THE LONG-MOODY CONSTRUCTION AND POLYNOMIAL FUNCTORS

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Abstract

In 1994, Long and Moody gave a construction on representations of braid groups which associates a representation of $\mathbf{B}_n$ with a representation of $\mathbf{B}_{n+1}$. In this paper, we prove that this construction is functorial and can be extended: it inspires endofunctors, called Long-Moody functors, between the category of functors from Quillen's bracket construction associated with the braid groupoid to a module category. Then we study the effect of Long-Moody functors on strong polynomial functors: we prove that they increase by one the degree of very strong polynomiality.

Introduction

Linear representations of the Artin braid group on n strands $\mathbf{B}_n$ is a rich subject which appears in diverse contexts in mathematics (see for example [5] or [19] for an overview). Even if braid groups are of wild representation type, any new result which allows us to gain a better understanding of them is a useful contribution.

In 1994, as a result of a collaboration with Moody in [17], Long gave a method to construct from a linear representation $\rho : \mathbf{B}_{n+1} \to GL(V)$ a new linear representation $\mathcal{LM}(\rho) : \mathbf{B}_n \to GL(V^{\oplus n})$ of $\mathbf{B}_n$ (see [17, Theorem 2.1]). Moreover, the construction complicates in a sense the initial representation. For example, applying it to a one dimensional representation of $\mathbf{B}_{n+1}$, the construction gives a mild variant of the unreduced Burau representation of $\mathbf{B}_n$. This method was in fact already implicitly present in two previous articles of Long dated 1989 (see [15, 16]). In the article [3] dating from 2008, Bigelow and Tian consider the Long-Moody construction from a matricial point of view. They give alternative and purely algebraic proofs of some results of [17], and they slightly extend some of them. In a survey on braid groups (see the Open Problem 7 in [5]), Birman and Brendle underline the fact that the Long-Moody construction should be studied in greater detail.

Our work focuses on the study of the Long-Moody construction $\mathcal{LM}$ from a functorial point of view. More precisely, we consider the category $\mathfrak{U}\boldsymbol{\beta}$ associated with braid groups. This category is an example of a general construction due to Quillen (see [9]) on the braid groupoid $\boldsymbol{\beta}$. In particular, the category $\mathfrak{U}\boldsymbol{\beta}$ has natural numbers $\mathbb{N}$ as objects. For each natural number n, the automorphism group $Aut_{\mathfrak{U}\boldsymbol{\beta}}(n)$ is the braid group $\mathbf{B}_n$. Let $\mathbb{K}\text{-}\mathfrak{Mod}$ be the category of $\mathbb{K}$-modules, with $\mathbb{K}$ a commutative ring, and $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ be the category of the functors from $\mathfrak{U}\boldsymbol{\beta}$ to $\mathbb{K}\text{-}\mathfrak{Mod}$. An object M of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ gives by evaluation a family of representations of braid groups $\{M_n : \mathbf{B}_n \to GL(M(n))\}_{n\in\mathbb{N}}$, which satisfies some compatibility properties (see Section 1.1). Randal-Williams and Wahl use the category $\mathfrak{U}\boldsymbol{\beta}$ in [20] to construct a general framework to prove homological stability for braid groups with twisted coefficients. Namely, they obtain the stability for twisted coefficients given by objects of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

In Proposition 2.21, we prove that a version of the Long-Moody construction is functorial. We fix two families of morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ and $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$, satisfying some coherence properties (see Section 2.1). Once this framework set, we show:

***Theorem A* (Proposition 2.21) .** *There is a functor* $\mathbf{LM}_{a,\varsigma} : \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, *called the Long-Moody functor with respect to coherent families of morphisms* $\{a_n\}_{n\in\mathbb{N}}$ *and* $\{\varsigma_n\}_{n\in\mathbb{N}}$, *which satisfies for* $\sigma \in \mathbf{B}_n$ *and* $M \in Obj(\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}))$

$$\mathbf{LM}_{a,\varsigma}(M)(\sigma) = \mathcal{LM}(M_n)(\sigma).$$

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Among the objects in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ the strong polynomial functors play a key role. This notion extends the classical one of polynomial functors, which were first defined by Eilenberg and Mac Lane in [8] for functors on module categories, using cross effects. This definition can also be applied to monoidal categories where the monoidal unit is a null object. Djament and Vespa introduce in [7] the definition of strong polynomial functors for symmetric monoidal categories with the monoidal unit being an initial object. Here, the category $\mathfrak{U}\boldsymbol{\beta}$ is neither symmetric, nor braided, but pre-braided in the sense of [20]. However, we show that the notion of strong polynomial functor extends to the wider context of pre-braided monoidal categories (see Definition 3.4). We also introduce the notion of very strong polynomial functor (see Definition 3.16). Strong polynomial functors turn out inter alia to be very useful for homological stability problems. For example, in [20], Randal-Williams and Wahl prove their homological stability results for twisted coefficients given by a specific kind of strong polynomial functors, namely coefficient systems of finite degree (see [20, Section 4.4]).

We investigate the effects of Long-Moody functors on very strong polynomial functors. We establish the following theorem, under some mild additional conditions (introduced in Section 4.1.1) on the families of morphisms $\{a_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$, which are then said to be reliable.

***Theorem B* (Corollary 4.27) .** *Let M be a very strong polynomial functor of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ of degree n and let $\{a_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$ be coherent reliable families of morphisms. Then, considering the Long-Moody functor $\mathbf{LM}_{a,\varsigma}$ with respect to the morphisms $\{a_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$, $\mathbf{LM}_{a,\varsigma}(M)$ is a very strong polynomial functor of degree $n+1$.*

Thus, iterating the Long-Moody functor on a very strong polynomial functor of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ of degree d, we generate polynomial functors of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, of any degree bigger than d. For instance, Randal-Williams and Wahl define in [20, Example 4.3] a functor $\mathfrak{Bur}_t : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ encoding the unreduced Burau representations. Similarly, we introduce a functor $\mathfrak{TYM}_t : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ corresponding to the representations considered by Tong, Yang and Ma in [22]. These functors $\mathfrak{Bur}_t$ and $\mathfrak{TYM}_t$ are very strong polynomial of degree one (see Proposition 3.25), and moreover, we prove that the functor $\mathfrak{Bur}_t$ is equivalent to a functor obtained by applying the Long-Moody construction. Thus, the Long-Moody functors will provide new examples of twisted coefficients corresponding to the framework of [20].

This construction is extended in the forthcoming work [21] for other families of groups, such as automorphism groups of free groups, braid groups of surfaces, mapping class groups of orientable and non-orientable surfaces or mapping class groups of 3-manifolds. The results proved here for (very) strong polynomial functors will also hold in the adapted categorical framework for these different families of groups.

The paper is organized as follows. Following [20], Section 1 introduces the category $\mathfrak{U}\boldsymbol{\beta}$ and gives first examples of objects of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Then, in Section 2, we introduce the Long-Moody functors, prove Theorem *A* and give some of their properties. In Section 3, we review the notion of strong polynomial functors and extend the framework of [7] to pre-braided monoidal categories. Finally, Section 4 is devoted to the proof of Theorem *B* and to some other properties of these functors. In particular, we tackle the Open Problem 7 of [5].

Notation 0.1. We will consider a commutative ring $\mathbb{K}$ throughout this work. We denote by $\mathbb{K}\text{-}\mathfrak{Mod}$ the category of $\mathbb{K}$-modules. We denote by $\mathfrak{Gr}$ the category of groups.

Let $\mathfrak{Cat}$ denote the category of small categories. Let $\mathfrak{C}$ be an object of $\mathfrak{Cat}$. We use the abbreviation $Obj(\mathfrak{C})$ to denote the objects of $\mathfrak{C}$. For $\mathfrak{D}$ a category, we denote by $\mathbf{Fct}(\mathfrak{C}, \mathfrak{D})$ the category of functors from $\mathfrak{C}$ to $\mathfrak{D}$. If 0 is initial object in the category $\mathfrak{C}$, then we denote by $\iota_A : 0 \to A$ the unique morphism from 0 to A. The maximal subgroupoid $\mathscr{G}\mathfrak{r}(\mathfrak{C})$ is the subcategory of $\mathfrak{C}$ which has the same objects as $\mathfrak{C}$ and of which the morphisms are the isomorphisms of $\mathfrak{C}$. We denote by $\mathscr{G}\mathfrak{r} : \mathfrak{Cat} \to \mathfrak{Cat}$ the functor which associates to a category its maximal subgroupoid.

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Contents

1 The category $\mathfrak{U}\boldsymbol{\beta}$

The aim of this section is to describe the category $\mathfrak{U}\boldsymbol{\beta}$ associated with braid groups that is central to this paper. On the one hand, we recall some notions and properties about Quillen's construction from a monoidal groupoid and pre-braided monoidal categories introduced by Randal-Williams and Wahl in [20]. On the other hand, we introduce examples of functors over the category $\mathfrak{U}\boldsymbol{\beta}$.

We recall that the braid group on $n \geq 2$ strands denoted by $\mathbf{B}_n$ is the group generated by σ_1, ..., σ_{n-1} satisfying the relations:

- $\forall i \in \{1, \ldots, n-2\}$, $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$;
- $\forall i, j \in \{1, \ldots, n-1\}$ such that $\mid i-j \mid \geq 2$, $\sigma_i \sigma_j = \sigma_j \sigma_i$.

$\mathbf{B}_0$ and $\mathbf{B}_1$ both are the trivial group. The family of braid groups is associated with the following groupoid.

Definition 1.1. The braid groupoid $\boldsymbol{\beta}$ is the groupoid with objects the natural numbers $n \in \mathbb{N}$ and morphisms (for $n, m \in \mathbb{N}$):

$$Hom_{\boldsymbol{\beta}}(n, m) = \begin{cases} \mathbf{B}_n & \text{if } n = m \\ \emptyset & \text{if } n \neq m. \end{cases}$$

Remark 1.2. The composition of morphisms $\circ$ in the groupoid $\boldsymbol{\beta}$ corresponds to the group operation of the braid groups. So we will abuse the notation throughout this work, identifying $\sigma \circ \sigma' = \sigma\sigma'$ for all elements σ and σ' of $\mathbf{B}_n$ with $n \in \mathbb{N}$ (with the convention that we read from the right to the left for the group operation).

1.1 Quillen's bracket construction associated with the groupoid $\boldsymbol{\beta}$

This section focuses on the presentation and the study of Quillen's bracket construction $\mathfrak{U}\boldsymbol{\beta}$ (see [9, p.219]) on the braid groupoid $\boldsymbol{\beta}$. It associates to $\boldsymbol{\beta}$ a monoidal category whose unit is initial. The category $\mathfrak{U}\boldsymbol{\beta}$ has further properties: Quillen's bracket construction on $\boldsymbol{\beta}$ is a pre-braided monoidal category (see Section 1.1.2) and $\boldsymbol{\beta}$ is its maximal subgroupoid. For an introduction to (braided) strict monoidal categories, we refer to [18, Chapter XI].

Notation 1.3. A strict monoidal category will be denoted by $(\mathfrak{C}, \natural, 0)$, where $\mathfrak{C}$ is the category, $\natural$ is the monoidal product and 0 is the monoidal unit.

1.1.1 Generalities

In [20], Randal-Williams and Wahl study a construction due to Quillen in [9, p.219], for a monoidal category S acting on a category X in the case $S = X = \mathfrak{G}$ where $\mathfrak{G}$ is a groupoid. It is called Quillen's bracket construction. Our study here is based on [20, Section 1] taking $\mathfrak{G} = \boldsymbol{\beta}$.

Definition 1.4. [18, Chapter XI, Section 4] A monoidal product $\natural : \boldsymbol{\beta} \times \boldsymbol{\beta} \longrightarrow \boldsymbol{\beta}$ is defined by the usual addition for the objects and laying two braids side by side for the morphisms. The object 0 is the unit of this monoidal product. The strict monoidal groupoid $(\boldsymbol{\beta}, \natural, 0)$ is braided, its braiding is denoted by $b^{\boldsymbol{\beta}}_{-,-}$. Namely, the braiding is defined for all natural numbers n and m such that $n + m \geq 2$ by:

$$b^{\boldsymbol{\beta}}_{n,m} = (\sigma_m \circ \cdots \circ \sigma_2 \circ \sigma_1) \circ \cdots \circ (\sigma_{n+m-2} \circ \cdots \circ \sigma_n \circ \sigma_{n-1}) \circ (\sigma_{n+m-1} \circ \cdots \circ \sigma_{n+1} \circ \sigma_n)$$

where $\{\sigma_i\}_{i \in \{1,\ldots,n+m-1\}}$ denote the Artin generators of the braid group $\mathbf{B}_{n+m}$.

We consider the strict monoidal groupoid $(\boldsymbol{\beta}, \natural, 0)$ throughout this section.

Definition 1.5. [20, Section 1.1] Quillen's bracket construction on the groupoid $\boldsymbol{\beta}$, denoted by $\mathfrak{U}\boldsymbol{\beta}$, is the category defined by:

- Objects: $Obj\,(\mathfrak{U}\boldsymbol{\beta}) = Obj\,(\boldsymbol{\beta}) = \mathbb{N}$;
- Morphisms: for n and n' two objects of $\boldsymbol{\beta}$, the morphisms from n to n' in the category $\mathfrak{U}\boldsymbol{\beta}$ are given by:

  $$Hom_{\mathfrak{U}\boldsymbol{\beta}}\left(n, n'\right) = \underset{\boldsymbol{\beta}}{colim}\left[Hom_{\boldsymbol{\beta}}\left(-\natural n, n'\right)\right].$$

  In other words, a morphism from n to n' in the category $\mathfrak{U}\boldsymbol{\beta}$, denoted by $[n'-n, f] : n \to n'$, is an equivalence class of pairs $(n'-n, f)$ where $n'-n$ is an object of $\boldsymbol{\beta}$, $f : (n'-n)\,\natural n \to n'$ is a morphism of $\boldsymbol{\beta}$, in other words an element of $\mathbf{B}_{n'}$. The equivalence relation $\sim$ is defined by $(n'-n, f) \sim (n'-n, f')$ if and only if there exists an automorphism $g \in Aut_{\boldsymbol{\beta}}\,(n'-n)$ such that the following diagram commutes.

  $$\begin{array}{ccc} (n'-n)\,\natural n & \xrightarrow{f} & n' \\ {\scriptstyle g\natural id_n}\downarrow & \nearrow_{f'} & \\ (n'-n)\,\natural n & & \end{array}$$

- For all objects n of $\mathfrak{U}\boldsymbol{\beta}$, the identity morphism in the category $\mathfrak{U}\boldsymbol{\beta}$ is given by $[0, id_n] : n \to n$.
- Let $[n'-n, f] : n \to n'$ and $[n''-n', g] : n' \to n''$ be two morphisms in the category $\mathfrak{U}\boldsymbol{\beta}$. Then, the composition in the category $\mathfrak{U}\boldsymbol{\beta}$ is defined by:

  $$\left[n''-n', g\right] \circ \left[n'-n, f\right] = \left[n''-n, g \circ (id_{n'-n}\natural f)\right].$$

The relationship between the automorphisms of the groupoid $\boldsymbol{\beta}$ and those of its associated Quillen's construction $\mathfrak{U}\boldsymbol{\beta}$ is actually clear. First, let us recall the following notion.

Definition 1.6. Let $(\mathfrak{G}, \natural, 0)$ be a strict monoidal category. It has no zero divisors if for all objects A and B of $\mathfrak{G}$, $A\natural B \cong 0$ if and only if $A \cong B \cong 0$.

The braid groupoid $(\boldsymbol{\beta}, \natural, 0)$ has no zero divisors. Moreover, by Definition 1.1, $Aut_{\boldsymbol{\beta}}(0) = \{id_0\}$. Hence, we deduce the following property from [20, Proposition 1.7].

Proposition 1.7. *The groupoid $\boldsymbol{\beta}$ is the maximal subgroupoid of $\mathfrak{U}\boldsymbol{\beta}$.*

In addition, $\mathfrak{U}\boldsymbol{\beta}$ has the additional useful property.

Proposition 1.8. *[20, Proposition 1.8 (i)] The unit* 0 *of the monoidal structure of the groupoid* $(\boldsymbol{\beta}, \natural, 0)$ *is an initial object in the category* $\mathfrak{U}\boldsymbol{\beta}$.

Remark 1.9. Let n be a natural number and $\phi \in Aut_{\boldsymbol{\beta}}(n)$. Then, as an element of $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n)$, we will abuse the notation $\phi = [0,\phi]$. This comes from the canonical functor:

$$\begin{array}{ccc} \boldsymbol{\beta} & \to & \mathfrak{U}\boldsymbol{\beta} \\ \phi \in Aut_{\boldsymbol{\beta}}(n) & \mapsto & [0,\phi]. \end{array}$$

Finally, we are interested in a way to extend an object of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ to an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. This amounts to studying the image of the restriction $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

Proposition 1.10. *Let* M *be an object of* $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. *Assume that for all* $n, n', n'' \in \mathbb{N}$ *such that* $n'' \geq n' \geq n$, *there exists an assignment* $M([n'-n, id_{n'}]) : M(n) \to M(n')$ *such that:*

$$M([n''-n', id_{n''}]) \circ M([n'-n, id_{n'}]) = M([n''-n, id_{n''}]) \tag{1}$$

Then, we define a functor $M : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ *(assigning* $M([n'-n,\sigma]) = M(\sigma) \circ M([n'-n, id_{n'}])$ *for all* $[n'-n,\sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$*) if and only if for all* $n, n' \in \mathbb{N}$ *such that* $n' \geq n$:

$$M([n'-n, id_{n'}]) \circ M(\sigma) = M(\psi \natural \sigma) \circ M([n'-n, id_{n'}]) \tag{2}$$

for all $\sigma \in \mathbf{B}_n$ *and all* $\psi \in \mathbf{B}_{n'-n}$.

Remark 1.11. Note that for $n' = n$, $M([n'-n, id_{n'}]) = Id_{M(n)}$.

Proof of Proposition 1.10. Let us assume that relation (2) is satisfied. We have to show that the assignment on morphisms is well-defined with respect to $\mathfrak{U}\boldsymbol{\beta}$. First, let us prove that our assignment conforms with the defining equivalence relation of $\mathfrak{U}\boldsymbol{\beta}$ (see Definition 1.5). For n and n' natural numbers such that $n' \geq n$, let us consider $[n'-n,\sigma]$ and $[n'-n,\sigma']$ in $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$ such that there exists $\psi \in \mathbf{B}_{n'-n}$ so that $\sigma' \circ (\psi \natural id_n) = \sigma$. Since M is a functor over $\boldsymbol{\beta}$, $M([n'-n,\sigma]) = M(\sigma') \circ (M(\psi \natural id_n) \circ M([n'-n, id_{n'}]))$. According to the relation (2) and since M satisfies the identity axiom, we deduce that $M([n'-n,\sigma]) = M(\sigma') \circ M(\psi \natural id_n) \circ M([n'-n, id_{n'}]) = M([n'-n,\sigma'])$.

Now, we have to check the composition axiom. Let n, n' and n'' be natural numbers such that $n'' \geq n' \geq n$, let $([n'-n,\sigma])$ and $([n''-n',\sigma'])$ be morphisms respectively in $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$ and in $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n',n'')$. By our assignment and composition in $\mathfrak{U}\boldsymbol{\beta}$ (see Definition 1.5) we have that:

$$M([n''-n',\sigma']) \circ M([n'-n,\sigma]) = M(\sigma') \circ (M([n''-n', id_{n''}]) \circ M(\sigma)) \circ M([n'-n, id_{n'}]).$$

According to the relation (2), we deduce that:

$$\begin{aligned} M([n''-n',\sigma']) \circ M([n'-n,\sigma]) &= M(\sigma') \circ (M([n''-n', id_{n''}]) \circ M(\sigma)) \circ M([n'-n, id_{n'}]). \\ &= M(\sigma') \circ (M(id_{n''-n'} \natural \sigma) \circ M([n''-n', id_{n''}])) \circ M([n'-n, id_{n'}]). \end{aligned}$$

Hence, it follows from relation (1) that:

$$M([n''-n',\sigma']) \circ M([n'-n,\sigma]) = M(\sigma' \circ (id_{n''-n'} \natural \sigma)) \circ M([n''-n, id_n]) = M([n''-n',\sigma'] \circ [n'-n,\sigma]).$$

Conversely, assume that the functor $M : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ is well-defined. In particular, composition axiom in $\mathfrak{U}\boldsymbol{\beta}$ is satisfied and implies that for all $n, n' \in \mathbb{N}$ such that $n' \geq n$, for all $\sigma \in \mathbf{B}_n$:

$$M([n'-n, id_{n'}]) \circ M(\sigma) = M([n'-n, id_{n'-n} \natural \sigma]).$$

It follows from the defining equivalence relation of $\mathfrak{U}\boldsymbol{\beta}$ (see Definition (1.5)) that for all $\psi \in \mathbf{B}_{n'-n}$:

$$M([n'-n, id_{n'}]) \circ M(\sigma) = M([n'-n, \psi \natural \sigma]).$$

We deduce from the composition axiom that relation (2) is satisfied. □

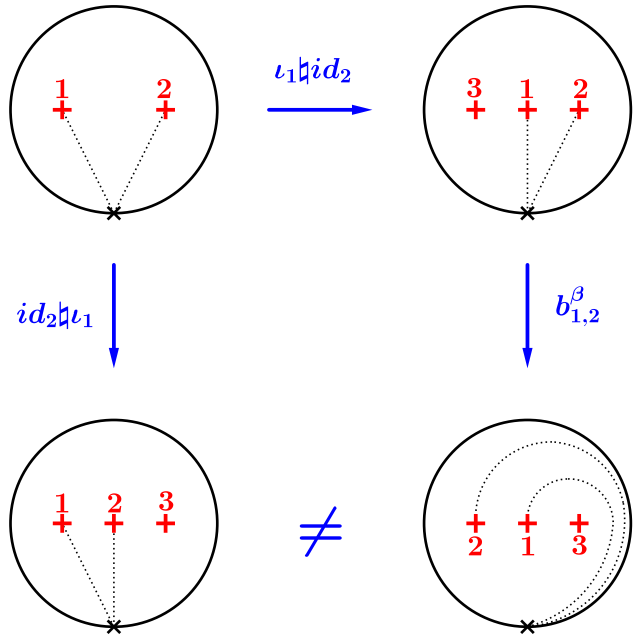


Figure 1: Failure of the braiding property

Proposition 1.12. *Let M and M' be objects of* $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$ *and $\eta : M \to M'$ a natural transformation in the category* $\mathbf{Fct}\left(\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$. *Then, η is a natural transformation in the category* $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$ *if and only if for all $n, n' \in \mathbb{N}$ such that $n' \geq n$:*

$$\eta_{n'} \circ M\left(\left[n'-n, id_{n'}\right]\right) = M'\left(\left[n'-n, id_{n'}\right]\right) \circ \eta_n. \tag{3}$$

Proof. The natural transformation η extends to the category $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$ if and only if for all $n, n' \in \mathbb{N}$ such that $n' \geq n$, for all $\left[n'-n,\sigma\right] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}\left(n,n'\right)$:

$$M'\left(\left[n'-n,\sigma\right]\right) \circ \eta_n = \eta_{n'} \circ M\left(\left[n'-n,\sigma\right]\right).$$

Since η is a natural transformation in the category $\mathbf{Fct}\left(\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$, we already have $\eta_{n'} \circ M\left(\sigma\right) = M'\left(\sigma\right) \circ \eta_{n'}$. Hence, this implies that the necessary and sufficient relation to satisfy is relation (3). $\square$

1.1.2 Pre-braided monoidal category

We present the notion of a pre-braided category, introduced by Randal-Williams and Wahl in [20]. This is a generalization of that of a braided monoidal category.

Definition 1.13. [20, Definition 1.5] Let $\left(\mathfrak{C},\natural,0\right)$ be a strict monoidal category such that the unit 0 is initial. We say that the monoidal category $\left(\mathfrak{C},\natural,0\right)$ is pre-braided if:

- The maximal subgroupoid $\mathscr{G}\mathfrak{r}\left(\mathfrak{C},\natural,0\right)$ is a braided monoidal category, where the monoidal structure is induced by that of $\left(\mathfrak{C},\natural,0\right)$.
- For all objects A and B of $\mathfrak{C}$, the braiding associated with the maximal subgroupoid $b^{\mathfrak{C}}_{A,B} : A\natural B \longrightarrow B\natural A$ satisfies:

  $$b^{\mathfrak{C}}_{A,B} \circ \left(id_A \natural \iota_B\right) = \iota_B \natural id_A : A \longrightarrow B\natural A.$$

  Recall that the notation $\iota_B : 0 \to B$ was introduced in Notation 0.1.

Since the groupoid $\left(\boldsymbol{\beta},\natural,0\right)$ is braided monoidal and it has no zero divisors, we deduce from [20, Proposition 1.8] the following properties.

Proposition 1.14. *The category $\mathfrak{U}\boldsymbol{\beta}$ is pre-braided monoidal. The monoidal structure $(\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$ is defined on objects as that of $(\boldsymbol{\beta}, \natural, 0)$ and defined on morphisms letting for $[n'-n, f] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$ and $[m'-m, g] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(m, m')$:*

$$[m'-m, g] \natural [n'-n, f] = \left[(m'-m) \natural (n'-n), (g \natural f) \circ \left(id_{m'-m} \natural \left(b^{\boldsymbol{\beta}}_{m,n'-n}\right)^{-1} \natural id_n\right)\right].$$

In particular, the canonical functor $\boldsymbol{\beta} \to \mathfrak{U}\boldsymbol{\beta}$ is monoidal.

Remark 1.15. The category $(\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$ is pre-braided monoidal, but not braided. Indeed, as Figure 1 shows, the pre-braiding defined on $\mathfrak{U}\boldsymbol{\beta}$ is not a braiding: Figure 1 shows that $b^{\boldsymbol{\beta}}_{1,2} \circ (\iota_1 \natural id_2) \neq id_2 \natural \iota_1$ whereas these two morphisms should be equal if $b^{\boldsymbol{\beta}}_{-,-}$ were a braiding.

1.2 Examples of functors associated with braid representations

Different families of representations of braid groups can be used to form functors over the pre-braided category $\mathfrak{U}\boldsymbol{\beta}$ to the category $\mathbb{K}$-$\mathfrak{Mod}$. Namely, considering $\{M_n : \mathbf{B}_n \to \mathbb{K}\text{-}\mathfrak{Mod}\}_{n\in\mathbb{N}}$ representations of braid groups, or equivalently an object M of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we are interested in the situations where Proposition 1.10 applies so as to define an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

Tong-Yang-Ma results In 1996, in the article [22], Tong, Yang and Ma investigated the representations of $\mathbf{B}_n$ where the i-th generator is sent to a matrix of the form $Id_{i-1} \oplus T \oplus Id_{n-i-1}$, with T a $m \times m$ non-singular matrix and $m \geq 2$. In particular, for $m = 2$, they prove that there exist up to equivalence only two non trivial representations of this type. We give here their result and an interpretation of their work from a functorial point of view, considering the representations over the ring of Laurent polynomials in one variable $\mathbb{C}\left[t^{\pm 1}\right]$.

Notation 1.16. Let $\mathfrak{gr}$ denote the full subcategory of $\mathfrak{Gr}$ of finitely generated free groups. The free product $* : \mathfrak{gr} \times \mathfrak{gr} \to \mathfrak{gr}$ defines a monoidal structure over $\mathfrak{gr}$, with 0 the unit, denoted by $(\mathfrak{gr}, *, 0)$.

Let $(\mathbb{N}, \leq)$ denote the category of natural numbers (natural means non-negative) considered as a poset. For all natural numbers n, we denote by γ_n the unique element of $Hom_{(\mathbb{N},\leq)}(n, n+1)$. For all natural numbers n and n' such that $n' \geq n$, we denote by $\gamma_{n,n'} : n \to n'$ the unique element of $Hom_{(\mathbb{N},\leq)}(n, n')$, composition of the morphisms $\gamma_{n'-1} \circ \gamma_{n'-2} \circ \cdots \circ \gamma_{n+1} \circ \gamma_n$. The addition defines a strict monoidal structure on $(\mathbb{N}, \leq)$, denoted by $((\mathbb{N}, \leq), +, 0)$.

Definition 1.17. Let $\mathbf{B}_- : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ and $GL_- : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ be the functors defined by:

- Objects: for all natural numbers n, $\mathbf{B}_-(n) = \mathbf{B}_n$ the braid group on n strands and $GL_-(n) = GL_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right)$ the general linear group of degree n.
- Morphisms: let n be a natural number. We define $\mathbf{B}_-(\gamma_n) = id_1 \natural - : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$ (where $\natural$ is the monoidal product introduced in Example 1.4). We define $GL_-(\gamma_n) : GL_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right) \hookrightarrow GL_{n+1}\left(\mathbb{C}\left[t^{\pm 1}\right]\right)$ assigning $GL_-(\gamma_n)(\varphi) = id_1 \oplus \varphi$ for all $\varphi \in GL_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right)$.

Notation 1.18. For all natural numbers $n \geq 2$, for all $i \in \{1, \ldots, n-1\}$, we denote by $incl^n_i : \mathbf{B}_2 \cong \mathbb{Z} \hookrightarrow \mathbf{B}_n$ the inclusion morphism induced by:

$$incl^n_i(\sigma_1) = \sigma_i.$$

Theorem 1.19. *[22, Part II] Let $\eta : \mathbf{B}_- \longrightarrow GL_-$ be a natural transformation. Assume that for all natural numbers $n \geq 2$, for all $i \in \{1, \ldots, n-1\}$, the following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{B}_n & \xrightarrow{\eta_n} & GL_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right) \\ \uparrow{\scriptstyle incl^n_i} & & \uparrow{\scriptstyle id_{i-1} \oplus - \oplus id_{n-i-1}} \\ \mathbf{B}_2 & \xrightarrow{\eta_2} & GL_2\left(\mathbb{C}\left[t^{\pm 1}\right]\right). \end{array}$$

Two such natural transformations η and η' are equivalent if there exists a natural equivalence $\mu : GL_- \longrightarrow GL_-$ such that $\mu \circ \eta = \eta'$. Then, η is equivalent to one of the following natural transformations.

1. *The trivial natural transformation, denoted by* $\mathfrak{id}$*: for every generator* σ_i *of* $\mathbf{B}_n$*,* $\mathfrak{id}_n(\sigma_i) = Id_{GL_n(\mathbb{C}[t^{\pm 1}])}$.

2. *The unreduced Burau natural transformation, denoted by* $\mathfrak{bur}$*: for all generators* σ_i *of* $\mathbf{B}_n$*,*

$$\mathfrak{bur}_{n,t}(\sigma_i) = Id_{i-1} \oplus B(t) \oplus Id_{n-i-1},$$

with $B(t) = \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}$.

3. *The natural transformation denoted by* $\mathfrak{tym}$*: for every generator* σ_i *of* $\mathbf{B}_n$ *if* $n \geq 2$*,*

$$\mathfrak{tym}_{n,t}(\sigma_i) = Id_{i-1} \oplus TYM(t) \oplus Id_{n-i-1},$$

with $TYM(t) = \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix}$*. We call it the Tong-Yang-Ma representation.*

The unreduced Burau representation (see [11, Section 3.1] or [5, Section 4.2] for more details about this family of representations) is reducible but indecomposable, whereas the Tong-Yang-Ma representation is irreducible (see [22, Part II]). We can also consider a natural transformation using the family of reduced Burau representations (see [11, Section 3.3] for more details about the associated family of representations): these are irreducible subrepresentations of the unreduced Burau representations.

Definition 1.20. Let $GL_{-\text{-}1} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ be the functor defined by:

- Objects: for all natural numbers n, $GL_{-\text{-}1}(n) = GL_{n-1}(\mathbb{C}[t^{\pm 1}])$ the general linear group of degree $n-1$.
- Morphisms: let n be a natural number. We define $GL_{-\text{-}1}(\gamma_n) : GL_{n-1}(\mathbb{C}[t^{\pm 1}]) \hookrightarrow GL_n(\mathbb{C}[t^{\pm 1}])$ assigning $GL_{-}(\gamma_n)(\varphi) = id_1 \oplus \varphi$ for all $\varphi \in GL_{n-1}(\mathbb{C}[t^{\pm 1}])$.

Definition 1.21. The reduced Burau natural transformation, denoted by $\overline{\mathfrak{bur}} : \mathbf{B}_{-} \to GL_{-\text{-}1}$ is defined by:

- For $n = 2$, one assigns $\overline{\mathfrak{bur}}(\sigma_1) = -t$.
- For all natural numbers $n \geq 3$, we define for every Artin generator σ_i of $\mathbf{B}_n$ with $i \in \{2, \ldots, n-2\}$:

  $$\overline{\mathfrak{bur}}_{n,t}(\sigma_i) = Id_{i-2} \oplus \overline{B}(t) \oplus Id_{n-i-2}$$

  with

  $$\overline{B}(t) = \begin{bmatrix} 1 & t & 0 \\ 0 & -t & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

  and

  $$\overline{\mathfrak{bur}}_{n,t}(\sigma_1) = \begin{bmatrix} -t & 0 \\ 1 & 1 \end{bmatrix} \oplus Id_{n-3} \quad ; \quad \overline{\mathfrak{bur}}_{n,t}(\sigma_{n-1}) = Id_{n-3} \oplus \begin{bmatrix} 1 & t \\ 0 & -t \end{bmatrix}.$$

Let us use these natural transformations to form functors over the category $\mathfrak{U}\boldsymbol{\beta}$. Indeed, a natural transformation $\eta : \mathbf{B}_{-} \to GL_{-}$ (or $GL_{-\text{-}1}$) provides in particular:

- a functor $\mathfrak{N} : \boldsymbol{\beta} \longrightarrow \mathbb{C}[t^{\pm 1}]\text{-}\mathfrak{Mod}$;
- morphisms $\mathfrak{N}([n'-n, id_{n'}]) : \mathfrak{N}(n) \to \mathfrak{N}(n')$ for all natural numbers $n' \geq n$, such that the relation (1) of Proposition 1.10 is satisfied.

Therefore, according to Proposition 1.10, it suffices to show that the relation (2) is satisfied to prove that $\mathfrak{N}$ is an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{C}[t^{\pm 1}]\text{-}\mathfrak{Mod})$.

Notation 1.22. Recall that 0 is a null object in the category of R-modules, and that the notation $\iota_G : 0 \to G$ was introduced in Notation 0.1. Let $n \in \mathbb{N}$. For all natural numbers n and n' such that $n' \geq n$, we define $\iota_{\mathbb{C}[t^{\pm 1}]^{\oplus n'-n}} \oplus id_{\mathbb{C}[t^{\pm 1}]^{\oplus n}} : \mathbb{C}[t^{\pm 1}]^{\oplus n} \hookrightarrow \mathbb{C}[t^{\pm 1}]^{\oplus n'}$ the embedding of $\mathbb{C}[t^{\pm 1}]^{\oplus n}$ as the submodule of $\mathbb{C}[t^{\pm 1}]^{\oplus n'}$ given by the n last copies of $\mathbb{C}[t^{\pm 1}]$.

Tong-Yang-Ma functor: This example is based on the family introduced by Tong, Yang and Ma (see Theorem 1.19). Let $\mathfrak{TYM}_t : \boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ be the functor defined on objects by $\mathfrak{TYM}_t(n) = \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n}$ for all natural numbers n, and for all numbers $n \geq 2$, for every Artin generator σ_i of $\mathbf{B}_n$, by $\mathfrak{TYM}_t(\sigma_i) = \mathfrak{tym}_{n,t}(\sigma_i)$ for morphisms. For all natural numbers n and n' such that $n' \geq n$, we assign $\mathfrak{TYM}_t([n'-n, id_{n'}]) : \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n} \hookrightarrow \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'}$ to be the embedding $\iota_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'-n}} \oplus id_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n}}$ (where these morphisms are introduced in Notation 1.22). For all natural numbers $n'' \geq n' \geq n$, for all Artin generators $\sigma_i \in \mathbf{B}_n$ and all $\psi_j \in \mathbf{B}_{n'-n}$, our assignments give:

$$\begin{aligned}\mathfrak{TYM}_t(\psi \natural \sigma) \circ \mathfrak{TYM}_t([n'-n, id_{n'}]) &= \left(Id_{j-1} \oplus TYM(t) \oplus Id_{(n'-n)-j-1} \oplus Id_{n'-n+i-1} \oplus TYM(t) \oplus Id_{n'-i-1}\right) \\ &\quad \circ \left(\iota_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'-n}} \oplus id_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n}}\right).\end{aligned}$$

Remark that $\left(Id_{j-1} \oplus TYM(t) \oplus Id_{(n'-n)-j-1}\right) \circ \iota_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus (n'-n)}} = \iota_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus (n'-n)}}$. Hence we deduce that

$$\mathfrak{TYM}_t(\psi \natural \sigma) \circ \mathfrak{TYM}_t([n'-n, id_{n'}]) = \mathfrak{TYM}_t([n'-n, id_{n'}]) \circ \mathfrak{TYM}_t(\sigma)$$

for all $\sigma \in \mathbf{B}_n$ and all $\psi \in \mathbf{B}_{n'-n}$. According to Proposition 1.10, our assignment defines a functor $\mathfrak{TYM}_t : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$, called the Tong-Yang-Ma functor.

Burau functors: Other examples naturally arise from the Burau representations.

Let $\mathfrak{Bur}_t : \boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ be the functor defined on objects by $\mathfrak{Bur}_t(n) = \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n}$ for all natural numbers n, and for all numbers $n \geq 2$, for every Artin generator σ_i of $\mathbf{B}_n$, by $\mathfrak{Bur}_t(\sigma_i) = \mathfrak{bur}_{n,t}(\sigma_i)$ for morphisms. For all natural numbers n and n' such that $n' \geq n$, we assign $\mathfrak{Bur}_t([n'-n, id_{n'}]) : \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n} \hookrightarrow \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'}$ to be the embedding $\iota_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'-n}} \oplus id_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n}}$ (where these morphisms are introduced in Notation 1.22).

As for the functor $\mathfrak{TYM}$, the assignment for $\mathfrak{Bur}$ implies that for all natural numbers $n'' \geq n' \geq n$, for all $\sigma \in \mathbf{B}_n$ and all $\psi \in \mathbf{B}_{n'-n}$, $\mathfrak{Bur}_t([n'-n, id_{n'}]) \circ \mathfrak{Bur}_t(\sigma) = \mathfrak{Bur}_t(\psi \natural \sigma) \circ \mathfrak{Bur}_t([n'-n, id_{n'}])$. According to Proposition 1.10, our assignment defines a functor $\mathfrak{Bur}_t : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$, called the unreduced Burau functor. This functor $\mathfrak{Bur}_t$ was already considered by Randal-Williams and Wahl in [20, Example 4.3].

Analogously, we can form a functor from the reduced Burau representations. Let $\overline{\mathfrak{Bur}}_t : \boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ be the functor defined on objects by $\overline{\mathfrak{Bur}}_t(0) = 0$ and $\overline{\mathfrak{Bur}}_t(n) = \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n-1}$ for all nonzero natural numbers n, and by $\overline{\mathfrak{Bur}}_t(\sigma_i) = \overline{\mathfrak{bur}}_{n,t}(\sigma_i)$ for morphisms for every Artin generator σ_i of $\mathbf{B}_n$ for all numbers $n \geq 2$.

For all natural numbers n and n' such that $n' \geq n$, we assign $\overline{\mathfrak{Bur}}_t([n'-n, id_{n'}]) : \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n-1} \hookrightarrow \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'-1}$ to be the embedding $\iota_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'-n}} \oplus id_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n-1}}$ (where these morphisms are introduced in Notation 1.22). Repeating mutadis mutandis the work done for the functor $\mathfrak{TYM}$, the assignment for $\overline{\mathfrak{Bur}}_t$ implies that for all natural numbers $n'' \geq n' \geq n$, for all $\sigma \in \mathbf{B}_n$ and all $\psi \in \mathbf{B}_{n'-n}$, $\overline{\mathfrak{Bur}}_t([n'-n, id_{n'}]) \circ \overline{\mathfrak{Bur}}_t(\sigma) = \overline{\mathfrak{Bur}}_t(\psi \natural \sigma) \circ \overline{\mathfrak{Bur}}_t([n'-n, id_{n'}])$. According to Proposition 1.10, our assignment defines a functor $\overline{\mathfrak{Bur}}_t : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$, called the reduced Burau functor.

Lawrence-Krammer functor: The family of Lawrence-Krammer representations was notably used to prove that braid groups are linear (see [2, 12, 13]). For this paragraph, we assign $\mathbb{K} = \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$ the ring of Laurent polynomials in two variables and consider the functor GL_{-} of Definition 1.17 with this assignment. Let $\mathfrak{LK} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]\text{-}\mathfrak{Mod}$ be the assignment:

- Objects: for all natural numbers $n \geq 2$, $\mathfrak{LK}(n) = \bigoplus_{1 \leq j < k \leq n} V_{j,k}$, with for all $1 \leq j < k \leq n$, $V_{j,k}$ is a free $\mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$-module of rank one. Hence, $\mathfrak{LK}(n) \cong \left(\mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]\right)^{\oplus n(n-1)/2}$ as $\mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$-modules. Moreover, one assigns $\mathfrak{LK}(1) = 0$ and $\mathfrak{LK}(0) = 0$.
- Morphisms:

– Automorphisms: for all natural numbers n, for every Artin generator σ_i of $\mathbf{B}_n$ (with $i \in \{1, \ldots, n-1\}$), for all $v_{j,k} \in V_{j,k}$ (with $1 \leq j < k \leq n$),

$$\mathfrak{LK}(\sigma_i)\, v_{j,k} = \begin{cases} v_{j,k} & \text{if } i \notin \{j-1, j, k-1, k\}, \\ t v_{i,k} + \left(t^2 - t\right) v_{i,i+1} + (1-t)\, v_{i+1,k} & \text{if } i = j-1, \\ v_{i+1,k} & \text{if } i = j \neq k-1, \\ t v_{j,i} + (1-t)\, v_{j,i+1} - \left(t^2 - t\right) q v_{i,i+1} & \text{if } i = k-1 \neq j, \\ v_{j,i+1} & \text{if } i = k, \\ -q t^2 v_{i,i+1} & \text{if } i = j = k-1. \end{cases}$$

– General morphisms: let $n, n' \in \mathbb{N}$, such that $n' \geq n$. For all natural numbers j and k such that $1 \leq j < k \leq n$, we define the embedding $\mathfrak{V}^{n,n'}_{j,k} : V_{j,k} \xrightarrow{\sim} V_{j+(n'-n),k+(n'-n)} \hookrightarrow \bigoplus_{1 \leq j < k \leq n'} V_{j,k}$ of free $\mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$-modules. Then we define $\mathfrak{LK}([n'-n, id_{n'}]) : \bigoplus_{1 \leq j < k \leq n} V_{j,k} \to \bigoplus_{1 \leq j < k \leq n'} V_{j,k}$ to be the embedding $\bigoplus_{1 \leq j < k \leq n} \mathfrak{V}^{n,n'}_{j,k}$.

Since we consider a family of representations of $\mathbf{B}_n$ (see [13]), the assignment $\mathfrak{LK}$ defines an object of $\mathbf{Fct}\left(\boldsymbol{\beta}, \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}\right)$.

Let n, n' and n'' be natural numbers such that $n'' \geq n' \geq n$. It follows directly from our definitions of $\mathfrak{LK}([n'-n, id_{n'}])$, $\mathfrak{LK}([n''-n', id_{n''}])$ and $\mathfrak{LK}([n''-n, id_{n''}])$ that relation (1) of Proposition 1.10 is satisfied.

According to the definition of $\mathfrak{LK}(\sigma_l)$ with σ_l an Artin generator of $\mathbf{B}_{n'-n}$, for all $v_{j,k} \in V_{j,k}$ with $1 + (n'-n) \leq j < k \leq n'$, $\mathfrak{LK}(\sigma_l)\, v_{j,k} = v_{j,k}$. Hence for all $\psi \in \mathbf{B}_{n'-n}$:

$$\mathfrak{LK}(\psi \natural id_n) \circ \mathfrak{LK}\left(\left[n'-n, id_{n'}\right]\right) = \mathfrak{LK}\left(\left[n'-n, id_{n'}\right]\right).$$

Note also that for all $l \in \{1, \ldots, n-1\}$, for all $v_{j,k} \in V_{j,k}$ with $1 + (n'-n) \leq j < k \leq n'$, it follows from the assignment of $\mathfrak{LK}$ that:

$$\mathfrak{LK}(id_{n'-n} \natural \sigma_l)\left(v_{(n'-n)+j,(n'-n)+k}\right) = \mathfrak{LK}(\sigma_{n'-n+l})\left(v_{(n'-n)+j,(n'-n)+k}\right) = \mathfrak{LK}\left(\left[n'-n, id_{n'}\right]\right)\left(\mathfrak{LK}(\sigma_l)\left(v_{j,k}\right)\right).$$

Therefore, this implies that for all $\sigma \in \mathbf{B}_n$, $\mathfrak{LK}([n'-n, id_{n'}]) \circ \mathfrak{LK}(\sigma) = \mathfrak{LK}(id_{n'-n} \natural \sigma) \circ \mathfrak{LK}([n'-n, id_{n'}])$. Hence, $\mathfrak{LK}$ satisfies the relation (2) of Proposition 1.10. Hence, the assignment defines a functor $\mathfrak{LK} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]\text{-}\mathfrak{Mod}$, called the Lawrence-Krammer functor.

2 Functoriality of the Long-Moody construction

The principle of the Long-Moody construction, corresponding to Theorem 2.1 of [17], is to build a linear representation of the braid group $\mathbf{B}_n$ starting from a representation $\mathbf{B}_{n+1}$. We develop a functorial version of this construction, which leads to the notion of Long-Moody functors (see Section 2.2). Beforehand, we need to introduce various tools, which are consequences of the relationships between braid groups and free groups (see Section 2.1). Finally, in Section 2.3, we investigate examples of functors which are recovered by Long-Moody functors.

2.1 Braid groups and free groups

This section recalls some relationships between braid groups and free groups. We also develop tools which will be used throughout our work of Sections 2.2 and 4.

We consider the free group on n generators, which we denote by $\mathbf{F}_n = \langle g_1, \ldots, g_n \rangle$.

Notation 2.1. We denote by $e_{\mathbf{F}_n}$ the unit element of the free group on n generators $\mathbf{F}_n$, for all natural numbers n.

Recall that the category of finitely generated free groups is monoidal using free product of groups (see Notation 1.16). The object 0 being null in the category $\mathfrak{gr}$, recall that $\iota_{\mathbf{F}_n} : 0 \to \mathbf{F}_n$ denotes the unique morphism from 0 to $\mathbf{F}_n$ as in Notation 0.1.

Definition 2.2. Let n be a natural number. We consider $\iota_{\mathbf{F}_1} * id_{\mathbf{F}_n} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n+1}$. This corresponds to the identification of $\mathbf{F}_n$ as the subgroup of $\mathbf{F}_{n+1}$ generated by the n last copies of $\mathbf{F}_1$ in $\mathbf{F}_{n+1}$. Iterating this morphism, we obtain for all natural numbers $n' \geq n$ the morphism $\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_n} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$.

Let $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ be a family of group morphisms from the free group $\mathbf{F}_n$ to the braid group $\mathbf{B}_{n+1}$, for all natural numbers n. We require these morphisms to satisfy the following crucial property.

Condition 2.3. For all elements $g \in \mathbf{F}_n$, for all natural numbers $n' \geq n$, the following diagram is commutative in the category $\mathfrak{U}\boldsymbol{\beta}$:

$$\begin{array}{ccc} 1\natural n & \xrightarrow{\varsigma_n(g)} & 1\natural n \\ {\scriptstyle id_1\natural\left[n'-n,id_{n'}\right]}\downarrow & & \downarrow{\scriptstyle id_1\natural\left[n'-n,id_{n'}\right]} \\ 1\natural n' & \xrightarrow[\varsigma_{n'}\left(e_{\mathbf{F}_{n'-n}}*g\right)]{} & 1\natural n'. \end{array}$$

Remark 2.4. Condition 2.3 will be used to prove that the Long-Moody functor is well defined on morphisms with respect to the tensor product structure in Theorem 2.21. Moreover, it will also be used in the proof of Propositions 4.14 and 4.18.

Lemma 2.5. *Condition 2.3 is equivalent to assume that for all natural numbers n, for all elements $g \in \mathbf{F}_n$, the morphisms $\{\varsigma_n\}_{n\in\mathbb{N}}$ satisfy the following equality in $\mathbf{B}_{n+2}$:*

$$\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right)\circ\left(id_1\natural\varsigma_n\left(g\right)\right)=\varsigma_{n+1}\left(e_{\mathbf{F}_1}*g\right)\circ\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right). \tag{4}$$

Proof. Let n and n' be natural numbers such that $n' \geq n$. The equality (4) implies that for all $1 \leq k \leq n'-n$, the following diagram in the category $\boldsymbol{\beta}$ is commutative :

$$\begin{array}{ccc} 1\natural n' & \xrightarrow{id_{n'-(n+k)}\natural\varsigma_{n+k-1}\left(e_{\mathbf{F}_{k-1}}*g\right)} & 1\natural n' \\ {\scriptstyle id_{n'-(n+k)}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{(k-1)+n}}\downarrow & & \downarrow{\scriptstyle id_{n'-(n+k)}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{(k-1)+n}} \\ 1\natural n' & \xrightarrow[id_{n'-(n+k)}\natural\varsigma_{n+k}\left(e_{\mathbf{F}_k}*g\right)]{} & 1\natural n'. \end{array}$$

Hence composing squares, we obtain that the following diagram is commutative in the category $\boldsymbol{\beta}$:

$$\begin{array}{ccccccc} 1\natural\cdots\natural\left(1\natural 1\right)\natural n & \xrightarrow{id_{n'-n-1}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n} & 1\natural\cdots\natural 1\natural\left(1\natural n\right) & \xrightarrow{id_{n'-n-2}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{1+n}} & \cdots & \xrightarrow{\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{n'-1}} & 1\natural n' \\ {\scriptstyle id_{n'}\natural\varsigma_n(g)}\downarrow & & {\scriptstyle id_{n'-1}\natural\varsigma_{n+1}\left(e_{\mathbf{F}_1}*g\right)}\downarrow & & & & \downarrow{\scriptstyle\varsigma_{n'}\left(e_{\mathbf{F}_1}*g\right)} \\ 1\natural\cdots\natural 1\natural n & \xrightarrow[id_{n'-n-1}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n]{} & 1\natural\cdots\natural 1\natural\left(1\natural n\right) & \xrightarrow[id_{n'-n-2}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{1+n}]{} & \cdots & \xrightarrow[\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{n'-1}]{} & 1\natural n'. \end{array}$$

By definition of the braiding (see Definition 1.1), we deduce that the composition of horizontal arrows is the morphism $\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n$ in $\boldsymbol{\beta}$. Recall from Proposition 1.14 that $id_1\natural\left[n'-n,\sigma\right]=\left[n'-n,\left(id_1\natural\sigma\right)\circ\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n\right)\right]$. Hence Condition 2.3 is satisfied if we assume that the equality (4) is satisfied for all natural numbers n.

Conversely, assume that Condition 2.3 is satisfied. Condition 2.3 with $n' = n+1$ ensures that:

$$\left[1,\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right)\circ\left(id_1\natural\varsigma_n\left(g\right)\right)\right]=\left[1,\varsigma_{n'}\left(e_{\mathbf{F}_1}*g\right)\circ\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right)\right].$$

Since $Aut_{\mathfrak{U}\beta}(1) = \mathbf{B}_1$ is the trivial group, we deduce from the defining equivalence relation of $\mathfrak{U}\beta$ (see Definition 1.5) the equality in $\mathbf{B}_{n+2}$:

$$\left(\left(b_{1,1}^{\beta}\right)^{-1} \natural id_n\right) \circ \left(id_1 \natural \varsigma_n(g)\right) = \varsigma_{1+n}\left(e_{\mathbf{F}_1} * g\right) \circ \left(\left(b_{1,1}^{\beta}\right)^{-1} \natural id_n\right).$$

□

Remark 2.6. It follows from Lemma 2.5 that, for $i \geq 2$, $\varsigma_n(g_i)$ is determined by $\varsigma_k(g_1)$ for $k \leq n$ by the equalities (4).

Example 2.7. The family $\varsigma_{n,1}$, based on what is called the pure braid local system in the literature (see [17, Remark p.223]), is defined by the following inductive assignment for all natural numbers $n \geq 1$.

$$\begin{array}{rcl} \varsigma_{n,1} : \mathbf{F}_n & \longrightarrow & \mathbf{B}_{n+1} \\ g_i & \longmapsto & \begin{cases} \sigma_1^2 & \text{if } i = 1 \\ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{i-1}^{-1} \circ \sigma_i^2 \circ \sigma_{i-1} \circ \cdots \circ \sigma_2 \circ \sigma_1 & \text{if } i \in \{2, \ldots, n\}. \end{cases} \end{array}$$

We assign $\varsigma_{0,1}$ to be the trivial morphism.

Proposition 2.8. *The family of morphisms $\{\varsigma_{n,1}\}_{n \in \mathbb{N}}$ satisfies Condition 2.3.*

Proof. Relation (4) is trivially satisfied for $n = 0$. Let $n \geq 1$ be a fixed natural number. By definition 1.4, we have $\left(b_{1,1}^{\beta}\right)^{-1} = \sigma_1^{-1}$. Moreover, for all $i \in \{2, \ldots, n\}$, we have $\varsigma_{n+1}\left(e_{\mathbf{F}_1} * g_{i-1}\right) = \varsigma_{n+1}(g_i)$ and

$$id_1 \natural \varsigma_{n,1}(g_{i-1}) = \sigma_2^{-1} \circ \cdots \circ \sigma_{i-1}^{-1} \circ \sigma_i^2 \circ \sigma_{i-1} \circ \cdots \circ \sigma_2.$$

We deduce that:

$$\left(\left(b_{1,1}^{\beta}\right)^{-1} \natural id_n\right) \circ \left(id_1 \natural \varsigma_{n,1}(g_{i-1})\right) \circ \left(b_{1,1}^{\beta} \natural id_n\right) = \varsigma_{n,1}(g_i).$$

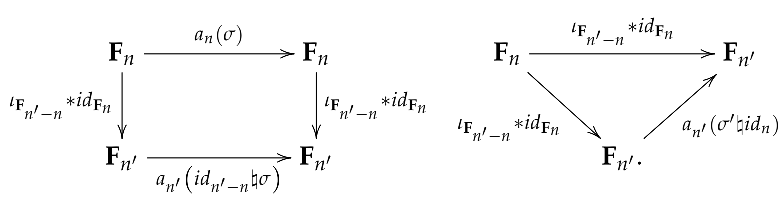
Hence Relation (4) of Lemma 2.5 is satisfied for all natural numbers. □

Example 2.9. Let us consider the trivial morphisms $\varsigma_{n,*} : \mathbf{F}_n \to 0_{\mathfrak{Gr}} \to \mathbf{B}_{n+1}$ for all natural numbers n. The relation of Lemma 2.5 being easily checked, this family of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n \in \mathbb{N}}$ satisfies Condition 2.3.

Action of braid groups on automorphism groups of free groups: There are several ways to consider the group $\mathbf{B}_n$ as a subgroup of $Aut(\mathbf{F}_n)$. For instance, the geometric point of view of topology gives us an action of $\mathbf{B}_n$ on the free group $\mathbf{F}_n$ (see for example [4] or [11]) identifying $\mathbf{B}_n$ as the mapping class group of a n-punctured disc $\Sigma_{0,1}^n$: fixing a point y on the boundary of the disc $\Sigma_{0,1}^n$, each free generator g_i can be taken as a loop of the disc based y turning around punctures. Each element σ of $\mathbf{B}_n$, as an automorphism up to isotopy of the disc $\Sigma_{0,1}^n$, induces a well-defined action on the fundamental group $\pi_1\left(\Sigma_{0,1}^n\right) \cong \mathbf{F}_n$ called Artin representation (see Example 2.15 for more details).

In the sequel, we fix a family of group actions of $\mathbf{B}_n$ on the free group $\mathbf{F}_n$: let $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n \in \mathbb{N}}$ be a family of group morphisms from the braid group $\mathbf{B}_n$ to the automorphism group $Aut(\mathbf{F}_n)$. For the work of Sections 2.2 and 4, we need the morphisms $a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)$ to satisfy more properties.

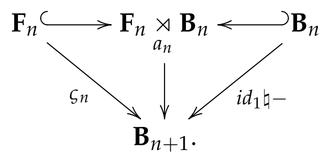
Condition 2.10. Let n and n' be natural numbers such that $n' \geq n$. We require $\left(\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_n}\right) \circ (a_n(\sigma)) = (a_{n'}(\sigma' \natural \sigma)) \circ \left(\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_n}\right)$ as morphisms $\mathbf{F}_n \to \mathbf{F}_{n'}$ for all elements σ of $\mathbf{B}_n$ and σ' of $\mathbf{B}_{n'-n}$, ie the following diagrams are commutative:



Remark 2.11. Condition 2.10 will be used to define the Long-Moody functor on morphisms in Theorem 2.21. Moreover, it will also be used for the proof of Propositions 4.14 and 4.18.

We will also require the families of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ to satisfy the following compatibility relations.

Condition 2.12. Let n be a natural number. We assume that the morphism given by the coproduct $\varsigma_n * (id_1 \natural -) : \mathbf{F}_n * \mathbf{B}_n \to \mathbf{B}_{n+1}$ factors across the canonical surjection to $\mathbf{F}_n \underset{a_n}{\rtimes} \mathbf{B}_n$. In other words, the following diagram is commutative:



where the morphism $\mathbf{F}_n \underset{a_n}{\rtimes} \mathbf{B}_n \to \mathbf{B}_{n+1}$ is induced by the morphism $\mathbf{F}_n * \mathbf{B}_n \to \mathbf{B}_{n+1}$ and the group morphism $id_1 \natural - : \mathbf{B}_n \to \mathbf{B}_{n+1}$ is induced by the monoidal structure. This is equivalent to requiring that, for all elements $\sigma \in \mathbf{B}_n$ and $g \in \mathbf{F}_n$, the following equality holds in $\mathbf{B}_{n+1}$:

$$(id_1 \natural \sigma) \circ \varsigma_n(g) = \varsigma_n(a_n(\sigma)(g)) \circ (id_1 \natural \sigma). \tag{5}$$

Remark 2.13. Condition 2.12 is essential in the definition of the Long-Moody functor on objects in Theorem 2.21.

We fix a choice for these families of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$.

Definition 2.14. The families $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ are said to be coherent if they satisfy conditions 2.3, 2.10 and 2.12.

Example 2.15. A classical family is provided by the Artin representations (see for example [4, Section 1]). For $n \in \mathbb{N}$, $a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)$ is defined for all elementary braids σ_i where $i \in \{1, \ldots, n-1\}$ by:

$$\begin{array}{rcl} a_{n,1}(\sigma_i) : \mathbf{F}_n & \longrightarrow & \mathbf{F}_n \\ g_j & \longmapsto & \begin{cases} g_{i+1} & \text{if } j = i \\ g_{i+1}^{-1} g_i g_{i+1} & \text{if } j = i+1 \\ g_j & \text{if } j \notin \{i, i+1\}. \end{cases} \end{array}$$

It clearly follows from their definitions that the morphisms $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ satisfy Condition 2.10.

Proposition 2.16. *The morphisms $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ together with the morphisms $\{\varsigma_{n,1} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ of Example 2.7 satisfy Condition 2.12.*

Proof. Let i be a fixed natural number in $\{1, \ldots, n-1\}$. We prove that the equality (5) of Condition 2.12 is satisfied for all Artin generator σ_i and all generator g_j of the free group (with $j \in \{1, \ldots, n\}$). First, it follows from the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ that:

$$\sigma_{1+i}^{-1} \circ \sigma_i^{-1} \circ \sigma_{1+i}^{-2} \circ \sigma_i^2 \circ \sigma_{1+i}^2 \circ \sigma_i \circ \sigma_{1+i} = \sigma_i^{-1} \circ \sigma_{1+i}^2 \circ \sigma_i,$$

and we deduce that:

$$\sigma_{1+i}^{-1} \circ \varsigma_{n,1}(a_{n,1}(\sigma_i)(g_{1+i})) \circ \sigma_{1+i} = \varsigma_{n,1}(g_{1+i}).$$

Also, the braid relation $\sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1} = \sigma_i \circ \sigma_{i+1} \circ \sigma_i$ implies that $\sigma_{i+1}^{-1} \circ \sigma_i^{-1} \circ \sigma_{i+1}^2 \circ \sigma_i \circ \sigma_{i+1} = \sigma_i^2$ and a fortiori:

$$\sigma_{1+i}^{-1} \circ \varsigma_{n,1}(a_{n,1}(\sigma_i)(g_i)) \circ \sigma_{1+i} = \varsigma_{n,1}(g_i).$$

Finally, for a fixed $j \notin \{i, i+1\}$, the commutation relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ and from the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ give directly:

$$\varsigma_{n,1}(g_j) = \sigma_{1+i}^{-1} \circ \varsigma_{n,1}(a_{n,1}(\sigma_i)(g_j)) \circ \sigma_{1+i}.$$

□

Corollary 2.17. *The families of morphisms* $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ *and* $\{\varsigma_{n,1} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ *are coherent.*

Example 2.18. Consider the family of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ of Example 2.9 and any family of morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$. Then Condition 2.12 is always satisfied. As a consequence, these families of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ are coherent if and only if the family of morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ satisfies Condition 2.10.

2.2 The Long-Moody functors

In this section, we prove that the Long-Moody construction of [17, Theorem 2.1] induces a functor

$$\mathbf{LM} : \mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}).$$

We fix families of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$, which are assumed to be coherent (see Definition 2.14).

We first need to make some observations and introduce some tools. Let F be an object of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$ and n be a natural number. A fortiori, the $\mathbb{K}$-module $F(n+1)$ is endowed with a left $\mathbb{K}[\mathbf{B}_{n+1}]$-module structure. Using the morphism $\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}$, $F(n+1)$ is a $\mathbb{K}[\mathbf{F}_n]$-module by restriction.

Let us consider the augmentation ideal of the free group $\mathbf{F}_n$, denoted by $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$. Since it is a (right) $\mathbb{K}[\mathbf{F}_n]$-module, one can form the tensor product $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+1)$. Also, for all natural numbers n and n' such that $n' \geq n$, the morphism $\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_n} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$ canonically induces a morphism $\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \hookrightarrow \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}$. In addition, the augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a $\mathbb{K}[\mathbf{B}_n]$-module too:

Lemma 2.19. *The action* $a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)$ *canonically induces an action of* $\mathbf{B}_n$ *on* $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ *denoted by* $a_n : \mathbf{B}_n \to Aut\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\right)$ *(abusing the notation).*

Proof. For any group morphism $H \to Aut(G)$, the group ring $\mathbb{K}[G]$ is canonically an H-module and so is the augmentation ideal $\mathcal{I}_G$, as a submodule of $\mathbb{K}[G]$. □

Remark 2.20. If the family of morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ is coherent with respect to the family of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$, the relation of Condition 2.10 remains true mutatis mutandis, for all natural numbers n and n', considering the induced morphisms $a_n : \mathbf{B}_n \to Aut\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\right)$ and $\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \to \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}$.

In the following theorem, we define an endofunctor of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$ corresponding to the Long-Moody construction. It will be called the Long-Moody functor with respect to $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$.

Theorem 2.21. *Recall that we have fixed coherent families of morphisms* $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ *and* $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$. *The following assignment defines a functor* $\mathbf{LM}_{a,\varsigma} : \mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$.

- *Objects: for* $F \in Obj(\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}))$, $\mathbf{LM}_{a,\varsigma}(F) : \mathfrak{U}\beta \to \mathbb{K}\text{-}\mathfrak{Mod}$ *is defined by:*
  - *Objects:* $\forall n \in \mathbb{N}$, $\mathbf{LM}_{a,\varsigma}(F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+1)$.
  - *Morphisms: for* $n, n' \in \mathbb{N}$, *such that* $n' \geq n$, *and* $[n'-n, \sigma] \in Hom_{\mathfrak{U}\beta}(n, n')$, *assign:*

$$\mathbf{LM}_{a,\varsigma}(F)([n'-n,\sigma])\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = a_{n'}(\sigma)\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i) \underset{\mathbb{K}[\mathbf{F}_{n'}]}{\otimes} F\left(id_1 \natural [n'-n,\sigma]\right)(v),$$

  for all $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ *and* $v \in F(n+1)$.

- *Morphisms: let* F *and* G *be two objects of* $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$, *and* $\eta : F \to G$ *be a natural transformation. We define* $\mathbf{LM}_{a,\varsigma}(\eta) : \mathbf{LM}_{a,\varsigma}(F) \to \mathbf{LM}_{a,\varsigma}(G)$ *for all natural numbers* n *by:*

$$\left(\mathbf{LM}_{a,\varsigma}(\eta)\right)_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} \eta_{n+1}.$$

In particular, the Long-Moody functor $\mathbf{LM}_{a,\varsigma}$ induces an endofunctor of the category $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

Notation 2.22. When there is no ambiguity, once the morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ are fixed, we omit them from the notation $\mathbf{LM}_{a,\varsigma}$ for convenience (especially for proofs).

Proof. For this proof, n, n' and n'' are natural numbers such that $n'' \geq n' \geq n$.

1. First let us show that the assignment of $\mathbf{LM}$ defines an endofunctor of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. The two first points generalize the proof of [17, Theorem 2.1]. Let F, G and H be objects of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

   (a) We first check the compatibility of the assignment $\mathbf{LM}(F)$ with respect to the tensor product. Consider $\sigma \in \mathbf{B}_n$ $g \in \mathbf{F}_n$, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n+1)$. Since $(id_1 \natural \sigma) \circ \varsigma_n(g) = \varsigma_n(a_n(\sigma)(g)) \circ (id_1 \natural \sigma)$ by Condition 2.12, we deduce that:

$$\begin{aligned}\mathbf{LM}(F)(\sigma)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(\varsigma_n(g))(v)\right) &= a_n(\sigma)(i) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(id_1 \natural \sigma)(F(\varsigma_n(g))(v)) \\ &= a_n(\sigma)(i) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} (F(\varsigma_n(a_n(\sigma)(g))) \circ F(id_1 \natural \sigma))(v) \\ &= a_n(\sigma)(i \cdot g) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(id_1 \natural \sigma)(v) \\ &= \mathbf{LM}(F)(\sigma)\left(i \cdot g \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} (v)\right).\end{aligned}$$

   (b) Let us prove that the assignment $\mathbf{LM}(F)$ defines an object of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. According to our assignment and since a_n and $id_1 \natural -$ are group morphisms, it follows from the definition that $\mathbf{LM}(F)(id_{\mathbf{B}_n}) = id_{\mathbf{LM}(F)(n)}$. Hence, it remains to prove that the composition axiom is satisfied. Let σ and σ' be two elements of $\mathbf{B}_n$, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n+1)$. From the functoriality of F over $\boldsymbol{\beta}$ and the compatibility of the monoidal structure $\natural$ with composition, we deduce that $F(id_1 \natural (\sigma')) \circ F(id_1 \natural (\sigma)) = F(id_1 \natural (\sigma' \circ \sigma))$. Since a_n is a group morphism, we have:

$$(a_n(\sigma' \circ \sigma))(i) = a_n(\sigma')(a_n(\sigma)(i)).$$

   Hence, it follows from the assignment of $\mathbf{LM}$ that:

$$\begin{aligned}\mathbf{LM}(F)(\sigma' \circ \sigma)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) &= (a_n(\sigma' \circ \sigma))(i) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(id_1 \natural (\sigma' \circ \sigma))(v) \\ &= a_n(\sigma')(a_n(\sigma)(i)) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} (F(id_1 \natural (\sigma')) \circ F(id_1 \natural (\sigma)))(v) \\ &= \mathbf{LM}(F)(\sigma') \circ \mathbf{LM}(F)(\sigma)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right).\end{aligned}$$

   (c) It remains to check the consistency of our definition of $\mathbf{LM}$ on morphisms of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Let $\eta : F \to G$ be a natural transformation. Hence, we have that:

$$G(id_1 \natural \sigma) \circ \eta_{n+1} = \eta_{n'+1} \circ F(id_1 \natural \sigma).$$

   Hence, it follows from the assignment of $\mathbf{LM}$ that:

$$\mathbf{LM}(G)(\sigma) \circ \mathbf{LM}(\eta)_n = \mathbf{LM}(\eta)_{n'} \circ \mathbf{LM}(F)(\sigma)$$

   Therefore $\mathbf{LM}(\eta)$ is a morphism in the category $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Denoting by $id_F : F \to F$ the identity natural transformation, it is clear that $\mathbf{LM}(id_F) = id_{\mathbf{LM}(F)}$. Finally, let us check the composition axiom. Let $\eta : F \to G$ and $\mu : G \to H$ be natural transformations. Let n be a natural number, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n)$. Now, since μ and η are morphisms in the category $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$:

$$\mathbf{LM}(\mu \circ \eta)_n\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} (\mu_{n+1} \circ \eta_{n+1})(v) = \mathbf{LM}(\mu)_n \circ \mathbf{LM}(\eta)_n\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right).$$

2. Let us prove that the assignment **LM** lifts to define an endofunctor of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Let F, G and H be objects of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

   (a) First, let us check the compatibility of the assignment $\mathbf{LM}(F)$ with respect to the tensor product. In fact, this compatibility being already done for automorphisms (see 1a), the remaining point to prove is the compatibility of $\mathbf{LM}(F)([n'-n, id_{n'}])$. Let $g \in \mathbf{F}_n$, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n+1)$. It follows from Condition 2.3 that in $\mathbf{B}_{n+1}$:

   $$id_1 \natural \left[n'-n, id_{n'-n} \natural \varsigma_n(g)\right] = \varsigma_{n'}\left(e_{\mathbf{F}_{n'-n}} * g\right) \circ \left(id_1 \natural \left[n'-n, id_{n'}\right]\right).$$

   Since $\left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i \cdot g) = \left(e_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} * i\right) \cdot \left(e_{\mathbf{F}_{n'-n}} * g\right)$, we deduce that:

   $$\begin{aligned} & \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(\varsigma_n(g))(v)\right) \\ = & \left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i) \underset{\mathbb{K}\left[\mathbf{F}_{n'}\right]}{\otimes} F\left(id_1 \natural \left[n'-n, id_{n'}\right]\right)\left(F(\varsigma_n(g))(v)\right) \\ = & \left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i \cdot g) \underset{\mathbb{K}\left[\mathbf{F}_{n'}\right]}{\otimes} F\left(id_1 \natural \left[n'-n, id_{n'}\right]\right)(v) \\ = & \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right)\left(i \cdot g \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right). \end{aligned}$$

   (b) Let us prove that the assignment $\mathbf{LM}(F)$ defines an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ using Proposition 1.10. Recall the compatibility of the monoidal structure $\natural$ with respect to composition and that F is an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Consider $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$. It follows from our assignment, that:

   $$\mathbf{LM}(F)\left(\left[n'-n, \sigma\right]\right) = \mathbf{LM}(F)(\sigma) \circ \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right).$$

   Moreover, the composition of morphisms introduced in Definition 2.2 implies that:

   $$\mathbf{LM}(F)\left(\left[n''-n, id_{n''}\right]\right) = \mathbf{LM}(F)\left(\left[n''-n', id_{n''}\right]\right) \circ \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right).$$

   Hence, the relation (1) of Proposition 1.10 is satisfied. Let $\sigma \in \mathbf{B}_n$ and $\psi \in \mathbf{B}_{n'-n}$. Since $(\iota_{n'-n} * id_n) \circ (a_n(\sigma)) = (a_{n'}(\psi \natural \sigma)) \circ (\iota_{n'-n} * id_n)$ by Condition 2.10, we deduce that:

   $$\mathbf{LM}(F)(\psi \natural \sigma) \circ \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right) = \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right) \circ \mathbf{LM}(F)(\sigma).$$

   Hence the relation (2) of Proposition 1.10 is also satisfied. Therefore, according to Proposition 1.10, since $\mathbf{LM}(F)$ is an object of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, the assignment $\mathbf{LM}(F)$ defines an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

   (c) Finally, let us check the consistency of our assignment for **LM** on morphisms. Let $\eta : F \to G$ be a natural transformation. We already proved in 1c that $\mathbf{LM}(\eta)$ is a morphism in the category $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Since η is a natural transformation between objects of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we have that:

   $$G\left(id_1 \natural \left[n'-n, id_{n'}\right]\right) \circ \eta_{n+1} = \eta_{n'+1} \circ F\left(id_1 \natural \left[n'-n, id_{n'}\right]\right).$$

   Hence, it follows from the assignment of **LM** that:

   $$\mathbf{LM}(G)\left(\left[n'-n, id_{n'}\right]\right) \circ \mathbf{LM}(\eta)_n = \mathbf{LM}(\eta)_{n'} \circ \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right).$$

   Hence the relation (3) of Proposition 1.12 is satisfied, and we deduce from this last proposition that $\mathbf{LM}(\eta)$ is a morphism in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. The verification of the composition axiom repeats mutatis mutandis the one of 1c.

□

Recall the following fact on the augmentation ideal of the free group $\mathbf{F}_n$ where $n \in \mathbb{N}$.

Proposition 2.23. *[25, Chapter 6, Proposition 6.2.6] The augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a free $\mathbb{K}[\mathbf{F}_n]$-module with basis the set $\{(g_i - 1) \mid i \in \{1, \ldots, n\}\}$.*

This result allows us to prove the following properties.

Proposition 2.24. *The functor $\mathbf{LM}_{a,\varsigma} : \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ is reduced and exact. Moreover, it commutes with all colimits and all finite limits.*

Proof. Let $0_{\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ denote the null functor. It follows from the definition of the Long-Moody functor that $\mathbf{LM}\left(0_{\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})}\right) = 0_{\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})}$.
Let n be a natural number. Since the augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a free $\mathbb{K}[\mathbf{F}_n]$-module (as stated in Proposition 2.23), it is therefore a flat $\mathbb{K}[\mathbf{F}_n]$-module. Then, the result follows from the fact that the functor $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} - : \mathbb{K}\text{-}\mathfrak{Mod} \to \mathbb{K}\text{-}\mathfrak{Mod}$ is an exact functor, the naturality for morphisms following from the definition of the Long-Moody functor (see Theorem 2.21).

Similarly, the fact that the functor $\mathbf{LM}_{a,\varsigma}$ commutes with all colimits is a formal consequence of the commutation with all colimits of the tensor products $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} -$ for all natural numbers n. The commutation result for finite limits is a property of exact functors (see for example [18, Chapter 8, section 3]). □

Remark 2.25. Let F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ and n a natural number. For all $k \in \{1, \ldots, n\}$, we denote $F(n+1)_k = \mathbb{K}[(g_k - 1)] \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+1)$ with g_k a generator of $\mathbf{F}_n$. We define an isomorphism

$$\begin{array}{rcl} \Lambda_{n,F} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+1) & \longrightarrow & \bigoplus_{k=1}^{n} F(n+1)_k \cong (F(n+1))^{\oplus n} \\ (g_k - 1) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v & \longmapsto & \left(0, \ldots, 0, \overbrace{v}^{k\text{-th}}, 0, \ldots, 0\right). \end{array}$$

Thus, for $\eta : F \to G$ a natural transformation, with Λ:

$$\forall n \in \mathbb{N},\ \Lambda_n\left((\mathbf{LM}(\eta))_n\right) = \eta_{n+1}^{\oplus n}.$$

Hence, we can have a matricial point of view on this construction (see [17, Theorem 2.2]). Similarly, the study of Bigelow and Tian in [3] is performed from a purely matricial point of view.

Case of trivial ς: Finally, let us consider the family of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ of Example 2.9.

Remark 2.26. As stated in Example 2.18, we only need to consider a family of morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ which satisfies Condition 2.10 so that the families $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ are coherent.

Notation 2.27. We denote by $\mathfrak{X} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ the constant functor such that $\mathfrak{X}(n) = \mathbb{K}$ for all natural numbers n.

We have the following remarkable property.

Proposition 2.28. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ a family of morphisms which satisfies Condition 2.10. Then, as objects of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, $\mathbf{LM}_{a,\varsigma_*}(F) \cong \mathbf{LM}_{a,\varsigma_*}(\mathfrak{X}) \underset{\mathbb{K}}{\otimes} F(1\natural -)$.*

Proof. Remark 2.25 shows that there is an isomorphism of $\mathbb{K}$-modules of the form:

$$\mathbf{LM}_{a,\varsigma_*}(F)(n) \xrightarrow{\Lambda_{n,F}} (F(n+1))^{\oplus n} \xrightarrow{\left(\Lambda_{n,\mathfrak{X}} \underset{\mathbb{K}}{\otimes} id_{F(1\natural n)}\right)^{-1}} \mathbf{LM}_{a,\varsigma_*}(\mathfrak{X})(n) \underset{\mathbb{K}}{\otimes} F(1\natural n).$$

It is straightforward to check that this isomorphism is natural if ς is trivial. □

2.3 Evaluation of the Long-Moody functor

A first step to understand the behaviour of a Long-Moody endofunctor is to investigate its effect on the constant functor $\mathfrak{X}$. This is indeed the most basic functor to study. Moreover, as Proposition 2.28 shows, the evaluation on this functor is the fundamental information to understand a given Long-Moody endofunctor when we consider the family of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ of Example 2.9.

Fixing coherent families of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$, we consider the Long-Moody functor

$$\mathbf{LM}_{a,\varsigma} : \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}).$$

For a fixed natural number n, using the isomorphism Λ_n of Remark 2.25, we observe that $\mathbf{LM}_{a,\varsigma}(\mathfrak{X})(n) \cong \mathbb{K}^{\oplus n}$.

Notation 2.29. Let y be an invertible element of $\mathbb{K}$. Let $y\mathfrak{X} : \boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ be the functor defined for all natural numbers n by $y\mathfrak{X}(n) = \mathbb{K}$ and such that:

- if $n = 0$ or $n = 1$, then $y\mathfrak{X}(id) = id_{\mathbb{K}}$;
- if $n \geq 2$, for every Artin generator σ_i of $\mathbf{B}_n$, $(y\mathfrak{X})(\sigma_i) : \mathbb{K} \to \mathbb{K}$ is the multiplication by y.

For an object F of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we denote the functor $y\mathfrak{X} \underset{\mathbb{K}}{\otimes} F : \boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ by yF.

2.3.1 Computations for $\mathbf{LM}_1$

Let us assume that $\mathbb{K} = \mathbb{C}\left[t^{\pm 1}\right]$. Let us consider the coherent families of morphisms $\{\varsigma_{n,1} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ (introduced in Example 2.7) and $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ (introduced in Example 2.15). We denote by $\mathbf{LM}_1$ the associated Long-Moody functor. We are interested in the behaviour of the functor $t^{-1}\mathbf{LM}_1(t\mathfrak{X}) : \boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ on automorphisms of the category $\mathfrak{U}\boldsymbol{\beta}$. Indeed, adding a parameter t is necessary to recover functors specifically associated with the category $\mathfrak{U}\boldsymbol{\beta}$, such as $\mathfrak{Bur}_t$ (see Section 1.2). Let us fix n a natural number and σ_i an Artin generator of $\mathbf{B}_n$.

Beforehand, let us understand the action $a_{n,1} : \mathbf{B}_n \longrightarrow Aut\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\right)$ induced by $a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)$. We compute:

$$\begin{array}{rcl} a_{n,1}(\sigma_i) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} & \longrightarrow & \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \\ g_j - 1 & \longmapsto & \begin{cases} g_{i+1} - 1 & \text{if } j = i \\ g_{i+1}^{-1} g_i g_{i+1} - 1 = [g_i - 1]\, g_{i+1} + [g_{i+1} - 1]\left(1 - g_{i+1}^{-1} g_i g_{i+1}\right) & \text{if } j = i+1 \\ g_j - 1 & \text{if } j \notin \{i, i+1\}. \end{cases} \end{array}$$

Notation 2.30. Let us fix the matrices $r_n = \overbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}}^{n}$ for all natural numbers n.

Hence, we have the following result.

Proposition 2.31. *The matrices $\{r_n\}_{n\in\mathbb{N}}$ define a natural equivalence $t^{-1}\mathbf{LM}_1(t\mathfrak{X}) \xrightarrow{r} \mathfrak{Bur}_{t^2}$ as objects of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.*

Proof. Using the isomorphism Λ_n of Remark 2.25, we obtain that for σ_i an Artin generator of $\mathbf{B}_n$:

$$t^{-1}\mathbf{LM}_1(t\mathfrak{X})(\sigma_i) = Id_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ t^2 & 1 - t^2 \end{bmatrix} \oplus Id_{n-i-1}.$$

We deduce that $r_n \circ \left(t^{-1}\mathbf{LM}_1(t\mathfrak{X})(\sigma_i)\right) \circ r_n^{-1} = \mathfrak{Bur}_{t^2}(\sigma_i)$. □

Recovering of the Lawrence-Krammer functor: Let us first introduce the following result due to Long in [17]. We assume that $\mathbb{K} = \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$. For this paragraph, we assume that $1 + qt = 0$, q has a square root, $q^2 \neq 1$ and $q^3 \neq 1$.

Notation 2.32. We denote by $\mathfrak{X}' : \boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]\text{-}\mathfrak{Mod}$ the constant functor such that $\mathfrak{X}'(n) = \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$ for all natural numbers n. Generally speaking, for F an object of $\mathbf{Fct}\left(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$ the representation of $\mathbf{B}_n$ induced by F will be denoted by $F_{|\mathbf{B}_n}$.

Proposition 2.33. *[17, special case of Corollary 2.10] Let n be a natural number such that $n \geq 4$. Then, the Lawrence-Krammer representation $\mathfrak{LK}_{|\mathbf{B}_n}$ is a subrepresentation of $q^{-1}\left(\mathbf{LM}_1\left(q\left(t^{-1}\mathbf{LM}_1\left(t\mathfrak{X}\right)\right)\right)\right)_{|\mathbf{B}_n}$.*

We first need to introduce new tools. Let n and m be two natural numbers. Let $\underline{w}_n = (w_1, \ldots, w_n) \in \mathbb{C}^n$ such that $w_i \neq w_j$ if $i \neq j$. We consider the configuration space:

$$Y_{\underline{w}_n, m} = \left\{(z_1, \ldots, z_m) \mid z_i \in \mathbb{C},\, z_i \neq w_k \text{ for } 1 \leq k \leq n,\, z_i \neq z_j \text{ if } i \neq j\right\}.$$

The two following results due to Long will be crucial to prove Proposition 2.33.

Proposition 2.34. *[17, Corollary 2.7] Let n be a natural number and $\rho : \mathbf{B}_{n+1} \to GL(V)$ be a representation of $\mathbf{B}_n$ with V a $\mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$-module. Then, the representation defined by Long in [17, Theorem 2.1], which we denote by $\mathcal{LM}$, is a group morphism:*

$$q^{-1}\mathcal{LM}\left(q\rho\right) : \mathbf{B}_n \to GL\left(H^1\left(Y_{\underline{w}_n, 1}, E_\rho\right)\right)$$

for E_ρ a flat vector bundle associated with ρ (see [17, p. 225-226]).

Lemma 2.35. *[17, Lemma 2.9] For all natural numbers m, there is an isomorphism of abelian groups:*

$$H^{m+1}\left(Y_{\underline{w}_n, m+1}, E_{\mathfrak{X}_{|\mathbf{B}_n}}\right) \cong H^1\left(Y_{\underline{w}_n, 1}, H^m\left(Y_{\underline{w}_{n+1}, m}, E_{\mathfrak{X}_{|\mathbf{B}_n}}\right)\right).$$

In particular, for $m = 1$, $H^2\left(Y_{\underline{w}_n, 2}, E_{\mathfrak{X}_{|\mathbf{B}_n}}\right) \cong H^1\left(Y_{\underline{w}_n, 1}, H^1\left(Y_{\underline{w}_{n+1}, 2}, E_{\mathfrak{X}_{|\mathbf{B}_n}}\right)\right)$.

Proof of Proposition 2.33. By Proposition 2.34, we can write as a representation:

$$q^{-1}\mathcal{LM}\left(q\left(t^{-1}\mathcal{LM}\left(t\mathfrak{X}\right)\right)\right) : \mathbf{B}_n \to GL\left(H^1\left(Y_{\underline{w}_n, 1}, E_{t^{-1}\mathcal{LM}(t\mathfrak{X})}\right)\right).$$

A fortiori by Lemma 2.35, $q^{-1}\mathcal{LM}\left(q\left(t^{-1}\mathcal{LM}\left(t\mathfrak{X}_{|\mathbf{B}_n}\right)\right)\right)$ is an action of $\mathbf{B}_n$ on $H^2\left(Y_{\underline{w}_n, 2}, E_{\mathfrak{X}_{|\mathbf{B}_n}}\right)$. In particular, for $m = 2$ and $n \geq 4$, according to [14, Theorem 5.1], the representation of $\mathbf{B}_n$ factoring through the Iwahori–Hecke algebra $H_n(t)$ corresponding to the Young diagram $(n-2, 2)$ is a subrepresentation of $q^{-1}\mathcal{LM}\left(q\left(t^{-1}\mathcal{LM}\left(t\mathfrak{X}_{|\mathbf{B}_n}\right)\right)\right)$. Moreover, this representation is equivalent to the Lawrence-Krammer representation by [1, Section 5]. By the definition of the Long-Moody construction (see [17, Theorem 2.1]), $q^{-1}\mathcal{LM}\left(q\left(t^{-1}\mathcal{LM}\left(t\mathfrak{X}_{|\mathbf{B}_n}\right)\right)\right)$ is the representation $q^{-1}\left(\tau_1\mathbf{LM}_1\right)\left(q\left(t^{-1}\mathbf{LM}_1\left(t\mathfrak{X}\right)\right)\right)_{|\mathbf{B}_n}$. □

We denote by $\mathfrak{LK}^{\geq 4} : \boldsymbol{\beta} \longrightarrow \left(\mathbb{C}\left[t^{\pm 1}\right]\right)\left[q^{\pm 1}\right]\text{-}\mathfrak{Mod}$ the subfunctor of the Lawrence-Krammer defined in Example 1.2 which is null on the objects such that $n < 4$. The result of Proposition 2.33 implies that:

Proposition 2.36. *The functor $\mathfrak{LK}^{\geq 4}$ is a subfunctor of $q^{-1}\left(\tau_1\mathbf{LM}_1\right)\left(q\left(t^{-1}\mathbf{LM}_1\left(t\mathfrak{X}\right)\right)\right)^{\geq 4}$.*

2.3.2 Computations for other cases

Let us introduce examples of Long-Moody functors which arise using other actions $a_n : \mathbf{B}_n \to Aut\left(\mathbf{F}_n\right)$.

Wada representations In 1992, Wada introduced in [24] a certain type of family of representations of braid groups. We give here a functorial approach to this work.

Definition 2.37. Let $Aut_{-} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ be the functor defined by:

- Objects: for all natural numbers n, $Aut_{-}(n) = Aut(\mathbf{F}_n)$ the automorphism group of the free group on n generators;
- Morphisms: let n be a natural number. We define $Aut_{-}(\gamma_n) : Aut(\mathbf{F}_n) \hookrightarrow Aut(\mathbf{F}_{n+1})$ assigning $Aut_{-}(\gamma_n)(\varphi) = id_1 * \varphi$ for all $\varphi \in Aut(\mathbf{F}_n)$, using the monoidal category $(\mathfrak{gr}, *, 0)$ recalled in Notation 1.16.

Definition 2.38. Let us consider two different non-trivial reduced words $W(g_1, g_2)$ and $V(g_1, g_2)$ on $\mathbf{F}_2$, such that:

- the assignments $g_1 \mapsto W(g_1, g_2)$ and $g_2 \mapsto V(g_1, g_2)$ define a automorphism of $\mathbf{F}_2$;
- the assignment $(W, V) : \mathbf{B}_2 \longrightarrow Aut(\mathbf{F}_2)$:

$$[(W,V)(\sigma_1)](g_j) = \begin{cases} W(g_1, g_2) & \text{if } j = 1 \\ V(g_1, g_2) & \text{if } j = 2 \end{cases}$$

  is a morphism.

Two morphisms $(W, V) : \mathbf{B}_2 \longrightarrow Aut(\mathbf{F}_2)$ and $(W', V') : \mathbf{B}_2 \to Aut(\mathbf{F}_2)$ are said to be swap-dual if $W'(g_1, g_2) = V(g_2, g_1)$ and $V'(g_1, g_2) = W(g_2, g_1)$, backward-dual if $W'(g_1, g_2) = \left(W\left(g_1^{-1}, g_2^{-1}\right)\right)^{-1}$ and $V'(g_1, g_2) = \left(V\left(g_1^{-1}, g_2^{-1}\right)\right)^{-1}$, inverse if $(W', V') = (W, V)^{-1}$.

Definition 2.39. [24] Let $W(g_1, g_2)$ and $V(g_1, g_2)$ be two words on $\mathbf{F}_2$. A natural transformation $\mathcal{W} : \mathbf{B}_{-} \to Aut_{-}$ is said to be of Wada-type if for all natural numbers n, for all $i \in \{1, \ldots, n-1\}$, the following diagram is commutative (we recall that $incl_i^n$ was introduced in Notation 1.18 and $Aut_{-}(\gamma_{2,i})$ in Definition 2.37):

$$\begin{array}{ccc} \mathbf{B}_n & \xrightarrow{\mathcal{W}_n} & Aut(\mathbf{F}_n) \\ {\scriptstyle incl_i^n}\uparrow & & \uparrow{\scriptstyle Aut_{-}(\gamma_{2,i}) * id_{\mathbf{F}_{n-i-1}}} \\ \mathbf{B}_2 & \xrightarrow[(W,V)]{} & Aut(\mathbf{F}_2). \end{array}$$

Remark 2.40. Note that therefore a Wada-type natural transformation is entirely determined by the choice of (W, V).

Wada conjectured a classification of these type of representations. This conjecture was proved by Ito in [10].

Theorem 2.41. *[10] There are seven classes of Wada-type natural transformation $\mathcal{W}$ up to the swap-dual, backward-dual and inverse equivalences, listed below.*

1. $(W, V)_{1,m}(g_1, g_2) = \left(g_2, g_2^m g_1 g_2^{-m}\right)$ *where* $m \in \mathbb{Z}$;
2. $(W, V)_2(g_1, g_2) = (g_1, g_2)$;
3. $(W, V)_3(g_1, g_2) = \left(g_2, g_1^{-1}\right)$;
4. $(W, V)_4(g_1, g_2) = \left(g_2, g_2^{-1} g_1^{-1} g_2\right)$;
5. $(W, V)_5(g_1, g_2) = \left(g_2^{-1}, g_1^{-1}\right)$;
6. $(W, V)_6(g_1, g_2) = \left(g_2^{-1}, g_2 g_1 g_2\right)$;

7. $(W,V)_7(g_1,g_2) = \left(g_1 g_2^{-1} g_1^{-1}, g_1 g_2^2\right)$.

Remark 2.42. Note that the action given by the first Wada representation with $m = 1$ is a generalization of the Artin representation.

Notation 2.43. The actions given by the k-th Wada-type natural transformation will be denoted by $a_{n,k} : \mathbf{B}_n \hookrightarrow Aut(\mathbf{F}_n)$. In particular, for $k = 1$ with $m = 1$, we recover the Artin representation (see Example 2.15).

For all $1 \leq k \leq 8$, it clearly follows from their definitions that the families of morphisms $\left\{a_{n,k} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\right\}_{n\in\mathbb{N}}$ satisfy Condition 2.10. Hence, for $1 \leq k \leq 8$, we consider a family of morphisms $\left\{\varsigma_{n,k} : \mathbf{F}_n \to \mathbf{B}_{n+1}\right\}$ assumed to be coherent with respect to the morphisms $\left\{a_{n,k} : \mathbf{B}_n \hookrightarrow Aut(\mathbf{F}_n)\right\}_{n\in\mathbb{N}}$ (in the sense of Definition 2.14). Such morphisms $\varsigma_{n,k}$ always exist because we could at least take the family of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}$ (see Example 2.18). We denote by $\mathbf{LM}_k : \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ the corresponding Long-Moody functor defined in Theorem 2.21 for $k \in \{1, \ldots, 8\}$.

Let us imitate the procedure of Section 2.3.1. We assume that $\mathbb{K} = \mathbb{C}\left[t^{\pm 1}\right]$. Let n be a fixed natural number. Let us consider the case of $k = 2$. Using the isomorphism Λ_n of Remark 2.25, we obtain the functor $\mathbf{LM}_2(\mathfrak{X}) : \boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$, defined for $\sigma_i \in \mathbf{B}_n$ by:

$$\mathbf{LM}_2(F)(\sigma_i) = (F(\sigma_i))^{\oplus n}.$$

For $k = 3$, using Λ_n, we compute that the functor $t^{-1}\mathbf{LM}_3(t\mathfrak{X}) : \boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ is defined for $\sigma_i \in \mathbf{B}_n$ by:

$$t^{-1}\mathbf{LM}_3(t\mathfrak{X})(\sigma_i) = Id_{i-1} \oplus \begin{bmatrix} 0 & -\varsigma_{n,3}(g_i) \\ 1 & 0 \end{bmatrix} \oplus Id_{n-i-1}.$$

Hence, the functor $t^{-1}\mathbf{LM}_3(t\mathfrak{X})$ is very similar to the one associated with the Tong-Yang-Ma representations (recall Definition 1.2). We deduce that the identity natural equivalence gives $t^{-1}\mathbf{LM}_3(t\mathfrak{X}) \cong \mathfrak{TYM}_{-\varsigma_{n,3}(g_i)}$ as objects of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

For the actions given by the Wada-type natural transformation 4, 5, 6 and 7 in Theorem 2.41, the produced functors $t^{-1}\mathbf{LM}_i(t\mathfrak{X}) : \boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ are mild variants of what is given by the case $i = 1$.

3 Strong polynomial functors

We deal here with the concept of a strong polynomial functor. This type of functor will be the core of our work in Section 4. We review (and actually extend) the definition and properties of a strong polynomial functor due to Djament and Vespa in [7] and also a particular case of coefficient systems of finite degree used by Randal-Williams and Wahl in [20].

In [7, Section 1], Djament and Vespa construct a framework to define strong polynomial functors in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, where $\mathfrak{M}$ is a symmetric monoidal category, the unit is an initial object and $\mathcal{A}$ is an abelian category. Here, we generalize this definition for functors from pre-braided monoidal categories having the same additional property. In particular, the notion of strong polynomial functor will be defined for the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. The keypoint of this section is Proposition 3.2, in so far as it constitutes the crucial property necessary and sufficient to extend the definition of strong polynomial functor to the pre-braided case.

3.1 Strong polynomiality

We first introduce the translation functor, which plays the central role in the definition of strong polynomiality.

Definition 3.1. Let $(\mathfrak{M}, \natural, 0)$ be a strict monoidal small category, let $\mathfrak{D}$ be a category and let x be an object of $\mathfrak{M}$. The monoidal structure defines the endofunctor $x\natural - : \mathfrak{M} \longrightarrow \mathfrak{M}$. We define the translation by x functor $\tau_x : \mathbf{Fct}(\mathfrak{M}, \mathfrak{D}) \to \mathbf{Fct}(\mathfrak{M}, \mathfrak{D})$ to be the endofunctor obtained by precomposition by the functor $x\natural -$.

The following proposition establishes the commutation of two translation functors associated with two objects of $\mathfrak{M}$. It is the keystone property to define strong polynomial functors.

Proposition 3.2. *Let $\mathfrak{D}$ be a category and $(\mathfrak{M}, \natural, 0)$ be a strict monoidal small category equipped with natural (in x and y) isomorphisms $x\natural y \cong y\natural x$. Let x and y be two objects of $\mathfrak{M}$. Then, there exists a natural isomorphism between functors from* $\mathbf{Fct}(\mathfrak{M}, \mathfrak{D})$ *to* $\mathbf{Fct}(\mathfrak{M}, \mathfrak{D})$*:*

$$\tau_x \circ \tau_y \cong \tau_y \circ \tau_x.$$

Proof. First, because of the associativity of the monoidal product $\natural$ and the strictness of $\mathfrak{M}$, we have that $\tau_x \circ \tau_y = \tau_{x\natural y}$ and $\tau_y \circ \tau_x = \tau_{y\natural x}$. We denote by $b^{\mathfrak{M}}_{-,-}$ the pre-braiding of $\mathfrak{M}$. The key point is the fact that as $b^{\mathfrak{M}}_{-,-}$ is a braiding on the maximal subgroupoid of $\mathfrak{M}$ (see Definition 1.13), $b^{\mathfrak{M}}_{x,y} : x\natural y \xrightarrow{\cong} y\natural x$ defines an isomorphism. Hence, precomposition by $b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}$ defines a natural transformation $\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^* : \tau_{x\natural y} \to \tau_{y\natural x}$. It is an isomorphism since we analogously construct an inverse natural transformation $\left(\left(b^{\mathfrak{M}}_{x,y}\right)^{-1}\natural id_{\mathfrak{M}}\right)^* : \tau_{y\natural x} \to \tau_{x\natural y}$. □

Remark 3.3. In Proposition 3.2, the natural isomorphism is not unique: as the proof shows, we could have used the morphism $\left(b^{\mathfrak{M}}_{y,x}\right)^{-1}\natural id_{\mathfrak{M}}$ instead to define an isomorphism between $\tau_{x\natural y}(F)$ and $\tau_{y\natural x}(F)$. In fact, a category only needs to be equipped with natural (in x and y) isomorphisms $x\natural y \cong y\natural x$ to satisfy the conclusion of Proposition 3.2.

Let us move on to the introduction of the evanescence and difference functors, which will characterize the (very) strong polynomiality of a functor in $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Recall that, if $\mathfrak{M}$ is a small category and $\mathcal{A}$ is an abelian category, then the functor category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is an abelian category (see [18, Chapter VIII]).

From now until the end of Section 3, we fix $(\mathfrak{M}, \natural, 0)$ a pre-braided strict monoidal category such that the monoidal unit 0 is an initial object, $\mathcal{A}$ an abelian category and x denotes an object of $\mathfrak{M}$.

Definition 3.4. For all objects F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, we denote by $i_x(F) : \tau_0(F) \to \tau_x(F)$ the natural transformation induced by the unique morphism $\iota_x : 0 \to x$ of $\mathfrak{M}$. This induces $i_x : Id_{\mathbf{Fct}(\mathfrak{M},\mathcal{A})} \to \tau_x$ a natural transformation of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Since the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is abelian, the kernel and cokernel of the natural transformation i_x exist. We define the functors $\kappa_x = \ker(i_x)$ and $\delta_x = \mathrm{coker}(i_x)$. The endofunctors κ_x and δ_x of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ are called respectively evanescence and difference functor associated with x.

The following proposition presents elementary properties of the translation, evanescence and difference functors. They are either consequences of the definitions, or direct generalizations of the framework considered in [7] where $\mathfrak{M}$ is symmetric monoidal.

Proposition 3.5. *Let y be an object of $\mathfrak{M}$. Then the translation functor τ_x is exact and we have the following exact sequence in the category of endofunctors of* $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$*:*

$$0 \longrightarrow \kappa_x \xrightarrow{\Omega_x} Id \xrightarrow{i_x} \tau_x \xrightarrow{\Delta_x} \delta_x \longrightarrow 0. \tag{6}$$

Moreover, for a short exact sequence $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$*, there is a natural exact sequence in the category* $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$*:*

$$0 \longrightarrow \kappa_x(F) \longrightarrow \kappa_x(G) \longrightarrow \kappa_x(H) \longrightarrow \delta_x(F) \longrightarrow \delta_x(G) \longrightarrow \delta_x(H) \longrightarrow 0. \tag{7}$$

In addition:

1. *The translation endofunctor τ_x of* $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ *commutes with limits and colimits.*
2. *The difference endofunctors δ_x and δ_y of* $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ *commute up to natural isomorphism. They commute with colimits.*
3. *The endofunctors κ_x and κ_y of* $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ *commute up to natural isomorphism. They commute with limits.*
4. *The natural inclusion $\kappa_x \circ \kappa_x \hookrightarrow \kappa_x$ is an isomorphism.*
5. *The translation endofunctor τ_x and the difference endofunctor δ_y commute up to natural isomorphism.*
6. *The translation endofunctor τ_x and the endofunctor κ_y commute up to natural isomorphism.*

7. *We have the following natural exact sequence in the category of endofunctors of* **Fct** $(\mathfrak{M},\mathcal{A})$*:*

$$0 \longrightarrow \kappa_y \longrightarrow \kappa_{x\natural y} \longrightarrow \tau_x\kappa_y \longrightarrow \delta_y \longrightarrow \delta_{x\natural y} \longrightarrow \tau_y\delta_x \longrightarrow 0. \tag{8}$$

Proof. In the symmetric monoidal case, this is [7, Proposition 1.4]: the numbered properties are formal consequences of the commutation property of the translation endofunctors given by Proposition 3.2. Hence, the proofs carry over mutatis mutandis to the pre-braided setting. □

Using Proposition 3.5, we can define strong polynomial functors.

Definition 3.6. We recursively define on $n \in \mathbb{N}$ the category $\mathcal{P}ol_n^{strong}(\mathfrak{M},\mathcal{A})$ of strong polynomial functors of degree less than or equal to n to be the full subcategory of **Fct** $(\mathfrak{M},\mathcal{A})$ as follows:

1. If $n<0$, $\mathcal{P}ol_n^{strong}(\mathfrak{M},\mathcal{A}) = \{0\}$;
2. if $n \geq 0$, the objects of $\mathcal{P}ol_n^{strong}(\mathfrak{M},\mathcal{A})$ are the functors F such that for all objects x of $\mathfrak{M}$, the functor $\delta_x(F)$ is an object of $\mathcal{P}ol_{n-1}^{strong}(\mathfrak{M},\mathcal{A})$.

For an object F of **Fct** $(\mathfrak{M},\mathcal{A})$ which is strong polynomial of degree less than or equal to $n \in \mathbb{N}$, the smallest $d \in \mathbb{N}$ ($d \leq n$) for which F is an object of $\mathcal{P}ol_d^{strong}(\mathfrak{M},\mathcal{A})$ is called the strong degree of F.

Remark 3.7. By Proposition 1.14, the category $(\mathfrak{U}\boldsymbol{\beta},\natural,0)$ is a pre-braided monoidal category such that 0 is initial object. This example is the first one which led us to extend the definition of [7]. Thus, we have a well-defined notion of strong polynomial functor for the category $\mathfrak{U}\boldsymbol{\beta}$.

The following three propositions are important properties of the framework in [7] adapted to the pre-braided case. Their proofs follow directly from those of their analogues in [7, Propositions 1.7, 1.8 and 1.9].

Proposition 3.8. *[7, Proposition 1.7] Let $\mathfrak{M}'$ be another pre-braided strict monoidal category and $\alpha : \mathfrak{M} \longrightarrow \mathfrak{M}'$ be a strong monoidal functor. Then, the precomposition by α provides a functor $\mathcal{P}ol_n^{strong}(\mathfrak{M},\mathcal{A}) \to \mathcal{P}ol_n^{strong}(\mathfrak{M}',\mathcal{A})$.*

Proposition 3.9. *[7, Proposition 1.8] The category $\mathcal{P}ol_n^{strong}(\mathfrak{M},\mathcal{A})$ is closed under the translation endofunctor τ_x, under quotient, under extension and under colimits. Moreover, assuming that there exists a set $\mathfrak{E}$ of objects of $\mathfrak{M}$ such that:*

$$\forall m \in Obj(\mathfrak{M}),\ \exists \{e_i\}_{i\in I} \in Obj(\mathfrak{E}) \text{ where } I \text{ is finite},\ m \cong \underset{i\in I}{\natural} e_i,$$

then, an object F of **Fct** $(\mathfrak{M},\mathcal{A})$ *belongs to $\mathcal{P}ol_n^{strong}(\mathfrak{M},\mathcal{A})$ if and only if $\delta_e(F)$ is an object of $\mathcal{P}ol_{n-1}^{strong}(\mathfrak{M},\mathcal{A})$ for all objects e of $\mathfrak{E}$.*

Corollary 3.10. *Let n be a natural number. Let F be a strong polynomial functor of degree n in the category* **Fct** $(\mathfrak{M},\mathcal{A})$*. Then a direct summand of F is necessarily an object of the category $\mathcal{P}ol_n^{strong}(\mathfrak{M},\mathcal{A})$.*

Proof. According to Proposition 3.9, the category $\mathcal{P}ol_n^{strong}(\mathfrak{M},\mathcal{A})$ is closed under quotients. □

Remark 3.11. The category $\mathcal{P}ol_n^{strong}(\mathfrak{M},\mathcal{A})$ is not necessarily closed under subobjects. For example, we will see in Section 3.3 that for $\mathfrak{M}=\mathfrak{U}\boldsymbol{\beta}$ and $\mathcal{A}=\mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$, the functor $\overline{\mathfrak{Bur}}_t$ is a subobject of $\tau_1\overline{\mathfrak{Bur}}_t$ (see Proposition 3.28), $\overline{\mathfrak{Bur}}_t$ is strong polynomial of degree 2 (see Proposition 3.28) whereas $\tau_1\overline{\mathfrak{Bur}}_t$ is strong polynomial of degree 1 (see Proposition 3.29). If we assume that the unit 0 is also a terminal object of $\mathfrak{M}$, then κ_x is the null endofunctor, δ_x is exact and commutes with all limits. In this case, the category $\mathcal{P}ol_n^{strong}(\mathfrak{M},\mathcal{A})$ is closed under subobjects.

Remark 3.12. If we consider $\mathfrak{M}=\mathfrak{U}\boldsymbol{\beta}$, then each object n (ie a natural number) is clearly $1^{\natural n}$. Hence, because of the last statement of Proposition 3.9, when we will deal with strong polynomiality of objects in **Fct** $(\mathfrak{U}\boldsymbol{\beta},\mathcal{A})$, it will suffice to consider τ_1.

Proposition 3.13. *[7, Proposition 1.9] Let F be an object of* **Fct** $(\mathfrak{M},\mathcal{A})$*. Then, the functor F is an object of $\mathcal{P}ol_0^{strong}(\mathfrak{M},\mathcal{A})$ if and only if it the quotient of a constant functor of* **Fct** $(\mathfrak{M},\mathcal{A})$*.*

Finally, let us point out the following property of the strong polynomial degree with respect to the translation functor.

Lemma 3.14. *Let d and k be natural numbers and F be an object of $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$ such that $\tau_k\left(F\right)$ is an object of $\mathcal{P}ol_d^{strong}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$. Then, F is an object of $\mathcal{P}ol_{d+k}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$.*

Proof. We proceed by induction on the degree of polynomiality of $\tau_k\left(F\right)$. First, assuming that $\tau_k\left(F\right)$ belongs to $\mathcal{P}ol_0^{strong}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$, we deduce from the commutation property 6 of Proposition 3.5 that $\tau_k\left(\delta_1 F\right)=0$. It follows from the definition of $\tau_k\left(F\right)$ (see Definition 3.1) that for all $n\geq 2$, $\delta_1\left(F\right)\left(n\right)=0$. Hence

$$\underbrace{\delta_1\cdots\delta_1\delta_1}_{k+1\text{ times}}\left(F\right)\cong 0$$

and therefore F is an object of $\mathcal{P}ol_k\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$. Now, assume that $\tau_k\left(F\right)$ is a strong polynomial functor of degree $d\geq 0$. Since $\left(\tau_k\circ\delta_1\right)\left(F\right)\cong\left(\delta_1\circ\tau_k\right)\left(F\right)$ by the commutation property 6 of Proposition 3.5, $\left(\tau_k\circ\delta_1\right)\left(F\right)$ is an object of $\mathcal{P}ol_{d-1}^{strong}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$. The inductive hypothesis implies that $\delta_1\left(F\right)$ is an object of $\mathcal{P}ol_{d+k}^{strong}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$. ☐

Remark 3.15. Let us consider the atomic functor $\mathfrak{A}_n$ (with $n>0$), which is strong polynomial of degree n (see Example 3.21). Then $\tau_k\left(\mathfrak{A}_n\right)\cong\mathfrak{A}_{n-k}^{\oplus n}$ is strong polynomial of degree $n-k$, for k a natural number such that $k\leq n$. This illustrates the fact that $d+k$ is the best boundary for the degree of polynomiality in Lemma 3.14.

3.2 Very strong polynomial functors

Let us introduce a particular type of strong polynomial functor, related to coefficient systems of finite degree (see Remark 3.17 below). We recall that we consider a pre-braided strict monoidal category $\left(\mathfrak{M},\natural,0\right)$ such that the monoidal unit 0 is an initial object and an abelian category $\mathcal{A}$.

Definition 3.16. We recursively define the category $\mathcal{VP}ol_n\left(\mathfrak{M},\mathcal{A}\right)$ of very strong polynomial functors of degree less than or equal to n to be the full subcategory of $\mathcal{P}ol_n^{strong}\left(\mathfrak{M},\mathcal{A}\right)$ as follows:

1. If $n<0$, $\mathcal{VP}ol_n\left(\mathfrak{M},\mathcal{A}\right)=\{0\}$;
2. if $n\geq 0$, a functor $F\in\mathcal{P}ol_n^{strong}\left(\mathfrak{M},\mathcal{A}\right)$ is an object of $\mathcal{VP}ol_n\left(\mathfrak{M},\mathcal{A}\right)$ if for all objects x of $\mathfrak{M}$, $\kappa_x\left(F\right)=0$ and the functor $\delta_x\left(F\right)$ is an object of $\mathcal{VP}ol_{n-1}\left(\mathfrak{M},\mathcal{A}\right)$.

For an object F of $\mathbf{Fct}\left(\mathfrak{M},\mathcal{A}\right)$ which is very strong polynomial of degree less than or equal to $n\in\mathbb{N}$, the smallest $d\in\mathbb{N}$ ($d\leq n$) for which F is an object of $\mathcal{VP}ol_d\left(\mathfrak{M},\mathcal{A}\right)$ is called the very strong degree of F.

Remark 3.17. A certain type of functor, called a coefficient system of finite degree, closely related to the strong polynomial one, is used by Randal-Williams and Wahl in [20, Definition 4.10] for their homological stability theorems, generalizing the concept introduced by van der Kallen for general linear groups [23]. Using the framework introduced by Randal-Williams and Wahl, a coefficient system in every object x of $\mathfrak{M}$ of degree n at $N=0$ is a very strong polynomial functor.

Remark 3.18. As we force κ_x to be null for all objects x of $\mathfrak{M}$, the category $\mathcal{VP}ol_n\left(\mathfrak{M},\mathcal{A}\right)$ is closed under kernel functors of the epimorphisms. In particular, this category is closed under direct summands. However, $\mathcal{VP}ol_n\left(\mathfrak{M},\mathcal{A}\right)$ is not necessarily closed under subobjects. For instance, as for Remark 3.11, we have that the functor $\overline{\mathfrak{Bur}}_t$ is strong polynomial of degree 2 (see Proposition 3.28), the functor $\tau_1\overline{\mathfrak{Bur}}_t$ is very strong polynomial of degree 1 (see Proposition 3.29), but $\overline{\mathfrak{Bur}}_t$ is a subobject of $\tau_1\overline{\mathfrak{Bur}}_t$ (see Proposition 3.28).

Proposition 3.19. *The category $\mathcal{VP}ol_n\left(\mathfrak{M},\mathcal{A}\right)$ is closed under the translation endofunctor τ_x, under kernel of epimorphism and under extension. Moreover, assuming that there exists a set $\mathfrak{E}$ of objects of $\mathfrak{M}$ such that:*

$$\forall m\in Obj\left(\mathfrak{M}\right),\ \exists\left\{e_i\right\}_{i\in I}\in Obj\left(\mathfrak{E}\right)\ (\text{where } I \text{ is finite}),\ m\cong\underset{i\in I}{\natural}e_i,$$

then, an object F of $\mathbf{Fct}\left(\mathfrak{M},\mathcal{A}\right)$ belongs to $\mathcal{VP}ol_n\left(\mathfrak{M},\mathcal{A}\right)$ if and only if $\kappa_e\left(F\right)=0$ and $\delta_e\left(F\right)$ is an object of $\mathcal{VP}ol_{n-1}\left(\mathfrak{M},\mathcal{A}\right)$ for all objects e of $\mathfrak{E}$.

Proof. The first assertion follows from the fact that for all objects x of $\mathfrak{M}$, the endofunctor τ_x commutes with the endofunctors δ_x and κ_x (see Proposition 3.5). For the second and third assertions, let us consider two short exact sequences of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$: $0 \longrightarrow G \longrightarrow F_1 \longrightarrow F_2 \longrightarrow 0$ and $0 \longrightarrow F_3 \longrightarrow H \longrightarrow F_4 \longrightarrow 0$ with F_i a very strong polynomial functor of degree n for all i. Let x be an object of $\mathfrak{M}$. We use the exact sequence (7) of Proposition 3.5 to obtain the two following exact sequences in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_x(G) \longrightarrow 0 \longrightarrow 0 \longrightarrow \delta_x(G) \longrightarrow \delta_x(F_1) \longrightarrow \delta_x(F_2) \longrightarrow 0;$$

$$0 \longrightarrow 0 \longrightarrow \kappa_x(H) \longrightarrow 0 \longrightarrow \delta_x(F_3) \longrightarrow \delta_x(H) \longrightarrow \delta_x(F_4) \longrightarrow 0.$$

Therefore, $\kappa_x(G) = \kappa_x(H) = 0$ and the result follows directly by induction on the degree of polynomiality. For the last point, we consider the long exact sequence (8) of Proposition 3.5 applied to an object F of $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ to obtain the following exact sequence in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_y(F) \longrightarrow \kappa_{x\natural y}(F) \longrightarrow \tau_x\kappa_y(F) \longrightarrow \delta_y(F) \longrightarrow \delta_{x\natural y}(F) \longrightarrow \tau_y\delta_x(F) \longrightarrow 0.$$

Hence, by induction on the length of objects as monoidal product of $\{e_i\}_{i\in I}$, we deduce that $\kappa_m(F) = 0$ for all objects m of $\mathfrak{M}$ if and only if $\kappa_e(F) = 0$ for all objects e of $\mathfrak{E}$. Moreover, since $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ is closed under extension and by the translation endofunctor τ_y, the result follows by induction on the degree of polynomiality n. ☐

Proposition 3.20. *Let F be an object of* $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. *The functor F is an object of $\mathcal{VP}ol_0(\mathfrak{M}, \mathcal{A})$ if and only if it is isomorphic to $\tau_k F$ for all natural numbers k.*

Proof. The result follows using the long exact sequence (6) of Proposition 3.5 applied to F. ☐

The following example show that there exist strong polynomial functors which are not very strong polynomial in any degree.

Example 3.21. Let us consider the categories $\mathfrak{U}\boldsymbol{\beta}$ and $\mathbb{K}$-$\mathfrak{Mod}$, and n a natural number. Let $\mathbb{K}$ be considered as an object of $\mathbb{K}$-$\mathfrak{Mod}$ and 0 be the trivial $\mathbb{K}$-module. Let $\mathfrak{A}_n$ be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, defined by:

- Objects: $\forall m \in \mathbb{N}, \mathfrak{A}_n(m) = \begin{cases} \mathbb{K} & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$.
- Morphisms: let $[j-i, f]$ with $f \in \mathbf{B}_n$ be a morphism from i to j in the category $\mathfrak{U}\boldsymbol{\beta}$. Then:

$$\mathfrak{A}_n(f) = \begin{cases} id_{\mathbb{K}} & \text{if } i = j = n \\ 0 & \text{otherwise.} \end{cases}$$

The functor $\mathfrak{A}_n$ is called an atomic functor in $\mathbb{K}$ of degree n. For coherence, we fix $\mathfrak{A}_{-1}$ to be the null functor of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Then, it is clear that $i_p(\mathfrak{A}_n)$ is the zero natural transformation. On the one hand, we deduce the following natural equivalence $\kappa_1(\mathfrak{A}_n) \cong \mathfrak{A}_n$ and a fortiori $\mathfrak{A}_n$ is not a very strong polynomial functor. On the other hand, it is worth noting the natural equivalence $\delta_1(\mathfrak{A}_n) \cong \tau_1(\mathfrak{A}_n)$ and the fact that $\tau_1(\mathfrak{A}_n) \cong \mathfrak{A}_{n-1}$. Therefore, we recursively prove that $\mathfrak{A}_n$ is a strong polynomial functor of degree n.

Remark 3.22. Contrary to $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$, a quotient of an object F of $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ is not necessarily a very strong polynomial functor. For example, for $\mathfrak{M} = \mathfrak{U}\boldsymbol{\beta}$ and $\mathcal{A} = \mathbb{K}$-$\mathfrak{Mod}$, let us consider the functor $\mathfrak{A}_0$ defined in Example 3.21, which we proved to be a strong polynomial functor of degree 0. Let $\mathfrak{A}$ be the constant object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ equal to $\mathbb{K}$. Then, we define a natural transformation $\alpha : \mathfrak{A} \to \mathfrak{A}_0$ assigning:

$$\forall n \in \mathbb{N},\ \alpha_n = \begin{cases} id_{\mathbb{K}} & \text{if } n = 0 \\ t_{\mathbb{K}} & \text{otherwise.} \end{cases}$$

Moreover, it is an epimorphism in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ since for all natural numbers n, $coker(\alpha_n) = 0_{\mathbb{K}\text{-}\mathfrak{Mod}}$. We proved in Example 3.21 that $\mathfrak{A}_0$ is not a very strong polynomial functor of degree 0 whereas $\mathfrak{A}$ is a very strong polynomial functor of degree 0 by Proposition 3.20.

Finally, let us remark the following behaviour of the translation functor with respect to very strong polynomial degree.

Lemma 3.23. *Let d and k be a natural numbers and F be an object of $\mathcal{VP}ol_d\left(\mathfrak{M},\mathbb{K}\text{-}\mathfrak{Mod}\right)$. Then the functor $\tau_k\left(F\right)$ is very strong polynomial of degree equal to that of F.*

Proof. We proceed by induction on the degree of polynomiality of F. First, if we assume that F belongs to $\mathcal{VP}ol_0\left(\mathfrak{M},\mathbb{K}\text{-}\mathfrak{Mod}\right)$, then according to Proposition 3.20, $\tau_k\left(F\right)\cong F$ is a degree 0 very strong polynomial functor. Now, assume that F is a very strong polynomial functor of degree $n\geq 0$. Using the commutation properties 5 and 6 of Proposition 3.5, we deduce that $\left(\kappa_1\circ\tau_k\right)\left(F\right)\cong\left(\tau_k\circ\kappa_1\right)\left(F\right)=0$ and $\left(\delta_1\circ\tau_k\right)\left(F\right)\cong\left(\tau_k\circ\delta_1\right)\left(F\right)$. Since the functor $\delta_1\left(F\right)$ is a degree $n-1$ very strong polynomial functor, the result follows from the inductive hypothesis. □

Remark 3.24. The previous proof does not work for strong polynomial functors since the initial step fails. Indeed, considering the atomic functor $\mathfrak{A}_1$, which is strong polynomial of degree 1 (see Example 3.21), then $\tau_2\left(\mathfrak{A}_0\right)=0$.

3.3 Examples of polynomial functors over $\mathfrak{U}\beta$

The different functors introduced in Section 1.2 are strong polynomial functors.

Very strong polynomial functors of degree one: Let us first investigate the polynomiality of the functors $\mathfrak{Bur}_t$ and $\mathfrak{TYM}_t$.

Proposition 3.25. *The functors $\mathfrak{Bur}_t$ and $\mathfrak{TYM}_t$ are very strong polynomial functors of degree* 1.

Proof. For the functor $\mathfrak{Bur}_t$, this is a consequence of [20, Example 4.15]. We will thus focus on the case of the functor $\mathfrak{TYM}_t$. Let n be a natural number. By Remark 3.12, it is enough to consider the application $i_1\mathfrak{TYM}_t\left(\left[0,id_n\right]\right)=\iota_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'-n}}\oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}}$. This map is a monomorphism and its cokernel is $\mathbb{C}\left[t^{\pm1}\right]$. Hence $\kappa_1\mathfrak{TYM}_t$ is the null functor of $\mathbf{Fct}\left(\mathfrak{U}\beta,\mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}\right)$. Let n' be a natural number such that $n'\geq n$ and let $\left[n'-n,\sigma\right]\in Hom_{\mathfrak{U}\beta}\left(n,n'\right)$. By naturality and the universal property of the cokernel, there exists a unique endomorphism of $\mathbb{C}\left[t^{\pm1}\right]$ such that the following diagram commutes, where the lines are exact. It is exactly the definition of $\delta_1\mathfrak{TYM}_t\left(\left[n'-n,\sigma\right]\right)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}\left[t^{\pm1}\right]^{\oplus n} & \xrightarrow{\iota_{\mathbb{C}\left[t^{\pm1}\right]}\oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}}} & \mathbb{C}\left[t^{\pm1}\right]^{\oplus n+1} & \xrightarrow{\pi_{n+1}} & \mathbb{C}\left[t^{\pm1}\right] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathfrak{TYM}\left(\left[n'-n,\sigma\right]\right)} & & \downarrow{\scriptstyle \tau_1\left(\mathfrak{TYM}\right)\left(\left[n'-n,\sigma\right]\right)} & & \vdots{\scriptstyle \exists!} & & \\
0 & \longrightarrow & \mathbb{C}\left[t^{\pm1}\right]^{\oplus n'} & \xrightarrow[\iota_{\mathbb{C}\left[t^{\pm1}\right]}\oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'}}]{} & \mathbb{C}\left[t^{\pm1}\right]^{\oplus n'+1} & \xrightarrow[\pi_{n'+1}]{} & \mathbb{C}\left[t^{\pm1}\right] & \longrightarrow & 0.
\end{array}$$

For all $\left(a,b\right)\in\mathbb{C}\left[t^{\pm1}\right]\oplus\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}=\mathbb{C}\left[t^{\pm1}\right]^{\oplus n+1}$, $\tau_1\left(\mathfrak{TYM}_t\right)\left(\left[n'-n,\sigma\right]\right)\left(a,b\right)=\left(a,\mathfrak{TYM}_t\left(\left[n'-n,\sigma\right]\right)\left(b\right)\right)$. Therefore, $\left(\pi_{n'+1}\circ\tau_1\left(\mathfrak{TYM}_t\right)\left(\left[n'-n,\sigma\right]\right)\right)\left(a,b\right)=a=\pi_{n+1}\left(a,b\right)$. Hence, $id_{\mathbb{C}\left[t^{\pm1}\right]}$ also makes the diagram commutative and thus $\delta_1\mathfrak{TYM}_t\left(\left[n'-n,\sigma\right]\right)=id_{\mathbb{C}\left[t^{\pm1}\right]}$. Hence, $\delta_1\mathfrak{TYM}_t$ is the constant functor equal to $\mathbb{C}\left[t^{\pm1}\right]$. A fortiori, because of Proposition 3.20, $\delta_1\mathfrak{TYM}_t$ is a very strong polynomial functor of degree 0. □

The particular case of $\overline{\mathfrak{Bur}_t}$:

Definition 3.26. Let $\mathcal{T}_1:\mathfrak{U}\beta\longrightarrow\mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}$ be the subobject of the constant functor $\mathfrak{X}$ (see Notation 2.27) such that $\mathcal{T}_1\left(0\right)=0$ and $\mathcal{T}_1\left(n\right)=\mathbb{C}\left[t^{\pm1}\right]$ for all non-zero natural numbers n.

Remark 3.27. It follows from Definition 3.26 that $\delta_1\mathcal{T}_1\cong\mathfrak{A}_0$ (where $\mathfrak{A}_0$ is introduced in Example 3.21). Therefore, $\mathcal{T}_1$ is a strong polynomial functor of degree 1, but is not very strong polynomial. Nevertheless, it is worth noting that $\kappa_1\mathcal{T}_1=0$.

Proposition 3.28. *The functor $\overline{\mathfrak{Bur}}$ is a strong polynomial functor of degree 2. This functor is not very strong polynomial. More precisely, we have the following short exact sequence in* $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}\right)$*:*

$$0 \longrightarrow \overline{\mathfrak{Bur}_t} \longrightarrow \tau_1\overline{\mathfrak{Bur}_t} \longrightarrow \mathcal{T}_1 \longrightarrow 0\,.$$

Proof. The natural transformation $i_1\left(\overline{\mathfrak{Bur}_t}\right)_n : \overline{\mathfrak{Bur}_t}\left(n\right) \to \tau_1\overline{\mathfrak{Bur}_t}\left(n\right)$ (introduced in Definition 3.4) is defined to be $\iota_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'-n}} \oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n-1}}$. Let $n \geq 2$ be a natural number. This map is a monomorphism (so $\kappa_1\overline{\mathfrak{Bur}_t} = 0$) and its cokernel is $\mathbb{C}\left[t^{\pm1}\right]$. Repeating mutatis mutandis the work done in the proof of Proposition 3.25, we deduce that for all $\left[n'-n,\sigma\right] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}\left(n,n'\right)$ (with $n' \geq n \geq 2$), $\delta_1\overline{\mathfrak{Bur}_t}\left(\left[n'-n,\sigma\right]\right) = Id_{\mathbb{C}\left[t^{\pm1}\right]}$. In addition, since $\overline{\mathfrak{Bur}_t}\left(1\right) = 0$ and $\tau_1\overline{\mathfrak{Bur}_t}\left(1\right) = \mathbb{C}\left[t^{\pm1}\right]$, we deduce that $\delta_1\overline{\mathfrak{Bur}_t}\left(1\right) = \mathbb{C}\left[t^{\pm1}\right]$ and for all $n' \geq 1$, for all $\left[n'-1,\sigma\right] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}\left(1,n'\right)$, $\delta_1\overline{\mathfrak{Bur}_t}\left(\left[n'-1,\sigma\right]\right) = Id_{\mathbb{C}\left[t^{\pm1}\right]}$. Hence, we prove that $\delta_1\overline{\mathfrak{Bur}_t} \cong \mathcal{T}_1$ where $\mathcal{T}_1$ is introduced in Definition 3.26. The results follow from the fact that $\delta_1\mathcal{T}_1 \cong \mathfrak{A}_0$ by Remark 3.27. ☐

For formal reasons (see Proposition 3.5), $\overline{\mathfrak{Bur}_t}$ is a subfunctor of $\tau_1\overline{\mathfrak{Bur}_t}$. The following proposition illustrates Remarks 3.11 and 3.18.

Proposition 3.29. *The functor $\tau_1\overline{\mathfrak{Bur}_t}$ is a very strong polynomial functor of degree* 1.

Proof. Repeating mutatis mutandis the work done in the proof of Proposition 3.28, we prove that $\delta_1\tau_1\overline{\mathfrak{Bur}_t}$ is the constant functor equal to $\mathbb{C}\left[t^{\pm1}\right]$ (denoted by $\mathfrak{X}$ in Notation 2.27). Since $\mathfrak{X}$ is a constant functor, $\delta_1\tau_1\overline{\mathfrak{Bur}_t}$ is by Proposition 3.20 a very strong polynomial functor of degree 0. ☐

A very strong polynomial functor of degree two: We could have defined the unreduced Burau functor of Example 1.2 assigning $\left(\left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]\right)^{\oplus n}$ to each object $n \in \mathbb{N}$.

Notation 3.30. Abusing the notation, $\left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right] : \mathfrak{U}\boldsymbol{\beta} \to \left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]\text{-}\mathfrak{Mod}$ denotes the constant functor at $\left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]$. The functor $\mathfrak{Bur}_t \underset{\mathbb{C}\left[t^{\pm1}\right]}{\otimes} \left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]$ is denoted by $\check{\mathfrak{Bur}}_t : \mathfrak{U}\boldsymbol{\beta} \to \left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]\text{-}\mathfrak{Mod}$.

Remark 3.31. These functors $\left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]$ and $\check{\mathfrak{Bur}}_t$ are also very strong polynomial of degree one (the proof is exactly the same as the one for $\mathfrak{Bur}_t$ in Proposition 3.27).

Lemma 3.32. *Considering the modified version of the unreduced Burau functor of Remark 3.30, then we have $\delta_1\mathfrak{LK} = \check{\mathfrak{Bur}}_t$.*

Proof. We consider the application $i_1\mathfrak{LK}\left(\left[0,id_n\right]\right)$. This map is a monomorphism and its cokernel is $\bigoplus_{2\leq l\leq n+1} V_{1,l}$. Let n and n' be two natural numbers such that $n' \geq n$. Let $\left[n'-n,\sigma\right] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}\left(n,n'\right)$. By naturality and because of the universal property of the cokernel, there exists a unique endomorphism of $\left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]$-modules such that the following diagram commutes, where the lines are exact. It is exactly the definition of $\delta_1\mathfrak{LK}\left(\left[n'-n,\sigma\right]\right)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{1\leq j<k\leq n} V_{j,k} & \xrightarrow{\mathfrak{LK}([1,id_{1+n}])} & \bigoplus_{1\leq i<l\leq n+1} V_{i,l} & \xrightarrow{\pi_n} & \bigoplus_{2\leq l\leq n+1} V_{1,l} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathfrak{LK}([n'-n,\sigma])} & & \downarrow{\scriptstyle \tau_1(\mathfrak{LK})([n'-n,\sigma])} & & \downarrow{\scriptstyle \exists!} & & \\
0 & \longrightarrow & \bigoplus_{1\leq j'<k'\leq n'} V_{j',k'} & \xrightarrow[\mathfrak{LK}([1,id_{1+n'}])]{} & \bigoplus_{1\leq l'\leq n'+1} V_{i',l'} & \xrightarrow[\pi_{n'}]{} & \bigoplus_{2\leq l'\leq n'+1} V_{1,l'} & \longrightarrow & 0.
\end{array}$$

Let $i \in \{1,\ldots,n-1\}$, $l \in \{2,\ldots,n+1\}$ and $v_{1,l}$ be an element of $V_{1,l}$. Then we compute:

$$\tau_1\mathfrak{LK}\left(\sigma_i\right)v_{1,l} = \mathfrak{LK}\left(\sigma_{1+i}\right)\left(v_{1,l}\right) = \begin{cases} v_{1,l} & \text{if } i+1 \notin \{l-1,l\}, \\ tv_{1,i+1} + \left(1-t\right)v_{1,i+2} - \left(t^2-t\right)qv_{i+1,i+2} & \text{if } i+2 = l, \\ v_{1,i+2} & \text{if } i+1 = l. \end{cases}$$

We deduce that in the canonical basis $\{\mathbf{e}_{1,2}, \mathbf{e}_{1,3}, \ldots, \mathbf{e}_{1,n+1}\}$ of $\bigoplus_{2\leq l\leq n+1} V_{1,l}$:

$$\delta_1 \mathfrak{LK}(\sigma_i) = Id_{i-1} \oplus \begin{bmatrix} 0 & t \\ 1 & 1-t \end{bmatrix} \oplus Id_{n-i-1} = \check{\mathfrak{Bur}}_t(\sigma_i).$$

So as to identify $\delta_1 \mathfrak{LK}$, it remains to consider the action on morphisms of type $[1, id_{n+1}]$. According to the definition of the Lawrence-Krammer functor, we have $\tau_1(\mathfrak{LK})([1, id_{n+1}]) = \mathfrak{LK}\left(\sigma_1^{-1}\right) \circ \mathfrak{LK}([1, id_{n+2}])$ and:

$$\mathfrak{LK}(\sigma_1)(v_{1,k}) = \begin{cases} v_{2,k} & \text{if } k \in \{3, \ldots, n+2\}, \\ -qt^2 v_{1,2} & \text{if } k = 2. \end{cases}$$

It follows that for all $v_{i,l} \in V_{i,l}$ with $1 \leq i < l \leq n+1$:

$$\pi_{n+1} \circ \tau_1(\mathfrak{LK})([1, id_{n+1}])(v_{i,l}) = \begin{cases} v_{1,l+1} & \text{if } i = 1 \text{ and } l \in \{2, \ldots, n+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we deduce that for all $2 \leq l \leq n+1$, $\delta_1 \mathfrak{LK}([1, id_{n+1}])(v_{1,l}) = v_{1,l+1} = \check{\mathfrak{Bur}}_t([1, id_{n+1}])(v_{1,l})$. □

Proposition 3.33. *The functor $\mathfrak{LK}$ is a very strong polynomial functor of degree* 2.

Proof. Let n be a natural number. By Remark 3.12, we only have to consider the application $i_1 \mathfrak{LK}([0, id_n])$. Since this map is a monomorphism with cokernel $\bigoplus_{1\leq i\leq n} V_{i,n+1}$, $\kappa_1 \mathfrak{LK}$ is the null constant functor. Since the functor $\check{\mathfrak{Bur}}_t$ is very strong polynomial of degree one (following exactly the same proof as the one of Proposition 3.25), we deduce from Lemma 3.32 that $\mathfrak{LK}$ is very strong polynomial of degree two. □

4 The Long-Moody functor applied to polynomial functors

Let us move on to the effect of the Long-Moody functors on (very) strong polynomial functors. For this purpose, it is enough by Remark 3.12 to consider the cokernel of the map $i_1\mathbf{LM}$. First, we decompose the functor $\tau_1 \circ \mathbf{LM}$ (see Proposition 4.19) so as to understand the behaviour of the image of $i_1\mathbf{LM}$ through this decomposition. This allows us to prove a splitting decomposition of the difference functor (see Theorem 4.23). This is the key point to prove our main results, namely Corollary 4.26 and Theorem 4.27. Finally, we give some additional properties of Long-Moody functors with respect to polynomial functors.

Let $\{\varsigma_n : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ be coherent families of morphisms (see Definition 2.14), with associated Long-Moody functor $\mathbf{LM}_{a,\varsigma}$ (see Theorem 2.21), which we fix for all the work of this section (in particular, we omit the "a, ς" from the notation).

4.1 Decomposition of the translation functor

We introduce two functors which will play a key role in the main result. First, let us recall the following crucial property of the augmentation ideal of a free product of groups, which follows by combining [6, Lemma 4.3] and [6, Theorem 4.7].

Proposition 4.1. *Let G and H be groups. Then, there is a natural $\mathbb{K}[G * H]$-module isomorphism:*

$$\mathcal{I}_{\mathbb{K}[G*H]} \cong \left(\mathcal{I}_{\mathbb{K}[G]} \underset{\mathbb{K}[G]}{\otimes} \mathbb{K}[G*H]\right) \oplus \left(\mathcal{I}_{\mathbb{K}[H]} \underset{\mathbb{K}[H]}{\otimes} \mathbb{K}[G*H]\right).$$

Remark 4.2. In the statement of Proposition 4.1, recall that the augmentation ideal $\mathcal{I}_{\mathbb{K}[G]}$ (respectively $\mathcal{I}_{\mathbb{K}[H]}$) is a free right $\mathbb{K}[G]$-module (respectively $\mathbb{K}[H]$-module) by Proposition 2.23. Moreover, the group ring $\mathbb{K}[G*H]$ is a left $\mathbb{K}[G]$-module (respectively left $\mathbb{K}[H]$-module) via the morphism $id_G * \iota_H : G \to G * H$ (respectively $\iota_G * id_H : H \to G * H$).

Notation 4.3. Let n and n' be natural numbers such that $n' \geq n$. We consider the morphism $id_{\mathbf{F}_n} * \iota_{\mathbf{F}_{n'-n}} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$. This corresponds to the identification of $\mathbf{F}_n$ as the subgroup of $\mathbf{F}_{n'}$ generated by the n first copies of $\mathbf{F}_1$ in $\mathbf{F}_{n'}$.

In addition, the group morphism $id_{\mathbf{F}_n} * \iota_{\mathbf{F}_{n'-n}} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$ canonically induces a $\mathbb{K}$-module morphism $id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} * \iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \hookrightarrow \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}$.

For F an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we consider the functor $(\tau_1 \circ \mathbf{LM})(F)$. For all natural numbers n, by Proposition 4.1, we have a $\mathbb{K}[\mathbf{F}_{1+n}]$-module isomorphism:

$$\begin{aligned}
&\mathcal{I}_{\mathbb{K}[\mathbf{F}_{1+n}]} \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} F(n+2) \\
&\cong \left(\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]} \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} \mathbb{K}[\mathbf{F}_{1+n}] \right) \oplus \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} \mathbb{K}[\mathbf{F}_{1+n}] \right) \right) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} F(n+2).
\end{aligned}$$

Now, by Remark 4.2, the $\mathbb{K}[\mathbf{F}_{n+1}]$-module $F(n+2)$ is a $\mathbb{K}[\mathbf{F}_1]$-module via

$$F\left(\varsigma_{1+n}\left(id_{\mathbf{F}_1} * \iota_{\mathbf{F}_n}\right)\right) : \mathbf{F}_1 \to Aut_{\mathbb{K}\text{-}\mathfrak{Mod}}(F(n+2))$$

and $\mathbb{K}[\mathbf{F}_n]$-module via

$$F\left(\varsigma_{1+n}\left(\iota_{\mathbf{F}_1} * id_{\mathbf{F}_n}\right)\right) : \mathbf{F}_n \to Aut_{\mathbb{K}\text{-}\mathfrak{Mod}}(F(n+2)).$$

Therefore, because of the distributivity of tensor product with respect to the direct sum, we have the following proposition.

Proposition 4.4. *Let $F \in Obj(\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}))$ and n be a natural number. Then, we have the following $\mathbb{K}$-module isomorphism:*

$$\tau_1 \mathbf{LM}(F)(n) \quad \cong \quad \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]} \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} F(n+2) \right) \oplus \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+2) \right). \tag{9}$$

Definition 4.5. For all natural numbers n and $F \in Obj(\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}))$, we denote by

- $\upsilon(F)_n$ the monomorphism of $\mathbb{K}$-modules $\left(id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * \iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} id_{F(n+2)} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_1]} \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} F(n+2) \hookrightarrow \tau_1 \mathbf{LM}(F)(n)$,
- $\xi(F)_n$ the monomorphism of $\mathbb{K}$-modules $\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} id_{F(n+2)} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+2) \hookrightarrow \tau_1 \mathbf{LM}(F)(n)$,

associated with the direct sum of Proposition 4.4.

The aim of this section is in fact to show that this $\mathbb{K}$-module decomposition leads to a decomposition of $\tau_1 \mathbf{LM}$ (see Theorem 4.23) as a functor.

4.1.1 Additional conditions

We need two additional conditions so as to make the decomposition of Proposition 4.4 functorial. First, we require the morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n \in \mathbb{N}}$ to satisfy the following property.

Condition 4.6. Let n and n' be natural numbers such that $n' \geq n$. We require $a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1} \natural id_n\right) \circ \left(\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_{n+1}}\right) \circ \left(id_{\mathbf{F}_1} * \iota_{\mathbf{F}_n}\right) = id_{\mathbf{F}_1} * \iota_{\mathbf{F}_{n'}}$. In other words, the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{F}_1 & \xrightarrow{\quad id_{\mathbf{F}_1} * \iota_{\mathbf{F}_{n'}} \quad} & \mathbf{F}_{1+n'} \\
{\scriptstyle id_{\mathbf{F}_1} * \iota_{\mathbf{F}_n}} \downarrow & & \uparrow {\scriptstyle a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1} \natural id_n\right)} \\
\mathbf{F}_{1+n} & \xrightarrow[\quad \iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_{1+n}} \quad]{} & \mathbf{F}_{n'-n} * \mathbf{F}_{1+n} \cong \mathbf{F}_{1+n'}.
\end{array}$$

Remark 4.7. Condition 4.6 will be used to define an intermediary functor (see Proposition 4.14).

In addition, we will assume that the morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ satisfy the following condition.

Condition 4.8. Let n and n' be natural numbers such that $n' \geq n$. We require $a_{n'}(id_{n'-n}\natural -) : \mathbf{B}_n \to Aut(\mathbf{F}_{n'})$ maps to the stabilizer of the homomorphism $id_{\mathbf{F}_{n'-n}} * \iota_{\mathbf{F}_n} : \mathbf{F}_{n'-n} \longrightarrow \mathbf{F}_{n'}$, ie for all element σ of $\mathbf{B}_n$ the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{F}_{n'-n} & \xrightarrow{id_{\mathbf{F}_{n'-n}} * \iota_{\mathbf{F}_n}} & \mathbf{F}_{n'} \\ & {\scriptstyle id_{\mathbf{F}_{n'-n}} * \iota_{\mathbf{F}_n}} \searrow \quad \nearrow {\scriptstyle a_{n'}(id_{n'-n}\natural\sigma)} & \\ & \mathbf{F}_{n'}. & \end{array}$$

Remark 4.9. Condition 4.8 will be used in the proof of Propositions 4.14 and 4.15.

Remark 4.10. The relations of Conditions 4.6 and 4.8 remain true mutatis mutandis, for all natural numbers n, considering the induced morphisms $a_n : \mathbf{B}_n \to Aut\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\right)$ and $id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} * \iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \hookrightarrow \mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'}\right]}$.

Definition 4.11. If the morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ also satisfy conditions 4.6 and 4.8, the coherent families of morphisms $\{\varsigma_n : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ are said to be reliable.

Proposition 4.12. *The coherent families of morphisms* $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ *and* $\{\varsigma_{n,1} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ *of Examples 2.7 and 2.15 are reliable.*

Proof. Recall from Definition 1.4 that $\left(b^{\beta}_{1,n'-n}\right)^{-1} = \sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{n'-n}^{-1}$. We consider the element $e_{\mathbf{F}_{n'-n}} * g_1 * e_{\mathbf{F}_n} = g_{n'-n+1} \in \mathbf{F}_{(n'-n)+1+n}$. The definition of $a_{n,1}$ gives that $a_{1+n',1}(\sigma_{n'-n})(g_{n'-n}) = g_{n'-n+1}$. Therefore, we have that:

$$a_{1+n',1}\left(\sigma_{n'-n}^{-1}\right)(g_{n'-n+1}) = g_{n'-n}.$$

Iterating this observation, we deduce that $a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n\right)(g_{n'-n+1}) = g_1 \in \mathbf{F}_{1+n'}$. Hence, the family of morphisms $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ satisfies Condition 4.6.

Similarly to Example 2.15 earlier, for all $g \in \mathbf{F}_{n'-n}$ and each Artin generator $\sigma_i \in \mathbf{B}_n$, $a_{n'}(id_{n'-n}\natural\sigma_i)(g * e_{\mathbf{F}_n}) = g * e_{\mathbf{F}_n}$. Hence, the family of morphisms $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ satisfies Condition 4.8. □

From now until the end of Section 4, we fix coherent reliable families of morphisms $\{\varsigma_n : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$.

4.1.2 The intermediary functors

The functor τ_2: Let us consider the factor $\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]} \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} F(n+2)$ of $\tau_1 \mathbf{LM}(F)(n)$ in the decomposition of Proposition 4.4.

Notation 4.13. For all objects F of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, for all natural numbers n, we denote $\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]} \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} F(n+2)$ by $\Upsilon(F)(n)$.

Recall the monomorphisms $\{v(F)_n : \Upsilon(F)(n) \hookrightarrow \tau_1 \mathbf{LM}(F)(n)\}_{n\in\mathbb{N}}$ of Definition 4.5.

Proposition 4.14. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. For all natural numbers n and n' such that $n' \geq n$, and for all $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$, assign:*

$$\Upsilon(F)([n'-n, \sigma]) = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} F\left(id_2 \natural [n'-n, \sigma]\right).$$

This defines a subfunctor $\Upsilon(F) : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ of $\tau_1 \mathbf{LM}(F)$, using the monomorphisms $\{v(F)_n\}_{n\in\mathbb{N}}$.

Proof. Let us check that the assignment $\Upsilon(F)$ is well defined with respect to the tensor product. Let n and n' be natural numbers such that $n' \geq n$, and $[n'-n,\sigma] \in Hom_{\mathfrak{U}\beta}(n,n')$ with $\sigma \in \mathbf{B}_{n'}$. Recall from Proposition 1.14 that $id_2 \natural [n'-n,\sigma] = \left[n'-n, (id_2\natural\sigma) \circ \left(\left(b^{\beta}_{2,n'-n}\right)^{-1} \natural id_n\right)\right]$. On the one hand, by Condition 2.12, we have:

$$(id_2\natural\sigma) \circ \varsigma_{1+n'}(g_1) = \varsigma_{1+n'}\left(a_{1+n'}(id_1\natural\sigma)(g_1)\right) \circ (id_2\natural\sigma).$$

Hence, it follows from Condition 4.8 that

$$(id_2\natural\sigma) \circ \varsigma_{1+n'}(g_1) = \varsigma_{1+n'}(g_1) \circ (id_2\natural\sigma). \tag{10}$$

On the other hand, Condition 4.6 gives that

$$g_1 = a_{2+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1} \natural id_{n+1}\right)(g_{n'-n+1})$$

and by Condition 4.8 we have

$$g_1 = a_{2+n'}\left(id_1 \natural \left(b^{\beta}_{1,n'-n}\right)^{-1} \natural id_n\right)(g_1).$$

By the definition of the braiding $b^{\beta}_{-,-}$ (see Definition 1.4), we deduce that:

$$\varsigma_{1+n'}(g_1) \quad = \quad \varsigma_{1+n'}\left(a_{2+n'}\left(\left(b^{\beta}_{2,n'-n}\right)^{-1} \natural id_n\right)(g_{n'-n+1})\right).$$

Then, it follows from the combination of Conditions 2.3 and 2.12 that as morphisms in $\mathfrak{U}\beta$:

$$\begin{aligned} &\left[n'-n, \varsigma_{1+n'}(g_1) \circ \left(\left(b^{\beta}_{2,n'-n}\right)^{-1} \natural id_n\right)\right] \\ &= \left[n'-n, \left(\left(b^{\beta}_{2,n'-n}\right)^{-1} \natural id_n\right) \circ \left(id_{n'-n}\natural\varsigma_{1+n}(g_1)\right)\right]. \end{aligned} \tag{11}$$

Hence, we deduce from the relations (10) and (11) that:

$$\begin{aligned} &\left[n'-n, \left((id_2\natural\sigma) \circ \left(\left(b^{\beta}_{2,n'-n}\right)^{-1} \natural id_n\right)\right) \circ \left(id_{n'-n}\natural\varsigma_{1+n}(g_1)\right)\right] \\ &= \left[n'-n, \varsigma_{1+n'}(g_1) \circ \left((id_2\natural\sigma) \circ \left(\left(b^{\beta}_{2,n'-n}\right)^{-1} \natural id_n\right)\right)\right]. \end{aligned}$$

A fortiori, $F(id_2\natural[n'-n,\sigma]) \circ F(\varsigma_{1+n}(g_1)) = F(\varsigma_{1+n'}(g_1)) \circ F(id_2\natural[n'-n,\sigma])$. Hence, our assignment is well defined with respect to the tensor product.

Let us prove that the subspaces $\Upsilon(F)(n)$ are stable under the action of $\mathfrak{U}\beta$. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}$ and $v \in F(n+2)$. We deduce from the definition of the monoidal structure morphisms of $\mathfrak{U}\beta$ (see Proposition 1.14) and from the definition of the Long-Moody functor (see Theorem 2.21) that, for all $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}$ and for all $v \in F(n+2)$:

$$\begin{aligned} &\left(\left(\tau_1\mathbf{LM}(F)\left([n'-n,\sigma]\right)\right) \circ v(F)_n\right)\left(i \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} v\right) \\ =&a_{1+n'}(id_1\natural\sigma)\left(a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1} \natural id_n\right)\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * \iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i)\right) \\ &\underset{\mathbb{K}[\mathbf{F}_{n'+1}]}{\otimes} F\left(id_1\natural id_1\natural[n'-n,\sigma]\right)(v). \end{aligned}$$

It follows from Condition 4.6 that:

$$a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1} \natural id_n\right)\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * \iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i) = \left(id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * \iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}}\right)(i).$$

Since by Condition 4.8, $a_{1+n'}(id_1 \natural \sigma)\left(id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * \iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}}\right)(i) = \left(id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * \iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}}\right)(i)$ for all elements σ of $\mathbf{B}_{n'}$, we deduce that:

$$\left(\tau_1 \mathbf{LM}(F)\left(\left[n'-n,\sigma\right]\right) \circ v(F)_n\right)\left(i \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} v\right) = \left(v(F)_{n'} \circ \Upsilon(F)\left(\left[n'-n,\sigma\right]\right)\right)\left(i \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} v\right).$$

Therefore, the functorial structure of $\tau_1 \mathbf{LM}(F)$ induces by restriction the one of $\Upsilon(F)$. □

Now, we can lift this link between $\Upsilon(F)$ *of* $\tau_1 \mathbf{LM}(F)$ to endofunctors of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$.

Proposition 4.15. *Let F and G be two objects of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$, and $\eta : F \to G$ be a natural transformation. For all natural numbers n, assign :*

$$(\Upsilon(\eta))_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} \eta_{n+2}.$$

Then we define a subfunctor $\Upsilon : \mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$ of $\tau_1 \mathbf{LM}$ using the monomorphisms $\{v(F)_n\}_{n\in\mathbb{N}}$.

Proof. The consistency of our definition follows repeating mutatis mutandis point 4 of the proof of Theorem 2.21. It directly follows from the definitions of $(\Upsilon(\eta))_n$, $v(G)_n$ and $\tau_1 \circ \mathbf{LM}$ (see Definition 2.2) that $v(G)_n \circ (\Upsilon)(\eta)_n = (\tau_1 \circ \mathbf{LM})(\eta)_n \circ v(F)_n$. □

In fact, we have an easy description of the functor Υ.

Proposition 4.16. *There is a natural equivalence $\Upsilon \cong \tau_2$ where τ_2 is the translation functor introduced in Definition 3.1.*

Proof. Let F be an object of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$. By Proposition 2.23, for all natural numbers n, we have an isomorphism:

$$\begin{array}{rcl} \chi_{n,F} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_1]} \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} F(n+2) & \overset{\cong}{\longrightarrow} & F(n+2). \\ (g_1 - 1) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v & \longmapsto & v \end{array}$$

It follows from Definition 3.1 and Proposition 4.14 that the isomorphisms $\{\chi_{n,F}\}_{n\in\mathbb{N}}$ define the desired natural equivalence $\Upsilon \overset{\chi}{\to} \tau_2$. □

The functor $\mathbf{LM} \circ \tau_1$: Now, let us consider the part $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+2)$ of $\tau_1 \circ \mathbf{LM}(F)(n)$ in the decomposition of Proposition 4.4. In fact, we are going to prove that these modules assemble to form a functor which identifies with $\mathbf{LM}(\tau_1 F)$. We recall from Theorem 2.21 and Definition 3.1 the following fact.

Remark 4.17. The functor $\mathbf{LM} \circ \tau_1 : \mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$ is defined by:

- for $F \in Obj(\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}))$, $\forall n \in \mathbb{N}$, $(\mathbf{LM} \circ \tau_1)(F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+2)$, where $F(n+2)$ is a left $\mathbb{K}[\mathbf{F}_n]$-module using $F(id_1 \natural \varsigma_n(-)) : \mathbf{F}_n \to Aut_{\mathbb{K}\text{-}\mathfrak{Mod}}(F(n+2))$. For $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n'-n,\sigma] \in Hom_{\mathfrak{U}\beta}(n,n')$:

$$(\mathbf{LM} \circ \tau_1)(F)\left(\left[n'-n,\sigma\right]\right) = a_{n'}(\sigma)\left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right) \underset{\mathbb{K}\left[\mathbf{F}_{n'}\right]}{\otimes} F\left(id_1 \natural id_1 \natural \left[n'-n,\sigma\right]\right).$$

- Morphisms: let F and G be two objects of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$, and $\eta : F \to G$ be a natural transformation. The natural transformation $(\mathbf{LM} \circ \tau_1)(\eta) : (\mathbf{LM} \circ \tau_1)(F) \to (\mathbf{LM} \circ \tau_1)(G)$ for all natural numbers n is given by:

$$((\mathbf{LM} \circ \tau_1)(\eta))_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} \eta_{n+2}.$$

Proposition 4.18. *For all $F \in Obj\left(\mathbf{Fct}\left(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}\right)\right)$, the monomorphisms $\left\{\xi\left(F\right)_n\right\}_{n\in\mathbb{N}}$ (see Definition 4.5) allow to define a natural transformation $\xi'\left(F\right) : \left(\mathbf{LM}\circ\tau_1\right)\left(F\right) \to \left(\tau_1\circ\mathbf{LM}\right)\left(F\right)$ where, for all natural numbers n:*

$$\xi'\left(F\right)_n = \left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} F\left(\left(b^{\beta}_{1,1}\right)^{-1} \natural id_n\right).$$

This yields a natural transformation $\xi' : \mathbf{LM}\circ\tau_1 \to \tau_1\circ\mathbf{LM}$.

Proof. Let n and n' be natural numbers such that $n' \geq n$, and $[n'-n,\sigma] \in Hom_{\mathfrak{U}\beta}\left(n,n'\right)$ with $\sigma \in \mathbf{B}_{n'}$. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, $v \in F\left(n+2\right)$ and $g \in \mathbf{F}_n$. By Condition 2.3 (using Lemma 2.5 with $n' = n+1$) the following equality holds in $\mathbf{B}_{n+2}$:

$$\left(\left(b^{\beta}_{1,1}\right)^{-1} \natural id_n\right) \circ \left(id_1 \natural \varsigma_n\left(g\right)\right) = \varsigma_{1+n}\left(e_{\mathbf{F}_1} * g\right) \circ \left(\left(b^{\beta}_{1,1}\right)^{-1} \natural id_n\right).$$

Recall that $F\left(n+2\right)$ is a $\mathbb{K}\left[\mathbf{F}_n\right]$-module via $F\left(\varsigma_{1+n}\circ\left(\iota_{\mathbf{F}_1} * id_{\mathbf{F}_n}\right)\right)$ and $\tau_1 F\left(n+1\right)$ is a $\mathbb{K}\left[\mathbf{F}_n\right]$-module via $F\left(id_1\natural\left(\varsigma_n\circ id_{\mathbf{F}_n}\right)\right)$. Then it follows that the assignment $\xi'\left(F\right)_n$ is well-defined with respect to the tensor product structures of $\left(\mathbf{LM}\circ\tau_1\right)\left(F\right)\left(n\right)$ and $\left(\tau_1\circ\mathbf{LM}\right)\left(F\right)\left(n\right)$. Moreover, we compute that:

$$\begin{aligned}
&\left(\left(\tau_1\circ\mathbf{LM}\right)\left(F\right)\left(\left[n'-n,\sigma\right]\right)\right)\circ\left(\xi'\left(F\right)_n\right)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right)\\
= \quad & a_{1+n'}\left(id_1\natural\sigma\right)\left(a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n\right)\left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{1+n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)\left(i\right)\right)\\
&\underset{\mathbb{K}\left[\mathbf{F}_{n'+1}\right]}{\otimes} F\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural\left[n'-n,\sigma\right]\right)\left(v\right).
\end{aligned}$$

It follows from Condition 2.10 that:

$$a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n\right)\circ\left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{1+n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)\left(i\right) = \left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{1+n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)\left(i\right).$$

Again by Condition 2.10, we deduce that:

$$a_{1+n'}\left(id_1\natural\sigma\right)\circ\left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{1+n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)\left(i\right) = \iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * a_{n'}\left(\sigma\right)\left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)\left(i\right).$$

Hence, we deduce that:

$$\left(\left(\tau_1\circ\mathbf{LM}\right)\left(F\right)\left(\left[n'-n,\sigma\right]\right)\right)\circ\left(\xi'\left(F\right)_n\right) = \left(\xi'\left(F\right)_{n'}\right)\circ\left(\left(\mathbf{LM}\circ\tau_1\right)\left(F\right)\left(\left[n'-n,\sigma\right]\right)\right).$$

Let $\eta : F \to G$ be a natural transformation in the category $\mathbf{Fct}\left(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}\right)$ and let n be a natural number. Since η is a natural transformation, we have:

$$G\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right)\circ\eta_{n+2} = \eta_{n+2}\circ F\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right).$$

Hence, we deduce from the definitions of $\tau_1\circ\mathbf{LM}$ (see Theorem 2.21) and of $\mathbf{LM}\circ\tau_1$ (see Remark 4.17) that:

$$\xi'\left(G\right)_n\circ\left(\mathbf{LM}\circ\tau_1\right)\left(\eta\right)_n = \left(\tau_1\circ\mathbf{LM}\right)\left(\eta\right)_n\circ\xi'\left(F\right)_n.$$

□

4.1.3 Splitting of the translation functor

Now, we can establish a decomposition result for the translation functor applied to a Long-Moody functor.

Proposition 4.19. *There is a natural equivalence of endofunctors of $\mathbf{Fct}\left(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}\right)$:*

$$\tau_1\circ\mathbf{LM} \cong \tau_2 \oplus \left(\mathbf{LM}\circ\tau_1\right).$$

Proof. Recall the natural transformations $\upsilon : \Upsilon \to \tau_1 \circ \mathbf{LM}$ (introduced in Proposition 4.15) and $\xi' : \mathbf{LM} \circ \tau_1 \to \tau_1 \circ \mathbf{LM}$ (defined in Proposition 4.18). The direct sum in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ (induced by the direct sum in the category $\mathbb{K}\text{-}\mathfrak{Mod}$) allows us to define a natural transformation:

$$\upsilon \oplus \xi' : \Upsilon \oplus (\mathbf{LM} \circ \tau_1) \longrightarrow (\tau_1 \circ \mathbf{LM})(F).$$

This is a natural equivalence since for all natural numbers n, we have an isomorphism of $\mathbb{K}$-modules according to Proposition 4.4: $\Upsilon(F)(n) \oplus (\mathbf{LM} \circ \tau_1)(F)(n) \cong (\tau_1 \circ \mathbf{LM})(F)(n)$. We conclude using Proposition 4.16. □

4.2 Splitting of the difference functor

Recall the natural transformation $i_1 : Id_{\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})} \to \tau_1$ of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Our aim is to study the cokernel of $i_1 \circ \mathbf{LM}$. We recall that for F an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, for all natural numbers n, $(i_1\mathbf{LM})(F)_n = \mathbf{LM}(F)([1, id_{1+n}])$ (see Definition 3.4).

Remark 4.20. Explicitly for all elements i of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, for all elements v of $F(n)$:

$$(i_1\mathbf{LM})(F)_n \left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v \right) = \left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} \right)(i) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} F(id_1 \natural \iota_1 \natural id_n)(v).$$

The natural transformation $\mathbf{LM} \circ i_1$: Let us consider the exact sequence (6) in the category of endofunctors of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ of Proposition 3.5:

$$0 \longrightarrow \kappa_1 \xrightarrow{\Omega_1} Id \xrightarrow{i_1} \tau_1 \xrightarrow{\Delta_1} \delta_1 \longrightarrow 0.$$

Since the Long-Moody functor is exact (see Proposition 2.24), we have the following exact sequence:

$$0 \longrightarrow \mathbf{LM} \circ \kappa_1 \xrightarrow{\mathbf{LM}(\Omega_1)} \mathbf{LM} \xrightarrow{\mathbf{LM}(i_1)} \mathbf{LM} \circ \tau_1 \xrightarrow{\mathbf{LM}(\Delta_1)} \mathbf{LM} \circ \delta_1 \longrightarrow 0. \tag{12}$$

Remark 4.21. From the definition of $\mathbf{LM}$ (see Theorem 2.21), we deduce that for F an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, for all natural numbers n, for all elements i of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, for all elements v of $F(n)$:

$$\mathbf{LM}(i_1)(F)_n \left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v \right) = i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(\iota_1 \natural id_1 \natural id_n)(v).$$

Recall the natural transformation $\xi' : \mathbf{LM} \circ \tau_1 \to \tau_1 \circ \mathbf{LM}$ introduced in 4.18.

Lemma 4.22. *As natural transformations from $\mathbf{LM}$ to $\tau_1 \circ \mathbf{LM}$, which are endofunctors of the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, the following equality holds:*

$$\xi' \circ (\mathbf{LM}(i_1)) = i_1\mathbf{LM}.$$

Proof. Let F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Let n be a natural number. Let i be an element of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and let v be an element of $F(n)$. Since $\left(b^{\boldsymbol{\beta}}_{1,1}\right)^{-1} \circ (\iota_1 \natural id_1) = id_1 \natural \iota_1$ by Definition 1.13, we deduce from Proposition 4.18, Remark 4.21 and Remark 4.20, that:

$$\left(\xi' \circ (\mathbf{LM}(i_1))\right)(F)_n \left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v \right) = (id_1 * i) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} F(id_1 \natural \iota_1 \natural id_n)(v) = (i_1\mathbf{LM})(F)_n \left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v \right).$$

□

Decomposition results: Lemma 4.22 leads to the following key results.

Theorem 4.23. *There is a natural equivalence in the category* $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$*:*

$$\delta_1 \circ \mathbf{LM} \cong \tau_2 \oplus \left(\mathbf{LM} \circ \delta_1\right).$$

Moreover, there is a natural isomorphism $\kappa_1 \circ \mathbf{LM} \cong \mathbf{LM} \circ \kappa_1$.

Proof. It follows from the definition of i_1 (see Proposition 3.5) and from Lemma 4.22 that the following diagram is commutative and the row is an exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \kappa_1 \circ \mathbf{LM} & \xrightarrow{\Omega_1 \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_1 \mathbf{LM}} & \tau_1 \circ \mathbf{LM} & \xrightarrow{\Delta_1 \mathbf{LM}} & \delta_1 \circ \mathbf{LM} \longrightarrow 0 \\
 & & & & \Vert & & \uparrow{\scriptstyle \xi' \text{ by Lemma 4.22}} & & \\
 & & & & \mathbf{LM} & \xrightarrow{\mathbf{LM}(i_1)} & \mathbf{LM} \circ \tau_1. & &
\end{array}$$

We denote by $i^{\oplus}_{\mathbf{LM}\circ\tau_1}$ the inclusion morphism $\mathbf{LM} \circ \tau_1 \hookrightarrow \tau_2 \oplus (\mathbf{LM} \circ \tau_1)$. The functor $\mathbf{LM} \circ \kappa_1$ is also the kernel of the natural transformation $i^{\oplus}_{\mathbf{LM}\circ\tau_1} \circ (\mathbf{LM} \circ i_1)$, as the inclusion morphism $i^{\oplus}_{\mathbf{LM}\circ\tau_1} : \mathbf{LM} \circ \tau_1 \hookrightarrow \tau_2 \oplus (\mathbf{LM} \circ \tau_1)$ is a monomorphism. Then, recalling the exact sequence (12), we obtain that the following diagram is commutative and that the two rows are exact:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \kappa_1 \circ \mathbf{LM} & \xrightarrow{\Omega_1 \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_1 \mathbf{LM}} & \tau_1 \circ \mathbf{LM} & \xrightarrow{\Delta_1 \mathbf{LM}} & \delta_1 \circ \mathbf{LM} & \longrightarrow & 0 \\
 & & & & \Vert & {\scriptstyle \cong \text{ by Proposition 4.19}} & \uparrow{\scriptstyle v \oplus \xi'} & & & & \\
0 & \longrightarrow & \mathbf{LM} \circ \kappa_1 & \xrightarrow[\mathbf{LM}(\Omega_1)]{} & \mathbf{LM} & \xrightarrow[i^{\oplus}_{\mathbf{LM}\circ\tau_1} \circ (\mathbf{LM}(i_1))]{} & \tau_2 \oplus (\mathbf{LM} \circ \tau_1) & \xrightarrow[id_{\tau_2} \oplus (\mathbf{LM}(\Delta_1))]{} & \tau_2 \oplus (\mathbf{LM} \circ \delta_1) & \longrightarrow & 0
\end{array}$$

A fortiori, by definition of δ_1 (see Definition 3.4) and the universal property of the cokernel, we deduce that:

$$\tau_2 \oplus (\mathbf{LM} \circ \delta_1) \cong \delta_1 \circ \mathbf{LM}.$$

Furthermore, by the unicity up to isomorphism of the kernel, we conclude that $\kappa_1 \circ \mathbf{LM} \cong \mathbf{LM} \circ \kappa_1$. □

4.3 Increase of the polynomial degree

The results formulated in Theorem 4.23 allow us to understand the effect of the Long-Moody functors on (very) strong polynomial functors.

Proposition 4.24. *Let F be a non-null object of* $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$*. If the functor F is strong polynomial of degree d, then:*

1. *the functor $\tau_2(F)$ belongs to* $\mathcal{P}ol_d^{strong}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$*;*
2. *the functor $\mathbf{LM}(F)$ belongs to* $\mathcal{P}ol_{d+1}^{strong}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$*.*

Proof. We prove these two results by induction on the degree of polynomiality. For the first result, it follows from the commutation property 5 of Proposition 3.5 for τ_2. For the second result, let us first consider F a strong polynomial functor of degree 0. By Theorem 4.23, we obtain that $\delta_1 \mathbf{LM}(F) \cong \tau_2(F)$. Therefore $\mathbf{LM}(F)$ is a strong polynomial functor of degree less than or equal to 1. Now, assume that F is a strong polynomial functor of degree $n \geq 0$. By Theorem 4.23: $\delta_1 \mathbf{LM}(F) \cong \mathbf{LM}(\delta_1 F) \oplus \tau_2(F)$. By the inductive hypothesis and the result on τ_2, we deduce that $\mathbf{LM}(F)$ is a strong polynomial functor of degree less than or equal to $n+1$. □

Corollary 4.25. *For all natural numbers d, the endofunctor $\mathbf{LM}$ of* $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$ *restricts to a functor:*

$$\mathbf{LM} : \mathcal{P}ol_d^{strong}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right) \longrightarrow \mathcal{P}ol_{d+1}^{strong}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right).$$

Corollary 4.26. *Let d be a natural number and F be an object of $\mathcal{P}ol_d^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ such that the strong polynomial degree of $\tau_2(F)$ is equal to d. Then, the functor $\mathbf{LM}(F)$ is a strong polynomial functor of degree equal to $d+1$.*

Theorem 4.27. *Let d be a natural number and F be an object of $\mathcal{VP}ol_d(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ of degree equal to d. Then, the functor $\mathbf{LM}(F)$ is a very strong polynomial functor of degree equal to $d+1$.*

Proof. Using Lemma 3.23, it follows from Corollary 4.26 that $\mathbf{LM}(F)$ is a strong polynomial functor of degree equal to $n+1$. Since the functor $\mathbf{LM}$ commutes with the evanescence functor κ_1 by Theorem 4.23, we deduce that $(\kappa_1 \circ \mathbf{LM})(F) \cong (\mathbf{LM} \circ \kappa_1)(F) = 0$. Moreover, using Theorem 4.23, we have:

$$(\kappa_1 \circ (\delta_1 \circ \mathbf{LM}))(F) \cong (\kappa_1 \circ \tau_2)(F) \bigoplus (\kappa_1 \circ (\mathbf{LM} \circ \delta_1))(F).$$

Therefore, the fact that τ_2 commutes with the evanescence functor κ_1 (see the commutation property 6 of Proposition 3.5) and Theorem 4.23 together imply that:

$$(\kappa_1 \circ (\delta_1 \circ \mathbf{LM}))(F) \cong (\tau_2 \circ \kappa_1)(F) \bigoplus (\mathbf{LM} \circ (\kappa_1 \circ \delta_1))(F).$$

The result then follows from the fact that F is an object of $\mathcal{VP}ol_n(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ and τ_2 is a reduced endofunctor of the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. ☐

Example 4.28. By Proposition 3.20, $\mathfrak{X}$ is a very strong polynomial functor of degree 0. Now applying the Long-Moody functor $\mathbf{LM}_1$, we proved in Proposition 2.31 that $t^{-1}\mathbf{LM}_1(t\mathfrak{X})$ is naturally equivalent to $\mathfrak{Bur}_{t^2}$, which is very strong polynomial of degree 1 by Proposition 3.25.

4.4 Other properties of the Long-Moody functors

We have proven in the previous section that a Long-Moody functor sends (very) strong polynomial functors to (very) strong polynomial functors. We can also prove that a (very) strong polynomial functor in the essential image of a Long-Moody functor is necessarily the image of another strong polynomial functor.

Proposition 4.29. *Let d be a natural number. Let F be a strong polynomial functor of degree d in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Assume that there exists an object G of the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ such that $\mathbf{LM}(G) = F$. Then, the functor G is a strong polynomial functor of degree less than or equal to $d+1$ in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.*

Proof. It follows from Theorem 4.23 that:

$$\delta_1 F \cong \tau_2(G) \oplus (\mathbf{LM} \circ \delta_1)(G).$$

According to Corollary 3.10, the functor $\tau_2(G)$ is an object of the category $\mathcal{P}ol_{d-1}^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, and because of Lemma 3.14 the functor G is an object of the category $\mathcal{P}ol_{d+1}^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. ☐

Proposition 4.30. *The Long-Moody functor $\mathbf{LM} : \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \longrightarrow \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ is not essentially surjective.*

Proof. Let l be a natural number. Let $E_l : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{K}\text{-}\mathfrak{Mod}$ be the functor which factorizes through the category $\mathbb{N}$, such that $E_l(n) = \mathbb{K}^{\oplus n^l}$ for all natural numbers n and for all $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$ (with n, n' natural numbers such that $n' \geq n$), $E_l([n'-n,\sigma]) = \iota_{\mathbb{C}[t^{\pm 1}]^{\oplus n'^l - n^l}} \oplus id_{\mathbb{C}[t^{\pm 1}]^{\oplus n^l}}$. In particular, for all natural numbers n, for every Artin generator σ_i of $\mathbf{B}_n$, $E_l(\sigma_i) = id_{\mathbb{K}^{\oplus n^l}}$. It inductively follows from this definition and direct computations that E_l is a very strong polynomial functor of degree l.

Let us assume that $\mathbf{LM}$ is essentially surjective. Hence, there exists an object F of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ such that $\mathbf{LM}(F) \cong E_l$. Because of the definition of $\mathbf{LM}(F)$ on morphisms (see Theorem 2.21), this implies that for all natural numbers n and for all $\sigma \in \mathbf{B}_n$, $a_n(\sigma) = id_n$. Also, if $\mathbf{LM}$ is essentially surjective, there exists an object T of the category $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ such that we can recover the Burau functor from $\mathbf{LM}(T)$, ie something like $\alpha\mathbf{LM}(T)$ (see Notation 2.29) with $\alpha \in \mathbb{K}$. We deduce from the definition of $\mathbf{LM}(T)$ on objects and morphisms that for all $n \geq 1$, $T(n) = \mathbb{K}$ and for all generator σ_i of $\mathbf{B}_n$:

$$\mathbf{LM}(T)(\sigma_i) = T(\sigma_i) \cdot Id_n.$$

Then necessarily, for all $i \in \{1, \ldots, n\}$, $T(\sigma_i) = \delta$ such that $\delta^2 = t$ and we consider $\delta^{-1}\mathbf{LM}(T)$. We deduce that there exists a natural transformation $\omega : \delta^{-1}\mathbf{LM}(T) \overset{\cong}{\to} \mathfrak{Bur}_t$. This contradicts the fact that for all $\sigma \in \mathbf{B}_n$, $a_n(\sigma) = id_n$. ☐

Remark 4.31. The proof of Proposition 4.30 shows in particular that a Long-Moody functor **LM** is not essentially surjective on very strong polynomial functors in any degree.

In [5, Section 4.7, Open Problem 7], Birman and Brendle ask "whether all finite dimensional unitary matrix representations of $\mathbf{B}_n$ arise in a manner which is related to the construction" recalled in Theorem 2.21. Since the Tong-Yang-Ma and unreduced Burau representations recalled in Theorem 1.19 are unitary representations, the proof of Proposition 4.30 shows that any Long-Moody functor (and especially the one based on the version of the construction of Theorem 2.21) cannot provide all the functors encoding unitary representations. Therefore, we refine the problem asking whether all functors encoding families of finite dimensional unitary representations of braid groups lie in the image of a Long-Moody functor.

Remark 4.32. Another question is to ask whether we can directly obtain the reduced Burau functor $\overline{\mathfrak{Bur}}_t$ by a Long-Moody functor. Recall that for all natural numbers n, $\overline{\mathfrak{Bur}}_t(n) = \mathbb{C}\left[t^{\pm1}\right]^{\oplus n-1}$ and $\mathbf{LM}(F)(n) \cong (F(n+1))^{\oplus n}$ for any Long-Moody functor **LM** and any object F of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ (see Remark 2.25). Therefore, for dimensional considerations on the objects, it is clear that we have to consider a modified version of the Long-Moody construction. This modification would be to take the tensor product with $\mathcal{I}_{\mathbf{F}_{n-1}}$ on $\mathbf{F}_{n-1}$, the $\mathbb{K}$-module $F(n+1)$ being a $\mathbb{K}[\mathbf{F}_{n-1}]$-module using a morphism $\mathbf{F}_{n-1} \to \left(\mathbf{F}_{n-1} \underset{a'_n}{\rtimes} \mathbf{B}_{n+1}\right) \to \mathbf{B}_{n+1}$ for all natural numbers n, where $a'_n : \mathbf{B}_{n+1} \to Aut(\mathbf{F}_{n-1})$ is a group morphism.

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