

GENERALIZED LONG-MOODY FUNCTORS

ARTHUR SOULIÉ

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Abstract

In this paper, we generalize the Long-Moody construction for representations of braid groups to other groups, such as mapping class groups of surfaces. Moreover, we introduce Long-Moody endofunctors over a functor category that encodes representations of a family of groups. In this context, notions of polynomial functor are defined; these play an important role in the study of homological stability. We prove that, under some additional assumptions, a Long-Moody functor increases the very strong (respectively weak) polynomial degree of functors by one.

Introduction

In 1994, as a result of a collaboration with Moody, Long [26] gave a method to construct a linear representation of \mathbf{B}_n from a representation of \mathbf{B}_{n+1} , where \mathbf{B}_n denotes the braid group on n strands. The underlying framework of this Long-Moody construction naturally arises in many situations in connection with topology: the first aim of this paper is to extend this construction to these situations.

Namely, for a family of groups $\{G_n\}_{n \in \mathbb{N}}$ equipped with injections $\gamma_n : G_n \hookrightarrow G_{n+1}$, we give a method to construct a representation of G_n from a -representation of G_{n+1} , which generalizes the underlying idea of the original Long-Moody construction. Achieving this requires another family of groups $\{H_n\}_{n \in \mathbb{N}}$, along with an action a_n of G_n on H_n for all natural numbers n . For instance, we can consider the following situation:

- take as the family of groups $\{G_n\}_{n \in \mathbb{N}}$ the family of the isotopy classes of homeomorphisms restricting to the identity on the boundary component $\{\Gamma_{n,1}\}_{n \in \mathbb{N}}$ (also known as mapping class groups) of smooth connected compact surfaces with genus n and one boundary component $\{\Sigma_{n,1}\}_{n \in \mathbb{N}}$;
- define the injection $\gamma_n : \Gamma_{n,1} \hookrightarrow \Gamma_{n+1,1}$ by extending an element $\varphi \in \Gamma_{n,1}$ to a mapping class of $\Sigma_{n+1,1}$ by the identity on the complement $\Sigma_{1,1}$ of $\Sigma_{n,1} \hookrightarrow \Sigma_{n+1,1}$;
- take as the family of groups $\{H_n\}_{n \in \mathbb{N}}$ to be the fundamental groups $\{\pi_1(\Sigma_{n,1}, p)\}_{n \in \mathbb{N}}$ of the surfaces $\{\Sigma_{n,1}\}_{n \in \mathbb{N}}$ (where p is a point in the boundary component);
- consider the natural action of $\Gamma_{n,1}$ on $\pi_1(\Sigma_{n,1}, p)$.

Let R be a commutative ring. For n a natural number, we denote by $\mathcal{I}_{R[H_n]}$ the augmentation ideal of the group H_n . Let M_{n+1} be an R -module and $\varrho_{n+1} : G_{n+1} \rightarrow GL_R(M_{n+1})$ a representation. Also, we need for the Long-Moody construction to consider M_{n+1} as an H_n -module: for this we use a group morphism $\zeta_n : H_n \rightarrow G_{n+1}$, given as part of the structure. The key idea of the Long-Moody construction is to give the tensor product

$$\mathcal{I}_{R[H_n]} \otimes_{R[H_n]} M_{n+1}$$

a G_n -module structure. The G_n -module structure for the left hand factor is then induced by the action a_n of G_n on H_n , and the one on the right hand factor is defined by ϱ_{n+1} precomposing by ζ_n : the latter requires technical compatibilities between the morphisms a_n , ζ_n and γ_n for the tensor product to be well-defined. These can be encoded by using the *Grothendieck construction* as follows.

We consider the groupoid \mathcal{G} with objects indexed by the natural numbers (and denoted by n) and with the groups $\{G_n\}_{n \in \mathbb{N}}$ as automorphism groups. The cornerstone to define the Long-Moody construction is

to assume that the families of morphisms $\{a_n\}_{n \in \mathbb{N}}$ and $\{\zeta_n\}_{n \in \mathbb{N}}$ assemble to define functors $\mathcal{A} : \mathcal{G} \rightarrow \mathfrak{Gr}$ and $\zeta : \int^{\mathcal{G}} \mathcal{A} \rightarrow \mathcal{G}$, where \mathfrak{Gr} denotes the category of groups and $\int^{\mathcal{G}} \mathcal{A}$ is the Grothendieck construction on \mathcal{A} (see Section 1). In addition, we require that the following diagram be commutative

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{i} & \int^{\mathcal{G}} \mathcal{A} \\ & \searrow \gamma & \downarrow \zeta \\ & & \mathcal{G} \end{array}$$

where i is the evident section of the projection functor induced by the Grothendieck construction and γ is the functor induced by the canonical injections $\{\gamma_n : G_n \hookrightarrow G_{n+1}\}_{n \in \mathbb{N}}$. The Grothendieck construction encodes semi-direct product structures, so this condition actually reflects the factorization of the injections γ_n through $H_n \rtimes_{a_n} G_n$.

Furthermore, instead of considering these constructions for only one group G_n , a natural question is how to extend these constructions to families of representations of $\{G_n\}_{n \in \mathbb{N}}$. We denote by $R\text{-Mod}$ the category of R -modules and by $\mathbf{Fct}(\mathcal{C}, R\text{-Mod})$ the category of functors from \mathcal{C} to $R\text{-Mod}$, for \mathcal{C} a small category. Hence, an object M of $\mathbf{Fct}(\mathcal{G}, R\text{-Mod})$ is a collection of linear representations

$$\{\varrho_n : G_n \rightarrow GL_R(M_n)\}_{n \in \mathbb{N}}.$$

Moreover, we require that considering the restriction of ϱ_{n+1} to G_n is ϱ_n . Namely, we assume that there exist maps $m_n : M_n \rightarrow M_{n+1}$, such that for all natural numbers n :

$$m_n \circ \varrho_n(g) = (\varrho_{n+1} \circ \gamma_n)(g) \quad (1)$$

for all $g \in G_n$. Thus, we say that the representations $\{M_n\}_{n \in \mathbb{N}}$ form a family of linear representations of the groups $\{G_n\}_{n \in \mathbb{N}}$. However, the extra information (1) on M is not encoded by the fact that M is an object of $\mathbf{Fct}(\mathcal{G}, R\text{-Mod})$. Quillen's bracket construction (see [20, p.219]) provides a new category \mathcal{UG} which resolves this failure: the groupoid \mathcal{G} is its maximal subgroupoid, \mathcal{UG} contains the canonical injections $\{\gamma_n\}_{n \in \mathbb{N}}$ and γ canonically extends to \mathcal{UG} . This category is also important since it provides a natural setting to study coefficient systems for homological stability (see [32, Section 4]).

For instance, the groupoid \mathcal{G} for the family of mapping class groups $\{\Gamma_{n,1}\}_{n \in \mathbb{N}}$ is introduced in Section 3.3.1 and is denoted by $\mathfrak{M}_2^{+,0}$. An example of object of $\mathbf{Fct}(\mathfrak{M}_2^{+,0}, R\text{-Mod})$ is given by the family of symplectic representations of the mapping class groups.

Therefore, our goal is to define the Long-Moody construction as an endofunctor of the functor category $\mathbf{Fct}(\mathcal{UG}, R\text{-Mod})$. We thus deal with an extension problem: we require the functor \mathcal{A} (respectively ζ) to extend along the inclusion $\mathcal{G} \hookrightarrow \mathcal{UG}$ (resp. along the inclusion $\int^{\mathcal{G}} \mathcal{A} \rightarrow \int^{\mathcal{UG}} \mathcal{A}$) so that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{UG} & \xrightarrow{i} & \int^{\mathcal{UG}} \mathcal{A} \\ & \searrow \gamma & \downarrow \zeta \\ & & \mathcal{UG} \end{array}$$

Under these assumptions, we prove:

Theorem A (Definition 2.12). *There is a right-exact functor $\mathbf{LM}_{\{\mathcal{G}, \zeta\}} : \mathbf{Fct}(\mathcal{UG}, R\text{-Mod}) \rightarrow \mathbf{Fct}(\mathcal{UG}, R\text{-Mod})$, the Long-Moody functor, that in particular assigns:*

$$\mathbf{LM}_{\{\mathcal{G}, \zeta\}}(F)(\underline{n}) = \mathcal{I}_{R[H_n]} \otimes_{R[H_n]} F(\underline{n+1})$$

for all objects F of $\mathbf{Fct}(\mathcal{UG}, R\text{-Mod})$ and \underline{n} objects of \mathcal{UG} , and:

$$\mathbf{LM}_{\{\mathcal{G}, \zeta\}}(F)(g) = a_n(g) \otimes_{R[H_n]} F(\gamma_n(g))$$

for all elements g of G_n .

In particular, if we take the groups $\{G_n\}_{n \in \mathbb{N}}$ to be the family of braid groups $\{\mathbf{B}_n\}_{n \in \mathbb{N}}$ and the groups $\{H_n\}_{n \in \mathbb{N}}$ to be the family of free groups $\{\mathbf{F}_n\}_{n \in \mathbb{N}}$, Theorem A recovers [33, Theorem A]. Additionally, the families of symmetric groups or mapping class groups of orientable and non-orientable surfaces also fit into this framework (see Section 3). Further families of groups also fit into the present framework such as mapping class groups of compact connected oriented 3-manifolds with boundary, or automorphism groups of free products of groups. These examples are not addressed in the present paper.

The necessary coherence conditions on $\{\zeta_n\}_{n \in \mathbb{N}}$ to define a Long-Moody functor are restrictive. Nevertheless, the trivial morphisms $\{\zeta_{n,t} : H_n \rightarrow G_{n+1}\}_{n \in \mathbb{N}}$ at least always satisfy the necessary technical condition. We denote by $\mathbf{LM}_{\{\mathcal{G}, \zeta_t\}}$ the associated Long-Moody functor and by $R : \mathcal{UG} \rightarrow R\text{-Mod}$ the constant functor at R . For example, $\mathbf{LM}_{\{\mathfrak{M}_2^{+,0}, \zeta_t\}}(R)$ encodes the family of symplectic representations of the mapping class groups $\{\Gamma_{n,1}\}_{n \in \mathbb{N}}$ (see Example 3.9). Furthermore, we prove (see Proposition 2.20) that for all objects F of $\mathbf{Fct}(\mathcal{UG}, R\text{-Mod})$, there is a natural equivalence for the associated Long-Moody functor:

$$\mathbf{LM}_{\{\mathcal{G}, \zeta_t\}}(F) \cong \mathbf{LM}_{\{\mathcal{G}, \zeta_t\}}(R) \otimes_R F(\gamma_n(-)). \quad (2)$$

The Long-Moody functor $\mathbf{LM}_{\{\mathcal{G}, \zeta_t\}}$ is thus defined as the tensor product of $F \circ \gamma_n$ with the functor $\mathbf{LM}_{\{\mathcal{G}, \zeta_t\}}(R)$. In general, the equivalence (2) does not hold: $\mathbf{LM}_{\{\mathcal{G}, \zeta_t\}}$ can then be considered as a “twisted” tensor product with the functor $\mathbf{LM}_{\{\mathcal{G}, \zeta_t\}}(R)$.

In addition, there are situations where non-trivial $\{\zeta_n\}_{n \in \mathbb{N}}$ arise naturally, in particular for some families of groups in connection with topology. For example, for a natural number $g \geq 2$, we denote by $\Sigma_{g,1}^m$ the surface $\Sigma_{g,1}$ with m punctures. We consider for the groups $\{G_n\}_{n \in \mathbb{N}}$ the family of mapping class groups $\{\Gamma_{g,1}^n\}_{n \in \mathbb{N}}$ of the surfaces $\{\Sigma_{g,1}^n\}_{n \in \mathbb{N}}$ (the mapping classes fix the boundary pointwise and permute the punctures), the associated groupoid is denoted by $\mathfrak{M}_2^{+,g}$, and the groups $\{H_n\}_{n \in \mathbb{N}}$ are the corresponding fundamental groups (see Section 3.4). Then, using a splitting of the Birman short exact sequence [15, Theorem 4.6] (see Definition 3.17), the morphisms

$$\{\zeta_{n,1} : \pi_1(\Sigma_{g,1}^n, p) \rightarrow \Gamma_{g,1}^{n+1}\}_{n \in \mathbb{N}}$$

give rise to a Long-Moody functor $\mathbf{LM}_{\{\mathfrak{M}_2^{g,0}, \zeta_1\}}$. The iterates of this procedure provide linear representations of the family of groups $\{\Gamma_{g,1}^n\}_{n \in \mathbb{N}}$ which, as far as the author knows, are unknown in the literature.

Furthermore, among the objects in the category $\mathbf{Fct}(\mathcal{UG}, R\text{-Mod})$ of particular importance are the *strong* and *very strong polynomial functors*. The first notions of polynomial functors date back to Eilenberg and Mac Lane in [13] for functors on module categories. Djament and Vespa introduced in [12] strong polynomial functors for symmetric monoidal categories in which the monoidal unit is initial. This definition was extended to pre-braided monoidal categories in which the monoidal unit is initial in [33, Section 3]. The notion of very strong polynomial functor in this context was introduced in [33, Section 3]; it is equivalent to that of coefficient systems of finite degree of [32, Section 4.4]. We show that these notions of strong and very strong polynomial functors extend to the more general context of the present paper (see Section 4.1).

One reason for our interest in very strong polynomial functors is their homological stability properties: in [32], Randal-Williams and Wahl prove homological stability results for certain families of groups $\{G_n\}_{n \in \mathbb{N}}$ with twisted coefficients given by very strong polynomial objects of $\mathbf{Fct}(\mathcal{UG}, \mathbb{Z}\text{-Mod})$. Their results hold for braid groups, automorphism groups of free products of groups, mapping class groups of orientable and non-orientable surfaces or mapping class groups of 3-manifolds. The representation theory of these groups is complicated and an active research topic (see for example [2, Section 4.6], [16], [23] or [29]). A fortiori, the very strong polynomial functors associated with these groups are not well-understood.

In addition, we are interested in *weak* polynomial functors, a notion introduced by Djament and Vespa in [12, Section 3.1] for symmetric monoidal categories and extended to the present framework in Section 5.3. This last notion is more appropriate for understanding the stable behaviour of a given functor.

We investigate the effects of Long-Moody functors on polynomial functors, and prove:

Theorem B (Theorems 5.18 and 5.21). *Assume the hypotheses of Theorem A are satisfied and that the groups*

$\{H_n\}_{n \in \mathbb{N}}$ are free. Under the Assumption 5.1, the Long-Moody functor $\mathbf{LM}_{\{\mathcal{G}, \mathfrak{C}\}}$ increases by one both the very strong and the weak polynomial degrees.

Assumption 5.1 is a technical (but quite natural) hypothesis that is required in the analysis of the behaviour of the Long-Moody functors with respect to polynomial degree. It is a further compatibility condition which is satisfied in many of the examples of interest, such as mapping class groups of surfaces (see Section 6). For the family of braid groups $\{\mathbf{B}_n\}_{n \in \mathbb{N}}$, Theorem B recovers [33, Theorem B].

Hence, the Long-Moody functors provide, by iteration, new families of (very) strong polynomial and weak polynomial functors of $\mathbf{Fct}(\mathcal{U}\mathcal{G}, R\text{-Mod})$ in any degree. Therefore, this result allows to gain a better understanding of polynomial functors for mapping class groups and extends the scope of twisted homological stability to more sophisticated sequences of representations. These methods also introduce new tools to clarify the structures of weak polynomial functors in this context (see Proposition 6.7).

The paper is organized as follows. In Section 1, we recall Quillen’s bracket construction. In Section 2, after setting up the general framework of the families of groups, we define the generalized Long-Moody functors and give some their properties. Section 3 is devoted to the application of Long-Moody functors to the mapping class groups of surfaces (recovering in particular the case of braid groups) and symmetric groups. Section 4 introduces the notions of strong, very strong and weak polynomial functors in the present framework. In Section 5, we consider the effect of Long-Moody functors on strong and weak polynomial functors, presenting in particular the keystone relations for the action of the *difference* and *evanescence* functors on Long-Moody functors. Finally, in Section 6, we explain the applications of the effect of Long-Moody functors on polynomiality, in particular their interest for homological stability results.

General notations. We fix a commutative unital ring R throughout this work. We denote by $R\text{-Mod}$ the category of R -modules. We denote by \mathfrak{Gr} the category of groups and by $*$ the coproduct in this category.

Let \mathfrak{Cat} denote the category of small categories. Let \mathfrak{C} be an object of \mathfrak{Cat} . We use the abbreviation $\text{Obj}(\mathfrak{C})$ to denote the set of objects of \mathfrak{C} . If there exists an initial object \emptyset in the category \mathfrak{C} , then we denote by $\iota_A : \emptyset \rightarrow A$ the unique morphism from \emptyset to A . If t is a terminal object in the category \mathfrak{C} , then we denote by $t_A : A \rightarrow t$ the unique morphism from A to t .

The maximal subgroupoid $\mathcal{Gr}(\mathfrak{C})$ is the subcategory of \mathfrak{C} which has the same objects as \mathfrak{C} and whose morphisms are the isomorphisms of \mathfrak{C} . We denote by $\mathcal{Gr} : \mathfrak{Cat} \rightarrow \mathfrak{Cat}$ the functor which associates to a category its maximal subgroupoid.

For \mathfrak{D} a category and \mathfrak{C} a small category, we denote by $\mathbf{Fct}(\mathfrak{C}, \mathfrak{D})$ the category of functors from \mathfrak{C} to \mathfrak{D} .

We take the convention that the natural numbers \mathbb{N} are the non-negative integers.

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1 Preliminaries on some categorical tools

The aim of this section is to introduce the categorical framework necessary for our study. In particular, we recall notions and properties of Quillen's bracket construction introduced in [20, p.219] and pre-braided monoidal categories. Our review here is based on [32, Section 1] to which we refer the reader for further details. Finally, we introduce a construction for functors from a small categories to the category of small categories, called the Grothendieck construction.

Beforehand, we take this opportunity to recall some terminology about strict monoidal categories. We refer to [27] for an introduction to (braided) strict monoidal categories. A strict monoidal category will be denoted by $(\mathcal{C}, \natural, 0)$, where \mathcal{C} is the category, \natural is the monoidal product and 0 is the monoidal unit. If it is braided, then its braiding is denoted by $b_{A,B}^{\mathcal{C}} : A \natural B \xrightarrow{\sim} B \natural A$ for all objects A and B of \mathcal{C} . **We fix a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ throughout this section.**

Quillen's bracket construction. The following definition is a particular case of a more general construction of [20].

Definition 1.1. [32, Section 1.1] Quillen's bracket construction on the groupoid \mathfrak{G} , denoted by $\mathfrak{L}\mathfrak{G}$, is the category defined by:

- Objects: $Obj(\mathfrak{L}\mathfrak{G}) = Obj(\mathfrak{G})$;
- Morphisms: for A and B objects of \mathfrak{G} :

$$Hom_{\mathfrak{L}\mathfrak{G}}(A, B) = colim_{\mathfrak{G}} [Hom_{\mathfrak{G}}(- \natural A, B)].$$

Thus, a morphism from A to B in the category $\mathfrak{L}\mathfrak{G}$ is an equivalence class of pairs (X, f) , where X is an object of \mathfrak{G} and $f : X \natural A \rightarrow B$ is a morphism of \mathfrak{G} ; this is denoted by $[X, f] : A \rightarrow B$.

- For all objects X of $\mathfrak{U}\mathfrak{G}$, the identity morphism in the category $\mathfrak{U}\mathfrak{G}$ is given by $[0, id_X] : X \longrightarrow X$.
- Let $[X, f] : A \longrightarrow B$ and $[Y, g] : B \longrightarrow C$ be two morphisms in the category $\mathfrak{U}\mathfrak{G}$. Then, the composition in the category $\mathfrak{U}\mathfrak{G}$ is defined by:

$$[Y, g] \circ [X, f] = [Y \natural X, g \circ (id_Y \natural f)].$$

Proposition 1.2. [32, Proposition 1.8 (i)] *The unit 0 of the monoidal structure of the groupoid $(\mathfrak{G}, \natural, 0)$ is an initial object in the category $\mathfrak{U}\mathfrak{G}$.*

Remark 1.3. Let X be an object of \mathfrak{G} . Let $\phi \in Aut_{\mathfrak{G}}(X)$. Then, as an element of $Hom_{\mathfrak{U}\mathfrak{G}}(X, X)$, we will abuse the notation and write ϕ for $[0, \phi]$. This comes from the (faithful) canonical functor $\mathfrak{c}_{\mathfrak{U}\mathfrak{G}} : \mathfrak{G} \hookrightarrow \mathfrak{U}\mathfrak{G}$ defined as the identity on objects and sending $\phi \in Aut_{\mathfrak{G}}(X)$ to $[0, \phi]$.

A natural question is the relationship between the automorphisms of the groupoid \mathfrak{G} and those of its associated Quillen bracket construction $\mathfrak{U}\mathfrak{G}$. Recall the following notion.

Definition 1.4. The strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors if, for all objects A and B of \mathfrak{G} , $A \natural B \cong 0$ if and only if $A \cong B \cong 0$.

Then, recall the result:

Proposition 1.5. [32, Proposition 1.7] *Assume that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$. Then, $\mathfrak{G} = \mathcal{G}\mathfrak{r}(\mathfrak{U}\mathfrak{G})$.*

Henceforth, we assume that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$.

A natural question is to wonder when an object of $\mathbf{Fct}(\mathfrak{G}, \mathcal{C})$ extends to an object of $\mathbf{Fct}(\mathfrak{U}\mathfrak{G}, \mathcal{C})$ for a category \mathcal{C} , which is the aim of the following lemma. Analogous statements can be found in [32, Proposition 2.4] and [33, Lemma 1.12] (for the category $\mathfrak{U}\beta$ for this last reference).

Lemma 1.6. *Let \mathcal{C} be a category and F an object of $\mathbf{Fct}(\mathfrak{G}, \mathcal{C})$. Assume that for $A, X, Y \in Obj(\mathfrak{G})$, there exist assignments $F([X, id_{X \natural A}]) : F(A) \rightarrow F(X \natural A)$ such that:*

$$F([Y, id_{Y \natural X \natural A}]) \circ F([X, id_{X \natural A}]) = F([Y \natural X, id_{Y \natural X \natural A}]). \quad (3)$$

Then, the assignments $F([X, \gamma]) = F(\gamma) \circ F([X, id_{X \natural A}])$ for all $[X, \gamma] \in Hom_{\mathfrak{U}\mathfrak{G}}(A, X \natural A)$ define a functor $F : \mathfrak{U}\mathfrak{G} \rightarrow \mathcal{C}$ if and only if for all $A, X \in Obj(\mathfrak{G})$, for all $\gamma'' \in Aut_{\mathfrak{G}}(A)$ and all $\gamma' \in Aut_{\mathfrak{G}}(X)$:

$$F([X, id_{X \natural A}]) \circ F(\gamma'') = F(\gamma' \natural \gamma'') \circ F([X, id_{X \natural A}]). \quad (4)$$

Proof. Assume that relation (4) is satisfied. Note that (3) implies that $F([0, id_A]) = id_{F(A)}$ for all objects A . First, let us prove that our assignment conforms with the defining equivalence relation of $\mathfrak{U}\mathfrak{G}$. Let $A, X \in Obj(\mathfrak{G})$. Let $\gamma, \gamma' \in Aut_{\mathfrak{G}}(X \natural A)$ such that there exists $\psi \in Aut_{\mathfrak{G}}(X)$ so that $\gamma' \circ (\psi \natural id_A) = \gamma$. According to the relation (4) and since F is a functor over \mathfrak{G} , we deduce that $F([X, \gamma]) = F(\gamma') \circ F([X, id_{X \natural A}]) \circ F(id_A) = F([X, \gamma'])$. Now, let us check the composition axiom. Let $A, X, Y \in Obj(\mathfrak{G})$, let $([X, \gamma]) \in Hom_{\mathfrak{U}\mathfrak{G}}(A, X \natural A)$ and $([Y, \gamma']) \in Hom_{\mathfrak{U}\mathfrak{G}}(X \natural A, Y \natural X \natural A)$. We deduce from relation (4) that:

$$F([Y, \gamma']) \circ F([X, \gamma]) = F(\gamma') \circ (F(id_Y \natural \gamma) \circ F([Y, id_{Y \natural X \natural A}])) \circ F([X, id_{X \natural A}]).$$

So, it follows from relation (3) that:

$$F([Y, \gamma']) \circ F([X, \gamma]) = F(\gamma' \circ (id_Y \natural \gamma)) \circ F([Y \natural X, id_{Y \natural X \natural A}]) = F([Y, \gamma'] \circ [X, \gamma]).$$

Conversely, assume that the functor $F : \mathfrak{U}\mathfrak{G} \rightarrow \mathcal{C}$ is well-defined. In particular, the composition axiom in $\mathfrak{U}\mathfrak{G}$ is satisfied and implies that for all $A, X \in Obj(\mathfrak{G})$, for all $\gamma \in Aut_{\mathfrak{G}}(A)$, $F([X, id_{X \natural A}]) \circ F(\gamma) = F([X, id_X \natural \gamma])$. So it follows from the defining equivalence relation of $\mathfrak{U}\mathfrak{G}$ that relation (4) is satisfied. \square

Similarly, we can find a criterion for extending a morphism in the category $\mathbf{Fct}(\mathfrak{G}, \mathcal{C})$ to a morphism in the category $\mathbf{Fct}(\mathfrak{U}\mathfrak{G}, \mathcal{C})$.

Lemma 1.7. Let \mathcal{C} be a category, F and G be objects of $\mathbf{Fct}(\mathfrak{U}\mathfrak{G}, \mathcal{C})$ and $\eta : F \rightarrow G$ a natural transformation in $\mathbf{Fct}(\mathfrak{G}, \mathcal{C})$. The restriction $\mathbf{Fct}(\mathfrak{U}\mathfrak{G}, \mathcal{C}) \rightarrow \mathbf{Fct}(\mathfrak{G}, \mathcal{C})$ is obtained by precomposing by the canonical inclusion $c_{\mathfrak{U}\mathfrak{G}}$ of Remark 1.3. Then, η is a natural transformation in the category $\mathbf{Fct}(\mathfrak{U}\mathfrak{G}, \mathcal{C})$ if and only if for all $A, B \in \text{Obj}(\mathfrak{G})$ such that $B \cong X \natural A$ with $X \in \text{Obj}(\mathfrak{G})$:

$$\eta_B \circ F([X, id_B]) = G([X, id_B]) \circ \eta_A. \quad (5)$$

Proof. The natural transformation η extends to the category $\mathbf{Fct}(\mathfrak{U}\mathfrak{G}, \mathcal{C})$ if and only if for all $A, B \in \text{Obj}(\mathfrak{G})$ such that $B \cong X \natural A$ with $X \in \text{Obj}(\mathfrak{G})$, for all $[X, \gamma] \in \text{Hom}_{\mathfrak{U}\mathfrak{G}}(A, B)$:

$$\eta_B \circ F([X, \gamma]) = G([X, \gamma]) \circ \eta_A.$$

Since η is a natural transformation in the category $\mathbf{Fct}(\mathfrak{G}, \mathcal{C})$, we already have $\eta_B \circ F(\gamma) = G(\gamma) \circ \eta_A$. So, η extends to the category $\mathbf{Fct}(\mathfrak{U}\mathfrak{G}, \mathcal{C})$ if and only if relation (5) is satisfied. \square

Pre-braided monoidal categories. If the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ is braided, Quillen's bracket construction $\mathfrak{U}\mathfrak{G}$ inherits a strict monoidal structure (see Proposition 1.10). However, the braiding $b_{-, -}^{\mathfrak{G}}$ does not extend in general to $\mathfrak{U}\mathfrak{G}$. First recall the notion of a pre-braided monoidal category, generalising that of a strict braided monoidal category, introduced by Randal-Williams and Wahl in [32].

Definition 1.8. [32, Definition 1.5] Let $(\mathfrak{C}, \natural, 0)$ be a strict monoidal category such that the unit 0 is initial. We say that the monoidal category $(\mathfrak{C}, \natural, 0)$ is pre-braided if:

- The maximal subgroupoid $\mathcal{G}\mathfrak{r}(\mathfrak{C})$ is a braided monoidal category, where the monoidal structure is induced by that of $(\mathfrak{C}, \natural, 0)$.
- For all objects A and B of \mathfrak{C} , the braiding associated with the maximal subgroupoid $b_{A, B}^{\mathcal{G}\mathfrak{r}(\mathfrak{C})} : A \natural B \rightarrow B \natural A$ satisfies:

$$b_{A, B}^{\mathcal{G}\mathfrak{r}(\mathfrak{C})} \circ (id_A \natural id_B) = \iota_B \natural id_A : A \rightarrow B \natural A.$$

(Recall that $\iota_B : 0 \rightarrow B$ denotes the unique morphism from 0 to B).

Remark 1.9. A braided monoidal category is automatically pre-braided. However, a pre-braided monoidal category is not necessarily braided (see for example [33, Remark 1.15]).

Finally, let us give the following key property when Quillen's bracket construction is applied on a strict braided monoidal groupoid $(\mathfrak{G}, \natural, 0, b_{-, -}^{\mathfrak{G}})$.

Proposition 1.10. [32, Proposition 1.8] Suppose that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $\text{Aut}_{\mathfrak{G}}(0) = \{id_0\}$. If the groupoid $(\mathfrak{G}, \natural, 0)$ is braided, then the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is pre-braided monoidal. If moreover $(\mathfrak{G}, \natural, 0, b_{-, -}^{\mathfrak{G}})$ is symmetric monoidal, then the category $(\mathfrak{U}\mathfrak{G}, \natural, 0, b_{-, -}^{\mathfrak{G}})$ is symmetric monoidal.

Remark 1.11. The monoidal structure on the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is defined on objects by that of $(\mathfrak{G}, \natural, 0)$ and defined on morphisms by letting for $[X, f] \in \text{Hom}_{\mathfrak{U}\mathfrak{G}}(A, B)$ and $[Y, g] \in \text{Hom}_{\mathfrak{U}\mathfrak{G}}(C, D)$:

$$[X, f] \natural [Y, g] = \left[X \natural Y, (f \natural g) \circ \left(id_X \natural \left(b_{A, Y}^{\mathfrak{G}} \right)^{-1} \natural id_C \right) \right].$$

In particular, the canonical functor $\mathfrak{G} \hookrightarrow \mathfrak{U}\mathfrak{G}$ is monoidal.

The Grothendieck construction. We present here the Grothendieck construction for a functor from a small category to the category of small categories. We refer the reader to [28, Chapter 1, Section 5] for further details.

Definition 1.12. Let \mathfrak{C} be a small category and $F : \mathfrak{C} \rightarrow \mathfrak{Cat}$ a functor. The Grothendieck construction for F (also known as the category of elements of F), denoted by $\int^{\mathfrak{C}} F$, is the category defined by:

- Objects: pairs (x, c) , where $c \in \text{Obj}(\mathfrak{C})$ and $x \in F(c)$;

- Morphisms: for objects (x, c) and (x', c') of $\int^{\mathfrak{C}} F$, a morphism in $\text{Hom}_{\int^{\mathfrak{C}} F}((x, c), (x', c'))$ is a pair (α, f) , where $f \in \text{Hom}_{\mathfrak{C}}(c, c')$ and $\alpha \in \text{Hom}_{F(c')}(F(f)(x), x')$;
- For $(\alpha, f) \in \text{Hom}_{\int^{\mathfrak{C}} F}((x_1, c_1), (x_2, c_2))$ and $(\beta, g) \in \text{Hom}_{\int^{\mathfrak{C}} F}((x_2, c_2), (x_3, c_3))$, the composition in the category $\int^{\mathfrak{C}} F$ is defined by $(\beta, g) \circ (\alpha, f) = [\beta \circ F(g)(\alpha), g \circ f]$.

There is a canonical projection functor $\int^{\mathfrak{C}} F \rightarrow \mathfrak{C}$, given by sending an object (x, c) to c .

Example 1.13. Let $\mathcal{A} : \mathfrak{C} \rightarrow \mathfrak{Gr}$ be a functor, then the Grothendieck construction $\int^{\mathfrak{C}} \mathcal{A}$ is defined by considering a group G as a category with one object (denoted by \cdot_G). Denoting by 0 the functor $\mathfrak{C} \rightarrow \mathfrak{Gr}$ sending all $c \in \text{Obj}(\mathfrak{C})$ to the trivial group $0_{\mathfrak{Gr}}$, there exists a unique natural transformation $0 \rightarrow \mathcal{A}$. Applying the Grothendieck construction, this induces a section $s_F : \mathfrak{C} = \int^{\mathfrak{C}} 0 \hookrightarrow \int^{\mathfrak{C}} \mathcal{A}$ to the projection functor $\int^{\mathfrak{C}} \mathcal{A} \rightarrow \mathfrak{C}$.

Also, if \mathfrak{C} is a groupoid, then $\int^{\mathfrak{C}} \mathcal{A}$ is a groupoid where the automorphism group of any object c is the semi-direct product $\mathcal{A}(c) \rtimes \text{Aut}_{\mathfrak{C}}(c)$ (the action of $\text{Aut}_{\mathfrak{C}}(c)$ on $\mathcal{A}(c)$ being given by \mathcal{A}).

2 The generalized Long-Moody functors

In this section, we introduce the notion of Long-Moody functors for an abstract family of groups, inspired by the Long-Moody construction for braid groups (see [26, Theorem 2.1]). First, in Section 2.1, we introduce a general construction using tensor product of functors and the required tools for our study. Then, we define the generalized Long-Moody functors and establish some of their first properties in Section 2.2. In addition of recovering all the results of [33, Section 2] (see Section 3.5), we give a new approach to the tools and conditions previously considered in [33], allowing a wider application and a deeper understanding of these constructions.

2.1 A general construction

In this first subsection, we present a general construction based on a tensor product for functor categories. The generalized Long-Moody functors introduced in Section 2.2 are particular cases of this construction. We refer the reader to [27, Section VII.3] for the notions of monoid objects and modules in a monoidal category, which will be used in this section.

2.1.1 Tensor product over a monoid functors

First, let us introduce the notion of tensor product over a monoid functor. **We fix a small category \mathfrak{C} throughout Section 2.1.**

Let \otimes_R be the pointwise tensor product in the functor category $\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$ and let R denote the constant functor at R . These endow $\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$ with a strict monoidal structure $(\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod}), \otimes_R, R)$. Let \mathcal{M} be a monoid object in $\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$. We denote by $\mathcal{M}\text{-}\mathfrak{Mod}$ (respectively $\mathfrak{Mod}\text{-}\mathcal{M}$) the category of left (resp. right) modules in $\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$ over \mathcal{M} . Hence, we introduce the tensor product over \mathcal{M} functor:

Definition 2.1. Let $- \otimes_{\mathcal{M}} - : \mathfrak{Mod}\text{-}\mathcal{M} \times \mathcal{M}\text{-}\mathfrak{Mod} \rightarrow \mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$ be the functor defined by:

- Objects: for $F \in \text{Obj}(\mathfrak{Mod}\text{-}\mathcal{M})$ and $G \in \text{Obj}(\mathcal{M}\text{-}\mathfrak{Mod})$, denoting ρ_F (respectively λ_G) the natural transformation action of \mathcal{M} on F (resp. on G), $F \otimes_{\mathcal{M}} G : \mathfrak{C} \rightarrow R\text{-}\mathfrak{Mod}$ is the coequalizer of the natural transformations $\rho_F \otimes_R id_G$ and $id_F \otimes_R \lambda_G$.
- Morphisms: let F_1 and F_2 (respectively G_1 and G_2) be two objects of $\mathfrak{Mod}\text{-}\mathcal{M}$ (resp. $\mathcal{M}\text{-}\mathfrak{Mod}$) and $f : F_1 \rightarrow F_2$ (resp. $g : G_1 \rightarrow G_2$) be a natural transformation in $\mathfrak{Mod}\text{-}\mathcal{M}$ (resp. $\mathcal{M}\text{-}\mathfrak{Mod}$). We define

$f \otimes_{\mathcal{M}} g : F_1 \otimes_{\mathcal{M}} G_1 \rightarrow F_2 \otimes_{\mathcal{M}} G_2$ to be the unique morphism induced from $f \otimes_R g : F_1 \otimes_R G_1 \rightarrow F_2 \otimes_R G_2$ by the universal property of the coequalizer $F_1 \otimes_{\mathcal{M}} G_1$.

The functor $- \otimes_{\mathcal{M}} -$ is called the tensor product functor over \mathcal{M} . In particular, fixing an object F of $\mathfrak{Mod}\text{-}\mathcal{M}$ defines a functor $F \otimes_{\mathcal{M}} - : \mathcal{M}\text{-}\mathfrak{Mod} \rightarrow \mathbf{Fct}(\mathcal{C}, R\text{-}\mathfrak{Mod})$.

2.1.2 Group algebra and augmentation ideal functors

From now on, we fix a functor $\mathcal{A} : \mathcal{C} \rightarrow \mathfrak{Gr}$ for the remainder of Section 2.1. We denote by $R\text{-}\mathfrak{Alg}$ the category of unital R -algebras.

For all objects G of \mathfrak{Gr} , the group rings $R[G]$ and augmentation ideals $\mathcal{I}_{R[G]}$ assemble to define the group algebra functor $R[-] : \mathfrak{Gr} \rightarrow R\text{-}\mathfrak{Alg}$ and the augmentation ideal functor $\mathcal{I}_{R[-]} : \mathfrak{Gr} \rightarrow R\text{-}\mathfrak{Mod}$ respectively. We thus introduce the following two functors:

Definition 2.2. Let $R[\mathcal{A}]$ be the composite functor $R[-] \circ \mathcal{A} : \mathcal{C} \rightarrow R\text{-}\mathfrak{Alg}$, called the group algebra functor induced by \mathcal{A} . Let $\mathcal{I}_{\mathcal{A}}$ be the composite functor $\mathcal{I}_{R[-]} \circ \mathcal{A} : \mathcal{C} \rightarrow R\text{-}\mathfrak{Mod}$, called the augmentation ideal functor induced by \mathcal{A} .

The unital R -algebra structures of $R[\mathcal{A}(c)]$ for all $c \in \text{Obj}(\mathcal{C})$ induce an associative unital monoid object structure on $R[\mathcal{A}]$ with respect to the monoidal structure $(\mathbf{Fct}(\mathcal{C}, R\text{-}\mathfrak{Mod}), \otimes_R, R)$. Furthermore:

Lemma 2.3. *The augmentation ideal functor $\mathcal{I}_{\mathcal{A}}$ is a right $R[\mathcal{A}]$ -module.*

Proof. The natural transformation $\mathcal{I}_{R[\mathcal{A}]} \otimes_R R[\mathcal{A}] \rightarrow \mathcal{I}_{R[\mathcal{A}]}$ is induced by the right $R[\mathcal{A}(c)]$ -module structure of the augmentation ideal $\mathcal{I}_{R[\mathcal{A}(c)]}$ for all objects c of \mathcal{C} , the associativity and unit axioms of a module over a monoid object being straightforward to check. \square

2.1.3 The tensorial construction

We present now a general construction for functor categories, using a tensor product functor (see Definition 2.1). First of all, we have the following key property:

Proposition 2.4. *The precomposition by the section $\mathfrak{s}_{\mathcal{A}}$ (see Example 1.13) induces an equivalence of categories*

$$\mathfrak{s}_{\mathcal{A}}^* : \mathbf{Fct} \left(\int^{\mathcal{C}} \mathcal{A}, R\text{-}\mathfrak{Mod} \right) \xrightarrow{\cong} R[\mathcal{A}]\text{-}\mathfrak{Mod}.$$

Proof. Let F be an object of $\mathbf{Fct} \left(\int^{\mathcal{C}} \mathcal{A}, R\text{-}\mathfrak{Mod} \right)$. For c and c' two objects of \mathcal{C} , a morphism from $(\cdot_{F(c)}, c)$ to $(\cdot_{F(c')}, c')$ in $\int^{\mathcal{C}} \mathcal{A}$ is of the form $(x, \varphi) = (x, id_{c'}) \circ (e_{\mathcal{A}(c')}, \varphi)$ where $x \in \mathcal{A}(c')$ and $\varphi \in \text{Hom}_{\mathcal{C}}(c, c')$. Hence, the morphisms $F(x, id_{c'})$ induce the $R[\mathcal{A}]$ -module structure natural transformation $\lambda_F : R[\mathcal{A}] \otimes_R \mathfrak{s}_{\mathcal{A}}^*(F) \rightarrow \mathfrak{s}_{\mathcal{A}}^*(F)$, the naturality following from the fact that F is a functor on $\int^{\mathcal{C}} \mathcal{A}$. The naturality with respect to F follows straightforwardly from these assignments. Conversely, to extend a left $R[\mathcal{A}]$ -module G to a functor \widehat{G} with $\int^{\mathcal{C}} \mathcal{A}$ as source category sending the object $(\cdot_{\mathcal{A}(c)}, c)$ to $G(c)$ for all $c \in \text{Obj}(\mathcal{C})$, it is enough to define $\widehat{G}((\alpha, id_c)) = \lambda_G \left(\alpha \otimes_R id_c \right)$, where $\lambda_G : R[\mathcal{A}] \otimes_R G \rightarrow G$ is the associated left module natural transformation. Also, the naturality with respect to G follows from the naturality with respect to λ_G of a functor between two $R[\mathcal{A}]$ -modules. Then, it follows from these definitions $\widehat{}$ is a functor from $R[\mathcal{A}]\text{-}\mathfrak{Mod}$ to $\mathbf{Fct} \left(\int^{\mathcal{C}} \mathcal{A}, R\text{-}\mathfrak{Mod} \right)$ and the inverse of $\mathfrak{s}_{\mathcal{A}}^*$. \square

Now, we can introduce the construction:

Definition 2.5. Let $\zeta : \int^{\mathcal{C}} \mathcal{A} \rightarrow \mathcal{D}$ be a functor where \mathcal{D} is another small category and we denote by ζ^* the precomposition functor induced by ζ . We define $\mathfrak{T}_{\mathcal{A}, \zeta} : \mathbf{Fct}(\mathcal{D}, R\text{-Mod}) \rightarrow \mathbf{Fct}(\mathcal{C}, R\text{-Mod})$ to be the composite:

$$\mathbf{Fct}(\mathcal{D}, R\text{-Mod}) \xrightarrow{\mathfrak{s}_{\mathcal{A} \circ \zeta}^*} R[\mathcal{A}]\text{-Mod} \xrightarrow{\mathcal{I}_{\mathcal{A}} \otimes_{R[\mathcal{A}]} -} \mathbf{Fct}(\mathcal{C}, R\text{-Mod}).$$

The functor $\mathfrak{T}_{\mathcal{A}, \zeta}$ is called the tensorial construction by the functor \mathcal{A} along ζ .

Let us give some immediate properties of a tensorial construction.

Proposition 2.6. *The tensorial construction by the functors $\mathcal{A} : \mathcal{C} \rightarrow \mathfrak{Gr}$ and $\zeta : \int^{\mathcal{C}} \mathcal{A} \rightarrow \mathcal{D}$ is additive, right exact and commutes with all colimits.*

Proof. Let $0 : \mathcal{C} \rightarrow R\text{-Mod}$ denote the null functor for a small category \mathcal{C} . It follows from the definition that $\mathfrak{T}_{\mathcal{A}, \zeta}(0) = 0$.

As the precomposition functors ζ^* and $\mathfrak{s}_{\mathcal{A}}^*$ are exact, the right-exactness and commutation property with all colimits of the functor $\mathcal{I}_{R[\mathcal{A}(c)]} \otimes_{R[\mathcal{A}(c)]} - : R[\mathcal{A}(c)]\text{-Mod} \rightarrow R\text{-Mod}$ for all $c \in \text{Obj}(\mathcal{C})$ (see for example [38,

Application 2.6.2]) induce the same properties for $\mathfrak{T}_{\mathcal{A}, \zeta}$, the naturality for morphisms following from the definition of a tensorial functor. \square

2.2 The Long-Moody functors

Using the tensorial construction of Section 2.1, we introduce here the generalized Long-Moody functors, inspired from the Long-Moody construction [26]. While the original construction was associated with braid groups, a large variety of groups falls within the following framework (see Section 3).

2.2.1 Categorical framework

First, we require the following categorical framework to define generalized Long-Moody functors. Let $(\mathcal{G}', \natural, 0_{\mathcal{G}'}, b_{-, -}^{\mathcal{G}'})$ be a braided monoidal small groupoid with no zero divisors and such that $\text{Aut}_{\mathcal{G}'}(0_{\mathcal{G}'}) = \{id_{0_{\mathcal{G}'}}\}$. Recall from Proposition 1.10 that Quillen's bracket construction $(\mathcal{U}\mathcal{G}', \natural, 0_{\mathcal{G}'})$ is a pre-braided monoidal category such that the unit $0_{\mathcal{G}'}$ is an initial object. Let $\underline{0}$ and 1 be two objects of \mathcal{G}' .

Notation 2.7. For all natural numbers n , we denote the object $1^{\natural n} \underline{0}$ of \mathcal{G}' by \underline{n} and the object $1^{\natural n}$ of \mathcal{G}' by n . Note that $m \natural \underline{n} = \underline{m+n}$ for all natural numbers m and n .

Definition 2.8. Let \mathcal{G} be the full subgroupoid of \mathcal{G}' on the objects $\{\underline{n}\}_{n \in \mathbb{N}}$. Let $\mathcal{U}\mathcal{G}$ be the full subcategory of Quillen's bracket construction $\mathcal{U}\mathcal{G}'$ on the objects $\{\underline{n}\}_{n \in \mathbb{N}}$.

Notation 2.9. We denote by G_n the automorphism group $\text{Aut}_{\mathcal{G}}(\underline{n})$ for all natural numbers n , and by $1 \natural - : \mathcal{U}\mathcal{G} \rightarrow \mathcal{U}\mathcal{G}$ the functor defined by $(1 \natural -)(\underline{n}) = 1 \natural \underline{n}$ for all $\underline{n} \in \text{Obj}(\mathcal{G})$ and $(1 \natural -)([n' - n, g]) = id_1 \natural [n' - n, g]$ for all morphism $[n' - n, g]$ of $\mathcal{U}\mathcal{G}$.

Remark 2.10. Warning: the category $\mathcal{U}\mathcal{G}$ is not in general Quillen's bracket construction of Definition 1.1 but depends on the ambient groupoid \mathcal{G}' on \mathcal{G} . Note that $\mathcal{U}\mathcal{G}$ is Quillen's bracket construction on \mathcal{G} if and only if $0_{\mathcal{G}'} = \underline{0}$.

The present framework allows to handle families of groups such as mapping class groups of surfaces with non-zero (orientable or non-orientable) genus (see Section 3.3). For instance, in the various situations of Section 3, the groupoids $\mathfrak{M}_2^{+,s}$, $\mathfrak{M}_2^{-,s}$ and $\mathfrak{M}_2^{g,c}$ (see Sections 3.3 and 3.4) are full subgroupoids of the braided monoidal groupoid $(\mathfrak{M}_2, \natural, \Sigma_{0,0,1}^0, b_{-, -}^{\mathfrak{M}_2})$ (see Proposition 3.5).

We fix the groupoids \mathcal{G}' , \mathcal{G} and a functor $\mathcal{A} : \mathcal{U}\mathcal{G} \rightarrow \mathfrak{Gr}$ for the remainder of Section 2.2.

2.2.2 Definition of the Long-Moody functors

The idea to define a Long-Moody functor is to use the tensorial construction by the functor \mathcal{A} , along some ζ such that the composition the section $\mathfrak{s}_{\mathcal{A}}$ is the functor $1_{\mathfrak{G}}$. We have the choice to consider \mathcal{A} over \mathcal{G} or $\mathfrak{U}\mathcal{G}$. Actually, this choice leads to an extension problem: the restriction along the canonical functor $\mathcal{G} \hookrightarrow \mathfrak{U}\mathcal{G}$ (see Remark 1.3) defines the Grothendieck construction $\int^{\mathcal{G}} \mathcal{A}$ together with an inclusion functor $\int^{\mathcal{G}} \mathcal{A} \hookrightarrow \int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$, so that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\quad} & \mathfrak{U}\mathcal{G} \\ \downarrow & & \downarrow \\ \int^{\mathcal{G}} \mathcal{A} & \xrightarrow{\quad} & \int^{\mathfrak{U}\mathcal{G}} \mathcal{A}. \end{array}$$

This is not a pushout of categories: given $\mathcal{G} \hookrightarrow \int^{\mathcal{G}} \mathcal{A} \rightarrow \mathfrak{E}$ a functor for \mathfrak{E} a small category, we deal with the extension problem

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\quad} & \mathfrak{U}\mathcal{G} \\ \downarrow & & \downarrow \\ \int^{\mathcal{G}} \mathcal{A} & \xrightarrow{\quad} & \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \\ & \searrow & \swarrow \\ & & \mathfrak{E}. \end{array}$$

In the current situation, \mathfrak{E} is taken to be $\mathfrak{U}\mathcal{G}$ and the composites $\mathcal{G} \rightarrow \int^{\mathcal{G}} \mathcal{A} \rightarrow \mathfrak{E}$ and $\mathfrak{U}\mathcal{G} \rightarrow \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \rightarrow \mathfrak{E}$ to be $1_{\mathfrak{G}}$. This motivates the following:

Definition 2.11. A Long-Moody system is $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$, where $\zeta : \int^{\mathcal{G}} \mathcal{A} \rightarrow \mathcal{G}$ is a functor such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\mathfrak{s}_{\mathcal{A}}} & \int^{\mathcal{G}} \mathcal{A} \\ & \searrow & \downarrow \zeta \\ & & \mathcal{G}. \end{array} \quad (6)$$

The functor \mathcal{A} equipped with such functor ζ is said to define a Long-Moody system, denoted by $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$. If ζ extends along the inclusion $\int^{\mathcal{G}} \mathcal{A} \hookrightarrow \int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$ to define a functor $\zeta : \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \rightarrow \mathfrak{U}\mathcal{G}$, so that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{U}\mathcal{G} & \xrightarrow{\mathfrak{s}_{\mathcal{A}}} & \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \\ & \searrow & \downarrow \zeta \\ & & \mathfrak{U}\mathcal{G}, \end{array} \quad (7)$$

then the Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ is then said to be coherent.

Then, we can introduce the main concept of Section 2:

Definition 2.12. The Long-Moody functor associated with the Long-Moody system (respectively coherent Long-Moody system) $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$, denoted by $\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}}$ (resp. $\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}}^{\mathfrak{U}}$), is the tensorial construction $\mathfrak{T}_{\mathcal{A}, \zeta}$ by the functors $\mathcal{A} : \mathfrak{U}\mathcal{G} \rightarrow \mathfrak{G}$ and $\zeta : \int^{\mathcal{G}} \mathcal{A} \rightarrow \mathcal{G}$ (resp. $\zeta : \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \rightarrow \mathfrak{U}\mathcal{G}$) of Definition 2.5.

Notation 2.13. When there is no ambiguity, once the Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ is fixed, we omit it from the notation. Also, if the Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ is assumed to be coherent, we omit it the \mathfrak{U} from the notation if there is no risk of confusion.

It is worth noting that non-trivial coherent Long-Moody systems arise naturally in many situations, in particular for families of groups in connection with topology (see Section 3). We give here a first example:

Example 2.14. Let us fix $(\mathcal{G}', \natural, 0_{\mathcal{G}'}) = (\mathcal{G}, \natural, 0_{\mathcal{G}}) = (\beta, \natural, 0)$, where β is the braid groupoid. It has the natural numbers as objects the natural numbers and its automorphisms are the braid groups $\{\mathbf{B}_n\}_{n \in \mathbb{N}}$. The strict monoidal structure \natural is defined by the usual addition for the objects and laying two braids side by side for the morphisms (see [27, Chapter XI, Section 4] for more details).

In this case, the Artin representations $\{a_{n,1} : \mathbf{B}_n \rightarrow \text{Aut}(\mathbf{F}_n)\}_{n \in \mathbb{N}'}$ defined by the action \mathbf{B}_n on the fundamental group $\pi_1(\Sigma_{0,1}^n) \cong \mathbf{F}_n$ for all natural numbers n , assemble to define a functor $\mathcal{A}_1^\beta : \mathcal{U}\beta \rightarrow \mathfrak{G}\tau$ (see [33, Example 2.3] or Section 3.2). Moreover, there exists a family of non-trivial morphisms $\{\zeta_{n,1} : \mathbf{F}_n \rightarrow \mathbf{B}_{n+1}\}_{n \in \mathbb{N}}$ (see [33, Example 2.1] or Definition 3.17) such that the morphism given by the coproduct $\zeta_{n,1} * (1_{\natural-}) : \mathbf{F}_n * \mathbf{B}_n \rightarrow \mathbf{B}_{n+1}$ factors across the canonical surjection to the semidirect product $\mathbf{F}_n \rtimes_{\mathcal{A}_{n,1}^\beta} \mathbf{B}_n$ and such that

the corresponding diagram (9) is commutative (see [33, Propositions 2.6 and 2.10] or Section 3.4). We thus define a non-trivial $\zeta_1 : \int^{\mathcal{U}\beta} \mathcal{A}_1^\beta \rightarrow \mathcal{U}\beta$ so that we have a coherent Long-Moody system $\{\mathcal{A}_1^\beta, \beta, \beta, \zeta_1\}$. We refer to Section 3.5 for more details.

Also, Definition 2.11 recovers its analogue of [33, Definition 2.14] and Theorem 2.12 recovers [33, Theorem 2.19].

2.2.3 Effect of a Long-Moody functors on a trivial functor

We denote by $R : \mathcal{U}\mathcal{G} \rightarrow R\text{-}\mathfrak{M}\text{o}\text{d}$ the constant functor at R . Recall that the homology group $H_1(-, R)$ defines a functor from the category $\mathfrak{G}\tau$ to the category $R\text{-}\mathfrak{M}\text{o}\text{d}$ (see for example [5, Section 8]). It describes the effect of a Long-Moody functor on the trivial functor R :

Proposition 2.15. *Let $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ be a coherent Long-Moody system. Then as functor $\mathcal{U}\mathcal{G} \rightarrow R\text{-}\mathfrak{M}\text{o}\text{d}$:*

$$\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}}^{\mathcal{U}}(R) \cong H_1(\mathcal{A}, R) \quad (8)$$

where $H_1(\mathcal{A}, R)$ denotes the functor composite $H_1(-, R) \circ \mathcal{A}$.

Proof. Applying classical homological algebra (see [38, Theorem 6.1.11]), we deduce that for all natural numbers n :

$$\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}}^{\mathcal{U}}(R)(n) \cong H_1(\mathcal{A}(\underline{n}), R).$$

The naturality follows from the fact that the assignments of the functor $H_1(\mathcal{A}, R)$ and $\mathcal{I}_{\mathcal{A}}$ on morphisms of $\mathcal{U}\mathcal{G}$ are both induced by the functor \mathcal{A} . \square

Remark 2.16. Assuming that the Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ is not coherent, the proof of Proposition 2.15 shows that $\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}}^{\mathcal{U}}(R) \cong H_1(\mathcal{A}, R)$ as objects of $\mathbf{Fct}(\mathcal{G}, R\text{-}\mathfrak{M}\text{o}\text{d})$.

2.2.4 Equivalent characterization of Long-Moody systems

We give now an equivalent description of the functor ζ introduced in Definition 2.11 to define a Long-Moody system.

For all natural numbers n , we denote by $\mathcal{A}_n : G_n \rightarrow \text{Aut}_{\mathfrak{G}\tau}(\mathcal{A}(\underline{n}))$ the group morphisms induced by the functor \mathcal{A} . By Example 1.13, considering the functor $\zeta : \int^{\mathcal{G}} \mathcal{A} \rightarrow \mathcal{G}$ is equivalent to considering a family of group morphisms $\left\{ \mathcal{A}(\underline{n}) \rtimes_{\mathcal{A}_n} G_n \rightarrow G_{n+1} \right\}_{n \in \mathbb{N}}$. Hence, we deduce:

Lemma 2.17. *Considering a functor $\zeta : \int^{\mathcal{G}} \mathcal{A} \rightarrow \mathcal{G}$ so that the diagram (6) is commutative is equivalent to considering a family of group morphisms $\{\zeta_n : \mathcal{A}(\underline{n}) \rightarrow G_{n+1}\}_{n \in \mathbb{N}}$ such that the morphism given by the coproduct $\zeta_n * (1_{\natural-}) : \mathcal{A}(\underline{n}) * G_n \rightarrow G_{n+1}$ factors across the canonical surjection to the semidirect product $\mathcal{A}(\underline{n}) \rtimes_{\mathcal{A}_n} G_n$.*

Proof. By Example 1.13, considering the functor $\zeta : \int^{\mathcal{G}} \mathcal{A} \rightarrow \mathcal{G}$ is equivalent to considering a family of group morphisms $\left\{ \mathcal{A}(\underline{n}) \rtimes_{\mathcal{A}_n} G_n \rightarrow G_{n+1} \right\}_{n \in \mathbb{N}}$. Moreover, the commutation of the diagram (6) is equivalent to the following equality in G_{n+1} :

$$(id_1 \natural g) \circ \zeta_n(h) = \zeta_n(\mathcal{A}_n(g)(h)) \circ (id_1 \natural g),$$

for all $g \in G_n$ and $h \in \mathcal{A}(\underline{n})$. This is exactly the definition of the fact that $\zeta_n * (1 \natural -)$ factors across the semidirect product $\mathcal{A}(\underline{n}) \rtimes_{\mathcal{A}_n} G_n$. \square

Also, the following result highlights the underlying subtleties when extending a Long-Moody system to a coherent one.

Proposition 2.18. *Let $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ be a Long-Moody system. If the functor ζ extends to the Grothendieck construction $\int^{\mathcal{U}\mathcal{G}} \mathcal{A}$ so that the diagram (7) is commutative, then the extension is unique.*

*Moreover, the functor ζ extends to define a **coherent** Long-Moody system if and only if the family of group morphisms $\{\zeta_n : \mathcal{A}(\underline{n}) \rightarrow G_{n+1}\}_{n \in \mathbb{N}}$ induced by ζ (see Lemma 2.17) is such that the following diagram is commutative in the category $\mathcal{U}\mathcal{G}$:*

$$\begin{array}{ccc} 1 \natural \underline{n} & \xrightarrow{\zeta_n(h)} & 1 \natural \underline{n} \\ id_1 \natural [n'-n, id_{\underline{n}'}] \downarrow & & \downarrow id_1 \natural [n'-n, id_{\underline{n}'}] \\ 1 \natural \underline{n}' & \xrightarrow{\zeta_{n'}(\mathcal{A}([n'-n, id_{\underline{n}'}])(h))} & 1 \natural \underline{n}' \end{array} \quad (9)$$

for all elements $h \in \mathcal{A}(\underline{n})$, for all natural numbers n and n' such that $n' \geq n$. In particular, it is enough to require that the following equality holds in G_{n+2} :

$$\left((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}} \right) \circ (id_1 \natural \zeta_n(h)) = \zeta_{n+1}(\mathcal{A}([1, id_{\underline{n+1}}])(h)) \circ \left((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}} \right), \quad (10)$$

for the diagram (9) to be commutative. Finally, if $Aut_{\mathcal{U}\mathcal{G}}(1) = \{id_1\}$, then the commutation of the diagram (9) is equivalent to the equality (10).

Proof. It follows from the definition of the Grothendieck construction $\int^{\mathcal{U}\mathcal{G}} \mathcal{A}$ (see Definition 1.12) that an extension of $\zeta : \int^{\mathcal{G}} \mathcal{A} \rightarrow \mathcal{G}$ is defined by $\zeta((\underline{n}, \underline{n})) = 1 \natural \underline{n}$ for all $\underline{n} \in Obj(\mathcal{G})$, $\zeta((h, id_{\underline{n}})) = \zeta_n(h)$ for all $h \in \mathcal{A}(\underline{n})$ and $\zeta\left(\left(e_{\mathcal{A}(\underline{n}'), [n'-n, \varphi]}\right)\right) = id_1 \natural [n'-n, \varphi]$ for all $[n'-n, \varphi] \in Hom_{\mathcal{U}\mathcal{G}}(\underline{n}, \underline{n}')$. In particular, the uniqueness of the extension follows from these assignments. Actually, ζ extends to define a coherent Long-Moody system if and only if these assignments on morphisms satisfy the composition axiom for a functor. Hence, the additional composition axiom which has to be checked for extending ζ to $\int^{\mathcal{U}\mathcal{G}} \mathcal{A}$ is for the composition of type $\left(e_{\mathcal{A}(\underline{n}'), [n'-n, id_{\underline{n}'}]}\right) \circ (h, id_{\underline{n}})$: namely, ζ extends to $\int^{\mathcal{U}\mathcal{G}} \mathcal{A}$ if and only if

$$\zeta\left(\left(e_{\mathcal{A}(\underline{n}'), [n'-n, id_{\underline{n}'}]}\right)\right) \circ \zeta((h, id_{\underline{n}})) = \zeta\left(\left(e_{\mathcal{A}(\underline{n}'), [n'-n, id_{\underline{n}'}]}\right) \circ (h, id_{\underline{n}})\right)$$

for all natural numbers $n' \geq n$ and $h \in \mathcal{A}(\underline{n})$. The second statement is then a direct consequence of the composition rule in $\int^{\mathcal{U}\mathcal{G}} \mathcal{A}$.

For the third statement, note that by definition of the braiding $b_{-, -}^{\mathcal{G}'}$, we have:

$$\left(b_{1, n'-n}^{\mathcal{G}'}\right)^{-1} \natural id_n = \left(\left(b_{1,1}^{\mathcal{G}'}\right)^{-1} \natural id_{n'-1}\right) \circ \left(id_1 \natural \left(b_{1, (n'-n)-1}^{\mathcal{G}'}\right)^{-1} \natural id_n\right).$$

Hence, a straightforward recursion proves that the commutation of the diagram (9) is equivalent to assuming that for all elements $h \in H_n$, for all natural numbers n , the morphisms $\{\zeta_n\}_{n \in \mathbb{N}}$ satisfy the following equality, as morphisms in the category $\mathcal{U}\mathcal{G}$:

$$\left[1, \left(\left(b_{1,1}^{\mathcal{G}'}\right)^{-1} \natural id_{\underline{n}}\right) \circ (id_1 \natural \zeta_n(h))\right] = \left[1, \zeta_{n+1}((\iota_H * id_{H_n})(h)) \circ \left(\left(b_{1,1}^{\mathcal{G}'}\right)^{-1} \natural id_{\underline{n}}\right)\right].$$

Hence, the third and fourth statements follow from the equivalence relation for morphisms in $\mathcal{U}\mathcal{G}$. \square

Remark 2.19. In Section 5, we will have to assume that the stronger equality (10) holds (see Assumption 5.1).

2.2.5 Case of trivial ζ

There always exists at least one functor $\zeta : \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \rightarrow \mathfrak{U}\mathcal{G}$ which so that the diagram (7) is commutative. Indeed, we can consider the functor $\zeta_t : \int^{\mathcal{G}} \mathcal{A} \rightarrow \mathcal{G}$ induced by the family of morphisms $\{\zeta_{n,t} : \mathcal{A}(\underline{n}) \rightarrow 0_{\mathfrak{G}\tau} \rightarrow G_{n+1}\}_{n \in \mathbb{N}}$ factoring across the trivial group $0_{\mathfrak{G}\tau}$ (considered as a category with one object). This functor ζ_t trivially extends to $\int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$, a fortiori defining a coherent Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta_t\}$. Moreover, we have the following property:

Proposition 2.20. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{M}\text{o}\text{d})$. Then, as objects of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{M}\text{o}\text{d})$:*

$$\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta_t\}}^{\mathfrak{U}}(F) \cong \mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta_t\}}^{\mathfrak{U}}(R) \otimes_R F(1 \natural -).$$

Proof. Let n be a natural number. The action induced by $\zeta_{n,t} : \mathcal{A}(\underline{n}) \rightarrow G_{n+1}$ makes $F(\underline{1+n}) = F(1 \natural \underline{n})$ a trivial $R[\mathcal{A}(\underline{n})]$ -module. A fortiori, there is an R -module isomorphism:

$$\mathcal{I}_{R[\mathcal{A}(\underline{n})]} \otimes_{R[\mathcal{A}(\underline{n})]} F(\underline{n+1}) \cong \left(\mathcal{I}_{R[\mathcal{A}(\underline{n})]} \otimes_{R[\mathcal{A}(\underline{n})]} R \right) \otimes_R F(1 \natural \underline{n}).$$

It is straightforward to check that this isomorphism is natural. \square

2.2.6 Case of free groups

Finally, assuming that the groups $\mathcal{A}(\underline{n})$ free groups for all natural numbers n , we obtain more properties for the associated Long-Moody functors. First, recall the following result.

Lemma 2.21. *Let G be a group. The augmentation ideal $\mathcal{I}_{R[G]}$ is a projective $R[G]$ -module if and only if G is a free group.*

Proof. Let us assume that $\mathcal{I}_{R[G]}$ is a projective $R[G]$ -module. The following short exact sequence is a projective resolution of R as a $R[G]$ -module.

$$0 \longrightarrow \mathcal{I}_{R[G]} \longrightarrow R[G] \longrightarrow R \longrightarrow 0$$

Hence the homological dimension of G is one. Thus, according to a theorem due to Swan [36, Theorem A], G is a free group. The converse is a classical result of homological algebra (see [38, Corollary 6.2.7]). \square

Corollary 2.22. *If $\mathcal{A}(\underline{n})$ is a free group for all natural numbers n , then \mathbf{LM} (respectively $\mathbf{LM}^{\mathfrak{U}}$) associated with the (resp. coherent) Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ is exact and commutes with all finite limits.*

Proof. Let n be a natural number. Since the augmentation ideal $\mathcal{I}_{R[\mathcal{A}(\underline{n})]}$ is a projective $R[\mathcal{A}(\underline{n})]$ -module (by Lemma 2.21), it is a flat $R[\mathcal{A}(\underline{n})]$ -module. Then, the result follows from the fact that the functor $\mathcal{I}_{R[\mathcal{A}(\underline{n})]} \otimes_{R[\mathcal{A}(\underline{n})]} - : R\text{-}\mathfrak{M}\text{o}\text{d} \rightarrow R\text{-}\mathfrak{M}\text{o}\text{d}$ is an exact functor, the naturality for morphisms following from the definition of the Long-Moody functor (see Theorem 2.12). The commutation result for finite limits is a general property of exact functors (see for example [27, Chapter 8, section 3]). \square

3 Examples

The groups $\{G_n\}_{n \in \mathbb{N}}$ for which it is natural to define the first generalized Long-Moody functors are mapping class groups of surfaces. In this section, we present various coherent Long-Moody systems which are defined for several families of mapping class groups, surfaces braid groups and finally symmetric groups. In particular, we recover the Long-Moody functors of [33] in Section 3.5. We will also focus on exhibiting the functors that we recover by applying the Long-Moody functors on the constant functor R . We are interested in these functors for two reasons. First, R is the most basic functor to study. Secondly, considering the particular case of the family of trivial morphisms $\{\zeta_{n,t}\}_{n \in \mathbb{N}}$, understanding $\mathbf{LM}(R)$ allows us to describe completely $\mathbf{LM}(F)$ for all objects F of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{M}\text{o}\text{d})$ by Proposition 2.20.

3.1 The monoidal groupoid associated with surfaces

Let us first introduce a suitable category for our work, inspired by [32, Section 5.6]. Namely:

Definition 3.1. The decorated surfaces groupoid \mathcal{M}_2 is the groupoid defined by:

- Objects: decorated surfaces (S, I) , where S is a smooth connected compact surface with one boundary component denoted by $\partial_0 S$ with $I : [-1, 1] \hookrightarrow \partial S$ a parametrized interval in the boundary and $p = 0 \in I$ a basepoint, where a finite number of points is removed from the interior of S (in other words with punctures);
- Morphisms: the isotopy classes of homeomorphisms restricting to the identity on a neighborhood of the parametrized interval I , freely moving the punctures, denoted by $\pi_0 \text{Homeo}^I(S, \{\text{punctures}\})$.

Remark 3.2. A homeomorphism of a surface which fixes an interval in a boundary component is isotopic to a homeomorphism which fixes pointwise the boundary component of the surface. Denote by \hat{S} the surface obtained from $S \in \text{Obj}(\mathcal{M}_2)$ removing a disc on a neighbourhood of each puncture. Note from [15, Section 1.4.2] that $\pi_0 \text{Homeo}^I(S, \{\text{punctures}\})$ identifies with the group $\pi_0 \text{Diff}^{\partial_0}(\hat{S})$ of isotopy classes of diffeomorphisms of \hat{S} fixing pointwise the boundary component ∂_0 and moving freely the other boundary components.

When the surface S is orientable, the orientation on S is induced by the orientation of I . The isotopy classes of homeomorphisms then automatically preserve that orientation as they restrict to the identity on a neighbourhood of I .

Notation 3.3. When there is no ambiguity, we omit the parametrized interval I from the notation.

We denote by $\Sigma_{0,0,1}^0$ a disc. We fix a unit disc with one puncture denoted by $\Sigma_{0,0,1}^1$, a torus with one boundary component denoted by $\Sigma_{1,0,1}^0$ and a Möbius band denoted by $\Sigma_{0,1,1}^0$. Let S be an object of the groupoid \mathcal{M}_2 . By the classification of surfaces, there exist $g, s, c \in \mathbb{N}$ such that there is an homeomorphism:

$$S \simeq \left(\natural_s \Sigma_{0,0,1}^1 \right) \natural_g \left(\natural_g \Sigma_{1,0,1}^0 \right) \natural_c \left(\natural_c \Sigma_{0,1,1}^0 \right).$$

Moreover, if $c = 0$, then g and s are unique.

The groupoid \mathcal{M}_2 has a monoidal structure induced by gluing; for completeness, the definition is outlined below (see [32, Section 5.6.1] for technical details). For two decorated surfaces (S_1, I_1) and (S_2, I_2) , the boundary connected sum $(S_1, I_1) \natural (S_2, I_2) = (S_1 \natural S_2, I_1 \natural I_2)$ is defined with $S_1 \natural S_2$ the surface obtained from gluing S_1 and S_2 along the half-interval I_1^+ and the half-interval I_2^- , and $I_1 \natural I_2 = I_1^- \cup I_2^+$. The homeomorphisms being the identity on a neighbourhood of the parametrized intervals I_1 and I_2 , we canonically extend the homeomorphisms of S_1 and S_2 to $S_1 \natural S_2$. The braiding of the monoidal structure $b_{(S_1, I_1), (S_2, I_2)}^{\mathcal{M}_2} : (S_1, I_1) \natural (S_2, I_2) \rightarrow (S_2, I_2) \natural (S_1, I_1)$ is given by doing half a Dehn twist in a pair of pants neighbourhood of ∂S_1 and ∂S_2 (see Figure 1).

By [32, Proposition 5.18], the boundary connected sum \natural induces a strict braided monoidal structure $(\mathcal{M}_2, \natural, (\Sigma_{0,0,1}^0, I), b_{-, -}^{\mathcal{M}_2})$. There are no zero divisors in the category \mathcal{M}_2 and $\text{Aut}_{\mathcal{M}_2}(\Sigma_{0,0,1}^0) = \{id_{\Sigma_{0,0,1}^0}\}$.

Definition 3.4. Let \mathfrak{M}_2 be the full subgroupoid of \mathcal{M}_2 of the boundary connected sum on the objects $\Sigma_{0,0,1}^0$, $\Sigma_{0,0,1}^1$, $\Sigma_{1,0,1}^0$ and $\Sigma_{0,1,1}^0$.

Proposition 3.5. The groupoid $(\mathfrak{M}_2, \natural, \Sigma_{0,0,1}^0, b_{-, -}^{\mathfrak{M}_2})$ is small braided monoidal with no zero divisors and such that $\text{Aut}_{\mathfrak{M}_2}(\Sigma_{0,0,1}^0) = \{id_{\Sigma_{0,0,1}^0}\}$.

By Definition 1.1, we denote by $\mathcal{U}\mathfrak{M}_2$ Quillen's bracket construction on the groupoid $(\mathfrak{M}_2, \natural, \Sigma_{0,0,1}^0)$; by Proposition 1.10, we obtain a strict pre-braided monoidal category $(\mathcal{U}\mathfrak{M}_2, \natural, \Sigma_{0,0,1}^0)$.

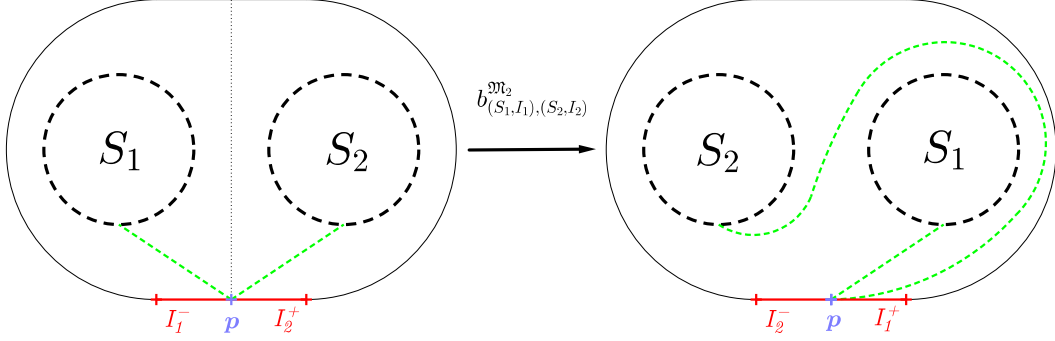


Figure 1: Braiding

3.2 Fundamental group functor

Let us introduce a non-trivial functor with \mathfrak{M}_2 as source category. The isotopy classes of the homeomorphisms of a surface $S \in \text{Obj}(\mathfrak{M}_2)$ act on its fundamental group (see for example [15, Chapter 3]). We denote this action by a_S . So, we define a functor

$$\pi_1(-, p) : (\mathfrak{M}_2, \natural, \Sigma_{0,1}^0) \rightarrow \mathfrak{gr}$$

assigning the fundamental groups $\pi_1(\Sigma_{0,0,1}^1, p)$, $\pi_1(\Sigma_{1,0,1}^0, p)$ and $\pi_1(\Sigma_{0,1,1}^0, p)$ on the objects $\Sigma_{0,0,1}^1$, $\Sigma_{1,0,1}^0$ and $\Sigma_{0,1,1}^0$, and then inductively $\pi_1(-, p)(S \natural S') = \pi_1(S, p) * \pi_1(S', p)$ for $S, S' \in \text{Obj}(\mathfrak{M}_2)$; and assigning the morphism $a_S(\varphi)$ for all $\varphi \in \pi_0 \text{Homeo}^I(S, \{\text{punctures}\})$. By Van Kampen's theorem, the group $\pi_1(-, p)(S \natural S')$ is isomorphic to the fundamental group of the surface $S \natural S'$, hence our assignment on morphisms is consistent. Here, we fix maps

$$\pi_0 \text{Homeo}^I(S, \{\text{punctures}\}) \rightarrow \text{Aut}_{\mathfrak{G}\tau}(\pi_1(S, p)).$$

Note that we could make other choices of such morphisms so that the following study still works.

Notation 3.6. Let \mathfrak{gr} denote the full subcategory of $\mathfrak{G}\tau$ of finitely-generated free groups. The free product $*$: $\mathfrak{gr} \times \mathfrak{gr} \rightarrow \mathfrak{gr}$ defines a monoidal structure on \mathfrak{gr} , with 0 the unit, denoted by $(\mathfrak{gr}, *, 0)$.

Lemma 3.7. *The functor $\pi_1(-, p) : (\mathfrak{M}_2, \natural, \Sigma_{0,0,1}^0) \rightarrow (\mathfrak{gr}, *, 0_{\mathfrak{G}\tau})$ is strict monoidal.*

Proof. By our assignments, $\pi_1(S' \natural S, p) = \pi_1(S', p) * \pi_1(S, p)$ for $S, S' \in \text{Obj}(\mathfrak{M}_2)$. It is clear that $\pi_0 \text{Homeo}^I(S, \{\text{punctures}\})$ (respectively $\pi_0 \text{Homeo}^I(S', \{\text{punctures}\})$) acts trivially on $\pi_1(S', p)$ (resp. $\pi_1(S, p)$) in $\pi_1(S' \natural S, p)$. Therefore, $id_{\pi_1(-, p)} * id_{\pi_1(-, p)}$ is a natural equivalence. \square

As the object $0_{\mathfrak{G}\tau}$ is null in the category of groups $\mathfrak{G}\tau$, $\iota_G : 0_{\mathfrak{G}\tau} \rightarrow G$ denotes the unique morphism from $0_{\mathfrak{G}\tau}$ to the group G .

Proposition 3.8. *The functor $\pi_1(-, p)$ of Lemma 3.7 extends to a functor $\pi_1(-, p) : \mathfrak{M}_2 \rightarrow \mathfrak{gr}$ by assigning for all $S, S' \in \text{Obj}(\mathfrak{M}_2)$:*

$$\pi_1(-, p)([S', id_{S' \natural S}]) = \iota_{\pi_1(S', p)} * id_{\pi_1(S, p)}.$$

Proof. It follows from the definitions that relation (3) of Lemma 1.6 is satisfied for

$$\pi_1(-, p)\left(\left[\Sigma_{0,0,1}^1, id_{\Sigma_{0,0,1}^1 \natural S}\right]\right), \pi_1(-, p)\left(\left[\Sigma_{1,0,1}^0, id_{\Sigma_{1,0,1}^0 \natural S}\right]\right) \text{ and } \pi_1(-, p)\left(\left[\Sigma_{0,1,1}^0, id_{\Sigma_{0,1,1}^0 \natural S}\right]\right).$$

Let S and S' be objects of \mathfrak{M}_2 . Let $\varphi \in \pi_0 \text{Homeo}^I(S, \{\text{punctures}\})$ and $\varphi' \in \pi_0 \text{Homeo}^I(S', \{\text{punctures}\})$. According to Lemma 3.7:

$$\pi_1(-, p)(\varphi' \natural \varphi) \circ \pi_1(-, p)([S', id_{S' \natural S}]) = (\pi_1(-, p)(\varphi') * \pi_1(-, p)(\varphi)) \circ \pi_1(-, p)([S', id_{S' \natural S}]).$$

Hence, by definition of the morphism $\iota_{\pi_1(S', p)}$, we have:

$$\pi_1(-, p)(\varphi' \natural \varphi) \circ \pi_1(-, p)([S', id_{S' \natural S}]) = \pi_1(-, p)([S', id_{S' \natural S}]) \circ \pi_1(-, p)(\varphi).$$

Relation (4) of Lemma 1.6 is thus satisfied, which implies the desired result. \square

3.3 Modifying the orientable or non-orientable genus

We fix the number s of punctures throughout Section 3.3. For all natural numbers n , we denote by $\Sigma_{n,0,1}^s$ the surface $(\Sigma_{1,0,1}^0)^{\natural n} \natural \Sigma_{0,0,1}^s$ and by $\Sigma_{0,n,1}^s$ the surface $(\Sigma_{0,1,1}^0)^{\natural n} \natural \Sigma_{0,0,1}^s$.

3.3.1 Orientable surfaces

Let $\mathfrak{M}_2^{+,s}$ be the full subgroupoid of \mathfrak{M}_2 on the objects $\{\Sigma_{n,0,1}^s\}_{n \in \mathbb{N}}$. We denote the mapping class group $\pi_0 \text{Homeo}^I(\Sigma_{n,0,1}^s, \{\text{punctures}\})$ by $\Gamma_{n,1}^s$, for all $n \in \mathbb{N}$.

Let H be the group $\pi_1(\Sigma_{1,0,1}^0, p) \cong \mathbf{F}_2$, H_0 be the group $\pi_1(\Sigma_{0,0,1}^s, p) \cong \mathbf{F}_s$ and $H_n = H^{*n} * H_0 \cong \pi_1(\Sigma_{n,0,1}^s, p)$ for all natural numbers n . Using Proposition 3.8, we denote by $\pi_1(\Sigma_{-,0,1}^s, p) : \mathfrak{M}_2^{+,s} \rightarrow \mathfrak{G}\mathfrak{r}$ the associated functor sending $\Sigma_{n,0,1}^s$ to H_n for all natural numbers n . Thus, we deduce from Section 2.2.6 that $\{\pi_1(\Sigma_{-,0,1}^s, p), \mathfrak{M}_2^{+,s}, \mathfrak{M}_2, \zeta_t\}$ is a coherent Long-Moody system, where $\zeta_{n,t} : \pi_1(\Sigma_{n,0,1}^s, p) \rightarrow \Gamma_{n+1,1}^s$ is the trivial morphism for all n .

Example 3.9. For each n , as $\Sigma_{n,0,1}^s$ is a classifying space of $\pi_1(\Sigma_{n,0,1}^s, p)$, the singular homology of $\Sigma_{n,0,1}^s$ is naturally isomorphic to the homology of $\pi_1(\Sigma_{n,0,1}^s, p)$. Hence, we denote by $H_1(\Sigma_{-,0,1}^s, R)$ the composite functor $H_1(-, R) \circ \pi_1(\Sigma_{-,0,1}^s, p)$. Note that if $s = 0$, the action of $\Gamma_{n,1}^0$ on $H_1(\Sigma_{n,0,1}^0, R)$ is the symplectic representation of the mapping class group $\Gamma_{n,1}^0$. We deduce from Proposition 2.15 that:

$$H_1(\Sigma_{-,0,1}^s, R) \cong \mathbf{LM}_{\{\pi_1(\Sigma_{-,0,1}^s, p), \mathfrak{M}_2^{+,s}, \mathfrak{M}_2, \zeta_t\}}(R). \quad (11)$$

This functor was introduced by Cohen and Madsen in [10] and by Boldsen in [4]. Furthermore, the homology of the mapping class groups $\Gamma_{n,1}$ for a large natural number n with coefficients $H_1(\Sigma_{n,0,1}^0, R)$ were computed by Harer in [22, Section 7] (see also the forthcoming work [34]).

As the morphisms $\{\zeta_{n,t}\}_{n \in \mathbb{N}}$ are trivial, the computation of the Long-Moody functor associated with the system $\{\pi_1(\Sigma_{-,0,1}^s, p), \mathfrak{M}_2^{+,s}, \mathfrak{M}_2, \zeta_t\}$ on any object is given by Proposition 2.20. However, there is an alternative to this choice for the family $\{\zeta_n\}_{n \in \mathbb{N}}$ to define a Long-Moody system in the current situation. The key point of this alternative is the use of *pure surface framed braids*: we refer the reader to [1] for a general introduction to these groups which are required for this section. From now until the end of Section 3.3.1, we assume that $s = 0$ and as the non-orientable genus of the surfaces is null for surfaces in $\mathfrak{M}_2^{+,0}$, we omit them from the notations.

Let us fix a natural number $n \geq 1$. Let x be a marked point in the interior of $\Sigma_{n,1}$. There is no loss of generality in endowing the surface $\Sigma_{n,1}$ with an arbitrary smooth structure and a Riemannian metric. We denote by $U\Sigma_{n,1}$ the total space of the unit tangent bundle and fix a unit tangent vector v of $\Sigma_{n,1}$ at x . Recall that the pure framed braid group $FP_1(\Sigma_{n,1})$ is the fundamental group $\pi_1(U\Sigma_{n,1}, v)$. Following [1, Theorem 5], as the surface has a boundary component, $\Sigma_{n,1}$ is parallelizable and the unit tangent bundle $U\Sigma_{n,1}$ is

homeomorphic to $\mathbb{S}^1 \times \Sigma_{n,1}$ (where \mathbb{S}^1 is the 1-dimensional unit sphere): then, using the long exact sequence in homotopy for the fibration $U\Sigma_{n,1} \rightarrow \Sigma_{n,1}$ and the contractibility results of [19, Théorème 5], we deduce that $FP_1(\Sigma_{n,1})$ is isomorphic to $\mathbb{Z} \times \pi_1(\Sigma_{n,1}, x)$ and denote this isomorphism by χ_n . We denote by $a_{\Sigma_{n,1}}^x$ the action of $\Gamma_{n,1}$ on $\pi_1(\Sigma_{g,1}, x)$. In particular, with a trivial action on \mathbb{Z} , $a_{\Sigma_{n,1}}^x$ induces an action through χ_n of $\Gamma_{n,1}$ on $FP_1(\Sigma_{n,1})$, denoted in the same way (abusing the notation).

For each natural number n , let $\Sigma_{n,2}$ be a smooth connected compact surface with two boundary components, one of the two being marked by the parametrized interval I . Denoting by $Diff^{+, \partial}(\Sigma_{n,2})$ the group of orientation preserving diffeomorphisms of $\Sigma_{n,2}$ which are the identity on the boundary, we denote by $\Gamma_{n,2}$ the mapping class group $\pi_0 Diff^{+, \partial}(\Sigma_{n,2})$. We denote by $cap : \Gamma_{n,2} \rightarrow \Gamma_{n,1}$ the morphism induced by capping the boundary component of $\Sigma_{n,2}$ without the interval I with a disc $\Sigma_{0,1}$. Recall from [1, Corollary 3] that $FP_1(\Sigma_{n,1})$ is the kernel of cap and a fortiori the following sequence is exact:

$$1 \longrightarrow FP_1(\Sigma_{n,1}) \longrightarrow \Gamma_{n,2} \xrightarrow{cap} \Gamma_{n,1} \longrightarrow 1. \quad (12)$$

Note that the boundary connected sum $\Sigma_{0,2} \natural \Sigma_{n,1}$ along I is diffeomorphic to $\Sigma_{n,2}$, thus defining a map $\Sigma_{n,1} \rightarrow \Sigma_{n,2}$ and inducing a morphism $id_{\Sigma_{0,2} \natural -} : \Gamma_{n,1} \rightarrow \Gamma_{n,2}$ by extending the diffeomorphisms by the identity on the added part. This is the same morphism as the one induced by gluing a pair of pant to the boundary component of $\Sigma_{n,1}$. We refer the reader to [4, Section 3] or [22, Section 0] if technical details are required for this definition. Moreover, $id_{\Sigma_{0,2} \natural -}$ is a section of cap and a fortiori the short exact sequence (12) splits. Hence, $\Gamma_{n,2}$ is isomorphic to $FP_1(\Sigma_{n,1}) \rtimes_{a_{\Sigma_{n,1}}^x} \Gamma_{n,1}$ (this group is also known as the *pure framed punctured*

mapping class group, see [1, Section 2]). On another note, gluing a torus with a boundary component \mathbb{T} along the boundary component of $\Sigma_{n,2}$ without the interval I defines a continuous map: extending the diffeomorphisms by the identity on the added part induces a morphism $id_{\mathbb{T} \natural -} : \Gamma_{n,2} \rightarrow \Gamma_{1+n,1}$. Again, there is an alternative of this definition using a gluing of a pair of pant on the boundary components of $\Sigma_{n,2}$; we refer to [4, Section 3] or [22, Section 0] for this alternative.

Finally, let α_f^γ be the simple closed curve based at x of $\Sigma_{n,1}$ representative of a generator f of H_n , obtained by moving the curve α_f along a path γ from p to x : this defines an isomorphism $\alpha_n^x : H_n \xrightarrow{\cong} \pi_1(\Sigma_{n,1}, x)$. There is no loss of generality in assuming that the path γ is in a disc $\Sigma_{0,1}^{[x]}$ with the marked point x , so that we have the diffeomorphism $\Sigma_{0,1}^{[x]} \natural \Sigma_{n,1} \simeq \Sigma_{n,1}$. As $\Gamma_{n,1}$ acts trivially on $\Sigma_{0,1}^{[x]}$, composing with χ_n , the isomorphism α_n^x induces an injection

$$\widehat{\chi}_n : H_n \rtimes_{a_{\Sigma_{n,1}}} \Gamma_{n,1} \cong \pi_1(\Sigma_{n,1}, x) \rtimes_{a_{\Sigma_{n,1}}^x} \Gamma_{n,1} \xrightarrow{\chi_n} \pi_1(\Sigma_{n,1}, x) \rtimes_{a_{\Sigma_{n,1}}^x} \Gamma_{n,1} \cong \Gamma_{n,2}.$$

Definition 3.10. We assign $\zeta_{0,fr} : 0_{\mathfrak{G}\tau} \rightarrow \Gamma_{1,1}$ to be the trivial morphism. For $n \geq 1$ a natural number, we define the morphism $\zeta_{n,fr} : H_n \rightarrow \Gamma_{1+n,1}$ to be the composition:

$$H_n \hookrightarrow H_n \rtimes_{a_{\Sigma_{n,1}}} \Gamma_{n,1} \xrightarrow{\widehat{\chi}_n} \Gamma_{n,2} \xrightarrow{id_{\mathbb{T} \natural -}} \Gamma_{1+n,1}.$$

Lemma 3.11. *The setting $\left\{ \pi_1(\Sigma_{-1}, p), \mathfrak{M}_2^{+,0}, \mathfrak{M}_2, \zeta_{fr} \right\}$ is a Long-Moody system.*

Proof. To prove that the diagram (6) of Definition 2.11 is commutative, we use Lemma 2.17. If $n \geq 1$, note that the injection $\Gamma_{n,1} \hookrightarrow H_n \rtimes_{a_{\Sigma_{n,1}}} \Gamma_{n,1} \xrightarrow{\widehat{\chi}_n} \Gamma_{n,2}$ is equal to $id_{\Sigma_{0,2} \natural -}$. Then, it is straightforward to check that the composite $(id_{\mathbb{T} \natural -}) \circ (id_{\Sigma_{0,2} \natural -})$ is equal to the morphism $id_{\Sigma_{1,1} \natural -} : \Gamma_{n,1} \rightarrow \Gamma_{1+n,1}$ induced by the boundary connected sum $\Sigma_{1,1} \natural \Sigma_{n,1}$. Hence, the following diagram is commutative (where ζ_n denotes the composite

$(id_{\mathbb{T}} \# -) \circ \widehat{\chi}_n$:

$$\begin{array}{ccccc}
H_n & \hookrightarrow & H_n \times \Gamma_{n,1} & \longleftarrow & \Gamma_{n,1} \\
& \searrow & \downarrow a_{\Sigma_{n,1}} & \swarrow & \\
& & \Gamma_{1+n,1} & & \\
& \searrow \zeta_{n,fr} & \downarrow \zeta_n & \swarrow id_{\Sigma_{1,1}} \# - & \\
& & \Gamma_{1+n,1} & &
\end{array}$$

If $n = 0$, the mapping class groups $\Gamma_{0,1}$ being the trivial, the commutativity of the diagram (6) is easily checked. \square

Example 3.12. As for Example 3.9, the evaluation on the constant functor R of the Long-Moody functor associated with the Long-Moody system $\{\pi_1(\Sigma_{-,1}, p), \mathfrak{M}_2^{+,0}, \mathfrak{M}_2, \zeta_{fr}\}$ is $H_1(\Sigma_{-,1}, R)$. Assuming that $R = \mathbb{C}$, since the morphisms $\Gamma_{n+1,1} \rightarrow Aut(\pi_1(\Sigma_{n+1,1}, p))$ are non-trivial for natural numbers $n \geq 2$, the action of $\Gamma_{n,1}$ on $\mathbf{LM}_{\{\pi_1(\Sigma_{-,1}, p), \mathfrak{M}_2^{+,0}, \mathfrak{M}_2, \zeta_{fr}\}}(R)(n)$ is not trivial for $n \geq 3$. So this result is consistent with [24, Theorem 1] asserting that for $n \geq 3$, a non-trivial linear representation of $\Gamma_{n,1}$ of dimension $2n$ is equivalent to the symplectic representation.

However, contrary to the situation of Example 3.9, since the morphisms $\zeta_{n,fr}$ are not trivial, the computation of the Long-Moody functor on an object F of $\mathbf{Fct}(\mathfrak{LM}_2^{+,0}, R\text{-Mod})$ is not given by Proposition 2.20. We thus provide new families of representations of the mapping class groups $\{\Gamma_{n,1}\}_{n \in \mathbb{N}}$ by iterating $\mathbf{LM}_{\{\pi_1(\Sigma_{-,1}, p), \mathfrak{M}_2^{+,0}, \mathfrak{M}_2, \zeta_{fr}\}}$.

3.3.2 Non-orientable surfaces

Let $\mathfrak{M}_2^{-,s}$ be the full subgroupoid of \mathfrak{M}_2 on the objects $\{\Sigma_{0,n,1}^s\}_{n \in \mathbb{N}}$. We denote the mapping class group $\pi_0 \mathbf{Homeo}^l(\Sigma_{0,n,1}^s, \{\text{punctures}\})$ by $\mathcal{N}_{n,1}^s$ for all $n \in \mathbb{N}$.

Let H be the group $\pi_1(\Sigma_{0,1,1}^0, p) \cong \mathbf{F}_1$, H_0 be $\pi_1(\Sigma_{0,0,1}^s, p) \cong \mathbf{F}_s$ and $H_n = H^{*n} * H_0 \cong \pi_1(\Sigma_{0,n,1}^s, p)$ for all natural numbers n . Using Proposition 3.8, we denote by $\pi_1(\Sigma_{0,-,1}^s, p) : \mathfrak{LM}_2^{-,s} \rightarrow \mathfrak{Gr}$ the associated functor sending $\Sigma_{0,n,1}^s$ to H_n for all natural numbers n . Hence, $\{\pi_1(\Sigma_{0,-,1}^s, p), \mathfrak{M}_2^{-,s}, \mathfrak{M}_2, \zeta_{-,t}\}$ is a coherent Long-Moody system by Section 2.2.6, where $\zeta_{n,t} : \pi_1(\Sigma_{0,n,1}^s, p) \rightarrow \mathcal{N}_{n+1,1}^s$ is the trivial morphism for all n .

Example 3.13. For each n , as $\Sigma_{0,n,1}^s$ is a classifying space of $\pi_1(\Sigma_{0,n,1}^s, p)$, we denote by $H_1(\Sigma_{0,-,1}^s, R)$ the composite functor $H_1(-, R) \circ \pi_1(\Sigma_{0,-,1}^s, p)$. We deduce from Proposition 2.15 that:

$$H_1(\Sigma_{0,-,1}^s, R) \cong \mathbf{LM}_{\{\pi_1(\Sigma_{0,-,1}^s, p), \mathfrak{M}_2^{-,s}, \mathfrak{M}_2, \zeta_{n,t}\}}(R).$$

Proposition 2.20 ensures that the functor $\mathbf{LM}_{\{\pi_1(\Sigma_{0,-,1}^s, p), \mathfrak{M}_2^{-,s}, \mathfrak{M}_2, \zeta_{n,t}\}}$ is determined by $H_1(\Sigma_{0,-,1}^s, R)$. In [35], Stukow computes the homology groups $H_1(\mathcal{N}_{n,1}, H_1(\Sigma_{0,n,1}^0, \mathbb{Z}))$ for all natural numbers n .

3.4 Modifying the number of punctures

From now until the end of Section 3, we assume that the non-orientable genus of the surfaces is null and omit it from the notations. We also fix a natural number g for the orientable genus. For all natural numbers n , we denote by $\Sigma_{g,1}^n$ the surface $(\Sigma_{0,1}^1)^{\natural n} \natural \Sigma_{g,1}^0$.

Let $\mathfrak{M}_2^{+,g}$ be the full subgroupoid of \mathfrak{M}_2 on the objects $\{\Sigma_{g,1}^n\}_{n \in \mathbb{N}}$. Let H be the group $\pi_1(\Sigma_{0,1}^1, p) \cong \mathbf{F}_1$, H_0 be the group $\pi_1(\Sigma_{g,1}^0, p) \cong \mathbf{F}_{2g}$ and $H_n = H^{*n} * H_0 \cong \pi_1(\Sigma_{g,1}^n, p)$ for all natural numbers n . We denote by $\pi_1(\Sigma_{g,-,1}^+, p) : \mathfrak{LM}_2^{+,g} \rightarrow \mathfrak{Gr}$ the associated functor sending $\Sigma_{g,1}^n$ to H_n (defined using Proposition 3.8).

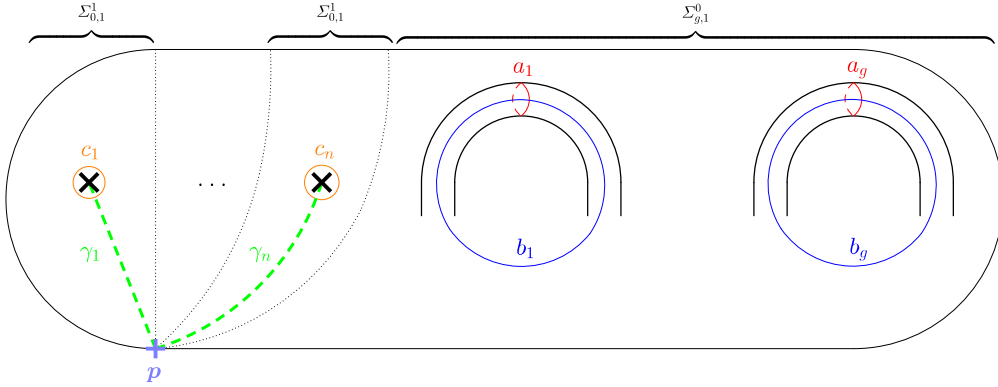


Figure 2: Generators and paths

Let $\{a_i, b_i\}_{i \in \{1, \dots, g\}}$ be a system of meridians and parallels of the surface $\Sigma_{g,1}^0$ and c be a closed curve encircling the puncture $\Sigma_{0,1}^1$ (see Figure 2). For n a natural number and $j \in \{1, \dots, n\}$, c_j thus denotes the corresponding curve c of the j -th copy of $\Sigma_{0,1}^1$ of in $\Sigma_{0,1}^1 \natural \dots \natural \Sigma_{0,1}^1 \natural \Sigma_{g,1}^0$. A generator f of H (respectively H_0) in H_n is the homotopy class of a simple closed curve α_f of $\Sigma_{0,1}^1$ (resp. $\Sigma_{g,1}^0$) in $\Sigma_{g,1}^n$ based at p and encircling the corresponding curve c in H (resp. $\{a_i, b_i\}_{i \in \{1, \dots, g\}}$ in H_0). From now on, we fix a choice of such simple closed curves α_f as generators of H_n .

To define the group morphisms $\{\zeta_n : \pi_1(\Sigma_{g,1}^n, p) \rightarrow \Gamma_{g,1}^n\}_{n \in \mathbb{N}}$ considered in this section, we first need to introduce additional tools and recall some classical facts about mapping class groups of surfaces.

For the unit disc with one puncture $\Sigma_{0,1}^1$, we consider x_1 a marked point filling in the puncture and denote by $\Sigma_{0,1}^{[x_1]}$ the obtained surface. Moreover, we fix γ_1 a path in $\Sigma_{0,1}^{[x_1]}$ connecting the point $p \in I$ to x_1 . For n a natural number and $j \in \{1, \dots, n\}$, x_j (respectively γ_j) denotes the corresponding filling point x_1 (resp. the corresponding path γ_1) of the j -th copy of $\Sigma_{0,1}^1$ of in $\Sigma_{0,1}^1 \natural \dots \natural \Sigma_{0,1}^1 \natural \Sigma_{g,1}^0$ (see Figure 2). For all natural numbers m , the surface $\Sigma_{0,1}^{[x_1]} \natural \Sigma_{0,1}^m$ is denoted by $\Sigma_{g,1}^{[x_1], m}$. Note that we relate the groups H_n to the fundamental groups of $\Sigma_{g,1}^n$ with x_1 as basepoint:

Definition 3.14. Let $\alpha_f^{\gamma_1}$ be the simple closed curve based at x_1 of $\Sigma_{g,1}^{[x_1], n}$ representative of a generator f of H_n , obtained by moving the curve α_f along the path γ_1 from p to x_1 . This defines an isomorphism $\alpha_n^{x_1} : H_n \xrightarrow{\cong} \pi_1(\Sigma_{g,1}^n, x_1)$.

In addition, we introduce the following group, which will be useful for our further work:

Definition 3.15. For all natural numbers n , let $\Gamma_{g,1}^{[1], n}$ be the subgroup of the mapping class group

$$\pi_0 \text{Homeo}^I(\Sigma_{0,1}^1 \natural \Sigma_{g,1}^n, \{\text{punctures}\}) \cong \Gamma_{g,1}^{1+n}$$

where the puncture of the first copy of the surface $\Sigma_{0,1}^1$ in $\Sigma_{0,1}^1 \natural \Sigma_{g,1}^n$ is sent to itself. Hence, we define a canonical embedding $\mathcal{E}_n : \Gamma_{g,1}^{[1], n} \hookrightarrow \Gamma_{g,1}^{1+n}$. In particular, the group $\Gamma_{g,1}^{[1], n}$ is isomorphic to the isotopy classes of homeomorphisms of the surface $\Sigma_{g,1}^{[x_1], n}$ restricting to the identity on the boundary component, freely moving the punctures and the marked point x (see [15, Section 1.1.1]).

Let us fix a natural number n . We consider the surface $\Sigma_{g,1}^n$ as the complement of the disc with one marked $\Sigma_{0,1}^{[x_1]}$ in the surface $\Sigma_{g,1}^{[x_1], n}$. Let β_n be a simple closed curve in $\Sigma_{g,1}^{[x_1], n}$ based at x_1 representative of a

generator h of $\pi_1 \left(\Sigma_{g,1}^n, x_1 \right)$. Let $N(\beta_h) \stackrel{\phi}{\cong} \mathbb{S}^1 \times [-1, 1]$ be a tubular neighbourhood of the curve β_h . Denote by β_h^- and β_h^+ the isotopy classes of the curves $\phi^{-1}(\mathbb{S}^1 \times \{-1\})$ and $\phi^{-1}(\mathbb{S}^1 \times \{1\})$. The group morphism $Push : \pi_1 \left(\Sigma_{g,1}^n, x_1 \right) \rightarrow \Gamma_{g,1}^{[1],n}$ is defined by sending h to $\tau_{\beta_h^-} \circ \tau_{\beta_h^+}^{-1}$ (see [15, Fact 4.7]), where τ_ε denotes the Dehn twist along the simple closed curve ε . Recall from [15, Theorem 4.6] that if n is such that $2g + n \geq 2$, then the following sequence is exact:

$$1 \longrightarrow \pi_1 \left(\Sigma_{g,1}^n, x_1 \right) \xrightarrow{Push} \Gamma_{g,1}^{[1],n} \xrightarrow{Forget} \Gamma_{g,1}^n \longrightarrow 1 \quad (13)$$

where the map $Forget : \Gamma_{g,1}^{[1],n} \rightarrow \Gamma_{g,1}^n$ is induced by forgetting that the point x_1 is marked. It is called Birman exact sequence: it uses the map $Push$ to describe the effect of forgetting a marked point fixed by the mapping class group.

Lemma 3.16. *Let n be a natural number such that $2g + n \geq 2$. The Birman exact sequence (13) splits, hence inducing an isomorphism $\Gamma_{g,1}^{[1],n} \stackrel{\mathcal{B}_n}{\cong} H_n \rtimes_{a_{\Sigma_{g,1}^n}} \Gamma_{g,1}^n$.*

Proof. We denote by $Emb \left(\left(\widehat{\Sigma}_{0,1}^1, I_1^+ \right), \left(\widehat{\Sigma}_{0,1}^1 \natural \widehat{\Sigma}_{g,1}^n, I_1^+ \right) \right)$ the space of embeddings taking I_1^- to I_2^- and such that the complement of $\widehat{\Sigma}_{0,1}^1$ in $\widehat{\Sigma}_{0,1}^1 \natural \widehat{\Sigma}_{g,1}^n$ is diffeomorphic to $\widehat{\Sigma}_{g,1}^n$. Using the long exact sequence of homotopy groups associated to the fibration sequence (see [7, II 2.2.2 Corollaire 2])

$$Diff^{f\partial_0} \left(\widehat{\Sigma}_{g,1}^n \right) \longrightarrow Diff^{f\partial_0} \left(\widehat{\Sigma}_{0,1}^1 \natural \widehat{\Sigma}_{g,1}^n \right) \longrightarrow Emb \left(\left(\widehat{\Sigma}_{0,1}^1, I_1^+ \right), \left(\widehat{\Sigma}_{0,1}^1 \natural \widehat{\Sigma}_{g,1}^n, I_1^+ \right) \right), \quad (14)$$

we deduce from the contractibility results of [19, Théorème 5] that the induced morphism $id_{\widehat{\Sigma}_{0,1}^1} \natural - : \Gamma_{g,1}^n \rightarrow \Gamma_{g,1}^{1+n}$ is injective (see the proof of [32, Proposition 5.18] if more details are required). As the elements of $\Gamma_{g,1}^n$ fix the first puncture, the injection $id_{\widehat{\Sigma}_{0,1}^1} \natural -$ factors across the subgroup $\Gamma_{g,1}^{[1],n}$ using the canonical embedding \mathcal{E}_n of Definition 3.15. We thus define an induced injection $\overline{id_{\widehat{\Sigma}_{0,1}^1} \natural -} : \Gamma_{g,1}^n \hookrightarrow \Gamma_{g,1}^{[1],n}$. This morphism provides a splitting of the exact sequence (13). We denote by $a_{\Sigma_{g,1}^n}^{x_1}$ the action of $\Gamma_{g,1}^n$ on $\pi_1 \left(\Sigma_{g,1}^n, x_1 \right)$. Hence, we have an isomorphism:

$$\Gamma_{g,1}^{[1],n} \cong \pi_1 \left(\Sigma_{g,1}^n, x_1 \right) \rtimes_{a_{\Sigma_{g,1}^n}^{x_1}} \Gamma_{g,1}^n.$$

Recall that the path γ_1 connecting the point p to x_1 is in the subsurface $\Sigma_{0,1}^{[x_1]}$ of $\Sigma_{0,1}^{[x_1]} \natural \Sigma_{g,1}^n$. Since the mapping class group $\Gamma_{g,1}^n$ of $\Sigma_{g,1}^n$ acts trivially on the disc $\Sigma_{0,1}^{[x_1]}$ with the marked point x_1 in $\Sigma_{0,1}^{[x_1]} \natural \Sigma_{g,1}^n$, the isomorphism $\alpha_n^{x_1}$ of Definition 3.14 induces the required isomorphism. \square

Definition 3.17. Let n be a natural number such that $2g + n \geq 2$. We define the morphism $\zeta_{n,1} : H_n \rightarrow \Gamma_{g,1}^{1+n}$ to be the composition:

$$H_n \hookrightarrow H_n \rtimes_{a_{\Sigma_{g,1}^n}} \Gamma_{g,1}^n \stackrel{\mathcal{B}_n}{\cong} \Gamma_{g,1}^{[1],n} \xrightarrow{\mathcal{E}_n} \Gamma_{g,1}^{1+n}.$$

There are two cases with $2g + n < 2$: when $g = 0$ and $n = 0$, we define $\zeta_{0,1} : \pi_1 \left(\Sigma_{0,1}^0, p \right) \rightarrow 0_{\mathfrak{B}_\tau}$ to be the trivial morphism; when $g = 0$ and $n = 1$ we define $\zeta_{1,1} : \pi_1 \left(\Sigma_{0,1}^1, p \right) \rightarrow \mathbf{B}_2$ to be the morphism sending the generator f_1 of $\pi_1 \left(\Sigma_{0,1}^1, p \right)$ to σ_1^2 (where σ_1 denotes the Artin generator of the braid group on two strands \mathbf{B}_2).

Remark 3.18. Let n be a natural number such that $2g + n \geq 2$ and f a generator of H_0 or of one of the copies of H in H_n . Using the notations of Definition 3.14, the morphism $\zeta_{n,1}$ sends f to $\tau_{(\alpha_f^{\gamma_1})^-} \circ \tau_{(\alpha_f^{\gamma_1})^+}^{-1}$.

Lemma 3.19. *The setting $\left\{ \pi_1 \left(\Sigma_{g,1}^-, p \right), \mathfrak{M}_2^{g,0}, \mathfrak{M}_{2, \zeta_1} \right\}$ is a Long-Moody system.*

Proof. To prove that the diagram (6) of Definition 2.11 is commutative, we use Lemma 2.17. It is clear from our assignments that if $2g + n \geq 2$, then the composition $\Gamma_{g,1}^n \hookrightarrow \pi_1 \left(\Sigma_{g,1}^n, p \right) \times_{a_{\Sigma_{g,1}^n}} \Gamma_{g,1}^n \xrightarrow{\mathcal{B}_n} \Gamma_{g,1}^{[1],n}$ is the morphism $\overline{id_{\Sigma_{0,0,1}^1} \natural}^- : \Gamma_{g,1}^n \rightarrow \Gamma_{g,1}^{[1],n}$. Hence, the following diagram is commutative:

$$\begin{array}{ccccc} H_n & \hookrightarrow & H_n \times \Gamma_{g,1}^n & \longleftarrow & \Gamma_{g,1}^n \\ & \searrow & \downarrow a_{\Sigma_{g,1}^n} & \swarrow & \downarrow \mathcal{E}_n \circ \mathcal{B}_n \\ & \searrow \zeta_{n,1} & \Gamma_{g,1}^{1+n} & \swarrow id_{\Sigma_{0,1}^1} \natural^- & \downarrow \mathcal{E}_n \circ \mathcal{B}_n \end{array}$$

If $g = 0$ and $n \leq 1$, the braid groups \mathbf{B}_0 and \mathbf{B}_1 being the trivial group, the commutativity of the diagram (6) is easily checked. \square

Moreover, we have the stronger property:

Proposition 3.20. *With the previous assignments and notation, $\left\{ \pi_1 \left(\Sigma_{g,1}^-, p \right), \mathfrak{M}_2^{+,g}, \mathfrak{M}_{2, \zeta_1} \right\}$ is a coherent Long-Moody system.*

Proof. By Proposition 2.18 and Lemma 3.19, it is enough to prove that the morphism $\zeta_{n,1}$ satisfies the equality (10) for any natural number n .

If $g = 0$ and $n \leq 1$, the result follows from [33, Proposition 2.8]. Assume that $2g + n \geq 2$. Let f be a generator of H_0 or of one of the copies of H in H_n . Note that $b_{\Sigma_{0,1}^1, \Sigma_{0,1}^1}^{\mathfrak{M}_2}$ being defined doing half a Dehn twist in a pair of pants neighbourhood of $\partial \Sigma_{0,1}^1$ and $\partial \Sigma_{0,1}^1$, the morphism $\left(b_{\Sigma_{0,1}^1, \Sigma_{0,1}^1}^{\mathfrak{M}_2} \natural id_n \right)$ is the element $\sigma_1 \in \mathbf{B}_2 \hookrightarrow \Gamma_{g,1}^{2+n}$ which exchanges the punctures of the two first copies of $\Sigma_{0,1}^1$ in $\Sigma_{g,1}^{2+n}$. Hence, it is enough to prove that, as elements of $\Gamma_{g,1}^{2+n}$:

$$\sigma_1 \circ \zeta_{n+1,1} \left(id_{\pi_1(\Sigma_{0,1}^1, p)} * f \right) \circ \sigma_1^{-1} = id_{\Sigma_{0,1}^1} \natural \zeta_{n,1} (f). \quad (15)$$

On the one hand, denoting $id_{\pi_1(\Sigma_{0,1}^1, p)} * f$ by f' , using Definition 3.14, it follows from [15, Fact 3.7] that:

$$\begin{aligned} \sigma_1 \circ \zeta_{n+1,1} \left(id_{\pi_1(\Sigma_{0,1}^1, p)} * f \right) \circ \sigma_1^{-1} &= \left(\sigma_1 \circ \tau_{(\alpha_{f'}^{\gamma_1})^-} \circ \sigma_1^{-1} \right) \circ \left(\sigma_1 \circ \tau_{(\alpha_{f'}^{\gamma_1})^+} \circ \sigma_1^{-1} \right) \\ &= \tau_{\sigma_1 \left((\alpha_{f'}^{\gamma_1})^- \right)} \circ \tau_{\sigma_1 \left((\alpha_{f'}^{\gamma_1})^+ \right)}^{-1}. \end{aligned}$$

On the other hand, we deduce from Definition 3.14 that

$$id_{\Sigma_{0,1}^1} \natural \zeta_{n,1} (f) = id_{\Sigma_{0,1}^1} \natural \left(\tau_{(\alpha_f^{\gamma_1})^-} \circ \tau_{(\alpha_f^{\gamma_1})^+}^{-1} \right) = \tau_{(\alpha_f^{\gamma_2})^-} \circ \tau_{(\alpha_f^{\gamma_2})^+}^{-1}.$$

Since the image of the curve γ_1 by σ_1 is isotopic to γ_2 (by the definition of the braiding, see Figure 1) and as σ_1 exchanges the two first punctures of $\Sigma_{g,1}^{2+n}$, it follows that the image of $\alpha_{f'}^{\gamma_1}$ by σ_1 is isotopic to $\alpha_f^{\gamma_2}$.

Therefore, as isotopy classes of curves, $(\alpha_f^{\gamma_2})^- = \sigma_1 \left((\alpha_{f'}^{\gamma_1})^- \right)$ and $(\alpha_f^{\gamma_2})^+ = \sigma_1 \left((\alpha_{f'}^{\gamma_1})^+ \right)$. A fortiori, we deduce from [15, Fact 3.6] that the equality (15) is satisfied. \square

Example 3.21. For each n , as $\Sigma_{g,1}^n$ is a classifying space of $\pi_1 \left(\Sigma_{g,1}^n, p \right)$, we denote by $H_1 \left(\Sigma_{g,1}^-, R \right)$ the composite functor $H_1 \left(-, R \right) \circ \pi_1 \left(\Sigma_{g,1}^-, p \right)$. We deduce from Proposition 2.15 that:

$$H_1 \left(\Sigma_{g,1}^-, R \right) \cong \mathbf{LM}_{\left\{ \pi_1 \left(\Sigma_{g,1}^-, p \right), \mathfrak{M}_2^{+g}, \mathfrak{M}_{2,\zeta_1} \right\}} \left(R \right).$$

Contrary to the cases of Section 3.3, since the morphisms $\zeta_{n,1}$ are not trivial, the computation of the Long-Moody functor on an object F of $\mathbf{Fct} \left(\mathfrak{U}\mathfrak{M}_2^{+g}, R\text{-}\mathfrak{Mod} \right)$ is not given by Proposition 2.20. We thus provide new families of representations of the mapping class groups $\left\{ \Gamma_{g,1}^n \right\}_{n \in \mathbb{N}}$ by iterating $\mathbf{LM}_{\left\{ \pi_1 \left(\Sigma_{g,1}^-, p \right), \mathfrak{M}_2^{+g}, \mathfrak{M}_{2,\zeta_1} \right\}}$ for all natural numbers g .

3.5 Surface braid groups

We fix a natural number g throughout Section 3.5; let \mathfrak{B}_2 (respectively \mathfrak{B}_2^g) be the subgroupoid of \mathfrak{M}_2 (resp. $\mathfrak{M}_2^{g,0}$) with the same objects and with morphisms those that become trivial forgetting all the punctures. Namely, for all objects $\Sigma_{g,0,1}^n$ of \mathfrak{M}_2 , we have the following short exact sequence (see for example [21, Section 2.4]):

$$1 \longrightarrow \mathbf{B}_n^g \longrightarrow \Gamma_{g,1}^n \longrightarrow \Gamma_{g,1}^0 \longrightarrow 1$$

where $\mathbf{B}_n^g = \mathbf{B} \left(\Sigma_{g,1}^n \right)$ denotes the braid group of the surface $\Sigma_{g,1}^n$. The monoidal structure $(\mathfrak{M}_2, \natural, 0)$ restricts to a braided monoidal structure on the subgroupoid \mathfrak{B}_2 , denoted in the same way $(\mathfrak{B}_2, \natural, 0)$.

As in Section 3.4, let H be the free group $\pi_1 \left(\Sigma_{0,1}^1, p \right) \cong \mathbf{F}_1$, H_0 be the free group $\pi_1 \left(\Sigma_{g,1}^0, p \right) \cong \mathbf{F}_{2g}$ and $H_n = H^{*n} * H_0 \cong \pi_1 \left(\Sigma_{g,1}^n, p \right)$ for all natural numbers n . Precomposing by $\mathfrak{U}\mathfrak{B}_2^g \rightarrow \mathfrak{U}\mathfrak{M}_2^{g,0}$, we consider the restriction of the functor $\pi_1 \left(\Sigma_{g,1}^-, p \right) : \mathfrak{U}\mathfrak{M}_2^{+g} \rightarrow \mathbf{gr}$ of Section 3.4 to $\mathfrak{U}\mathfrak{B}_2^g$ and obtain the associated functor $\pi_1 \left(\Sigma_{g,1}^-, p \right)^b : \mathfrak{U}\mathfrak{B}_2^g \rightarrow \mathfrak{Gr}$ sending $\Sigma_{g,1}^n$ to H_n . For all natural numbers n , we denote by $a_{\Sigma_{g,1}^n}^b$ the morphism induced by $a_{\Sigma_{g,1}^n}$ using the precomposition $\mathfrak{U}\mathfrak{B}_2^g \rightarrow \mathfrak{U}\mathfrak{M}_2^{+g}$.

For all natural numbers n , we denote by $\mathbf{B}_{[1],n}^g$ the subgroup of $\mathbf{B} \left(\Sigma_{0,1}^1 \natural \Sigma_{g,1}^n \right)$ where the puncture of $\Sigma_{0,1}^1$ is fixed. This group $\mathbf{B}_{[1],n}^g$ is also known as the intertwining $(1, n)$ -braid group on the surface $\Sigma_{g,1}^n$, which is the kernel of the morphism $\Gamma_{g,1}^{[1],n} \rightarrow \Gamma_{g,1}^{[1]}$ defined by filling in the n last punctures. Hence, there is a canonical embedding:

$$\mathcal{E}_n^b : \mathbf{B}_{[1],n}^g \hookrightarrow \ker \left(\Gamma_{g,1}^{1+n} \rightarrow \Gamma_{g,1}^0 \right) \cong \mathbf{B}_{1+n}^g.$$

The following lemma (also stated in [14, 18]) is the key to define a Long-Moody system in the current situation.

Lemma 3.22. *For all natural numbers n , there is an isomorphism:*

$$\mathbf{B}_{[1],n}^g \stackrel{\mathcal{B}_n}{\cong} \pi_1 \left(\Sigma_{g,1}^n, p \right) \underset{a_{\Sigma_{g,1}^n}^b}{\times} \mathbf{B}_n^g.$$

Proof. Recall the isomorphism $\Gamma_{g,1}^{[1],n} \stackrel{\mathcal{B}_n}{\cong} \pi_1 \left(\Sigma_{g,1}^n, p \right) \underset{a_{\Sigma_{g,1}^n}^b}{\times} \Gamma_{g,1}^n$ of Lemma 3.16. The result is a consequence of the universal property of the kernel of the morphism $\Gamma_{g,1}^{[1],n} \rightarrow \Gamma_{g,1}^{[1]}$. \square

Definition 3.23. Let n be a natural number such that $2g + n \geq 2$. We define the morphism $\zeta_{n,1}^b : H_n \rightarrow \mathbf{B}_{1+n}^g$ to be the composition:

$$H_n \hookrightarrow H_n \times_{a_{\Sigma_{g,1}^n}^b} \mathbf{B}_n^g \xrightarrow{\mathcal{B}_n^g} \mathbf{B}_{[1],n}^g \xrightarrow{\mathcal{C}_n^b} \mathbf{B}_{1+n}^g.$$

If $g = 0$, we define $\zeta_{0,1} : \pi_1(\Sigma_{0,1}^0, p) \rightarrow 0_{\mathfrak{G}\tau}$ to be the trivial morphism and $\zeta_{1,1} : \pi_1(\Sigma_{0,1}^1, p) \rightarrow \mathbf{B}_2$ to be the morphism sending the generator f_1 of $\pi_1(\Sigma_{0,1}^1, p)$ to σ_1^2 (where σ_1 denotes the Artin generator of the braid group on two strands \mathbf{B}_2).

Proposition 3.24. *With the previous assignments and notations, $\left\{ \pi_1(\Sigma_{g,1}^-, p)^b, \mathfrak{B}_2^g, \mathfrak{B}_2, \zeta_1^b \right\}$ is a coherent Long-Moody system.*

Proof. First, following mutatis mutandis the proof of Lemma 3.19., the setting $\left\{ \pi_1(\Sigma_{g,1}^-, p)^b, \mathfrak{B}_2^g, \mathfrak{B}_2, \zeta_1^b \right\}$ is a Long-Moody system. Then, as \mathbf{B}_n^g is a subgroup of $\Gamma_{g,0,1}^n$, repeating mutatis mutandis the proof of Proposition 3.20, the morphisms $\zeta_{n,1}^b$ satisfy the equality (10) of Proposition 2.18 for all natural numbers n . \square

Example 3.25. We denote by $H_1(\Sigma_{g,1}^-, R)_{\mathfrak{U}\mathfrak{B}_2}$ the restriction of the functor induced by the functor $H_1(\Sigma_{g,1}^-, R)$ of Example 3.21 to the subcategory $\mathfrak{U}\mathfrak{B}_2^g$ of $\mathfrak{U}\mathfrak{M}_2^{+\mathfrak{g}}$. We deduce from Proposition 2.15 that:

$$H_1(\Sigma_{g,1}^-, R)_{\mathfrak{B}_2} \cong \mathbf{LM} \left\{ \pi_1(\Sigma_{g,1}^-, p)^b, \mathfrak{B}_2^g, \mathfrak{B}_2, \zeta_1^b \right\} (R).$$

As for Example 3.21, since the morphisms $\left\{ \zeta_{n,1}^b \right\}_{n \in \mathbb{N}}$ are not trivial, the computation of $\mathbf{LM} \left\{ \pi_1(\Sigma_{g,1}^-, p)^b, \mathfrak{B}_2^g, \mathfrak{B}_2, \zeta_1^b \right\}$ on an object F of $\mathbf{Fct}(\mathfrak{U}\mathfrak{B}_2^g, R\text{-Mod})$ is not a priori determined by $H_1(\Sigma_{g,1}^-, R)_{\mathfrak{B}_2}$ using Proposition 2.20. Hence, the iterates of the Long-Moody functor $\mathbf{LM} \left\{ \pi_1(\Sigma_{g,1}^-, p)^b, \mathfrak{B}_2^g, \mathfrak{B}_2, \zeta_1^b \right\}$ define new representations for surface braid groups. As far as the author knows, there are very few explicit examples of representations of surfaces braid groups for $g \geq 1$.

The case of braid groups: Assuming that $g = 0$, we recover the results of [33]. Indeed, in this case we consider the category $\mathfrak{U}\mathfrak{B}_2^0 = \mathfrak{U}\mathfrak{B}$, which is Quillen's bracket construction on the braid groupoid \mathfrak{B} . The choice $\zeta_{n,1} = \zeta_{n,1}^b : \mathbf{F}_n \rightarrow \mathbf{B}_{n+1}$ of Definitions 3.17 and 3.23 corresponds to the morphism introduced in [33, Example 2.7]:

$$\begin{aligned} \zeta_{n,1} : \mathbf{F}_n &\longrightarrow \mathbf{B}_{n+1} \\ g_i &\longmapsto \begin{cases} \sigma_1^2 & \text{if } i = 1 \\ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \dots \circ \sigma_{i-1}^{-1} \circ \sigma_i^2 \circ \sigma_{i-1} \circ \dots \circ \sigma_2 \circ \sigma_1 & \text{if } i \in \{2, \dots, n\}. \end{cases} \end{aligned}$$

In Section 3.2, we fixed the actions

$$a_{\Sigma_{0,1}^n} : \mathbf{B}_n \cong \Gamma_{0,1}^n \rightarrow \text{Aut}_{\mathfrak{G}\tau}(\pi_1(\Sigma_{0,1}^n, p)),$$

which correspond to Artin's representations for all natural numbers n . By [33, Section 2.3.1] we obtain:

Proposition 3.26. $\mathbf{LM} \left\{ \pi_1(\Sigma_{0,1}^-, p)^b, \mathfrak{B}_2^0, \mathfrak{B}_2, \zeta_1^b \right\} = \mathbf{LM}_1$ where \mathbf{LM}_1 denotes the Long-Moody functor of [33, Section 2.3.1].

In particular, if $R = \mathbb{C} [t^{\pm 1}]$, by [33, Proposition 2.31] we have:

$$t^{-1} \cdot \mathbf{LM} \left\{ \pi_1 \left(\Sigma_{s,0,1}^-, p \right)^b, \mathfrak{B}_2^0, \mathfrak{B}_2, \zeta_1^b \right\} \left(t \cdot \mathbb{C} [t^{\pm 1}] \right) \cong \mathfrak{Bur}_{t^2},$$

where $\mathfrak{Bur}_{t^2} : \mathfrak{U}\mathfrak{B} \rightarrow \mathbb{C} [t^{\pm 1}]$ - \mathfrak{Mod} denotes the functor associated with the family of unreduced Burau representations with parameter t^2 (see [33, Section 1.2]).

We could have chosen other actions $a_n : \mathbf{B}_n \rightarrow \text{Aut}(\mathbf{F}_n)$ and morphisms $\zeta_n : \mathbf{F}_n \rightarrow \mathbf{B}_{n+1}$ so that the framework of Section 2 is satisfied. Hence, we can recover all the Long-Moody functors introduced in [33]. In addition, the new framework developed in the present paper recovers even more families of representations of braid groups that the work of [33] could not obtain:

Example 3.27. Let n be a natural number. Using the terminology of [37], there is a classical geometric embedding $\mathcal{W}_n : \mathbf{B}_{2n+1} \hookrightarrow \Gamma_{n,0,1}^0$ that sends the standard generators of the braid group to Dehn twists around a fixed system of meridians and parallels on the surface $\Sigma_{n,1}^0$ (we refer to [3, Section 1] for more details about this embedding). Let \mathcal{W} be the subgroupoid of $\mathfrak{M}_2^{+,0}$ defined by the embeddings $\{\mathcal{W}_n\}_{n \in \mathbb{N}}$. We assign H to be the group $\pi_1 \left(\Sigma_{n,1}^0, p \right)$ and H_0 to be the trivial group.

Hence, the functor $\pi_1 \left(\Sigma_{-,0,1}^0, p \right)$ of Section 3.3.1 provides a functor $\pi_1 \left(\Sigma_{-,1}^0, p \right)^{b,2} : \mathfrak{U}\mathcal{W} \rightarrow \mathfrak{U}\mathfrak{M}_2^{+,0} \rightarrow \mathfrak{G}\mathfrak{T}$ by restriction. According to Section 2.2.6, $\left\{ \pi_1 \left(\Sigma_{-,1}^0, p \right)^{b,2}, \mathcal{W}, \mathfrak{M}_2^{+,0}, \zeta_t \right\}$ is a coherent Long-Moody system. Then, by Lemma 2.15 that:

$$H_1 \left(\Sigma_{-,1}^0, R \right)_{\mathfrak{U}\mathcal{W}} \cong \mathbf{LM} \left\{ \pi_1 \left(\Sigma_{-,1}^0, p \right)^{b,2}, \mathcal{W}, \mathfrak{M}_2^{+,0}, \zeta_t \right\} (R)$$

where $H_1 \left(\Sigma_{-,1}^0, R \right)_{\mathfrak{U}\mathcal{W}}$ denotes the restriction of the functor $H_1 \left(\Sigma_{-,1}^0, R \right)$ to the category $\mathfrak{U}\mathcal{W}$. In [6], Callegaro and Salvetti compute the homology of braid groups with twisted coefficients given by the functor $H_1 \left(\Sigma_{-,1}^0, \mathbb{Z} \right)_{\mathfrak{U}\mathcal{W}}$. In [33], H_n is the free group on n generators \mathbf{F}_n . A fortiori, for dimensional considerations on the objects, it was impossible to directly recover the functor of Example 3.27 applying a Long-Moody functor with this setting.

3.6 Symmetric groups

Let Σ be the skeleton of the groupoid of finite sets and bijections. Its automorphism groups are the symmetric groups \mathfrak{S}_n . The disjoint union of finite sets \sqcup induces a monoidal structure $(\Sigma, \sqcup, 0)$, the unit 0 being the empty set. This groupoid is symmetric monoidal, the symmetry being given by the canonical bijection $b_{n_1, n_2}^\Sigma : n_1 \sqcup n_2 \rightarrow n_2 \sqcup n_1$ for all natural numbers n_1 and n_2 . This symmetric monoidal groupoid has no zero divisors and $\text{Aut}_\Sigma(0_\Sigma) = \{id_{0_\Sigma}\}$. The category $\mathfrak{U}\Sigma$ is equivalent to the category of finite sets and injections FI , studied in [8]. The classical surjections $\{p_n : \mathbf{B}_n \twoheadrightarrow \mathfrak{S}_n\}_{n \in \mathbb{N}}$, sending each $\sigma_i \in \mathbf{B}_n$ to the transposition $\tau_i \in \mathfrak{S}_n$, define a strict monoidal functor $\mathfrak{P} : (\mathfrak{U}\mathfrak{B}, \natural, 0) \rightarrow (\mathfrak{U}\Sigma, \sqcup, 0)$. For all natural numbers n , we denote by $a_n^\mathfrak{S} : \mathfrak{S}_n \rightarrow \text{Aut}(\mathbf{F}_n)$ the morphism defined by $a_n^\mathfrak{S}(\sigma)(f_i) = f_{\sigma(i)}$ for all $\sigma \in \mathfrak{S}_n$ and generator f_i of \mathbf{F}_n . We thus define functors $\mathcal{A}_\mathfrak{S} : \Sigma \rightarrow \mathfrak{gr}$ assigning $\mathcal{A}_\mathfrak{S}(n) = \mathbb{Z}^n$ on objects and for all $\sigma \in \mathfrak{S}_n$, $\mathcal{A}_\mathfrak{S}(\sigma) = a_n^\mathfrak{S}(\sigma)$.

Lemma 3.28. *The functor $\mathcal{A}_\mathfrak{S} : (\Sigma, \sqcup, 0) \rightarrow (\mathfrak{gr}, *, 0_{\mathfrak{G}\mathfrak{T}})$ is symmetric strict monoidal. It extends to define a functor $\mathcal{A}_\mathfrak{S} : \mathfrak{U}\Sigma \rightarrow \mathfrak{gr}$ assigning $\mathcal{A}_\mathfrak{S}([n_1, id_{n_1 \sqcup n_2}]) = \iota_{n_1} \oplus id_{n_2}$ for all natural numbers n_1 and n_2 .*

Proof. For n_1 and n_2 two natural numbers, the group \mathfrak{S}_{n_1} (resp. \mathfrak{S}_{n_2}) acting trivially on $\mathcal{A}_\mathfrak{S}(n_2)$ (resp. $\mathcal{A}_\mathfrak{S}(n_1)$) in $\mathcal{A}_\mathfrak{S}(n_1 \sqcup n_2)$, $id_{\mathcal{A}_\mathfrak{S}(n_1)} * id_{\mathcal{A}_\mathfrak{S}(n_2)}$ is a natural equivalence. It is easy to check from the assignments that Relations (3) and (4) of Lemma 1.6 is satisfied, which implies the result. \square

Hence, we deduce from Section 2.2.6:

Corollary 3.29. *With the previous assignments and notations, $\{\mathcal{A}_\mathfrak{S}, \Sigma, \Sigma, \zeta_t\}$ is a coherent Long-Moody system.*

The functor $\mathbf{LM}_{\{\mathcal{A}_{\mathfrak{S}}, \Sigma, \Sigma, \zeta_t\}}$ is closely related to the functor \mathbf{LM}_1 for braid groups (see Proposition 3.26) introduced in [33, Section 1.3]:

Proposition 3.30. *We denote by $(\mathfrak{P})^*$ the precomposition by the functor \mathfrak{P} . The following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{Fct}(\mathfrak{A}\beta, R\text{-}\mathfrak{M}\text{o}\mathfrak{d}) & \xrightarrow{\mathbf{LM}_1} & \mathbf{Fct}(\mathfrak{A}\beta, R\text{-}\mathfrak{M}\text{o}\mathfrak{d}) \\ (\mathfrak{P})^* \uparrow & & \uparrow (\mathfrak{P})^* \\ \mathbf{Fct}(\mathfrak{A}\Sigma, R\text{-}\mathfrak{M}\text{o}\mathfrak{d}) & \xrightarrow{\mathbf{LM}_{\{\mathcal{A}_{\mathfrak{S}}, \Sigma, \Sigma, \zeta_t\}}} & \mathbf{Fct}(\mathfrak{A}\Sigma, R\text{-}\mathfrak{M}\text{o}\mathfrak{d}), \end{array}$$

Proof. First, it follows from $\mathfrak{p}_{n+1}(\sigma_i^2) = 1_{\mathfrak{S}_n}$ (where $1_{\mathfrak{S}_n}$ is the neutral element of \mathfrak{S}_n) that $\mathfrak{p}_{n+1} \circ \zeta_{n,1} = \zeta_{n,t}$. A fortiori, as \mathfrak{P} is strict monoidal and $\pi_1(\Sigma_{g,0,1}^-)^b([n' - n, id_{n'}]) = \mathcal{A}_{\mathfrak{S}}([n' - n, id_{n'}])$ for all natural numbers $n' \geq n$, it is enough to prove that for all object F of $\mathbf{Fct}(\mathfrak{A}\Sigma, R\text{-}\mathfrak{M}\text{o}\mathfrak{d})$, for all Artin generators $\sigma_i \in \mathbf{B}_n$ and all a natural numbers n :

$$\mathcal{I}_{\pi_1(\Sigma_{g,0,1}^-)^b}(\sigma_i) \otimes_{R[\mathbb{F}_n]} (F \circ \mathfrak{P})(\sigma_{1+i}) = \mathcal{I}_{\mathcal{A}_{\mathfrak{S}}}(\tau_i) \otimes_{R[\mathbb{F}_n]} F(\tau_{1+i}). \quad (16)$$

First, we deduce from the strict monoidal property of \mathfrak{P} that $(F \circ \mathfrak{P})(id_1 \natural \sigma_i) = F(id_1 \sqcup \mathfrak{P}(\sigma_i))$. It follows from the definition of Artin representation (see [33, Section 2.3.1]) that:

$$\begin{aligned} \mathcal{I}_{\pi_1(\Sigma_{g,0,1}^-)^b} : \mathcal{I}_{\mathbb{K}[\mathbb{F}_n]} &\longrightarrow \mathcal{I}_{\mathbb{K}[\mathbb{F}_n]} \\ f_j - 1 &\longmapsto \begin{cases} f_{i+1} - 1 & \text{if } j = i \\ f_{i+1}^{-1} f_i f_{i+1} - 1 = [f_i - 1] f_{i+1} + [f_{i+1} - 1] (1 - f_{i+1}^{-1} f_i f_{i+1}) & \text{if } j = i + 1 \\ f_j - 1 & \text{if } j \notin \{i, i + 1\}. \end{cases} \end{aligned}$$

The equality (16) follows from the relations $\mathfrak{p}_{n+1} \circ \zeta_{n,1} = \zeta_{n,t}$ and $\mathcal{I}_{\mathcal{A}_{\mathfrak{S}}}(\mathfrak{P}(\sigma_i))(f_{i+1} - 1) = f_i - 1$ for $j = i + 1$. The others cases are clear. \square

For all natural numbers, we denote by $\mathfrak{P}\text{erm}_n$ the permutation representation of the symmetric group to $GL_n(R)$, defined assigning for every transposition $\sigma_i \in \mathfrak{S}_n$ (with $i \in \{1, \dots, n - 1\}$):

$$\mathfrak{P}\text{erm}_n(\sigma_i) = Id_{R^{\oplus i}} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus Id_{R^{\oplus n-i-1}}$$

It is a well-known fact (see for example [8]) that the permutation representations $\{\mathfrak{P}\text{erm}_n\}_{n \in \mathbb{N}}$ assemble to form a functor $\mathfrak{P}\text{erm} : \mathfrak{A}\Sigma \rightarrow R\text{-}\mathfrak{M}\text{o}\mathfrak{d}$. For $R : \mathfrak{A}\Sigma \rightarrow R\text{-}\mathfrak{M}\text{o}\mathfrak{d}$ the constant functor, we compute that $\mathbf{LM}_{\{\mathcal{A}_{\mathfrak{S}}, \Sigma, \Sigma, \zeta_t\}}(R) \cong \mathfrak{P}\text{erm}$. By Proposition 2.20, all the iterations of $\mathbf{LM}_{\{\mathcal{A}_{\mathfrak{S}}, \Sigma, \Sigma, \zeta_t\}}$ on an object F of $\mathbf{Fct}(\mathfrak{A}\Sigma, R\text{-}\mathfrak{M}\text{o}\mathfrak{d})$ are determined by $\mathfrak{P}\text{erm}$. We conclude the study for symmetric groups giving the following result, obtained as a Corollary of [26, Theorem 4.3]:

Proposition 3.31. *Let m be a natural number. Consider the iteration $\mathbf{LM}_{\{\mathcal{A}_{\mathfrak{S}}, \Sigma, \Sigma, \zeta_t\}}^{\circ(m+1)}(R)$ of the Long-Moody functor $\mathbf{LM}_{\{\mathcal{A}_{\mathfrak{S}}, \Sigma, \Sigma, \zeta_t\}}$. Then, all the irreducible representations of the symmetric group \mathfrak{S}_m are subrepresentations of the induced representation*

$$\mathbf{LM}_{\{\mathcal{H}_{\mathfrak{S}}, \Sigma, \zeta_{n,t}\}}^{\circ(m+1)}(R)|_{\mathfrak{S}_m} : \mathfrak{S}_m \rightarrow GL_M(R)$$

where $M = \frac{(2m+1)!}{m!}$.

4 Strong and weak polynomial functors

This section introduces the notions of (very) strong and weak polynomial functors with respect to the framework of the present paper. Namely, the first subsection presents strong and very strong polynomial functors and their basic properties. In the second subsection, we introduce weak polynomial functors for some subcategories of pre-braided monoidal categories with an initial object, generalising the previous notion of [12, Section 1]. We also detail some first properties of these functors.

4.1 Strong and very strong polynomial functors

For the remainder of Section 4.1, $(\mathfrak{M}', \natural, 0)$ is a pre-braided strict monoidal small category where the unit 0 is an initial object. We consider \mathfrak{M} a full subcategory of $(\mathfrak{M}', \natural, 0)$. Finally, we fix \mathcal{A} an abelian category.

In this section, we introduce the notions of strong and very strong polynomiality for objects in the functor category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. In [33, Section 3], a framework is given for defining these notions in the category $\mathbf{Fct}(M, \mathcal{A})$, where M is a pre-braided monoidal category where the unit is an initial object. It generalizes the previous work of Djament and Vespa in [12, Section 1]. We also refer to [30] for a comparison of the various instances of the notions of twisted coefficient system and polynomial functor. This section thus extends the definitions and properties of [33, Section 3] to the present larger framework, the various proofs being direct generalizations of this previous work.

Notation 4.1. We denote by $Obj(\mathfrak{M}')_{\natural}$ the set of objects m' of \mathfrak{M}' such that $m' \natural m$ is an object of \mathfrak{M} for all objects m of \mathfrak{M} .

Let $m \in Obj(\mathfrak{M}')_{\natural}$. We denote by $\tau_m : \mathbf{Fct}(\mathfrak{M}, \mathcal{A}) \rightarrow \mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ the translation functor defined by $\tau_m(F) = F(m \natural -)$, $i_m : Id \rightarrow \tau_m$ the natural transformation of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ induced by the unique morphism $i_m : 0 \rightarrow m$. We define $\delta_m = \text{coker}(i_m)$ the difference functor and $\kappa_m = \text{ker}(i_m)$ the evanescence functor. The following basic properties are direct generalizations of [33, Propositions 3.2 and 3.5]:

Proposition 4.2. *Let $m, m' \in Obj(\mathfrak{M}')_{\natural}$. Then the translation functor τ_m is exact and we have the following exact sequence of endofunctors of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:*

$$0 \longrightarrow \kappa_m \xrightarrow{\Omega_m} Id \xrightarrow{i_m} \tau_m \xrightarrow{\Delta_m} \delta_m \longrightarrow 0. \quad (17)$$

Moreover, for a short exact sequence $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$ in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, there is a natural exact sequence in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_m(F) \longrightarrow \kappa_m(G) \longrightarrow \kappa_m(H) \longrightarrow \delta_m(F) \longrightarrow \delta_m(G) \longrightarrow \delta_m(H) \longrightarrow 0. \quad (18)$$

In addition, the functors τ_m and $\tau_{m'}$ commute up to natural isomorphism and they commute with limits and colimits; the difference functors δ_m and $\delta_{m'}$ commute up to natural isomorphism and they commute with colimits; the functors κ_m and $\kappa_{m'}$ commute up to natural isomorphism and they commute with limits; the functor τ_m commutes with the functors δ_m and κ_m up to natural isomorphism.

We can define the notions of strong and very strong polynomial functors using Proposition 4.2. Namely:

Definition 4.3. We recursively define on $d \in \mathbb{N}$ the categories $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ and $\mathcal{V}Pol_d(\mathfrak{M}, \mathcal{A})$ of strong and very strong polynomial functors of degree less than or equal to d to be the full subcategories of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ as follows:

1. If $d < 0$, $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A}) = \mathcal{V}Pol_d(\mathfrak{M}, \mathcal{A}) = \{0\}$;
2. if $d \geq 0$, the objects of $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ are the functors F such that for all $m \in Obj(\mathfrak{M}')_{\natural}$, the functor $\delta_m(F)$ is an object of $\mathcal{P}ol_{d-1}^{strong}(\mathfrak{M}, \mathcal{A})$; the objects of $\mathcal{V}Pol_d(\mathfrak{M}, \mathcal{A})$ are the objects F of $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$ such that $\kappa_m(F) = 0$ and the functor $\delta_m(F)$ is an object of $\mathcal{V}Pol_{d-1}(\mathfrak{M}, \mathcal{A})$ for all $m \in Obj(\mathfrak{M}')_{\natural}$.

For an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ which is strong (respectively very strong) polynomial of degree less than or equal to $n \in \mathbb{N}$, the smallest natural number $d \leq n$ for which F is an object of $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ (resp. $\mathcal{V}Pol_d(\mathfrak{M}, \mathcal{A})$) is called the strong (resp. very strong) degree of F .

Finally, we recall useful properties of the categories associated with strong and very strong polynomial functors. Beforehand, we introduce the following terminology:

Definition 4.4. Let $(\mathcal{C}, \natural, 0)$ be a strict monoidal category. A full subcategory \mathcal{D} of \mathcal{C} is said to be finitely generated by the monoidal structure if there exists a finite set E of objects of the category \mathcal{C} such that for all objects d of \mathcal{D} , d is isomorphic to a finite monoidal product of objects of E .

The following properties are direct generalizations of [33, Propositions 3.9 and 3.19]. Hence, the proofs carry over mutatis mutandis to the present framework.

Proposition 4.5. *We assume that the category \mathfrak{M} is finitely generated by the monoidal structure in $(\mathfrak{M}', \natural, 0)$. We denote by E a finite generating set of \mathfrak{M} .*

Let d be a natural number. The category $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ is closed under the translation functor, under quotient, under extension and under colimits. The category $\mathcal{V}Pol_d(\mathfrak{M}, \mathcal{A})$ is closed under the translation functor, under normal subobjects and under extension.

Moreover, an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ belongs to $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ (respectively to $\mathcal{V}Pol_d(\mathfrak{M}, \mathcal{A})$) if and only if $\delta_e(F)$ is an object of $\mathcal{P}ol_{d-1}^{strong}(\mathfrak{M}, \mathcal{A})$ (resp. $\kappa_e(F) = 0$ and $\delta_e(F)$ is an object of $\mathcal{V}Pol_{n-1}(\mathfrak{M}, \mathcal{A})$), for all objects e of $E \cap \text{Obj}(\mathfrak{M}')$.

Finally, an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ belongs to $\mathcal{P}ol_0^{strong}(\mathfrak{M}, \mathcal{A})$ if and only if it the quotient of a constant object of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$.

4.2 Weak polynomial functors

We deal here with the concept of weak polynomial functor. It is introduced by Djament and Vespa in [12, Section 1] in the category $\mathbf{Fct}(S, A)$ where S is a symmetric monoidal category where the unit is an initial object, and A is a Grothendieck category. Weak polynomial functors form a thick subcategory of $\mathbf{Fct}(S, A)$ (see Definition 4.12 and Proposition 4.13). In particular, this notion happens to be more appropriate to study the stable behaviour for objects of the category $\mathbf{Fct}(S, A)$ (see [12, Section 5] and [11]).

We adapt the definition and properties of weak polynomial functors to the present larger setting. In particular, the notion of weak polynomial functor will be well-defined for the category $\mathbf{Fct}(\mathcal{U}\mathcal{G}, R\text{-Mod})$ where $\mathcal{U}\mathcal{G}$ is Quillen's bracket construction applied to the groupoid \mathcal{G} given by a reliable Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$. We refer the reader to [17, Chapitres II et III] for general notions on abelian categories and quotient abelian category which are required for this section. A Grothendieck category is a cocomplete abelian category which admits a generator and such that direct limits are exact.

For the remainder of Section 4.2, we assume that the abelian category \mathcal{A} is a Grothendieck category. We recall that we consider $(\mathfrak{M}', \natural, 0)$ a strict pre-braided monoidal small category where the unit 0 is an initial object and \mathfrak{M} a full subcategory of $(\mathfrak{M}', \natural, 0)$ finitely generated by the monoidal structure. Therefore, the functor category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is a Grothendieck category (see [17]). We recall that we introduced $\text{Obj}(\mathfrak{M}')_{\natural}$ (a particular set of objects of \mathfrak{M}') in Notation 4.1.

Definition 4.6. [12, Definition 1.10] Let F be an object of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. The subfunctor $\sum_{m \in \text{Obj}(\mathfrak{M}')_{\natural}} \kappa_m F$ of F is denoted by $\kappa(F)$. The functor F is said to be stably null if $\kappa(F) = F$. Stably null objects of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ define a full subcategory of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, denoted by $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$.

We have the following basic properties:

Lemma 4.7. *The functor κ is left exact. Moreover, the functor $\kappa(F)$ is an object of $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ for all objects F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$.*

Proof. A filtration on the evanescence functors $\{\kappa_m\}_{m \in \text{Obj}(\mathfrak{M}')_{\natural}}$ is given by the inclusions $\kappa_{n'} \hookrightarrow \kappa_{n' \natural n}$ and $\kappa_n \hookrightarrow \kappa_{n' \natural n}$ induced by the morphisms $n \rightarrow n' \natural n$ and $n' \rightarrow n' \natural n$. Hence, the functor κ is left exact as the filtered colimit of the left exact functors $\{\kappa_m\}_{m \in \text{Obj}(\mathfrak{M}')_{\natural}}$. For F an object of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, $\kappa_m F$ is an object of $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ for all $m \in \text{Obj}(\mathfrak{M}')_{\natural}$ since filtered colimit commute with finite limits (see [27, Chapter IX, section 2, Theorem 1]). Hence, the second result follows from the commutation of κ with filtered colimits since it is a filtered colimit (see [27, Chapter IX, section 2]). \square

The following proposition is the key property to define weak polynomial functors. It extends the result [12, Corollary 1.15], its proof is although quite different.

Proposition 4.8. *The category $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is a thick subcategory of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ and it is closed under colimits.*

Proof. Recall that the functor κ commutes with filtered colimits (see Lemma 4.7). Hence, the category $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under filtered colimits. As $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is a Grothendieck category, the category $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under colimits (see [27, Chapters V and IX]).

Let us prove that $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is a thick subcategory of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Let B be a subfunctor of F . As $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is a Grothendieck category, we denote by F/B the quotient. Hence, since κ is left exact, the following diagram (where the lines are exact and the vertical arrows are the inclusions) is commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \kappa(B) & \longrightarrow & \kappa(F) & \longrightarrow & \kappa(F/B) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & F & \longrightarrow & F/B \longrightarrow 0. \end{array}$$

It follows from the five lemma that the inclusion $\kappa(B) \hookrightarrow B$ is an equality: $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is thus closed under subobject.

Let $f : F \rightarrow Q \rightarrow 0$ be an epimorphism of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Consider the following commutative diagram where the vertical arrows are the inclusions:

$$\begin{array}{ccccc} \kappa(F) & \xrightarrow{\kappa(f)} & \kappa(Q) & & \\ \downarrow & & \downarrow & & \\ F & \xrightarrow{f} & Q & \longrightarrow & 0. \end{array}$$

Thus, if $\kappa(F) = F$, then the arrow $\kappa(Q) \hookrightarrow Q$ is also an epimorphism and a fortiori an equality. Hence, $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under quotient.

Finally, let $0 \rightarrow B \rightarrow F \rightarrow Q \rightarrow 0$ be a short exact sequence of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ with $B, Q \in \text{Obj}(\mathcal{S}n(\mathfrak{M}, \mathcal{A}))$. Let m be an object of $\text{Obj}(\mathfrak{M}')_{\natural}$. Let F_m be the pullback of the morphisms $F \rightarrow Q$ and $\kappa_m(Q) \hookrightarrow Q$: F is thus the filtered colimit (with respect to the inclusions) of the pullbacks $\{F_m\}_{m \in \text{Obj}(\mathfrak{M}')_{\natural}}$. Let B_m be the kernel of $F_m \rightarrow \kappa_m(Q)$. Recall that κ commutes with filtered colimits and that filtered colimits in \mathcal{A} are exact (since it is a Grothendieck category). Hence, it is enough to prove that F_m is in $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ for all $m \in \text{Obj}(\mathfrak{M}')_{\natural}$ to show that $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under extension. As κ_m is the kernel of a natural transformation between the identity functor and a left exact functor, $\kappa_m \circ \kappa_m$ is isomorphic to κ_m and therefore $i_m(\kappa_m(Q)) = 0$. By the universal property of the kernel, there exists a unique morphism φ_m such that the following diagram is commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_m & \xrightarrow{\alpha} & F_m & \longrightarrow & \kappa_m(Q) \longrightarrow 0 \\ & & \downarrow i_m(B_m) & \swarrow \varphi_m & \downarrow i_m(F_m) & & \downarrow i_m(\kappa_m(Q))=0 \\ 0 & \longrightarrow & \tau_m(B_m) & \longrightarrow & \tau_m(F_m) & \longrightarrow & \tau_m(\kappa_m(Q)) \longrightarrow 0. \end{array}$$

For all $n \in \text{Obj}(\mathfrak{M}')_{\natural}$, let $\varphi_m^{-1}(\tau_m(\kappa_n(B_m)))$ be the pullback of the morphisms $\varphi_m : F_m \rightarrow \tau_m(B_m)$ and $\tau_m(\kappa_n(B_m)) \hookrightarrow \tau_m(B_m)$. As a pullback commutes with a filtered colimit in an abelian category and since τ_m commutes with filtered colimits, we deduce that

$$\text{Colim}_{n \in \text{Obj}(\mathfrak{M}')_{\natural}} \left(\varphi_m^{-1}(\tau_m(\kappa_n(B_m))) \right) = F_m.$$

In addition, for all $n \in \text{Obj}(\mathfrak{M}')_{\natural}$, the following diagram is commutative:

$$\begin{array}{ccccc} & & F_m & & \\ & \swarrow \varphi_m & \downarrow i_n(F_m) & \searrow i_{n\sharp m}(F_m) & \\ \tau_m(B_m) & \xrightarrow{\tau_m(\alpha)} & \tau_m(F_m) & \xrightarrow{i_n(\tau_m(F_m))} & \tau_n \tau_m(F_m) \cong \tau_{n\sharp m}(F_m) \\ \tau_m(i_n(B_m)) \swarrow & \downarrow i_n(\tau_m(B_m)) & \searrow \tau_n \tau_m(\alpha) & & \\ \tau_n \tau_m(B_m) & \xrightarrow{\cong} & \tau_n \tau_m(B_m) & & \end{array}$$

We deduce from the previous commutative diagram and the universal property of the kernel that there exists an inclusion morphism $\varphi_m^{-1}(\tau_m(\kappa_n(B_m))) \hookrightarrow \kappa_{n \downarrow m}(F_m)$ for all $n \in \text{Obj}(\mathfrak{M}')_{\natural}$. Using the definition of κ as a filtered colimit (see Definition 4.6), we deduce that $\text{Colim}_{n \in \text{Obj}(\mathfrak{M}')_{\natural}} (\varphi_m^{-1}(\tau_m(\kappa_n(B_m))))$ is a subobject of $\kappa(F_m)$. Hence, we have $\kappa(F_m) = F_m$ and thus $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under extension. \square

Remark 4.9. We see here why we require the category \mathcal{A} to have more properties than just being an abelian category: it is necessary for the proof of Proposition 4.8 to assume that the filtered colimits in the category \mathcal{A} are exact, which is the case for a Grothendieck category.

The thickness property of Proposition 4.8 ensures that we can consider the quotient category of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ by $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ (see [17, Chapter III]) as in [12, Definition 1.16].

Definition 4.10. Let $\mathbf{St}(\mathfrak{M}, \mathcal{A})$ be the quotient category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A}) / \mathcal{S}n(\mathfrak{M}, \mathcal{A})$. The canonical functor associated with this quotient is denoted by $\pi_{\mathfrak{M}} : \mathbf{Fct}(\mathfrak{M}, \mathcal{A}) \rightarrow \mathbf{Fct}(\mathfrak{M}, \mathcal{A}) / \mathcal{S}n(\mathfrak{M}, \mathcal{A})$, it is exact, essentially surjective and commutes with all colimits (see [17, Chapter 3]). The right adjoint functor of $\pi_{\mathfrak{M}}$ is denoted by $s_{\mathfrak{M}} : \mathbf{Fct}(\mathfrak{M}, \mathcal{A}) / \mathcal{S}n(\mathfrak{M}, \mathcal{A}) \rightarrow \mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ and called the section functor (see [17, Section 3.1]).

The following proposition introduces the induced translation and difference functors on the category $\mathbf{St}(\mathfrak{M}, \mathcal{A})$. Its proof is analogous to that of [12, Proposition 1.19], using Proposition 4.2.

Proposition 4.11. *Let $m \in \text{Obj}(\mathfrak{M}')_{\natural}$. The translation functor τ_m and the difference functor δ_m of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ respectively induce an exact endofunctor of $\mathbf{St}(\mathfrak{M}, \mathcal{A})$ which commute with colimits, respectively again called the translation functor τ_m and the difference functor δ_m . In addition:*

1. *The following relations hold: $\delta_m \circ \pi_{\mathfrak{M}} = \pi_{\mathfrak{M}} \circ \delta_m$ and $\tau_m \circ \pi_{\mathfrak{M}} = \pi_{\mathfrak{M}} \circ \tau_m$.*
2. *The exact sequence (17) induces a short exact sequence of endofunctors of $\mathbf{St}(\mathfrak{M}, \mathcal{A})$:*

$$0 \longrightarrow \text{Id} \xrightarrow{i_m} \tau_m \xrightarrow{\Delta_m} \delta_m \longrightarrow 0. \quad (19)$$

3. *For another object m' of \mathfrak{M} , the endofunctors $\delta_m, \delta_{m'}, \tau_m$ and $\tau_{m'}$ of $\mathbf{St}(\mathfrak{M}, \mathcal{A})$ pairwise commute up to natural isomorphism.*

We can now introduce the notion of a weak polynomial functor, thus extending that of [12, Definition 1.22].

Definition 4.12. We recursively define on $d \in \mathbb{N}$ the category $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$ of polynomial functors of degree less than or equal to d to be the full subcategory of $\mathbf{St}(\mathfrak{M}, \mathcal{A})$ as follows:

1. *If $d < 0$, $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A}) = \{0\}$;*
2. *if $d \geq 0$, the objects of $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$ are the functors F such that the functor $\delta_x(F)$ is an object of $\mathcal{P}ol_{d-1}(\mathfrak{M}, \mathcal{A})$ for all $x \in \text{Obj}(\mathfrak{M}')_{\natural}$.*

For an object F of $\mathbf{St}(\mathfrak{M}, \mathcal{A})$ which is polynomial of degree less than or equal to $d \in \mathbb{N}$, the smallest natural number $n \leq d$ for which F is an object of $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$ is called the degree of F . An object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is weak polynomial of degree at most d if its image $\pi_{\mathfrak{M}}(F)$ is an object of $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$. The degree of polynomiality of $\pi_{\mathfrak{M}}(F)$ is called the (weak) degree of F .

Let us give some important properties of the categories of weak polynomial functors used in Sections 5 and 6. Their proofs follow mutatis mutandis their analogues in [12, Section 1].

Proposition 4.13. [12, Propositions 1.24-1.26] *For d a natural number, the subcategory $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$ of $\mathbf{St}(\mathfrak{M}, \mathcal{A})$ is thick and closed under limits and colimits. Furthermore, there is an equivalence of categories $\mathcal{A} \simeq \mathcal{P}ol_0(\mathfrak{M}, \mathcal{A})$.*

Finally, we assume that the category \mathfrak{M} is finitely generated by the monoidal structure in $(\mathfrak{M}', \natural, 0)$. We denote by E a finite generating set of \mathfrak{M} . Let F be an object of $\mathbf{St}(\mathfrak{M}, \mathcal{A})$. Then, the functor F is an object of $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$ if and only if the functor $\delta_e(F)$ is an object of $\mathcal{P}ol_{d-1}(\mathfrak{M}, \mathcal{A})$ for all objects e of $E \cap \text{Obj}(\mathfrak{M}')_{\natural}$.

Finally, if the category $(\mathfrak{M}', \natural, 0)$ is symmetric monoidal as in [12], we have an equivalent definition of stably null functor of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Namely, following mutatis mutandis [12, Section 1.2] and the proof of [12, Proposition 1.13], we have:

Lemma 4.14. *We assume that the category $(\mathfrak{M}', \natural, 0)$ is symmetric monoidal and that there exist two objects e and e' of \mathfrak{M}' such that for all objects m of the category \mathfrak{M} , there exists a natural number n such that $m \cong e^{\natural n} \natural e'$. Then, an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is stably null if and only if $\text{Colim}_{n \in (\mathbb{N}, \leq)} \left(F \left(e^{\natural n} \natural e' \right) \right) = 0$. Here, (\mathbb{N}, \leq) is considered as a subcategory of \mathfrak{M} using the functor $(\mathbb{N}, \leq) \rightarrow \mathfrak{M}$ sending a natural number n to $e^{\natural n} \natural e'$ and assigning $\iota_e \natural id_{e^{\natural n} \natural e'}$ to the unique morphism $\gamma_n : n \rightarrow n + 1$.*

Remark 4.15. In some situations, this alternative definition is more convenient than the original one of Definition 4.6. This is the case for example for the proof of Lemma 5.19.

5 Behaviour of the generalized Long-Moody functors on polynomial functors

In this section, we study the effect of some generalized Long-Moody functors on (very) strong and weak polynomial functors. Indeed, under some additional assumptions, they have the property to increase by one both the very strong and the weak polynomial degrees (see Theorems 5.18 and 5.21).

For all the work of this section, we fix a coherent Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ (see Definition 2.11). Let \mathfrak{Gr}_{H, H_0} be the full subcategory of \mathfrak{Gr} of the finite free products on the objects $0_{\mathfrak{Gr}}$, H and H_0 . The free product $*$ defines a symmetric strict monoidal product on \mathfrak{Gr}_{H, H_0} , with $0_{\mathfrak{Gr}}$ the unit. The symmetry of the monoidal structure is given by the canonical bijection $A * B \cong B * A$ which permutes the two terms of the free product, for A and B two objects of \mathfrak{Gr}_{H, H_0} . Let $\mathcal{G}'_{(0,1)}$ be the full subgroupoid of $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ of the finite monoidal products on the objects $0_{\mathcal{G}'}$, 0 and 1 of \mathcal{G}' . Note that the monoidal structure \natural restricts to give $\mathcal{G}'_{(0,1)}$ a braided monoidal structure. We assume that the functors $\mathcal{A} : \mathcal{UG} \rightarrow \mathfrak{Gr}$ and $\zeta : \int^{\mathcal{UG}} \mathcal{A} \rightarrow \mathcal{UG}$ satisfy the following properties:

Assumption 5.1. *There exist two groups H_0 and H , with H non-trivial, such that:*

- for all objects \underline{n} of \mathcal{G} , $\mathcal{A}(\underline{n}) = H^{*n} * H_0$.
- $\mathcal{A}([1, id_{\underline{n+1}}]) = \iota_H * id_{\mathcal{A}(\underline{n})}$ for all natural numbers n (where $\iota_H : 0_{\mathfrak{Gr}} \rightarrow H$ denotes the unique morphism from $0_{\mathfrak{Gr}}$ to the group H).

Moreover, the functor \mathcal{A} extends to define a strict braided monoidal functor $\mathcal{A} : (\mathcal{G}'_{(0,1)}, \natural, 0_{\mathcal{G}'}) \rightarrow (\mathfrak{Gr}_{H, H_0}, *, 0_{\mathfrak{Gr}})$.

Finally, the family of group morphisms $\{\zeta_n : H_n \rightarrow G_{n+1}\}_{n \in \mathbb{N}}$ induced by ζ (see Lemma 2.17) satisfies the equality (10) of Proposition 2.18, namely in G_{n+2} :

$$\left(\left(b_{1,1}^{\mathcal{G}'} \right)^{-1} \natural id_{\underline{n}} \right) \circ (id_1 \natural \zeta_n(h)) = \zeta_{n+1}((\iota_H * id_{H_n})(h)) \circ \left(\left(b_{1,1}^{\mathcal{G}'} \right)^{-1} \natural id_{\underline{n}} \right), \quad (20)$$

for all elements $h \in H_n$, for all natural numbers n .

Remark 5.2. Some of the results presented in Section 5.1 still hold without the hypotheses of Assumption 5.1. However, these additional properties are necessary to prove Proposition 5.5 (see Remark 5.6) and Proposition 5.12 (see Remark 5.10).

Notation 5.3. For all natural numbers m , the free product $H^{*m} * H_0$ is denoted by H_m . Also, we denote by e_H (respectively e_{H_0}) the identity element of the group H (resp. H_0).

Definition 5.4. A coherent Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ is said to be reliable if Assumption 5.1 is satisfied.

We assume that the fixed coherent Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ is reliable. Consequences of Assumption 5.1 will be heavily used in our study. Note that such functors \mathcal{A} and ζ always exist: we can at least consider the functor \mathcal{A}_{id} defined assigning $\mathcal{A}(g) = id_{\mathcal{A}(\underline{n})}$ for all $g \in G_n$ and for all natural numbers n and the trivial functor ζ_t of Section 2.2.5.

We consider the associated Long-Moody functor $\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}}$ (see Theorem 2.12), which is fixed throughout this section (in particular, we omit the “ $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta_n\}$ ” from the notation most of the time). Since the category \mathcal{UG} is generated by the objects 0 and 1 using the monoidal product \natural (see Section 2.1), it is enough for our work to only consider the translation functor τ_1 by Propositions 4.5 and 4.13.

5.1 Relation with evanescence and difference functors

In this section, we first describe the decomposition of the Long-Moody functor $\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}}$ with respect to the translation functor τ_1 (see Corollary 5.14). Then, we establish the crucial results stated in Theorem 5.15, describing the behaviour of the Long-Moody functor $\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}}$ with respect to the evanescence and difference functors.

5.1.1 Factorization of the natural transformation $i_1 \mathbf{LM}$ by $\mathbf{LM}(i_1)$

Recall from Proposition 4.2 the exact sequence in the category of endofunctors of $\mathbf{Fct}(\mathcal{UG}, R\text{-}\mathfrak{Mod})$, which defines the natural transformation i_1 :

$$0 \longrightarrow \kappa_1 \xrightarrow{\Omega_1} Id \xrightarrow{i_1} \tau_1 \xrightarrow{\Delta_1} \delta_1 \longrightarrow 0. \quad (21)$$

As we are interested in the effect of the Long-Moody functor $\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}}$ on (very) strong and weak polynomial functors, our objective is to study the cokernel of the natural transformation $i_1 \mathbf{LM} : \mathbf{LM} \rightarrow \tau_1 \circ \mathbf{LM}$. We recall that for F an object of $\mathbf{Fct}(\mathcal{UG}, R\text{-}\mathfrak{Mod})$, for all natural numbers n , this is defined by the morphisms:

$$(i_1 \mathbf{LM})(F)_n = \mathbf{LM}(F)(\iota_1 \natural id_n) = \mathbf{LM}(F)([1, id_{1+n}]) : \mathbf{LM}(F)(\underline{n}) \rightarrow \tau_1 \mathbf{LM}(F)(\underline{n}).$$

Also, since the associated Long-Moody functor is right-exact (see Proposition 2.6), we have the following exact sequence:

$$\mathbf{LM} \xrightarrow{\mathbf{LM}(i_1)} \mathbf{LM} \circ \tau_1 \xrightarrow{\mathbf{LM}(\Delta_1)} \mathbf{LM} \circ \delta_1 \longrightarrow 0. \quad (22)$$

Moreover, if the groups H_0 and H are free, as the associated Long-Moody functor is then exact (see Corollary 2.22), note that the following sequence is exact:

$$0 \longrightarrow \mathbf{LM} \circ \kappa_1 \xrightarrow{\mathbf{LM}(\Omega_1)} \mathbf{LM} \xrightarrow{\mathbf{LM}(i_1)} \mathbf{LM} \circ \tau_1 \xrightarrow{\mathbf{LM}(\Delta_1)} \mathbf{LM} \circ \delta_1 \longrightarrow 0. \quad (23)$$

First, we prove that the functor $i_1 \mathbf{LM}$ factors through $\mathbf{LM}(i_1)$ (see Proposition 5.7).

Proposition 5.5. *For all $F \in \text{Obj}(\mathbf{Fct}(\mathcal{UG}, R\text{-}\mathfrak{Mod}))$, for all natural numbers n , the following monomorphisms $\mathcal{I}(\underline{n}) \otimes_{R[H_n]} F(\underline{n+2}) \hookrightarrow \tau_1 \mathbf{LM}(F)(\underline{n})$*

$$\zeta(F)_n = \left(\mathcal{I}_{\mathcal{A}}([1, id_{n+1}]) \otimes_{R[H_{1+n}]} F \left(\left(b_{1,1}^{\mathcal{G}'} \right)^{-1} \natural id_n \right) \right)$$

define a natural transformation $\zeta(F) : (\mathbf{LM} \circ \tau_1)(F) \rightarrow (\tau_1 \circ \mathbf{LM})(F)$. This yields a natural transformation $\zeta : \mathbf{LM} \circ \tau_1 \rightarrow \tau_1 \circ \mathbf{LM}$.

Proof. Let n be a natural number. The $R[G_{n+2}]$ -module $F(\underline{n+2})$ is also a $R[H_n]$ -module via $\zeta_{n+1} \circ (\iota_H * id_{H_n}) : H_n \rightarrow G_{n+2}$. The fact that the assignments $\zeta(F)_n$ are well-defined with respect to the tensor product structures of $(\mathbf{LM} \circ \tau_1)(F)(\underline{n})$ and $(\tau_1 \circ \mathbf{LM})(F)(\underline{n})$ is a direct consequence of the relation (20) of Assumption 5.1.

Let us show that $\zeta(F)$ is a natural transformation. Let n and n' be natural numbers such that $n' \geq n$, and $[n' - n, g] \in \text{Hom}_{\mathcal{U}\mathcal{G}}(\underline{n}, \underline{n}')$. Since $\mathcal{I}_{\mathcal{A}}$ is a functor and by the defining equivalence relation of $\mathcal{U}\mathcal{G}'$ (see Definition 1.1), we have:

$$\begin{aligned} \mathcal{I}_{\mathcal{A}}(id_1 \natural [n' - n, g]) \circ \mathcal{I}([1, id_{\underline{n+1}}]) &= \mathcal{I}_{\mathcal{A}}([n' - n + 1, (id_1 \natural g)]) \\ &= \mathcal{I}_{\mathcal{A}}([1, id_{\underline{n'+1}}]) \circ \mathcal{I}([n' - n, g]). \end{aligned}$$

Hence, we deduce that:

$$((\tau_1 \circ \mathbf{LM})(F)([n' - n, g])) \circ (\zeta(F)_{\underline{n}}) = (\zeta(F)_{\underline{n}'}) \circ ((\mathbf{LM} \circ \tau_1)(F)([n' - n, g])).$$

That ζ is a natural transformation follows from the definitions of $\tau_1 \circ \mathbf{LM}$ and $\mathbf{LM} \circ \tau_1$. \square

Remark 5.6. We stress that the condition (20) of Assumption 5.1 is required for Proposition 5.5.

The natural transformation ζ defines desired factorization:

Proposition 5.7. *As natural transformations from \mathbf{LM} to $\tau_1 \circ \mathbf{LM}$, the following equality holds:*

$$\zeta \circ (\mathbf{LM}(i_1)) = i_1 \mathbf{LM}.$$

Moreover, there exists a unique natural transformation $\mathbf{LM} \circ \delta_1 \rightarrow \delta_1 \circ \mathbf{LM}$ such that the following diagram is commutative and the rows are exact sequences in the category of endofunctors of $\mathbf{Fct}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{LM} \circ \tau_1 & \xrightarrow{\zeta} & \tau_1 \circ \mathbf{LM} & \longrightarrow & \text{Coker}(\zeta) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbf{LM} \circ \delta_1 & \longrightarrow & \delta_1 \circ \mathbf{LM} & \longrightarrow & \text{Coker}(\zeta) \longrightarrow 0. \end{array}$$

Proof. Let F be an object of $\mathbf{Fct}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ and n be a natural number. Since $(b_{1,1}^{\mathcal{G}'})^{-1} \circ (\iota_1 \natural id_1) = id_1 \natural \iota_1$ by Definition 1.8, we deduce from Proposition 5.5 that:

$$(\zeta \circ (\mathbf{LM}(i_1)))(F)_{\underline{n}} = \mathcal{I}_{\mathcal{A}}([1, id_{\underline{n+1}}]) \otimes_{R[H_{1+n}]} F(id_1 \natural \iota_1 \natural id_{\underline{n}}) = (i_1 \mathbf{LM})(F)_{\underline{n}}.$$

Then, it follows from the definition of i_1 that the following diagram is commutative and the rows are exact sequences in the category of endofunctors of $\mathbf{Fct}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \kappa_1 \circ \mathbf{LM} & \xrightarrow{\Omega_1 \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_1 \mathbf{LM}} & \tau_1 \circ \mathbf{LM} & \xrightarrow{\Delta_1 \mathbf{LM}} & \delta_1 \circ \mathbf{LM} & \longrightarrow & 0 \\ & & & & \downarrow \mathbf{LM}(i_1) & & \parallel & & & & \\ & & 0 & \longrightarrow & \mathbf{LM} \circ \tau_1 & \xrightarrow[\text{by Lemma 5.7}]{\zeta} & \tau_1 \circ \mathbf{LM} & \xrightarrow{\varrho} & \text{Coker}(\zeta) & \longrightarrow & 0. \end{array}$$

The result is thus a consequence of the universal property of the cokernel. \square

5.1.2 Study of $\text{Coker}(\zeta)$

Let F be an object of $\mathbf{Fct}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ and n be a natural number. From [9, Section 4, Lemma 4.3 and Theorem 4.7] and the distributivity of the tensor product with respect to direct sum, we deduce that we have the R -module isomorphism:

$$\tau_1 \mathbf{LM}(F)(n) \cong \left(\mathcal{I}_{R[H]} \otimes_{R[H]} F(\underline{n+2}) \right) \oplus \left(\mathcal{I}_{\mathcal{A}}(\underline{n}) \otimes_{R[H_n]} F(\underline{n+2}) \right). \quad (24)$$

Recall that the $R[G_{n+2}]$ -module $F(\underline{n+2})$ is a $R[H]$ -module via $\zeta_{n+1} \circ (id_H * \iota_{H_n}) : H \rightarrow G_{n+2}$ and a $R[H_n]$ -module via $\zeta_{n+1} \circ (\iota_H * id_{H_n}) : H_n \rightarrow G_{n+2}$. Hence, the R -module $\text{Coker}(\zeta)(F(\underline{n}))$ is isomorphic to the factor $\mathcal{I}_{R[H]} \otimes_{R[H]} F(\underline{n+2})$.

Notation 5.8. Recall that $t_G : G \rightarrow 0_{\mathfrak{G}\tau}$ denotes the unique morphism from the group G to $0_{\mathfrak{G}\tau}$. Let n and n' be natural numbers such that $n' \geq n$. We denote by $\overline{\mathcal{I}}_{\mathcal{A}}([n' - n, id_{n'}]) : \mathcal{I}_{R[H^{*n}]} \rightarrow \mathcal{I}_{\mathcal{A}}(\underline{n}')$ the R -module morphism induced by the group morphism $id_{H^{*n}} * \iota_{H_{n'-n}} : H^{*n} \rightarrow H_{n'}$ and by $\overline{\mathcal{I}}_{\mathcal{A}}^{-1}([n' - n, id_{n'}]) : \mathcal{I}_{\mathcal{A}}(\underline{n}') \rightarrow \mathcal{I}_{R[H^{*n}]}$ the R -module morphism induced by the group morphism $id_{H^{*n}} * t_{H_{n'-n}} : H_{n'} \rightarrow H^{*n}$.

Considering the natural transformation $\varrho : \tau_1 \circ \mathbf{LM} \rightarrow \text{Coker}(\zeta)$ of Proposition 5.7, it follows from the isomorphism (24) and Proposition 5.7 that for all $F \in \text{Obj}(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{M}\text{od}))$ and for all natural numbers n and n' such that $n' \geq n$:

$$\varrho(F)_{\underline{n}} = \overline{\mathcal{I}}_{\mathcal{A}}^{-1}([n, id_{n+1}]) \otimes_{R[H_{1+n}]} id_{F(\underline{n+2})}.$$

This leads ineluctably to wonder if the isomorphism (24) is functorial.

Identification with a translation functor: First of all, since $\mathcal{A} : (\mathcal{G}'_{(0,1)}, \natural, 0_{\mathcal{G}'}) \rightarrow (\mathfrak{G}\tau_{H, H_0}, *, 0_{\mathfrak{G}\tau})$ is strict braided monoidal (see Assumption 5.1), we deduce the following relations:

Lemma 5.9. *For all natural numbers m and n , for all $g \in G_n$, $\mathcal{A}(id_m \natural g) \circ (id_{H^{*m}} * \iota_{H_n}) = (id_{H^{*m}} * \iota_{H_n})$ and $\mathcal{A}(b_{m,n}^{\mathcal{G}'}) = b_{H^{*m}, H^{*n}}^{\mathfrak{G}\tau_{H, H_0}}$. Moreover, for all natural numbers n' such that $n' \geq n$:*

- $\mathcal{I}_{\mathcal{A}}(id_{n'-n} \natural g) \circ \overline{\mathcal{I}}_{\mathcal{A}}([n' - n, id_{n'}]) = \overline{\mathcal{I}}_{\mathcal{A}}([n' - n, id_{n'}]);$
- $\mathcal{I}_{\mathcal{A}}\left(\left(b_{1, n'-n}^{\mathcal{G}'}\right)^{-1} \natural id_{\underline{n}}\right) \circ \left(\mathcal{I}_{\mathcal{A}}([n' - n, id_{n'+1}]) \circ \overline{\mathcal{I}}_{\mathcal{A}}([n, id_{n+1}])\right) = \overline{\mathcal{I}}_{\mathcal{A}}([n, id_{n+1}]).$

Remark 5.10. These relations will be used in the proof of Proposition 5.12: this highlights the importance to assume that \mathcal{A} is strict braided monoidal in Assumption 5.1.

Now, we can prove that the isomorphism $\text{Coker}(\zeta)(\underline{n}) \cong \mathcal{I}_{R[H]} \otimes_{R[H]} (\tau_2 F)(\underline{n})$ is functorial.

Lemma 5.11. *For F an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{M}\text{od})$, the R -modules $\left\{ \mathcal{I}_{R[H]} \otimes_{R[H]} (\tau_2 F)(\underline{n}) \right\}_{n \in \mathbb{N}}$ assemble to form a functor $\mathcal{I}_{R[H]} \otimes_{R[H]} (\tau_2 F) : \mathfrak{U}\mathcal{G} \rightarrow R\text{-}\mathfrak{M}\text{od}$. Assigning $id_{\mathcal{I}_{R[H]}} \otimes_{R[H]} \tau_2(\eta)$ for any natural transformation η of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{M}\text{od})$, we define an endofunctor:*

$$\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 : \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{M}\text{od}) \rightarrow \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{M}\text{od}).$$

Proof. The result is clear from the functoriality of F . □

For all natural numbers n , let $v(F)_n : \mathcal{I}_{R[H]} \otimes_{R[H]} F(\underline{n+2}) \hookrightarrow \tau_1 \mathbf{LM}(F)(\underline{n})$ be the monomorphism of R -modules $\overline{\mathcal{I}}_{\mathcal{A}}([n, id_{n+1}]) \otimes_{R[H_{1+n}]} id_{F(\underline{n+2})}$. Then:

Proposition 5.12. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{M}\text{od})$. The monomorphisms $\{v(F)_n\}_{n \in \mathbb{N}}$ define a natural transformation $v(F) : \mathcal{I}_{R[H]} \otimes_{R[H]} (\tau_2 F) \rightarrow (\tau_1 \circ \mathbf{LM})(F)$. This yields a natural transformation $v : \mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \rightarrow \tau_1 \circ \mathbf{LM}$.*

Proof. First, we check the consistency of $v(F)$ with respect to the tensor product. Let n and n' be natural numbers such that $n' \geq n$, let $[n' - n, g] \in \text{Hom}_{\mathfrak{U}\mathcal{G}}(\underline{n}, \underline{n}')$ and $h \in H$. Using Lemma 5.9, we deduce that:

$$\mathcal{A}\left(id_{\natural} \natural \left(\left(b_{1, n'-n}^{\mathcal{G}'}\right)^{-1} \natural id_{\underline{n}}\right)\right) \left(id_H * \iota_{H_{1+n'}}\right) = id_H * \iota_{H_{1+n'}} = \mathcal{A}\left(\left(b_{1, n'-n}^{\mathcal{G}'}\right)^{-1} \natural id_{\underline{n}}\right) \circ \left(\iota_{H^{*n'-n}} * id_H * \iota_{H_{1+n}}\right)$$

as morphisms $H \rightarrow H_{2+n'}$. Hence, we deduce from Lemma 2.17 that:

$$\left(id_1 \natural \left(\left(b_{1,n'-n}^{\mathcal{G}'} \right)^{-1} \natural id_{\underline{n}} \right) \right) \circ \zeta_{n'+1} \left(h * e_{H_{1+n'}} \right) \quad (25)$$

$$= \zeta_{n'+1} \left(h * e_{H_{1+n'}} \right) \circ \left(id_1 \natural \left(\left(b_{1,n'-n}^{\mathcal{G}'} \right)^{-1} \natural id_{\underline{n}} \right) \right). \quad (26)$$

Also, it follows from the relation (20) that:

$$\zeta_{n'+1} \left(h * e_{H_{1+n'}} \right) \circ \left(\left(b_{1,n'-n}^{\mathcal{G}'} \right)^{-1} \natural id_{\underline{n}'} \right) = \left(\left(b_{1,n'-n}^{\mathcal{G}'} \right)^{-1} \natural id_{\underline{n}'} \right) \circ \left(id_{n'-n} \natural \zeta_n \left(h * e_{H_{1+n'}} \right) \right) \quad (27)$$

We deduce from the relations (25) and (27) that as morphisms in \mathfrak{UG} :

$$\begin{aligned} & \left[n' - n, \zeta_{n'+1} \left(h * e_{H_{1+n'}} \right) \circ \left(\left(b_{2,n'-n}^{\mathcal{G}'} \right)^{-1} \natural id_{\underline{n}} \right) \right] \\ &= \left[n' - n, \left(\left(b_{2,n'-n}^{\mathcal{G}'} \right)^{-1} \natural id_{\underline{n}} \right) \circ \left(id_{n'-n} \natural \zeta_n \left(h * e_{H_{1+n'}} \right) \right) \right]. \end{aligned} \quad (28)$$

Since by Lemma 2.17

$$(id_2 \natural g) \circ \zeta_{n'+1} \left(h * e_{H_{1+n'}} \right) = \zeta_{n'+1} \left(\mathcal{A} (id_2 \natural g) \left(h * e_{H_{1+n'}} \right) \right) \circ (id_2 \natural g),$$

it follows from Lemma 5.9 that

$$(id_2 \natural g) \circ \zeta_{n'+1} \left(h * e_{H_{1+n'}} \right) = \zeta_{n'+1} \left(h * e_{H_{1+n'}} \right) \circ (id_2 \natural g). \quad (29)$$

Hence, it follows from the combination of the relations (28) and (29) that:

$$(id_2 \natural [n' - n, g]) \circ \zeta_{n+1} \left(h * e_{H_{1+n'}} \right) = \zeta_{n+1} \left(h * e_{H_{1+n'}} \right) \circ (id_2 \natural [n' - n, g]).$$

A fortiori, the assignments $v(F)_n$ are well-defined with respect to the tensor product structures of $\mathcal{I}_{R[H]} \otimes_{R[H]} (\tau_2 F)(n)$ and $(\tau_1 \circ \mathbf{LM})(F)(n)$.

To prove that $v(F)$ is a natural transformation, remark that the relations of Corollary 5.9 imply that:

$$\mathcal{I}_{\mathcal{A}} (id_1 \natural g) \circ \mathcal{I} \left(\left(b_{1,n'-n}^{\mathcal{G}'} \right)^{-1} \natural id_{\underline{n}} \right) \circ \left(\mathcal{I}_{\mathcal{A}} \left([n' - n, id_{\underline{n}+1}] \right) \circ \overline{\mathcal{I}_{\mathcal{A}}} \left([n, id_{\underline{n}+1}] \right) \right) = \overline{\mathcal{I}_{\mathcal{A}}} \left([n, id_{\underline{n}+1}] \right).$$

We then deduce from the definition of the generalized Long-Moody functor that:

$$(\tau_1 \mathbf{LM}(F) ([n' - n, g]) \circ v(F)_n) = v(F)_{n'} \circ \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right) (F) ([n' - n, g]).$$

The proof that v is a natural transformation follows from the definitions of $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ and $\tau_1 \circ \mathbf{LM}$. \square

Remark 5.13. Assume that H is a free group. Let M be a $R[H]$ -module. Since H is free, $\mathcal{I}_{R[H]}$ is a free $R[H]$ -module of rank $\text{rank}(H)$, hence there are isomorphisms of R -modules:

$$\mathcal{I}_{R[H]} \otimes_{R[H]} M \cong \left(\mathcal{I}_{R[H]} \otimes_{R[H]} R \right) \otimes_R M \cong M^{\oplus \text{rank}(H)}.$$

We denote by $\Lambda_{\text{rank}(H), M}$ the composition of these isomorphisms.

Since $\overline{\mathcal{I}_{\mathcal{A}}}^{-1} ([n, id_{\underline{n}+1}]) \circ \overline{\mathcal{I}_{\mathcal{A}}} ([n, id_{\underline{n}+1}]) = id_{\mathcal{I}_{R[H]}}$ for all natural numbers n , $v : \mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \rightarrow \tau_1 \circ \mathbf{LM}$ is a right inverse of the natural transformation $\varrho : \tau_1 \circ \mathbf{LM} \rightarrow \text{Coker}(\xi)$. Hence:

Corollary 5.14. For $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ a reliable Long-Moody system, there is an isomorphism $\text{Coker}(\xi) \cong \mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ as endofunctors of $\mathbf{Fct}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, and there is a natural equivalence of endofunctors of $\mathbf{Fct}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$:

$$\tau_1 \circ \mathbf{LM} \cong \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right) \oplus (\mathbf{LM} \circ \tau_1). \quad (30)$$

Furthermore, if we assume that the groups H_0 and H are free, the isomorphisms $\Lambda_{\text{rank}(H), M}$ of Remark 5.13 provide a natural equivalence $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \cong \tau_2^{\oplus \text{rank}(H)}$.

5.1.3 Key relations with the difference and evanescence functors

This section presents the key commutation relations of the generalized Long-Moody functors with the evanescence and difference functors. Lemma 5.7 and Corollary 5.14 lead to the following result.

Theorem 5.15. Let $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ be a reliable Long-Moody system. There is a natural equivalence in the category $\mathbf{Fct}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$:

$$\delta_1 \circ \mathbf{LM} \cong \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right) \oplus (\mathbf{LM} \circ \delta_1). \quad (31)$$

Moreover, if we assume that the groups H_0 and H are free, then the evanescence endofunctor κ_1 commutes with the endofunctor \mathbf{LM} and the isomorphisms $\Lambda_{\text{rank}(H), M}$ of Remark 5.13 provide a natural equivalence:

$$\delta_1 \circ \mathbf{LM} \cong \tau_2^{\oplus \text{rank}(H)} \oplus (\mathbf{LM} \circ \delta_1). \quad (32)$$

Proof. We denote by $i_{\mathbf{LM} \circ \tau_1}^{\oplus}$ the inclusion morphism $\mathbf{LM} \circ \tau_1 \hookrightarrow \tau_2 \oplus (\mathbf{LM} \circ \tau_1)$. Then, recalling the exact sequence (22), we obtain that the following diagram is commutative and that the two rows are exact:

$$\begin{array}{ccccccc} \mathbf{LM} & \xrightarrow{i_1 \circ \mathbf{LM}} & \tau_1 \mathbf{LM} & \xrightarrow{\Delta_1 \circ \mathbf{LM}} & \delta_1 \circ \mathbf{LM} & \longrightarrow & 0 \\ & & \uparrow \cong \text{by Corollary 5.14} \quad v \oplus \xi & & & & \\ \mathbf{LM} & \xrightarrow{i_{\mathbf{LM} \circ \tau_1}^{\oplus} \circ (\mathbf{LM} \circ i_1)} & \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right) \oplus (\mathbf{LM} \circ \tau_1) & \xrightarrow{id_{\tau_2} \oplus (\mathbf{LM} \circ \Delta_1)} & \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right) \oplus (\mathbf{LM} \circ \delta_1) & \longrightarrow & 0. \end{array}$$

A fortiori, by the universal property of the cokernel and 5-lemma, we deduce that $\tau_2 \oplus (\mathbf{LM} \circ \delta_1) \cong \delta_1 \circ \mathbf{LM}$. Furthermore, assuming that the groups H_0 and H are free, we have the exact sequence (23). We thus obtain the following commutative diagram, in which the two rows are exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \kappa_1 \circ \mathbf{LM} & \xrightarrow{\Omega_1 \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_1 \circ \mathbf{LM}} & \tau_1 \mathbf{LM} \\ & & & & \parallel & & \uparrow \cong \quad v \oplus \xi \\ 0 & \longrightarrow & \mathbf{LM} \circ \kappa_1 & \xrightarrow{\mathbf{LM}(\Omega_1)} & \mathbf{LM} & \xrightarrow{i_{\mathbf{LM} \circ \tau_1}^{\oplus} \circ (\mathbf{LM} \circ i_1)} & \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right) \oplus (\mathbf{LM} \circ \tau_1). \end{array}$$

By the universal property of the kernel, we conclude that $\kappa_1 \circ \mathbf{LM} \cong \mathbf{LM} \circ \kappa_1$. \square

Remark 5.16. Let $m \geq 1$ be a natural number. Assume that the groups H_0 and H are free. Repeating mutatis mutandis the work of Section 5.1, we prove that the evanescence endofunctor κ_m commutes with the Long-Moody functor following the proof of Theorem 5.15. This property is used to prove Lemma 5.19.

5.1.4 Generalizations

All the methods developed in Section 5.1 apply in the following more general context: instead of considering a reliable Long-Moody system as done in this section, we can establish the same kind of behaviour for a tensorial functor

$$\mathcal{I}_{\mathcal{A}} \otimes_{R[\mathcal{A}]} - : R[\mathcal{A}]\text{-}\mathfrak{Mod} \rightarrow \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$$

with respect to the translation functor τ_1 , for a functor $\mathcal{A} : \mathfrak{U}\mathcal{G} \rightarrow \mathfrak{Gr}$ satisfying the corresponding properties of Assumption 5.1. Namely, the analogous decomposition

$$\tau_1 \left(\mathcal{I}_{\mathcal{A}} \otimes_{R[\mathcal{A}]} M \right) \cong \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 M \right) \oplus \mathcal{I}_{\mathcal{A}} \otimes_{R[\mathcal{A}]} (\tau_1 M)$$

holds for M an object of $R[\mathcal{A}]\text{-}\mathfrak{Mod}$, under the assumption that the left module natural transformation $\lambda_M : R[\mathcal{A}] \otimes_R M \rightarrow M$ commutes with the translation functor τ_1 :

$$\lambda_M \circ \left(\tau_1 \otimes_R \tau_1 \right) = \tau_1 \circ \lambda_M.$$

Then, the analogous results of Theorem 5.15 hold for $\mathcal{I}_{\mathcal{A}} \otimes_{R[\mathcal{A}]} M$. This will be developed elsewhere.

5.2 Effect on strong polynomial functors

In this section, we focus on the behaviour of the generalized Long-Moody functor on (very) strong polynomial functors. We recover in particular the results of [33, Section 4] when $(\mathfrak{U}\mathcal{G}', \mathfrak{h}, 0_{\mathcal{G}'}) = (\mathfrak{U}\mathcal{G}, \mathfrak{h}, 0_{\mathcal{G}}) = (\mathfrak{U}\beta, \mathfrak{h}, 0)$. First, we have the following property:

Lemma 5.17. *The functor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ commutes with the difference functor δ_1 . Moreover, if H is free, then $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ commutes with the evanescence functor κ_m for all natural numbers $m \geq 1$.*

Proof. The commutation result with the difference functor δ_1 is a consequence of the right-exactness of the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} - : R\text{-}\mathfrak{Mod} \rightarrow R\text{-}\mathfrak{Mod}$ and of the exactness and the commutation property of the translation functor τ_2 (see Proposition 4.2). Assuming that the group H is free, the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} - : R\text{-}\mathfrak{Mod} \rightarrow R\text{-}\mathfrak{Mod}$ is exact (as a consequence of Lemma 2.21). Hence, the claim follows from the commutation of the evanescence functor κ_m with the translation functor τ_2 (see Proposition 4.2). \square

Theorem 5.18. *Let d be a natural number and F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. Recall that we consider a reliable Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$. If the functor F is strong polynomial of degree d , then:*

- the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ belongs to $\mathcal{P}ol_d^{strong}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$;
- the functor $\mathbf{LM}(F)$ belongs to $\mathcal{P}ol_{d+1}^{strong}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$.

Moreover, if the groups H_0 and H are free and F is very strong polynomial of degree d , then the functor $\mathbf{LM}(F)$ is a very strong polynomial functor of degree equal to $d + 1$.

Proof. By induction on the polynomial degree, the result on $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ follows from Lemma 5.17 and we deduce the first result on $\mathbf{LM}(F)$ using the relation (31) of Theorem 5.15.

Assume now that the groups H_0 and H are free groups. Recall that H is non-trivial. For a very strong polynomial functor F of degree d , an easy induction on the polynomial degree proves that $\tau_2^{rank(H)}(F)$ is very strong polynomial of degree d . A fortiori, the result follows from the relation (32) of Theorem 5.15. \square

5.3 Effect on weak polynomial functors

We investigate the effect on weak polynomial functors of the Long-Moody functor associated with the reliable Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$. The first step of this study consists in defining the Long-Moody functor on the quotient category $\mathbf{St}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. First, note the following property.

Lemma 5.19. *Let F be an object of $\mathbf{Fct}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. Assume that the groups H_0 and H are free, or that the groupoid $(\mathcal{G}', \natural, 0)$ is symmetric monoidal. If the functor F is in $\mathcal{S}n(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, then the functors $\mathbf{LM}(F)$ and $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ are in $\mathcal{S}n(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$.*

Proof. Assume that H_0 and H are free. Recall from Remark 5.16 and Lemma 5.17 that the endofunctors \mathbf{LM} and $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ commute with the evanescence functor κ_m for all natural numbers $m \geq 1$. It follows from Proposition 2.6 and the commutation with all colimits of $\mathcal{I}_{R[H]} \otimes_{R[H]} - : R\text{-}\mathfrak{Mod} \rightarrow R\text{-}\mathfrak{Mod}$ that if F is in $\mathcal{S}n(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, then:

$$\kappa(\mathbf{LM}(F)) = \mathbf{LM}(\kappa(F)) = \mathbf{LM}(F) \quad \text{and} \quad \kappa\left(\mathcal{I}_{R[H]} \otimes_{R[H]} F\right) = \mathcal{I}_{R[H]} \otimes_{R[H]} \kappa(F) = \mathcal{I}_{R[H]} \otimes_{R[H]} F.$$

If one of H_0 or H is not free, the hypothesis that \mathcal{G}' is symmetric monoidal allows Lemma 4.14 to be applied. For all natural numbers n and n' such that $n' \geq n$, recall that $\mathbf{LM}(F)([n' - n, id_{\underline{n}}])$ is the unique morphism induced by the universal property of the tensor product with respect to the map

$$\mathcal{I}_{R[H_n]} \times F(\underline{1+n}) \xrightarrow{\mathcal{I}([n'-n, id_{\underline{n}}]) \times F(id_1 \natural [n'-n, id_{\underline{n}}])} \mathcal{I}_{R[H_{n'}]} \times F(\underline{1+n'}) \xrightarrow{\mathcal{I}_{R[H_{n'}]} \otimes_{R[H_{n'}]} F(\underline{1+n'})} \mathcal{I}_{R[H_{n'}]} \otimes_{R[H_{n'}]} F(\underline{1+n'}).$$

For a fixed natural number n , let $i \in \mathcal{I}_{R[H_n]}$ and let $x \in F(\underline{1+n})$. We assume that F is in $\mathcal{S}n(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. Since the translation functor τ_1 commutes with all the evanescence functors (see Proposition 4.2), $\tau_1 F$ is in $\mathcal{S}n(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. Recall that by Lemma 4.14, $\text{Colim}_{n \in (\mathbb{N}, \leq)} (\tau_1 F(\underline{n})) = 0$. This is equivalent to the fact that for all

natural numbers n , for all $x \in F(\underline{1+n})$, there exists a natural number m_x such that $F(id_1 \natural [m_x - n, id_{\underline{m_x}}])(x) = 0$ and a fortiori:

$$\mathbf{LM}(F)\left([m_x - n, id_{\underline{m_x}}]\right)\left(i \otimes_{R[H_n]} x\right) = 0.$$

Hence, $\text{Colim}_{n \in (\mathbb{N}, \leq)} (\mathbf{LM}(F)(\underline{n})) = 0$. The result for $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ follows using previous argument. \square

From now until the end of Section 5.3, we assume that the groups H_0 and H are free, or that the groupoid $(\mathcal{G}', \natural, 0)$ is symmetric monoidal. By Lemma 5.19, the endofunctors \mathbf{LM} and $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ induce two functors on the quotient category $\mathbf{St}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, denoted by

$$\mathbf{LM}_{\mathbf{St}} : \mathbf{St}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \rightarrow \mathbf{St}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \quad \text{and} \quad \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2\right)_{\mathbf{St}} : \mathbf{St}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \rightarrow \mathbf{St}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod}).$$

The behaviour of the Long-Moody functor of Theorem 5.15 and $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ of Lemma 5.17 with respect to the difference functor remain true for the induced functors in the category $\mathbf{St}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$.

Proposition 5.20. *Let F be an object of $\mathbf{St}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. Then, as objects of $\mathbf{St}(\mathcal{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, there are natural equivalences:*

$$\delta_1 \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2\right)_{\mathbf{St}}(F) \cong \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2\right)_{\mathbf{St}}(\delta_1 F), \quad (33)$$

$$\delta_1 \mathbf{LM}_{\mathbf{St}}(F) \cong \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right)_{\mathbf{St}} (F) \oplus \mathbf{LM}_{\mathbf{St}}(\delta_1 F). \quad (34)$$

Proof. As a consequence of the definitions of the induced difference functor (see Proposition 4.11) and of the induced functors $\left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right)_{\mathbf{St}}$ and $\mathbf{LM}_{\mathbf{St}}$, we have natural equivalences:

$$\delta_1 \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right)_{\mathbf{St}} \cong \left(\delta_1 \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right) \right)_{\mathbf{St}} \quad \text{and} \quad \delta_1 \mathbf{LM}_{\mathbf{St}} \cong (\delta_1 \circ \mathbf{LM})_{\mathbf{St}}.$$

Hence, the result follows from Lemma 5.17 and Theorem 5.15. \square

Theorem 5.21. *Let d be a natural number and F be an object of $\mathbf{Fct}(\mathfrak{UG}, R\text{-Mod})$. Assume that the groups H_0 and H are free, or that the groupoid $(\mathcal{G}', \mathfrak{I}, 0)$ is symmetric monoidal. Assume that F is weak polynomial of degree d . Then the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ is a weak polynomial functor of degree less than or equal to d and the functor $\mathbf{LM}(F)$ is a weak polynomial functor of degree less than or equal to $d + 1$.*

Moreover, if H is free, then the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ is a weak polynomial functor of degree d and the functor $\mathbf{LM}(F)$ is a weak polynomial functor of degree $d + 1$.

Proof. The first result for $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ is a direct consequence of the relation (33) of Proposition 5.20.

If H is a free group, we proceed by induction on the degree of polynomiality of F . Beforehand, note that the isomorphisms $\Lambda_{\text{rank}(H), M}$ of Remark 5.13 provide a natural equivalence $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \cong \tau_2^{\oplus \text{rank}(H)}$. Thus, if the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ is in $\mathcal{Sn}(\mathfrak{UG}, R\text{-Mod})$, then the functor F is in $\mathcal{Sn}(\mathfrak{UG}, R\text{-Mod})$. A fortiori, the

induced functor $\left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right)_{\mathbf{St}}$ is equivalent to the functor $\tau_2^{\oplus \text{rank}(H)}$.

If F is weak polynomial of degree 0, then according to Proposition 4.13, there exists a constant functor C of $\mathbf{St}(\mathfrak{UG}, R\text{-Mod})$ such that $\pi_{\mathfrak{UG}}(F) \cong C$. Hence, we deduce from the above observation that

$$\left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right)_{\mathbf{St}} (C) \cong C^{\oplus \text{rank}(H)}$$

which is a degree 0 weak polynomial functor. Now, assume that F is weak polynomial functor of degree $n \geq 0$. Then, the result follows from the relation (33) of Proposition 5.20 and the inductive hypothesis.

For \mathbf{LM} , we also proceed by induction. Assume that F is a weak polynomial functor of degree 0. So $\pi_{\mathfrak{UG}}(F)$ is a constant functor according to Proposition 4.13. By the equivalence (34) of Proposition 5.20, we have:

$$\delta_1(\pi_{\mathfrak{UG}}(\mathbf{LM}(F))) \cong \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right) (\pi_{\mathfrak{UG}}(F)).$$

According to the result on $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$, this is weak polynomial functor of degree less than or equal to 0, and if H is free the degree is exactly 0. Therefore, $\mathbf{LM}(F)$ is a weak polynomial functor of degree less than or equal to 1. Now, assume that F is a weak polynomial functor of degree $d \geq 1$. By the equivalence (34):

$$\delta_1(\pi_{\mathfrak{UG}}(\mathbf{LM}(F))) \cong \left(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \right) (\pi_{\mathfrak{UG}}(F)) \oplus \mathbf{LM}_{\mathbf{St}}(\delta_1(\pi_{\mathfrak{UG}}(F))).$$

The result follows from the inductive hypothesis and the result on $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$. \square

6 Examples and applications

This last section presents applications of the results of Section 5. Namely, the generalized Long-Moody functors provide very strong and weak polynomial functors in any degree for the families of groups of Section 3. In particular, they give twisted coefficients for which homological stability is satisfied (see Section 6.1) and introduce a new tool for classifying weak polynomial functors with \mathcal{UG} as source category.

6.1 Strong polynomial functors

Proposition 6.1. *The coherent Long-Moody systems of Sections 3.3, 3.4 and 3.5 are reliable (see Definition 5.4).*

Proof. For the families of trivial morphisms $\{\zeta_{n,t}\}_{n \in \mathbb{N}}$, relation (20) of Assumption 5.1 is automatically satisfied. For the families of morphisms $\{\zeta_{n,1}\}_{n \in \mathbb{N}}$ and $\{\zeta_{n,1}^b\}_{n \in \mathbb{N}}$, the equality (15) of Lemma 3.20 implies that relation (20) of Assumption 5.1 is satisfied.

Recall from Lemma 3.7 that the functor $\pi_1(-, p)$ is strict monoidal and it is clear that the symmetry $b_{\pi_1(S,p), \pi_1(S',p)}^{\text{gr}}$ is equal to $\pi_1(b_{S,S'}^{\text{m}_2}, p)$. Hence the functor $\pi_1(-, p)$ is strict braided monoidal and a fortiori Assumption 5.1 is satisfied. The analogous argument holds for the functor \mathcal{A}_Σ which is therefore strict symmetric monoidal on Σ . \square

Hence, applying a Long-Moody functor on the constant functor R , we prove:

Corollary 6.2. *The functors introduced in Examples 3.9, 3.21, 3.21, 3.27 and 3.25 are very strong polynomial of degree one.*

In [32, Section 5], Randal-Williams and Wahl prove homological stability for the families of mapping class groups of surfaces and of symmetric groups considered in Section 3, with twisted coefficients given by very strong polynomial functors. Namely, for all the coherent Long-Moody systems $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ introduced in the examples of Section 3, they show:

Theorem 6.3. [32, Theorem A] *If $F : \mathcal{UG} \rightarrow \mathbb{Z}\text{-Mod}$ is a very strong polynomial functor of degree d , then the canonical maps*

$$H_*(G_n, F(\underline{n})) \rightarrow H_*(G_{n+1}, F(\underline{n+1}))$$

are isomorphisms for $N(, r) \leq n$ with $N(*, r) \in \mathbb{N}$ depending on $*$ and r .*

Remark 6.4. This framework is generalized by Krannich to a topological setting in [25]. Also, for the case of surface braid groups, similar results are established by Palmer in [31]: in this case, the twisted coefficients are functors $\mathcal{B}(\mathbb{R}^2, *) \rightarrow \mathbb{Z}\text{-Mod}$ satisfying a polynomial condition, where $\mathcal{B}(\mathbb{R}^2, *)$ is a certain category with the braid groups as its automorphism groups (in particular, there is a functor $\mathcal{UB} \rightarrow \mathcal{B}(\mathbb{R}^2, *)$ which preserves the polynomial degree).

As the representation theory of mapping class groups of surfaces is wild and an active research topic (see for example [2, Section 4.6], [16] or [23]), there are very few known examples of very strong polynomial functors over \mathcal{UG} . Using Long-Moody functors (and their iterates), we thus construct very strong polynomial functors in any degree for these families of groups. Homological stability is thus satisfied for these functors by Theorem 6.3.

6.2 Weak polynomial functors

By Proposition 4.13, the constant functor R is weak polynomial of degree 0 (as $\pi_{\mathcal{UG}}(R) = R$). Examples of weak polynomial functors for mapping class groups of surfaces are thus given by Theorem 5.21. Namely, applying a Long-Moody functor to the constant functor R we obtain:

Proposition 6.5. *The functors of Corollary 6.2 are weak polynomial of degree one.*

A strong polynomial functor of degree d is always weak polynomial of degree less than or equal to d by the first property of Proposition 4.11. The converse is false (see [12, Example 4.4] for a counterexample). Also, the weak polynomial degree of a strong polynomial functor can be strictly smaller than its strong polynomial degree as the following example shows. Recall from [33, Section 1.3] the functor $\overline{\mathfrak{B}ur} : \mathfrak{U}\beta \rightarrow \mathbb{C}[t^{\pm 1}]\text{-}\mathfrak{Mod}$ which encodes the family of reduced Burau representations.

Proposition 6.6. *The functor $\overline{\mathfrak{B}ur} : \mathfrak{U}\beta \rightarrow \mathbb{C}[t^{\pm 1}]\text{-}\mathfrak{Mod}$ is a strong polynomial functor of degree 2 and weak polynomial of degree 1.*

Proof. The strong polynomial degree result is proved in [33, Proposition 3.28], using the following short exact sequence in $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{C}[t^{\pm 1}]\text{-}\mathfrak{Mod})$:

$$0 \longrightarrow \overline{\mathfrak{B}ur}_t \longrightarrow \tau_1 \overline{\mathfrak{B}ur}_t \longrightarrow R_{\geq 1} \longrightarrow 0,$$

where $R_{\geq 1}$ is the subfunctor of R which is null at 0 and equal to R elsewhere. Since $\pi_{\mathfrak{U}\mathcal{G}}$ is exact, we deduce that:

$$\delta_1(\pi_{\mathfrak{U}\mathcal{G}}(\overline{\mathfrak{B}ur}_t)) \cong \pi_{\mathfrak{U}\mathcal{G}}(R_{\geq 1}).$$

The functor $R_{\geq 1}$ is a subfunctor of a weak polynomial functor of degree 0 and it is not stably null. So, we deduce from Proposition 4.13 that $R_{\geq 1}$ is weak polynomial of degree 0 and therefore the functor $\overline{\mathfrak{B}ur}_t$ is weak polynomial of degree 1. \square

A fundamental reason for the notion of weak polynomial functors to be introduced in [12] is that, contrary to the category $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ (see [33, Remark 3.18]), the category $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$ is localizing (see Proposition 4.13). This allows the quotient categories

$$\mathcal{P}ol_{d+1}(\mathfrak{M}, \mathcal{A}) / \mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$$

to be considered. Remark that as a consequence of Theorem 5.21, we obtain:

Proposition 6.7. *The Long-Moody functor defined by the reliable Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \zeta\}$ induces a functor:*

$$\mathcal{P}ol_d(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) / \mathcal{P}ol_{d-1}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \rightarrow \mathcal{P}ol_{d+1}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) / \mathcal{P}ol_d(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}),$$

if the groups H_0 and H are free, or if the groupoid $(\mathcal{G}', \mathfrak{h}, 0)$ is symmetric monoidal.

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IRMA, UNIVERSITÉ DE STRASBOURG, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE
E-mail address: `soulie@math.unistra.fr`