COMBINATORIAL TANGLE FLOER HOMOLOGY

(joint w/ Ina Petkova & Alex P. Ellis)

knot invariants

polynom | **Alexander polynomial** | **HOMFLY-PT polynomial** | **Jones polynomial**

$\Delta_K(q) = P_K(1,q)$ $\qquad$ $P_K(a,q)$ $\qquad$ $P_K(q,q) = V_K(q)$

$(q^{\frac{1}{2}}+q^{-\frac{1}{2}})^{\ell-1}\Delta_K(q)$ $\qquad$ $P_K(q^N, q)$ $\quad N>0$ $\qquad$ $(q^{\frac{1}{2}}+q^{-\frac{1}{2}})\cdot V_K(q)$

$\chi = \sum(-1)^i q^i \operatorname{rk} V_{i,j}$

$=1$ for $(-3,5,7)$ Pretzel knot

$=1$ for some linked links

$\overset{?}{=}1$ for not linked?

bigraded vectorspace | **knot Floer homology** | **Khovanov-Rozansky homology** | **Khovanov homology**

$\widehat{HFK}(K)$ $\qquad$ $Kh(K)$

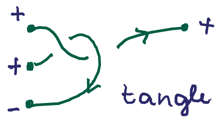
i: Alexander grading $\qquad$ i: Jones grading

j: Maslov grading $\qquad$ j: u-grading

definition holomorphic discs $\qquad$ resolutions

tangle invariants (Tanaev)

tangles:



tangle $T \in Mor(P_0, P_1) \xrightarrow{\text{functor}}$ some category

$P_0 \qquad P_1 \qquad \in Ob(Tan)$

sign-sequence

$U_q(sl_2)$-representations

$\phi \qquad \phi$

V ← vector representation

$\otimes$

$V \xrightarrow{\mathcal{J}(T)} V$ $\qquad$ Reshetikhin-Turaev intertwiner

$\otimes$

V^* $\qquad$ $\cdot \otimes C_{Kh}(T)$

$\mathcal{J}(T_1 \circ T_2) = \mathcal{J}(T_1) \circ \mathcal{J}(T_2)$ $\qquad$ $\mathbb{C}(q) \xrightarrow{\cdot V_K(q)} \mathbb{C}(q)$

Grothendieck group $\qquad$ finitely generated projective modules / $[A]-[B]+[C]=0$ if $0\to A\to B\to C\to 0$ $\qquad$ χ

Khovanov

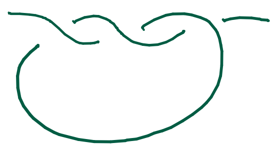
$H_{P_0} \quad C_{Kh}(T) \quad H_{P_1}$ $\qquad$ bimodule

graded ring

$C_{Kh}(T_1 \circ T_2) = C_{Kh}(T_1) \otimes C_{Kh}(T_2)$ $\qquad$ $\mathbb{C}(q) \xrightarrow{Kh(K)} \mathbb{C}(q)$

$U_q(gl_{1|1})$-representations

quantum enveloping algebra of the Lie superalgebra $gl_{1|1}$



V ← vector representation

$\otimes$

$V \xrightarrow{Q(T)} V$

$\otimes$

V^* $\qquad$ $\cdot \boxtimes CT(T)$

$Q(T_1 \circ T_2) = Q(T_1) \circ Q(T_2)$ $\qquad$ O $\qquad$ $V \xrightarrow{\cdot \Delta_K(q)} V$ $\qquad$ Schur's Lemma

$V_{P_0} = $ Grothendieck group

a model for derived tensor product $\qquad$ (Ellis - Petkova - V)

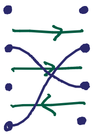
$\mathcal{A}_{P_0} \quad CT(T) \quad \mathcal{A}_{P_1}$

$CT(T_1 \circ T_2) = CT(T_1) \boxtimes CT(T_2)$ $\qquad$ shifted $CFK(K) \otimes (\mathbb{F}_2 \oplus \mathbb{F}_2)$, with gradings $-\frac{1}{2}$ and $\frac{1}{2}$ on the two $\mathbb{F}_2$ summands, $= CFK(K \cup U)$ $\qquad$ (Petkova - V)

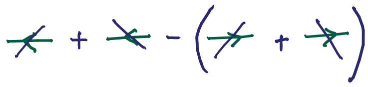
type DA - bimodule

DG-algebra

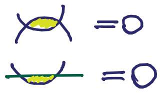
The algebra $\mathcal{A}(P)$: for $P = (+,+,-)$

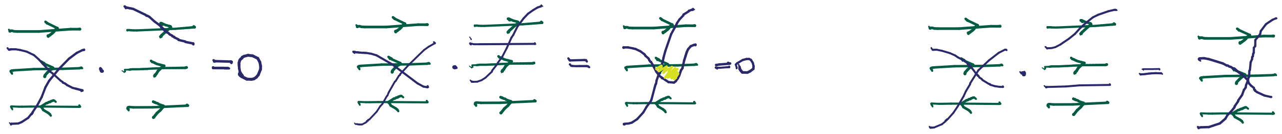


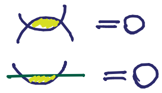
generators: partial bijections over $\mathbb{F}_2$

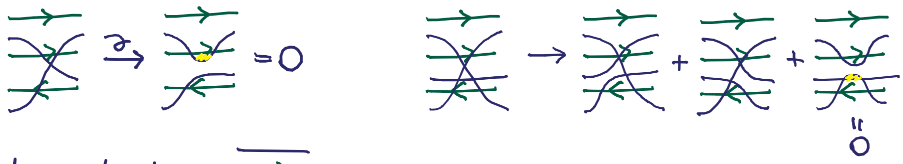
grading: $2A =$ 

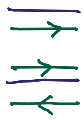
$M =$ 

multiplication: concatenation + pull it tight + relations: 



differential: resolving X + pull it tight + relations 



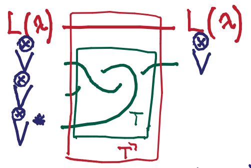
idempotents  $e_s \quad s \in 2^4$

Grothendieck group of compact derived category of left DG-modules over $\mathcal{A}(P)$

Thm: (Ellis–Petková–V): $K_0(\mathcal{A}(P))$ is generated by $[\mathcal{A}(P)e_s]$

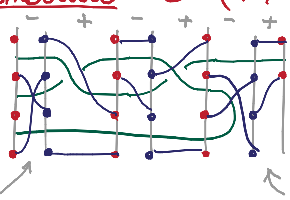
sign sequence $\rightsquigarrow$ λ integral weight for $\mathcal{U}_q(\mathfrak{gl}_{1|1})$ $\quad \langle h_1+h_2, \lambda\rangle = 1-\Sigma P$

$K_0(\mathcal{A}(P)) \otimes_{\mathbb{Z}[q^{\pm1}]} \mathbb{C}(q) \cong V_P \otimes L(\lambda)$



$K_0(\mathcal{A}(P_0)) \xrightarrow{\cdot \boxtimes CT(T)} K_0(\mathcal{A}(P_1)) \longleftrightarrow V_{P_0} \otimes L(\lambda) \xrightarrow{Q(T')} V_{P_1} \otimes L(\lambda)$

the bimodule $CT(T)$



+ : 

− : 

generators: sequences of partial matchings $\underline{x} = (x_1, x_2, \dots, x_6)$ s.t.

$$\text{image}(x_i) = S_i - \text{domain}(x_{i+1})$$

grading: $2A =$ 

on +: $M =$ 

on −: $M =$ 

right action: concatenation w/ algebra + pull tight + relations

left action: —

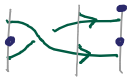
differential: $\partial \underline{x} = (\partial x_1, x_2, \dots, x_6) + (x_1, \partial x_2, \dots, x_6) + \dots + (x_1, x_2, \dots, \partial x_6) + \partial_{MIX}$

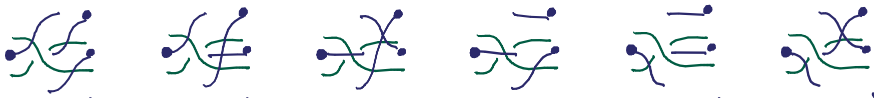
where ∂ on + is resolution of X

on − is introduction of X

can't loose crossings at +

can't gain crossin

Example: 



$(M,2A)$	$(-1,-2)$	$(-2,-4)$	$(-1,-2)$	$(0,0)$	$(-1,-2)$	$(0,-2)$	
	q^{-2}	q^{-4}	q^{-2}	1	q^{-2}	q^{-2}	$= 1+q^{-4}-2q^{-2} = (1-q^{-2})^2$

$$\Rightarrow \quad e_{\{0,2\}}\, CT(\chi)\, e_{\{1,2\}} = (1-q^{-2})^2$$

up to a factor of $(1-q^{-2})^2$:

$$e_\phi\, CT(\chi)\, e_\phi = (-q) \qquad \begin{array}{c|ccc} & e_0 & e_1 & e_2 \\ \hline e_0 & -q & 1 & 0 \\ e_1 & 0 & q^{-1} & 0 \\ e_2 & 0 & 1 & -q \end{array} \qquad \begin{array}{c|ccc} & e_{01} & e_{02} & e_{12} \\ \hline e_{01} & q^{-1} & 0 & 0 \\ e_{02} & 1 & -q & 1 \\ e_{12} & 0 & 0 & q^{-1} \end{array} \qquad e_{012}\, CT(\chi)\, e_{012} = (q^{-1})$$

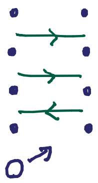
$U_q(\mathfrak{gl}_{1|1})$-action:

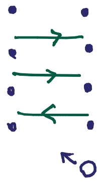
Thm (Ellis–Petkova–V) P sign sequence $\rightsquigarrow$ DG-bimodules $E(P), F(P)$ s.t.

- $\tilde{\otimes}\, E(P)$ on $K_0(\mathcal{A}(P))$ is E
- $\qquad\qquad\qquad\qquad\qquad\qquad F$
- $E(P) \tilde{\otimes} E(P) \simeq 0$
- $F(P) \tilde{\otimes} F(P) \simeq 0$
- there is a distinguished triangle:

$$\begin{tikzcd} E(P)\tilde{\otimes}F(P) \arrow[rr] & & A(P) \arrow[dl] \\ & F(P)\tilde{\otimes}E(P) \arrow[ul, "1"] & \end{tikzcd}$$

- $E(P_0) \tilde{\otimes} CT(T) \simeq \mathcal{A}(P_0) \boxtimes CT(T) \otimes E(P_1)$
- $F(P_0) \otimes CT(T) \simeq \mathcal{A}(P_0) \quad CT(T) \otimes F(P_1)$

$F(P)$ 

$E(P)$ 

$\mathfrak{gl}_{1|1}$: $= \mathrm{End}(\mathbb{C}^{1|1})$

↖ supervectorspace w gen. u_0, u_1 $\quad |u_0| = 0 \quad |u_1| = 1$

$$\underline{h}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \underline{h}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \underline{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \underline{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$|\underline{h}_1| = |\underline{h}_2| = 0 \qquad |\underline{e}| = |\underline{f}| = 1$

$[a,b] = ab - (-1)^{|a||b|} ba \qquad$ Cartan subalgebra $\langle \underline{h}_1, \underline{h}_2 \rangle = P^*$

$[\cdot,\cdot]$	$\underline{h}_1$	$\underline{h}_2$	$\underline{e}$	$\underline{f}$
$\underline{h}_1$	0	0	$\underline{e}$	$-\underline{f}$
$\underline{h}_2$		0	$-\underline{e}$	$\underline{f}$
$\underline{e}$			0	$\underline{h}_1 + \underline{h}_2$
$\underline{f}$				0

$U_q(\mathfrak{gl}_{1|1})$: superalgebra over $\mathbb{C}(q)$ generated by : E, F odd

$\{ q^{\underline{h}} : \underline{h} \in \mathbb{Z}^2 \}$ even, $\quad \underline{h} = (h_1, h_2)$

subject to the relations :

$$q^{\underline{0}} = 1$$
$$q^{\underline{h}} q^{\underline{h}'} = q^{\underline{h} + \underline{h}'}$$
$$q^{\underline{h}} E = q^{h_1 - h_2} E \, q^{\underline{h}}$$
$$q^{\underline{h}} F = q^{h_2 - h_1} F \, q^{\underline{h}}$$
$$EF + FE = \frac{q^{(1,1)} - q^{-(1,1)}}{q - q^{-1}}$$

representations $L(\lambda)$: $\lambda \in \mathbb{C}^2 \qquad \lambda_1 \neq \lambda_2$

$\lambda = (\lambda_1, \lambda_2) \qquad (\varepsilon_1, \varepsilon_2)$

duals to $\underline{h}_1$ & $\underline{h}_2$

generated by v_0^λ, v_1^λ :

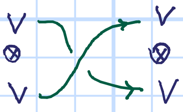
$$E v_0 = 0 \qquad F v_0 = \frac{q^{\lambda_1 + \lambda_2} - q^{-(\lambda_1 + \lambda_2)}}{q - q^{-1}} v_1$$
$$E v_1 = v_0 \qquad F v_1 = 0$$

$$q^{h_1} v_0 = q^{\lambda_1} v_0 \qquad q^{h_1} v_0 = q^{\lambda_1 - 1} v_0$$
$$q^{h_2} v_0 = q^{\lambda_2} v_0 \qquad q^{h_2} v_1 = q^{\lambda_2 - 1} v_1$$

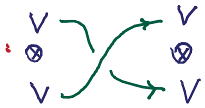
vectorrepresentation : $V = L(\varepsilon_1)$

$V^* = L(\varepsilon_2)$

interwiners :

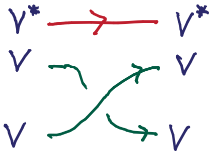


$$\begin{array}{c|c} & \begin{matrix} V_{00} & V_{01} & V_{10} & V_{11} \end{matrix} \\ \begin{matrix} V_{00} \\ V_{01} \\ V_{10} \\ V_{11} \end{matrix} & \begin{pmatrix} q^{-1} & & & \\ & & 1 & \\ & 1 & q^{-1} - q & \\ & & & -q \end{pmatrix} \end{array}$$



$$\begin{array}{c|cccc} & V_{00} & V_{01} & V_{10} & V_{11} \\ \hline V_{00} & q^{-1} & & & \\ V_{01} & & & 1 & \\ V_{10} & & 1 & q^{-1}-q & \\ V_{11} & & & & -q \end{array}$$

$\Sigma P = 1+1$ need λ : $\langle \underline{h}_1 + \underline{h}_2, \lambda \rangle = 1 - \Sigma P = -1 \Rightarrow \lambda = -\varepsilon_2$ is good



$$\begin{array}{c|cccccccc} & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ \hline 000 & q^{-1} & & & & & & & \\ 001 & & & 1 & & & & & \\ 010 & & 1 & q^{-1}-q & & & & & \\ 011 & & & & -q & & & & \\ 100 & & & & & q^{-1} & & & \\ 101 & & & & & & & 1 & \\ 110 & & & & & & 1 & q^{-1}-q & \\ 111 & & & & & & & & -q \end{array}$$