\mathbb{C}}^*$, and

$$g \mapsto \pi(g)(\lambda) v$$
 is C^{∞} , for all $(\lambda, v) \in \mathfrak{a}_{\mathbb{C}}^* \times V$.

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Example (Minimal principal series of G)

 $\forall \xi \in M^{\wedge}$, $(\pi_{\xi,\lambda})_{\lambda}$ is a holomorphic family of smooth rep-ns of G.

Tensoring with finite dimensional \mathcal{O}_0 -module

Let \mathcal{O}_0 be the algebra of germs at 0 of holomorphic functions defined on a neighborhood of 0 in $\mathfrak{a}_{\mathbb{C}}^*$ and \mathscr{M} its maximal ideal.

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Definition (Successive derivatives)

For any finite dimensional \mathcal{O}_0 -module E, any $\Phi \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*)$, define $\Phi^{(E)} \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*, End(E))$ by:

$$\forall_{\lambda \in \mathfrak{a}_c^*} \Phi^{(E)}(\lambda) := \text{the germ at } 0 \text{ of } \Phi(\cdot + \lambda)$$

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Using canonical identification, we can extend this definition to hol. f-ies of rep-ns (π,V) as follows: $(\pi^{(E)},V^{(E)})$ is the hol. f-y of rep-ns given by:

- $V^{(E)} := E \otimes V$
- $\bullet \ \pi^{(\mathrm{E})}(g,\lambda) := (\pi(g))^{(\mathrm{E})}(\lambda), \quad g \in \mathrm{G}, \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$

Some examples

• For any $u \in S(\mathfrak{a}_{\mathbb{C}}^*)$, one can canonically associate a f-dim \mathcal{O}_0 -module E and $\mu \in End(E)^*$ s.t.

$$\Phi(\lambda; u) = \mu \circ \Phi^{(E)}(\lambda), \quad \Phi \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*), \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

 $\bullet \ \, \text{For any} \,\, \eta \in \mathfrak{a}_{\mathbb{C}}^*, \, \text{let} \,\, \mathscr{I}_{\eta} := \{ \phi \in \mathscr{M} : \, \vartheta(\eta) \phi \in \mathscr{M} \}. \,\, \text{It is a cofinite ideal of} \,\, \mathscr{O}_0. \,\, \\ \text{Then} \,\,$

$$\Phi^{(\eta)} := \left(\begin{smallmatrix} \Phi(\cdot) & \Phi(\cdot; \eta) \\ 0 & \Phi(\cdot) \end{smallmatrix} \right) = \Phi^{(\mathcal{O}_0/\mathscr{I}_\eta)}.$$

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Definition and general facts

Let $\mathscr{H}(G,K)$ be the (convolution) algebra of left K-finite distributions on G with support contained in K.

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Let $\mathcal{H}(G,K)$ be the (convolution) algebra of left K-finite distributions on G with support contained in K.

Some properties

- $\mathcal{H}(G,K)$ is an associative algebra and has an approximation of identity.
- $\bullet \ \mathcal{H}(G,K) \simeq U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} \mathcal{H}(K) \ \text{(as linear spaces), given by:}$

$$\partial(u) * T \leftarrow u \otimes T$$

equivariant for left $U(\mathfrak{g}_{\mathbb{C}})$ -action and right $\mathscr{H}(K)$ -action.

- $\bullet \ \mathcal{H}(G,K) \simeq \mathcal{H}(K) \otimes_{\mathbf{U}(\mathfrak{k}_{\mathbb{C}})} \mathbf{U}(\mathfrak{g}_{\mathbb{C}}).$
- All elements of $\mathcal{H}(G,K)$ are bi-K-finite.

A Harish-Chandra module V is a $(\mathfrak{g},K)\text{-module}$

A Harish-Chandra module V is a (g, K)-module, i.e

- V is a complex vector space endowed with
- compatible $U(\mathfrak{g}_{\mathbb{C}})$ and K-module structures: $\forall v \in V, k \in K, u \in U(\mathfrak{g}_{\mathbb{C}}), X \in \mathfrak{k}$,

$$k \cdot (u \cdot v) = \operatorname{Ad}(k) u \cdot k \cdot v, \ \frac{d}{dt} (\exp(tX) \cdot v)_{|t=0} = X \cdot v),$$

all elements are K-finite

A Harish-Chandra module V is a (g, K)-module, s.t.

- $\bullet~V$ is a finitely generated $U(\mathfrak{g}_{\mathbb{C}})\text{-module}$
- V is admissible (each irreducible K-module occurs only finitely often in V).

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Example (Underlying (g, K)-module of the principal series)

Let $V_{\xi,\lambda}$ be the underlying (\mathfrak{g},K) -module $C^{\infty}(K:\xi)_K$ of $\pi_{\xi,\lambda}$.

Then, for any f. d. \mathscr{O}_0 -modules E, $V_{\xi,\lambda}^{(E)}$ is a Harish-Chandra module.

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Let $V_{\xi,\lambda}$ be the underlying (\mathfrak{g},K) -module $C^\infty(K:\xi)_K$ of $\pi_{\xi,\lambda}$. Then, for any f. d. \mathscr{O}_0 -modules $E,V_{\xi,\lambda}^{(E)}$ is a Harish-Chandra module.

Property

 $HC\text{-}mod \stackrel{\simeq}{\longleftrightarrow} f. g. adm. appr. unital \mathcal{H}(G,K)\text{-}mod$

A bi-commutant theorem

Let π be an admissible Fréchet representation of G with finite composition series and V its Harish-Chandra module.

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Definition

- $\pi(G)^{\perp} := \{ \psi \in \operatorname{End}(V)_{K \times K}^* : \forall_{g \in G}, \langle \psi, \pi(g) \rangle = 0 \}$
- $\pi(G)^{\perp \perp} := \{ \phi \in End(V)_{K \times K} : \forall_{\psi \in \pi(G)^{\perp}}, \langle \phi, \psi \rangle = 0 \}$
- $\operatorname{End}(V)^{\#} := \{ \varphi \in \operatorname{End}(V)_{K \times K} : \forall_n \forall_{U < V^{\oplus n}}, \varphi^{\oplus n}(U) \subset U \}.$

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- $\operatorname{End}(V)^{\#} := \{ \phi \in \operatorname{End}(V)_{K \times K} : \forall_n \forall_{U < V^{\oplus n}}, \phi^{\oplus n}(U) \subset U \}.$

Theorem (van den Ban & S.)

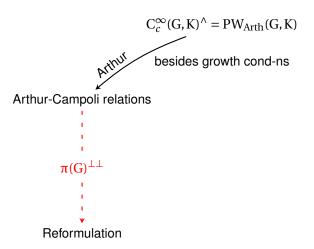
Let π and V as above. Then, as linear subspaces of $End(V)_{K\times K}$,

- End(V)# = $\pi(\mathcal{H}(G, K))$
- $\pi(G)^{\perp \perp} = \pi(\mathcal{H}(G, K)).$

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- Holomorphic families of representations and their successive derivatives
- igg(4) The generalized Hecke algebra for (G,K)
- 5 The Paley-Wiener theorem(s) for G and some reformulation
- Application

The Arthur Paley-Wiener theorem



Theorem (Arthur '82)

The Fourier transform is a topological isomorphism from $C_c^\infty(G,K)$ onto $PW_{Arth}(G,K)$ with

$$PW_{Arth}(G,K) = \{ \phi \in PW(\mathfrak{a}) \otimes (\oplus_{\xi \in M^{\wedge}} \mathcal{S}(\xi)) : \phi \ \textit{satisfies A-C} \}.$$

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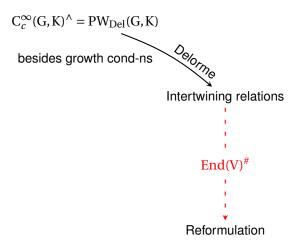
Lemma ("Algebraic" reformulation)

For any $\varphi \in \mathscr{F}$,

φ satisfies A-C

$$\iff \forall_{(E,\xi,\lambda) \in \mathscr{O}_0 - mod_{fd} \times M^{\wedge} \times \mathfrak{a}_{\mathbb{C}}^*}, \phi^{(E)}(\xi,\lambda) \in \pi_{\xi,\lambda}^{(E)}(G)^{\perp \perp}.$$

The Delorme Paley-Wiener theorem



Theorem (Delorme '06)

The Fourier transform is a topological isomorphism from $C_c^\infty(G,K)$ onto $PW_{Del}(G,K)$ with

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Recall that

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Then, as a corollary of

$$\operatorname{End}(V_{\xi,\lambda}^{(E)})^{\#} = \pi_{\xi,\lambda}^{(E)}(\mathscr{H}(G,K)) = \pi_{\xi,\lambda}^{(E)}(G)^{\perp\perp},$$

we get

Corollary

$$PW_{Del}(G, K) = PW_{Arth}(G, K).$$

Comparison with Helgason's theorem

Remark that

$$\mathscr{H}(G,K)^{\wedge} \subset P(\mathfrak{a}_{\mathbb{C}}^{*}) \otimes End(\bigoplus_{\xi \in M^{\wedge}} C^{\infty}(K : \xi)_{K \times K})$$

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Theorem (Reformulation of Arthur's theorem)

Let $\phi \in \bigoplus_{\xi \in M^{\wedge}} PW(\mathfrak{a}) \otimes \mathscr{S}(\xi)$. One has

$$\varphi \in PW_{Arth}(G, K)$$

$$\iff \forall_{(u_i,\xi_i,\lambda_i) \subset S(\mathfrak{a}_{\mathbb{C}}^*) \times \mathcal{M}^{\wedge} \times \mathfrak{a}_{\mathbb{C}}^*}, \exists_{p \in \mathcal{H}(G,K)^{\wedge}}, \forall_i \quad \varphi(\xi_i,\lambda_i;u_i) = p(\xi_i,\lambda_i;u_i).$$

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Lemma

Let $\delta_K \in \mathcal{H}(G,K)$ denote the distribution on G given by $1_K dk$. Then

- $(\operatorname{pr}_1 \mathcal{H}(G, K) * \delta_K)^{\wedge} = \operatorname{P}(\mathfrak{a}_{\mathbb{C}}^*)^W$
- $\bullet \ (\mathcal{H}(G,K)*\delta_K)^\wedge = \{\phi \in \mathrm{P}(\mathfrak{a}_{\mathbb{C}}^*) \otimes \mathrm{L}^2(K/M)_K : \ \forall_{w \in W}, \phi(w\lambda) = \mathrm{A}_w(\lambda)\phi(\lambda)\}$

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Question

- Give another description in terms of intertwining cond-ns.
- What will play the role of the Hecke algebra?