Paley-Wiener theorem(s) for real reductive Lie groups

Sofiane Souaifi [1]

[1] Institut de Recherche Mathématique Avancée
Université de Strasbourg

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joint work with E. P. van den Ban

Outline

1. Introduction
2. The Paley-Wiener theorem(s) for G
3. Holomorphic families of representations and their successive derivatives
4. The generalized Hecke algebra for (G, K)
5. The Paley-Wiener theorem(s) for G and some reformulation
6. Application

Outline

Paley-Wiener theorem?

Describe the image of the Fourier transform

Let G be a real reductive Lie group of the Harish-Chandra class,
e.g. G semisimple, connected, with finite center.

Let $\mathrm{K} = \mathrm{G}^{\theta}$, θ the associated Cartan involution.

Let $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, its Cartan decomposition w.r.t. θ.

Let $\mathfrak{a}$ be a fixed maximal abelian subspace of $\mathfrak{p}$, $\mathrm{A} = \exp(\mathfrak{a})$.

For instance, if $\mathrm{G} = \mathrm{SL}(n, \mathbb{R})$, then one can take $\mathrm{K} = \mathrm{SO}(n)$, $\mathfrak{a} \simeq \mathbb{R}^{n-1}$.

The classical Euclidean case

Consider the Euclidean space $\mathfrak{a}$, $\mathfrak{a}_{\mathbb{C}} := \mathfrak{a} \otimes \mathbb{C}$, $\mathfrak{a}_{\mathbb{C}}^*$ dual of $\mathfrak{a}_{\mathbb{C}}$.

Paley-Wiener thm

$$(C_c^\infty(\mathfrak{a}))^\wedge = PW(\mathfrak{a}).$$

with

$$PW(\mathfrak{a}) := \{\varphi \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*) : \exists_{R>0} \forall_{n>0} \exists_{C_n>0}, \\ |\varphi(\lambda)| \le C_n (1+|\lambda|)^{-n} e^{R|\mathrm{Re}\lambda|}\}$$

Fourier transform of $f \in C_c^\infty(\mathfrak{a})$: $\widehat{f}(\lambda) := \int_{\mathfrak{a}} f(X) \pi_{1,\lambda}(X)\, dX$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$
where $\pi_{1,\lambda}$ 1-dim. rep-n of $\mathfrak{a}$ (as an additive group): $\pi_{1,\lambda}(X) = e^{-\lambda(X)}$

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Paley-Wiener thm

$$(C_c^\infty(\mathfrak{a})_R)^\wedge = PW(\mathfrak{a})_R.$$

with

$$C_c^\infty(\mathfrak{a})_R := \{f \in C_c^\infty(\mathfrak{a}) : \|\mathrm{supp}(f)\| \leq R\}, \text{ and}$$

$$PW(\mathfrak{a})_R := \{\varphi \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*) : \forall_{n>0} \exists_{C_n>0}, \; |\varphi(\lambda)| \leq C_n (1+|\lambda|)^{-n} e^{R|\mathrm{Re}\lambda|}\}$$

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The (operator valued) Fourier transform

Let P be a fixed minimal parabolic subgroup, with $\mathrm{P} = \mathrm{MAN}$ its Langlands decomposition,

e.g., if $\mathrm{G} = \mathrm{SL}(n, \mathbb{R})$, P is the upper triangular group with A block-diagonal, N strictly upper triangular.

Let $\mathrm{M}^\wedge$ be the unitary dual of M.

Definition (Minimal principal series of G)

For $(\xi, \lambda) \in \mathrm{M}^\wedge \times \mathfrak{a}_{\mathbb{C}}^*$, $\overline{\pi}_{\xi,\lambda}$ is the right regular rep-n of G on $\mathrm{C}^\infty(\mathrm{G} : \xi \otimes \lambda)$ where:

$$\mathrm{C}^\infty(\mathrm{G} : \xi \otimes \lambda) := \{\psi \in \mathrm{C}^\infty(\mathrm{G}, \mathrm{V}_\xi) : \psi(manx) = a^{\lambda + \rho_{\mathrm{P}}} \xi(m) \psi(x)\}$$

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For $(\xi, \lambda) \in \mathrm{M}^\wedge \times \mathfrak{a}_{\mathbb{C}}^*$, $\pi_{\xi,\lambda}$ is the rep-n of the compact realization of the smooth minimal p-s of G on $\mathrm{C}^\infty(\mathrm{K}:\xi)$, given by transport of structure from $\overline{\pi}_{\xi,\lambda}$ under res-n to K:

$$\mathrm{C}^\infty(\mathrm{G}:\xi \otimes \lambda) \xrightarrow{\simeq} \mathrm{C}^\infty(\mathrm{K}:\xi)$$

Example

Consider $G = SL(2,\mathbb{R})$. Here $\widehat{M} = \{+,-\}$, $N = \{\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\}$ and $\mathfrak{a}^*_{\mathbb{C}} \simeq \mathbb{C}$. Under restriction of functions on G to $\theta(N)$, the rep-n $\pi_{\pm,\lambda}$ is equiv-t to $p_{\pm,\lambda}$

$$p_{\pm,\lambda}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(y) = \begin{cases} |by+d|^{1+\lambda} f\left(\frac{ay+c}{by+d}\right) & \text{if } + \\ \mathrm{sgn}(by+d)|by+d|^{1+\lambda} f\left(\frac{ay+c}{by+d}\right) & \text{if } -. \end{cases}$$

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Definition (Fourier transform)

For $f \in C_c^\infty(G)$, $\xi \in M^\wedge$,

$$\widehat{f}(\xi,\lambda) := \int_G f(x)\pi_{\xi,\lambda}(x)\,dx, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}.$$

$f \mapsto \widehat{f}$ maps $C_c^\infty(G)$ into $\oplus_{\xi\in M^\wedge}\mathscr{O}(\mathfrak{a}^*_{\mathbb{C}}) \otimes \operatorname{End}(C^\infty(K:\xi))$.

The Paley-Wiener theorems

Let $C_c^\infty(K\backslash G/K)$ denote the subspace of bi-K-invariant elements in $C_c^\infty(G)$.

Let W the Weyl group associated to $(\mathfrak{g}, \mathfrak{a})$.

Theorem (Helgason '66, estimates by Gangolli '71)

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Let $C_c^\infty(G/K)_K$ denote the subspace of right K-invariant and left K-finite elements in $C_c^\infty(G)$.

Theorem (Helgason '73)

$$(C_c^\infty(G/K)_K)^\wedge = PW(G/K)$$

$$\textit{with} \quad PW(G/K) = \{\varphi \in PW(\mathfrak{a}) \otimes L^2(K/M) : \varphi(w\lambda) = A_w(\lambda)\varphi(\lambda)\}$$

where $A_w(\lambda) \in \mathrm{End}(L^2(K/M)_K)$ normalized standard intertwining operator (with rational coeff-s in λ).

Let $C_c^\infty(G, K)$ denote the space of bi-K-finite elements in $C_c^\infty(G)$.
For $\xi \in M^\wedge$, let

$$\mathscr{S}(\xi) := \mathrm{End}(C^\infty(K:\xi)_{K\times K}).$$

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Theorem (Arthur '82, Campoli '80 for real rank one case)

$$C_c^\infty(G,K)^\wedge = PW_{Arth}(G,K)$$

$$\textit{with} \quad PW_{Arth}(G,K) = \{\varphi \in PW(\mathfrak{a}) \otimes (\oplus_{\xi\in M^\wedge}\mathscr{S}(\xi)) : \varphi \textit{ satisfies } A\text{-}C\}.$$

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Without using the Paley-Wiener theorems, we want to show

$$PW_{\mathrm{Arth}}(G,K) = PW_{\mathrm{Del}}(G,K)$$

Some notation

For any $\eta \in \mathfrak{a}_{\mathbb{C}}^*, \Phi \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*)$, let define

$$\Phi(\lambda;\eta) := \frac{d}{dz}\big(\Phi(\lambda + z\eta)\big)_{|z=0}, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

The map

$$\eta \mapsto [\Phi \mapsto \Phi(\cdot;\eta)]$$

extends uniquely to an algebra hom of $S(\mathfrak{a}_{\mathbb{C}}^*)$.

For any $\eta \in \mathfrak{a}_{\mathbb{C}}^*$ and $\Phi \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}(V))$ (V a F-space), let define $\Phi^{(\eta)} \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}(V \oplus V))$ by

$$\Phi^{(\eta)}(\lambda) := \begin{pmatrix} \Phi(\lambda) & \Phi(\lambda;\eta) \\ 0 & \Phi(\lambda) \end{pmatrix}.$$

By iterating the process, one can generalize this definition to any finite sequence in $\mathfrak{a}_{\mathbb{C}}^*$.

Arthur-Campoli relations

An Arthur-Campoli sequence is a finite sequence $(\xi_i, \psi_i, \lambda_i, u_i)_i$ with $\xi_i \in \mathrm{M}^\wedge$, $\psi \in \mathscr{S}(\xi_i)^*_{\mathrm{K}\times\mathrm{K}}$, $\lambda_i \in \mathfrak{a}^*_{\mathbb{C}}$ and $u_i \in \mathrm{S}(\mathfrak{a}^*_{\mathbb{C}})$ s.t

$$\sum_i \langle \pi_{\xi_i,\lambda_i;u_i}(x), \psi_i \rangle = 0, \quad x \in \mathrm{G}.$$

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$$\sum_i \langle \pi_{\xi_i,\lambda_i;u_i}(x), \psi_i \rangle = 0, \quad x \in G.$$

Let

$$\mathscr{F} := \oplus_{\xi \in M^\wedge} \mathscr{O}(\mathfrak{a}^*_{\mathbb{C}}) \otimes \mathscr{S}(\xi).$$

Definition

$\varphi \in \mathscr{F}$ satisfies the Arthur-Campoli relations if

$$\sum_i \langle \varphi(\xi_i, \lambda_i; u_i), \psi_i \rangle = 0, \quad \text{for any } (\xi_i, \psi_i, \lambda_i, u_i)_i \text{ A-C sequence.}$$

Intertwining relations

Let $\delta = (\xi, \lambda, \eta) \in \mathscr{D}$ with $\xi \in M^\wedge$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and η a finite sequence in $\mathfrak{a}_{\mathbb{C}}^*$.
Define the rep-n π_δ of G in $V_{\pi_\delta} := C^\infty(K:\xi)^{(\eta)}$ by $\pi_\delta := \pi_{\xi,\lambda}^{(\eta)}$.
Similarly, define for $\varphi \in \mathscr{F}$, $\varphi_\delta \in \mathrm{End}(C^\infty(K:\xi)^{(\eta)})$ by $\varphi_\delta := \varphi^{(\eta)}(\xi, \lambda)$.

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Definition (Intertwining relations)

A function $\varphi \in \mathscr{F}$ satisfies Delorme's intertwining relations if

- for every $N \in \mathbb{Z}_+$ and each $\delta \in \mathscr{D}^N$, φ_δ preserves all closed inv-nt s-spaces of π_δ;
- for all $N_1, N_2 \in \mathbb{Z}_+$, all $\delta_1 \in \mathscr{D}^{N_1}$ and $\delta_2 \in \mathscr{D}^{N_2}$, and any two sequences of closed invariant subspaces $U_j \subset V_j$ for π_{δ_j},

$$\begin{tikzcd} V_1/U_1 \arrow[r, "\overline{\varphi}_{\delta_1}"] \arrow[d, "T"'] & V_2/U_2 \arrow[d, "T"] \\ V_1/U_1 \arrow[r, "\overline{\varphi}_{\delta_2}"'] & V_2/U_2 \end{tikzcd}$$

The space of functions $\varphi \in \mathscr{F}$ satisfying (a) and (b) is denoted by $\mathscr{F}(\mathscr{D})$.

Some examples of rank one

where “non-classical” conditions occur already.

Example (PW for $G = SU(2,1)$ on 2-dim-l K-types)

Besides growth and symmetric cond-ns, one has extra cond-ns related to holomorphic families of intertwining op-rs between p. s. rep-ns.

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Example (PW for $G = SU(2,1)$ on 4-dim-l K-types)

Besides growth and symmetric cond-ns, one has extra cond-ns related to derivatives of holomorphic families of intertwining op-rs between p. s. rep-ns.

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$\pi(G)^{\perp\perp} = \pi(\mathscr{H}(G,K))$

$\pi(\mathscr{H}(G,K)) = \mathrm{End}(V)^\#$

Relations in terms of $\mathscr{H}(G,K)$

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Holomorphic families of representations

Definition (Holomorphic families of representations)

The pair (π, V) is a holomorphic family of smooth rep-ns of G over $\mathfrak{a}_{\mathbb{C}}^*$ if

- V is a Fréchet space
- $\pi : \mathrm{G} \to \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}(\mathrm{V}))$ is a group homomorphism, continuous, with rest-n to K constant on $\mathfrak{a}_{\mathbb{C}}^*$, and

$$g \mapsto \pi(g)(\lambda)v \text{ is } \mathrm{C}^\infty, \text{ for all } (\lambda, v) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathrm{V}.$$

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Example (Minimal principal series of G)

$\forall \xi \in M^\wedge$, $(\pi_{\xi,\lambda})_\lambda$ is a holomorphic family of smooth rep-ns of G.

Tensoring with finite dimensional $\mathscr{O}_0$-module

Let $\mathscr{O}_0$ be the algebra of germs at 0 of holomorphic functions defined on a neighborhood of 0 in $\mathfrak{a}_{\mathbb{C}}^*$ and $\mathscr{M}$ its maximal ideal.

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Definition (Successive derivatives)

For any finite dimensional $\mathscr{O}_0$-module E, any $\Phi \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*)$, define $\Phi^{(\mathrm{E})} \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}(\mathrm{E}))$ by:

$$\forall_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} \quad \Phi^{(\mathrm{E})}(\lambda) := \text{ the germ at } 0 \text{ of } \Phi(\cdot + \lambda) \quad \circlearrowright \quad \mathrm{E}$$

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Using canonical identification, we can extend this definition to hol. f-ies of rep-ns (π, V) as follows: $(\pi^{(\mathrm{E})}, \mathrm{V}^{(\mathrm{E})})$ is the hol. f-y of rep-ns given by:

- $\mathrm{V}^{(\mathrm{E})} := \mathrm{E} \otimes \mathrm{V}$
- $\pi^{(\mathrm{E})}(g, \lambda) := (\pi(g))^{(\mathrm{E})}(\lambda), \quad g \in \mathrm{G}, \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$

Some examples

- For any $u \in S(\mathfrak{a}_{\mathbb{C}}^*)$, one can canonically associate a f-dim $\mathscr{O}_0$-module E and $\mu \in \mathrm{End}(E)^*$ s.t.

$$\Phi(\lambda; u) = \mu \circ \Phi^{(E)}(\lambda), \quad \Phi \in \mathscr{O}(\mathfrak{a}_{\mathbb{C}}^*), \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

- For any $\eta \in \mathfrak{a}_{\mathbb{C}}^*$, let $\mathscr{I}_\eta := \{\varphi \in \mathscr{M} : \partial(\eta)\varphi \in \mathscr{M}\}$. It is a cofinite ideal of $\mathscr{O}_0$. Then

$$\Phi^{(\eta)} := \begin{pmatrix} \Phi(\cdot) & \Phi(\cdot;\eta) \\ 0 & \Phi(\cdot) \end{pmatrix} = \Phi^{(\mathscr{O}_0/\mathscr{I}_\eta)}.$$

Outline

Definition and general facts

Let $\mathscr{H}(\mathrm{G},\mathrm{K})$ be the (convolution) algebra of left K-finite distributions on G with support contained in K.

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Let $\mathscr{H}(\mathrm{G},\mathrm{K})$ be the (convolution) algebra of left K-finite distributions on G with support contained in K.

Some properties

- $\mathscr{H}(\mathrm{G},\mathrm{K})$ is an associative algebra and has an approximation of identity.
- $\mathscr{H}(\mathrm{G},\mathrm{K}) \simeq \mathrm{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathrm{U}(\mathfrak{k}_{\mathbb{C}})} \mathscr{H}(\mathrm{K})$ (as linear spaces), given by:

$$\partial(u) * \mathrm{T} \leftarrow u \otimes \mathrm{T}$$

  equivariant for left $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$-action and right $\mathscr{H}(\mathrm{K})$-action.
- $\mathscr{H}(\mathrm{G},\mathrm{K}) \simeq \mathscr{H}(\mathrm{K}) \otimes_{\mathrm{U}(\mathfrak{k}_{\mathbb{C}})} \mathrm{U}(\mathfrak{g}_{\mathbb{C}})$.
- All elements of $\mathscr{H}(\mathrm{G},\mathrm{K})$ are bi-K-finite.

$\mathscr{H}(G,K)$-modules and Harish-Chandra modules

A Harish-Chandra module V is a $(\mathfrak{g}, K)$-module

$\mathscr{H}(\mathrm{G},\mathrm{K})$-modules and Harish-Chandra modules

A Harish-Chandra module V is a $(\mathfrak{g},\mathrm{K})$-module, i.e

- V is a complex vector space endowed with
- compatible $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$ and K-module structures: $\forall\, v \in \mathrm{V}, k \in \mathrm{K}, u \in \mathrm{U}(\mathfrak{g}_{\mathbb{C}}), \mathrm{X} \in \mathfrak{k}$,
$$k \cdot (u \cdot v) = \mathrm{Ad}(k)u \cdot k \cdot v,\ \frac{d}{dt}\big(\exp(t\mathrm{X}) \cdot v\big)_{|t=0} = \mathrm{X} \cdot v),$$
- all elements are K-finite

$\mathscr{H}(G,K)$-modules and Harish-Chandra modules

A Harish-Chandra module V is a $(\mathfrak{g},K)$-module, s.t.

- V is a finitely generated $U(\mathfrak{g}_{\mathbb{C}})$-module
- V is admissible (each irreducible K-module occurs only finitely often in V).

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Example (Underlying $(\mathfrak{g},K)$-module of the principal series)

Let $V_{\xi,\lambda}$ be the underlying $(\mathfrak{g},K)$-module $C^{\infty}(K:\xi)_K$ of $\pi_{\xi,\lambda}$.
Then, for any f. d. $\mathscr{O}_0$-modules E, $V_{\xi,\lambda}^{(E)}$ is a Harish-Chandra module.

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Property

$$\textit{HC-mod} \xleftrightarrow{\ \simeq\ } \textit{f. g. adm. appr. unital } \mathscr{H}(\mathrm{G},\mathrm{K})\textit{-mod}$$

A bi-commutant theorem

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- $\pi(G)^{\perp} := \{\psi \in \mathrm{End}(V)^*_{K\times K} : \forall_{g\in G}, \langle \psi, \pi(g)\rangle = 0\}$
- $\pi(G)^{\perp\perp} := \{\varphi \in \mathrm{End}(V)_{K\times K} : \forall_{\psi\in\pi(G)^{\perp}}, \langle \varphi, \psi\rangle = 0\}$
- $\mathrm{End}(V)^{\#} := \{\varphi \in \mathrm{End}(V)_{K\times K} : \forall_n \forall_{U<V^{\oplus n}}, \varphi^{\oplus n}(U) \subset U\}$.

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Theorem (van den Ban & S.)

Let π *and* V *as above. Then, as linear subspaces of* $\mathrm{End}(V)_{K\times K}$,

- $\mathrm{End}(V)^{\#} = \pi(\mathscr{H}(G,K))$
- $\pi(G)^{\perp\perp} = \pi(\mathscr{H}(G,K))$.

Outline

The Arthur Paley-Wiener theorem

$$C_c^\infty(G,K)^\wedge = PW_{\mathrm{Arth}}(G,K)$$

Arthur

besides growth cond-ns

Arthur-Campoli relations

$\pi(G)^{\perp\perp}$

Reformulation

Theorem (Arthur '82)

The Fourier transform is a topological isomorphism from $C_c^\infty(G,K)$ *onto* $PW_{Arth}(G,K)$ *with*

$$PW_{Arth}(G,K) = \{\varphi \in PW(\mathfrak{a}) \otimes (\oplus_{\xi \in M^\wedge} \mathscr{S}(\xi)) : \varphi \textit{ satisfies } A\text{-}C\}.$$

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Lemma ("Algebraic" reformulation)

For any $\varphi \in \mathscr{F}$,

$$\varphi \textit{ satisfies } A\text{-}C$$

$$\iff \forall_{(\mathrm{E},\xi,\lambda)\in\mathscr{O}_0-mod_{fd}\times \mathrm{M}^\wedge\times\mathfrak{a}_{\mathbb{C}}^*}, \varphi^{(\mathrm{E})}(\xi,\lambda) \in \pi_{\xi,\lambda}^{(\mathrm{E})}(\mathrm{G})^{\perp\perp}.$$

The Delorme Paley-Wiener theorem

$$C_c^\infty(G,K)^\wedge = PW_{\mathrm{Del}}(G,K)$$

besides growth cond-ns

$$\xrightarrow{\text{Delorme}}$$

Intertwining relations

$$\color{red}{\mathrm{End}(V)^\#}$$

Reformulation

Theorem (Delorme '06)

The Fourier transform is a topological isomorphism from $\mathrm{C}_c^\infty(\mathrm{G},\mathrm{K})$ *onto* $\mathrm{PW}_{Del}(\mathrm{G},\mathrm{K})$ *with*

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Outline

Application

Recall that

$$\begin{aligned} &\varphi \text{ satisfies A-C} \\ \iff\ &\forall_{(E,\xi,\lambda)\in\mathscr{O}_0-\mathrm{mod}_{fd}\times M^\wedge\times\mathfrak{a}_{\mathbb{C}}^*}, \varphi^{(E)}(\xi,\lambda)\in\pi_{\xi,\lambda}^{(E)}(G)^{\perp\perp}. \end{aligned}$$

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Then, as a corollary of

$$\mathrm{End}(V_{\xi,\lambda}^{(E)})^{\#} = \pi_{\xi,\lambda}^{(E)}(\mathscr{H}(G,K)) = \pi_{\xi,\lambda}^{(E)}(G)^{\perp\perp},$$

we get

Corollary

$$PW_{Del}(G,K) = PW_{Arth}(G,K).$$

Comparison with Helgason's theorem

Remark that

$$\mathscr{H}(G, K)^\wedge \subset P(\mathfrak{a}^*_\mathbb{C}) \otimes \mathrm{End}(\oplus_{\xi \in M^\wedge} C^\infty(K:\xi)_{K\times K})$$

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Theorem (Reformulation of Arthur's theorem)

Let $\varphi \in \oplus_{\xi\in M^{\wedge}} PW(\mathfrak{a}) \otimes \mathscr{S}(\xi)$. *One has*

$$\varphi \in PW_{Arth}(G,K)$$

$$\iff \forall_{(u_i,\xi_i,\lambda_i)\subset S(\mathfrak{a}_{\mathbb{C}}^{*})\times M^{\wedge}\times\mathfrak{a}_{\mathbb{C}}^{*}}, \exists_{p\in\mathscr{H}(G,K)^{\wedge}}, \forall_i \quad \varphi(\xi_i,\lambda_i;u_i) = p(\xi_i,\lambda_i;u_i).$$

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Lemma

Let $\delta_{\mathrm{K}} \in \mathcal{H}(\mathrm{G},\mathrm{K})$ *denote the distribution on* G *given by* $1_{\mathrm{K}}\,dk$. *Then*

- $(\mathrm{pr}_1\mathcal{H}(\mathrm{G},\mathrm{K}) * \delta_{\mathrm{K}})^\wedge = \mathrm{P}(\mathfrak{a}_{\mathbb{C}}^*)^{\mathrm{W}}$
- $(\mathcal{H}(\mathrm{G},\mathrm{K}) * \delta_{\mathrm{K}})^\wedge = \{\varphi \in \mathrm{P}(\mathfrak{a}_{\mathbb{C}}^*) \otimes \mathrm{L}^2(\mathrm{K}/\mathrm{M})_{\mathrm{K}} : \forall_{w\in\mathrm{W}}, \varphi(w\lambda) = \mathrm{A}_w(\lambda)\varphi(\lambda)\}$

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Question

- Give another description in terms of intertwining cond-ns.
- What will play the role of the Hecke algebra?