The constant term of tempered functions on a real spherical space

Sofiane Souaifi (joint work with P. Delorme and B. Krötz)

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1 Motivation of this work: Plancherel formula for the real spherical spaces

Try to get the Plancherel formula for real spherical spaces following the steps of the work of Sakellaridis-Venkatesh (Bernstein maps). For this, one needs to get a theory of the constant term. We are interested in some class of functions (namely tempered) on a real spherical space $Z = G/H$ which are eigen under the action of the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$.
To such a function, we want to associate functions of the same nature on the boundary degenerations Z_I of Z. Historically,

- Harish-Chandra did it for $Z = H \times H/\mathrm{Diag}(H)$
- Carmona did it for Z a symmetric space.

To obtain such a theory, the ideas we use are basically the same, modulo some not only technical difficulties.

2 Real spherical spaces

1. $G = \underline{G}(\mathbb{R})$ where $\underline{G}$ is a connected reductive algebraic group defined over $\mathbb{R}$
   H closed connected subgroup of G
   $Z = G/H$
2. **Definition (Krötz-Sayag-Schlichtkrull, Brion for the complex case).** Z real spherical if:

$$\exists P_{min} \text{ minimal parabolic subgroup of } G \text{ s.t. } P_{min}H \text{ open in } G.$$

3. **Examples.**

(a) Symmetric spaces: $H = (G^\sigma)_0$ with σ rational involution of G, P_{min} minimal $\sigma\theta$-stable parabolic subgroup, with θ Cartan involution which commutes with σ
in particuliar $H \times H/\mathrm{Diag}(H) \simeq H$, $P_{min} = P \times \tau(P)$, P minimal parabolic subgroup of H, τ Cartan involution of H.

(b) Non-symmetric spaces:

- triple spaces: $H \times H \times H/\mathrm{Diag}(H)$
  P_{min} product of 3 minimal parabolics of H pairwise different.
- $Z = G/N$ with N unipotent radical of a parabolic of G.
- others: see classification by Knop-Krötz-Schlichtkrull

3 Some properties

3.1 Local structure Theorem (Knop-Krötz-Schlichtkrull)

$\exists! Q \subset G$ parabolic sg with $Q = LU$ Levi decomposition s.t.:

$$\begin{aligned} &P_{min} \subseteq Q \\ &P_{min}H = QH \\ &L_n \subseteq Q \cap H \subseteq L, \end{aligned}$$

with L_n product of non-abelian non-compact factors of L. One says that Q is Z-adapted to P_{min} (terminology of KKS).
Then

$$\mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{l} \cap \mathfrak{h})^c \oplus \mathfrak{u},$$

with $\mathfrak{l} = \mathrm{Lie}\, L$, $\mathfrak{u} = \mathrm{Lie}\, U$ and $(\mathfrak{l} \cap \mathfrak{h})^c$ is a supplementary of $\mathfrak{l} \cap \mathfrak{h}$ in $\mathfrak{l}$.
Let p denote the projection of $\mathfrak{g}$ on $(\mathfrak{l} \cap \mathfrak{h})^c \oplus \mathfrak{u}$ parallel to $\mathfrak{h}$. Hence

$$X - p(X) \in \mathfrak{h}, \quad X \in \mathfrak{g}.$$

3.2 Polar decomposition

1. One fixes

   - A_L maximal vector subgroup of the center of L
   - A_{min} maximal vector subgroup of L containing A_L
   - $L = K_L A_{min} N_L$ Iwasawa decomposition
   - K maximal compact subgroup of G s.t. $K \supset K_L$
     One has $G = K A_{min} N_{min}$ Iwasawa decomposition

2. Set

- A_H analytic subgroup of G s.t. $\mathfrak{a}_H = \mathrm{Lie}\, A_H = \mathfrak{a}_L \cap \mathfrak{h}$
- $A_Z = A_L / A_H$
- $\Sigma_{\mathfrak{u}}$: roots of $\mathfrak{a}_{min} = \mathrm{Lie}\, A_{min}$ in $\mathfrak{u} = \mathrm{Lie}\, U$.
- S : spherical roots, i.e. irreducible elements (which cannot be expressed as a sum of two) of the (additive with zero) monoid of $\mathbb{N}_0 \Sigma_{\mathfrak{u}}$ generated by:

$$\alpha + \beta, \quad \alpha \in \Sigma_{\mathfrak{u}}, \beta \in \Sigma_{\mathfrak{u}} \cup \{0\}, \text{ s.t. } \exists X_{-\alpha} \in \mathfrak{g}^{-\alpha} \text{ with } X_{\alpha,\beta} \neq 0,$$

  where $X_{\alpha,0} \in (\mathfrak{l} \cap \mathfrak{h})^c$ and $X_{\alpha,\beta} \in \mathfrak{g}^{\beta} \subset \mathfrak{u}$ such that:

$$p(X_{-\alpha}) = X_{\alpha,0} + \sum_{\beta \in \Sigma_{\mathfrak{u}}} X_{\alpha,\beta}.$$

  Actually $S \subset \mathfrak{a}_Z^*$.
- **Example.** Symmetric case. P_{min} minimal $\sigma\theta$ stable parabolic.
  For any $\alpha \in \Sigma_{P_{min}}$ and $X_{-\alpha} \in \mathfrak{g}^{-\alpha}$,

$$p(X_{-\alpha}) = \theta(X_{-\alpha}) \in \mathfrak{g}^{\alpha}.$$

  Hence S: irreducible elements the monoid generated by $\{2\alpha : \alpha \in \Sigma_Q\}$.

3. **Polar decomposition (Knop-Krötz-Sayag-Schlichtkrull).**
   Let $A_Z^- = \{a \in A_Z : a^{\alpha} \leqslant 1, \alpha \in S\}$ (compression cone).
   Then

   $\exists \mathcal{F}, \mathcal{W} \subset G$ finite s.t.:

$$Z = \mathcal{F} K A_Z^- \mathcal{W} \cdot H,$$

4 Temperedness

1. $Z(\mathfrak{g})$ center of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

2. **Z-tempered continuous linear forms.**
   Assume that Z is unimodular. In particular, ρ_Q (half sum of the roots of $\mathfrak{a}_L = \mathrm{Lie}\, A_L$ in $\mathfrak{u}$) defines a linear form on $\mathfrak{a}_Z$.
   Let (π, V) be a Harish-Chandra G-representation = smooth globalization of moderate growth of a Harish-Chandra $(\mathfrak{g}, K)$-module. In particular of finite length, which implies vectors are annihilated by a finite codimensional ideal of $Z(\mathfrak{g})$.
   Let $\eta \in V^*$ be continuous, H-invariant, is Z-tempered if the associated matrix coefficients $m_{\eta,v}$, $v \in V$, and their derivatives satisfy a temperedness inequality according to the polar decomposition. Namely

$$\exists N \in \mathbb{N}, \forall v \in V, \forall u \in U(\mathfrak{g}), \quad \sup_{\omega \in \mathcal{F}K, a \in A_Z^-} a^{-\rho_Q}(1 + \|\log a\|)^{-N} |(L_u m_{\eta,v})(\omega a H)| < \infty.$$

3. $\mathcal{A}_{temp}(Z) = \{m_{\eta,v} : \eta$ et v as above$\} \subset C^\infty(Z)$

5 The constant term

5.1 Boundary degenerations (Knop-Krötz-Schlichtkrull)

1. $I \subset S$, $\mathfrak{a}_I = \{X \in \mathfrak{a}_Z : \alpha(X) = 0, \alpha \in I\}$, $\mathfrak{a}_I^{--} = \{X \in \mathfrak{a}_I : \alpha(X) < 0, \alpha \in S \backslash I\}$ e.g. $\mathfrak{a}_\varnothing = \mathfrak{a}_Z$, $\mathfrak{a}_\varnothing^{--} = \mathfrak{a}_Z^{--}$, $\mathfrak{a}_S = \mathfrak{a}_S^{--} = \bigcap_{\alpha \in S} \operatorname{Ker} \alpha$

2. One fixes $X_I \in \mathfrak{a}_I^{--}$. One sets $\mathfrak{h}_I = \lim_{t\to\infty} e^{t\operatorname{ad}(X_I)}\mathfrak{h}$ in the Grassmanian of $\mathfrak{g}$, e.g. $\mathfrak{h}_S = \mathfrak{h}$, $\mathfrak{h}_\varnothing = \mathfrak{l} \cap \mathfrak{h} \oplus \mathfrak{u}^-$.

3. H_I analytic subgroup of G s.t. $\operatorname{Lie} H_I = \mathfrak{h}_I$, $Z_I = G/H_I$ real spherical.

4. **Properties.**

   (a) Local structure Theorem: Q is Z_I-adapted to P_{min}.
   Polar decomposition of Z_I (KKS):
   $$Z_I = \mathcal{F}_I K A_{Z_I}^- \mathcal{W}_I \cdot H_I, \tag{5.1}$$
   with $\mathcal{F}_I$, $\mathcal{W}_I$ finite subsets of G and $A_{Z_I}^- = \{a \in A_Z : a^\alpha \leqslant 1, \alpha \in I\}$.
   E.g. when $I = \varnothing$, $\mathcal{W}_I = \{1\}$. Indeed $A_{Z_\varnothing}^- = A_Z$ and $G = KA_{min}N_{min}$ imply that $Z_\varnothing = KA_Z \cdot H_\varnothing$ as $A_{min}N_{min} \subset A_Z H_\varnothing$.
   Let us remark that there is a complicated (not very explicit) relation between $\mathcal{W}$ and $\mathcal{W}_I$.

   (b) **Hypothesis: Z wave-front**
   $$Z \text{ wave-front if } \mathfrak{a}_Z^- = (\mathfrak{a}_{min}^- + \mathfrak{a}_H)/\mathfrak{a}_H$$
   where Σ^+ : roots of $\mathfrak{a}_{min}$ in $\mathfrak{n}_{min} = \operatorname{Lie} N_{min}$, $\Sigma_\mathfrak{u} \subset \Sigma^+$
   $\mathfrak{a}_{min}^- = \{X \in \mathfrak{a}_{min} : \alpha(X) \leqslant 0, \alpha$ simple roots in $\Sigma^+\}$,
   Examples: symmetric spaces, some triple spaces.
   But G/N is not wave-front.
   Property (KKS). One has

   Z wave-front implies that $\exists Q_I \supset Q$ parabolic of G s.t. Q_I^- interlaces H_I, i.e.

   $$Q_I \text{ has a Levi decomp. } Q_I = L_I U_I$$
   $$\text{s.t. } (L_I \cap H)_0 U_I^- \subset H_I \subset Q_I^-.$$

   e.g. one can take $Q_\varnothing = Q$, $L_\varnothing = L$.

5. **Example.** If Z symmetric, $Q_I = L_I U_I$ $\sigma\theta$-stable parabolic subgroup containing P_{min}. Then $Z_I = G/(L_I \cap H)_0 U_I^-$.

5.2 Main Theorem

To simplify the exposition, I will assume here:

$$\mathcal{W} = \{1\}.$$

But getting the non-trivial $\mathcal{W}$ case involves difficult techniques which actually occupies a big part of the paper.

Theorem (Delorme-S.). *For any $f \in \mathcal{A}_{temp}(Z)$, there exists a unique $f_I \in C^\infty(Z_I)$ s.t.*
$\forall g \in G$, $X \in \mathfrak{a}_I^{--}$:

(i) $\lim_{T \to +\infty} e^{-T\rho_Q(X)} \left(f(g\exp(TX)) - f_I(g\exp(TX))\right) = 0.$

(ii) $T \mapsto e^{-T\rho_Q(X)} f_I(g\exp(TX))$ *is an exponential polynomial with unitary characters, i.e. of the form* $\sum_{k=1}^n p_k(T) e^{i\nu_k T}$*, where the* p_k*'s are polynomial and the* ν_k*'s are real numbers.*

Let us assume $\mathcal{W}_I = \{1\}$. Then one has $f_I \in \mathcal{A}_{temp}(Z_I)$. Moreover the linear map $f \mapsto f_I$ is a continuous G-morphism, and $\forall \mathcal{C}$ compact in $\mathfrak{a}_I^{--}$, $\exists \varepsilon > 0$ and a continuous seminorm p on $\mathcal{A}_{temp,N}(Z)$ tq :

$$\begin{aligned}&|(a\exp TX)^{-\rho_Q} \left(f(\omega a\exp(TX)) - f_I(\omega a\exp(TX))\right)| \\ &\leqslant e^{-\varepsilon T} p(f)(1 + \|\log a\|)^N, \quad a \in A_Z^-, X \in \mathcal{C}, \omega \in \mathcal{F}K, T \geqslant 0.\end{aligned}$$

Remark. Our work for $\mathcal{W}_I$ non trivial is still in progress. There are some technical issues we still have to deal with.

5.3 Uniform estimate

As Harish-Chandra, we get some uniform estimates of the rest (the difference between the function and its constant term) when the functions are $Z(\mathfrak{g})$-eigenvectors with infinitesimal character for which we have a certain control of the real part. Would be useful to establish the Plancherel formula for Z.

5.4 Unicity of the constant term

It comes from the property (ii) of the constant term and the following fact :

> If an exponential polynomial $P(t)$ of one variable, with unitary characters, satisfies
> $$\lim_{t \to +\infty} P(t) = 0,$$
> then $P \equiv 0$.

6 Main steps of the proof of the existence (following Harish-Chandra)

We follow the arguments of Harish-Chandra. Contrary to his approach, one does not need to restrict ourselves to the case of τ-spherical functions (τ finite dimensional representation of K) thanks to the theory of moderate growth completions of Harish-Chandra modules (Bernstein-Krötz).

To obtain the existence of a constant term for f tempered, annihilated by $\mathcal{I} \lhd Z(\mathfrak{g})$ of finite codimension, one associates a vectorial function:

$$\Phi_f : A_Z \to \underline{W}^*,$$

which satisfies a linear differential equation, for any $X \in \mathfrak{a}_I$,

$$L_X \Phi_f = \Gamma^*(X)\Phi_f + \Psi_{f,X},$$

where $(\Gamma, \underline{W})$ is a finite dimensional $\mathfrak{a}_I$-module, $(\Gamma^*, \underline{W}^*)$ its contragredient module and $\Psi_{f,X}$ having "fast decay at ∞".

6.1 Differential equation for $I = \emptyset$

Let $\gamma : Z(\mathfrak{g}) \to Z(\mathfrak{l})$ be the Harish-Chandra homomorphism.
Fix $W \subset Z(\mathfrak{l})$ containing 1 s.t.

$$\gamma(Z(\mathfrak{g})) \otimes W \simeq Z(\mathfrak{l}).$$

Fix $\mathcal{I} \lhd Z(\mathfrak{g})$ of finite codimension and choose $V \subset \gamma(Z(\mathfrak{g}))$ finite dimensional subspace containing 1 s.t.

$$V \simeq \gamma(Z(\mathfrak{g}))/\gamma(\mathcal{I}).$$

One sets $\underline{W} = VW$. Then

$$Z(\mathfrak{l}) = \underline{W} \oplus \gamma(\mathcal{I})W.$$

One then defines the finite dimensional $\mathfrak{a}_Z$-module $(\Gamma, \underline{W})$ by:

$$\Gamma(X)v = \text{projection of } Xv \text{ on } \underline{W}, \quad X \in \mathfrak{a}_Z \text{“} \subset \text{”} Z(\mathfrak{l}), v \in \underline{W}.$$

For $f \in \mathcal{A}_{temp}(Z)$ annihilated by $\mathcal{I}$, let Φ_f be the function from A_Z into $\underline{W}^*$ defined by:

$$< \Phi_f(a), v >= a^{-\rho_Q}(L_v f)(a), \quad a \in A_Z, v \in \underline{W},$$

Then

$$L_X \Phi_f = \Gamma^*(X)\Phi_f + \Psi_{f,X}, \quad X \in \mathfrak{a}_Z,$$

where $\Psi_{f,X}$ is a $\underline{W}^*$-valued function on A_Z depending on f, satisfying

$\exists \beta$ function on $\mathfrak{a}_Z$ s.t. $\beta(Y) < 0$ for any $Y \in \mathfrak{a}_Z^{--}$ and

$$\|\Psi_{f,X}(\exp Y)\| \leqslant c_f(1 + \|Y\|)^N e^{\beta(Y)}.$$

The above model is close to the following one.

Let E be a finite dimensional complex vector space, $A \in \mathrm{End}(E)$, $\psi \in C^\infty([0,+\infty[, E)$. Consider the linear differential equation on $[0,+\infty[$:

$$\phi' = A\phi + \psi,$$

where ψ has an exponential decay at infinity, i.e.

$$\exists \beta < 0, \quad \|\psi(t)\| \leqslant e^{\beta t}, \quad t \geqslant 0. \tag{6.1}$$

Fact. *If ϕ is a bounded solution, then $\exists \tilde{\phi}$ an exponential polynomial with unitary characters s.t.:*

$$\lim_{t\to\infty} \phi(t) - \tilde{\phi}(t) = 0. \tag{6.2}$$

To get such a $\tilde{\phi}$, one discusses the asymptotical behavior of a solution ϕ according to the sign of $\mathrm{Re}\,(\lambda) - \beta$ with λ eigenvalue of A.

7 Application to the discrete series for Z

If we have the Theorem for any $I \subseteq S$, we will be able to say the following:

If we assume that Z is without center (i.e. $\mathfrak{a}_Z^- \cap (-\mathfrak{a}_Z^-) = \{0\}$) then, for any $f \in \mathcal{A}_{temp}(Z)$,

$$f_I = 0 \text{ for all } I \subsetneq S \quad \text{if and only if} \quad f \text{ is square integrable}$$

8 Constant term of a Z-tempered continuous linear form

Let ξ be a Z-tempered continuous linear form of a Harish-Chandra G-representation (π, V). The constant term gives us a unique linear form ξ_I on (π, V), Z_I-tempered if $\mathcal{W}_I = \{1\}$, s.t.:

$$(m_{\eta,v})_I = m_{\eta_I,v}, \quad v \in V.$$

9 Details on (6.2) : à raconter si le temps

1. By integration of the equation, one has, for $a, t \geqslant 0$,

$$\phi(a+t) = e^{tA}\phi(a) + \int_0^t e^{(t-s)A}\psi(a+s)\,ds.$$

2. Let ϕ be a bounded solution and $\lambda \in \mathbb{C}$ be an eigenvalue of A. Let ϕ_λ be the projection of ϕ on the generalized eigenspace of A for the eigenvalue λ. For $\operatorname{Re}\lambda > \beta$, the limit:

$$\phi_{\lambda,\infty}(a) := \lim_{t\to\infty} e^{-tA}\phi_\lambda(a+t)$$

exists and

$$\phi_{\lambda,\infty}(a) = \phi_\lambda(a) + \int_0^\infty (e^{-sA}\psi(a+s))_\lambda\,ds.$$

   (a) As ϕ bounded, if $\operatorname{Re}\lambda > 0$, one then has $\phi_{\lambda,\infty}(a) = 0$. Indeed

$$\|e^{-tA}\phi_\lambda(a+t)\| \leqslant e^{-t\operatorname{Re}\lambda}(1+t)^{\dim E}\|\phi(a)\|.$$

   By change of variables, one deduces that:

$$\phi_\lambda(a+t) = -\int_t^\infty (e^{-(s-t)A}\psi(a+s))_\lambda\,ds.$$

   (b) If $\operatorname{Re}\lambda \geqslant \beta/2$, then

$$\|\phi_\lambda(a+t)\| \leqslant e^{t\beta/2}\int_t^\infty e^{-s\beta/2}(1+(s-t))^{\dim E}\|\psi(a+s)\|\,ds.$$

   (c) If $\operatorname{Re}\lambda \leqslant \beta/2$, then

$$\|\phi_\lambda(a+t)\| \leqslant e^{t\beta/2}\left((1+t)^{\dim E}\|\phi(a)\| + \int_0^\infty e^{-s\beta/2}(1+|s-t|)^{\dim E}\|\psi(a+s)\|\,ds\right).$$

   (d) If $\operatorname{Re}\lambda = 0$, then

$$\|\phi_\lambda(a+t) - \phi_{\lambda,\infty}(a+t)\| \leqslant e^{t\beta/2}\int_0^\infty e^{-s\beta/2}(1+(|s-t|)^{\dim E}\|\psi(a+s)\|\,ds.$$

   (e) Set, for λ eigenvalue of A,

$$\tilde\phi_{\lambda,\infty} = \begin{cases} \phi_{\lambda,\infty}, & \text{if } \operatorname{Re}\lambda = 0 \\ 0, & \text{otherwise} \end{cases}$$

(f) Then (take in the following $\delta = 1/2$)

$$\begin{aligned}\|\phi_\lambda(a+t) - \tilde{\phi}_{\lambda,\infty}(a+t)\| &\leqslant e^{t\delta\beta}((1+t)^{\dim E}\|\phi(a)\| \\ &\quad + \int_0^\infty e^{-s\beta/2}(1+|s-t|)^{\dim E}\|\psi(a+s)\|\,ds).\end{aligned}$$

As ψ satisifes (6.1), one then has

$$\|\phi_\lambda(a+t) - \tilde{\phi}_{\lambda,\infty}(a+t)\| \leqslant e^{t\delta\beta}\left((1+t)^{\dim E}\|\phi(a)\| + c\int_0^\infty e^{s\beta/2}(1+|s-t|)^{\dim E}\,ds\right).$$

(g) Let

$$\tilde{\phi} = \sum_{\lambda \text{ eigenvalue of } A} \tilde{\phi}_{\lambda,\infty}.$$

(h) Hence one obtains from above:

$$\lim_{t\to\infty} \phi(t) - \tilde{\phi}(t) = 0.$$