TRANSVERSE INVARIANTS IN HEEGAARD FLOER HOMOLOGY

(joint work with J. Baldwin & D. S. Vela-Vick) WORK IN PROGRESS!!

<u>Thm</u>: T transverse knot $\to$ $\mathfrak{T}^-(T) = \vartheta^-(T)$

LOSS-invariant; invariant in combinatorial knot ($OSzT$)

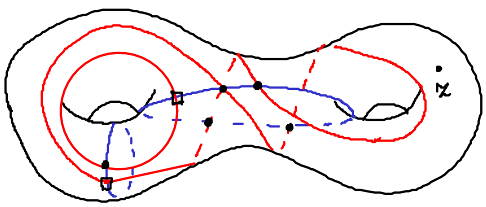
THE CONTACT INVARIANT IN HEEGAARD FLOER HOMOLOGY

<u>Heegaard Floer homology</u> (Ozsváth - Szabó)

Y closed 3-manifold $\rightsquigarrow$ $\widehat{HF}(Y) = \bigoplus_{s \in Spin^c(Y)} \widehat{HF}(Y, s)$ $\wedge$ $\mathbb{Z}_2$-vectorspace

Heegaard diagram

Homology (independent of all choices)



$\rightsquigarrow$ $\widehat{CF}$ is generated by g-tuples of intersection points one on each α- & β-curve

each intersection point has a Spin^c-structure

$\hat{\partial}$ counts holomorphic discs in $Sym^g(\Sigma)$ missing z

$$\hat{\partial}\,\underline{x} = \sum_{\substack{\phi \in \pi_2(\underline{x},\underline{y}) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} |\hat{\mathcal{M}}(\phi)|\,\underline{y}$$

(relative) Maslov grading $\mu(\underline{x}) - \mu(\underline{y}) = \mu(\phi) - 2n_z(\phi)$

<u>Knot Floer homology</u> (Ozsváth - Szabó, Rasmussen)

puting an extra basepoint on the Heegaard diagram describes a knot K in Y

w gives an extra filtration on $\widehat{CF}$:

$$\partial_K^- \underline{x} = \sum_{\substack{\phi \in \pi_2(\underline{x},\underline{y}) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} |\hat{\mathcal{M}}(\phi)|\, U^{n_w(\phi)}\, \underline{y} \rightsquigarrow HFK^-(Y,K)$$

$\mathbb{Z}_2[U]$-module

(Alexander grading $A(\underline{x}) - A(\underline{y}) = n_z(\phi) - n_w(\phi)$ $\phi \in \pi_2(x,y)$)

setting $U=0$: $\hat{\partial}_K\, \underline{x} = \sum_{\substack{\phi \in \pi_2(\underline{x},\underline{y}) \\ \mu(\phi)=1 \\ n_z(\phi)=n_w(\phi)=0}} |\hat{\mathcal{M}}(\phi)|\, \underline{y} \rightsquigarrow \widehat{HFK}(Y,K)$

$\mathbb{Z}_2$-vectorspace

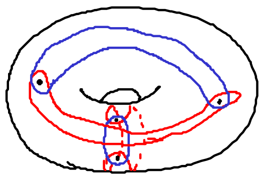
<u>Rmk</u>: A is well-defined only if K bounds a surface S in Y,

then we can do 0-surgery on K: $Y_0(K)$ if $\hat{S}$ is the capped surface

then $A(\underline{x}) = \langle c_1(\underline{s}_{\underline{x}}), [\hat{S}] \rangle$

Link Floer homology (Ozsváth - Szabó)

if we want to describe a link on a Heegaard diagram we need to introduce more basepoints: $2 \times (g + k \text{ basepoints})$



(multi Alexander filtration $A_i(\underline{x}) - A_i(\underline{y}) = (n_{z_i}(\phi) - n_{w_i}(\phi))$ $\phi \in \pi_2(\underline{x}, \underline{y})$
or we can add this up for different components)

if a sublink bounds the surface F then $\sum A_i(\underline{x}) = \langle c_1(s_{\underline{x}}), [\hat{F}] \rangle$

Contact structure $\xi = \ker \alpha$ 2 plane field so that $\alpha \wedge d\alpha > 0$

Open book $B \subset Y$ s.t $\begin{array}{c} Y - B \\ \downarrow F \\ S^1 \end{array}$ $\partial F_p = B$

can be described with F & φ (monodromy)

every closed 3-manifold admits an open book decomposition (Alexander)

an open book is compatible with a contact structure if

$\alpha > 0$ on B $(B \pitchfork \xi)$ (Giroux)
$d\alpha > 0$ on F ($d\alpha$ is a volume form on F)

Thm (Giroux, Thurston - Winkenkempler)

open book / positive stab $\longleftrightarrow$ contact structure / isotopy

The above Thm gives a good way to define invariants of ctct structures:

OSz: HKM

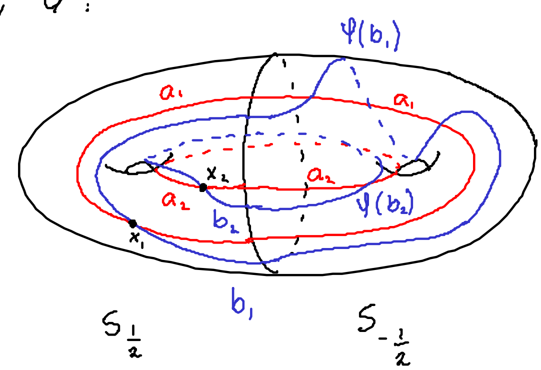
Given ξ pick a compatible open book (S, φ) with connected binding B

Thm: the bottommost filtration level $(-g)$ of $HFK(Y, B)$ is $\mathbb{Z}_2$

the image of its generator in $\widehat{HF}(Y)$ is independent of the open book decomposition choosen for ξ

& it is called the contact invariant $c(\xi)$

(S, φ) gives a Heegaard decomposition for Y:



Thm $[\underline{x} = (x_1, \dots, x_{2g})] \in \widehat{HF}(Y)$ is independent of the above choices

$x = c(\xi)$

Properties of the contact invariant

ξ overtwisted $\Rightarrow c(\xi) = 0$

ξ Stein fillable $\Rightarrow c(\xi) \neq 0$

ξ contains Giroux torsion $\Rightarrow c(\xi) = 0$ (Ghiggini - Honda - Van-Horn Morris)

Transverse / Legendrian Invariant in Heegaard Floer Homology

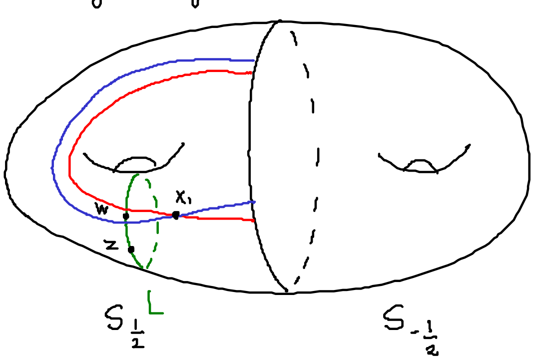
$L \subset Y$ is Legendrian if $T_pL \in \xi_p \quad \forall p \in L$

$T \subset Y$ is transverse if $T_pT \pitchfork \xi_p \quad \forall p \in T$ (the binding of an open book, Reeb orbits)

transverse knots / transverse isotopy $\longleftrightarrow$ Legendrian knots / Legendrian isotopy negative stabilization

Lisca - Ozsváth - Stipsicz - Szabó

place the Legendrian knot on the page of an open book so that it is homologically nontrivial



Thm: $[\underline{x} = (x_1, \dots, x_g)] \in \widehat{HFK}^-(Y, K)$ is independent of the above choices thus defines an invariant for Legendrian knots $\mathcal{L}^-(L)$

moreover $\hat{\mathcal{L}}^-(L)$ is invariant under negative stabilization giving an invariant for transverse knots: $\hat{\tau}^-(T)$

Properties of $\hat{\mathcal{L}}^-(L) / \hat{\tau}^-(T)$

$Y - L$ is overtwisted $\Rightarrow \hat{\mathcal{L}}(L) = 0$, $\mathcal{L}^-(L) \in \operatorname{im} U$

$Y - L$ contains Giroux torsion $\Rightarrow \hat{\mathcal{L}}(L) = 0$ (Stipsicz - V, Vela - Vick)
$\Rightarrow \mathcal{L}^-(L) \in \operatorname{im} U$ (Vela - Vick)

$\hat{\tau}(T) \neq 0$ for the binding of an open book (Vela - Vick)

A new transverse invariant

going to be an invariant of transverse braids.

Alexander theorem for transverse knots (Elena Pavelescu): Given (S,φ) & T transverse knot. T is transverse isotopic to a braid $B \pitchfork S$. (transverse braid)

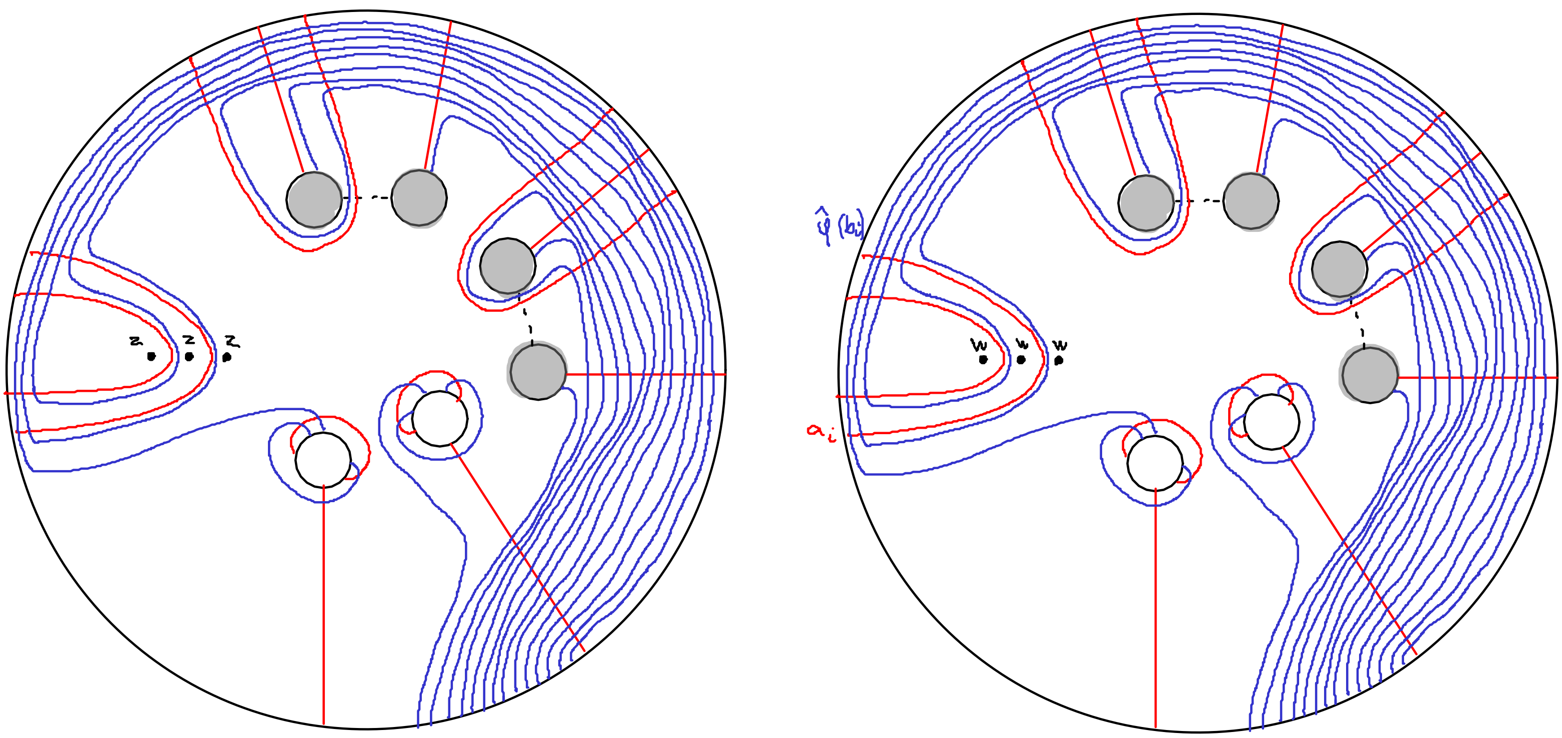
she also proved a Markov theorem for transverse knots, thus we have a complete understanding of transverse braids that give the same transverse knot T

Put T in transverse braid position for any open book (S,φ) compatible with ξ then the binding B gives an extra filtration on $HFK^-(K)$ and

Thm (Baldwin - Vela-Vick - V) The bottommost filtration level of $HFK^-(K)$ with respect to B is $\mathbb{Z}_2$. The image of its generator is independent of our choices. Thus it gives an invariant of transverse knots $t^-(T)$.

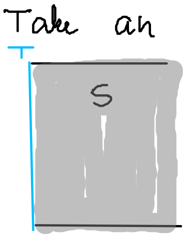
A construction for the Heegaard diagram

Given a transverse braid B in the open book (S,φ) that intersects the pages in n pts. Take a lift $\hat{\varphi}: (S,\{p_1,\dots,p_n\})^{\circlearrowleft}$ of φ so that the cylinder of $\{p_1,\dots,p_n\}$ gives B. Then one can take the Heegaard diagram:

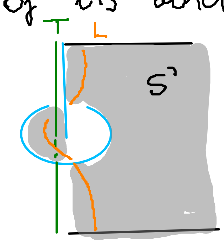


Thm (Baldwin - Vela-Vick - V) $t^-(T) = \tau^-(T)$

Proof: Take an open book so that one of its binding components is T



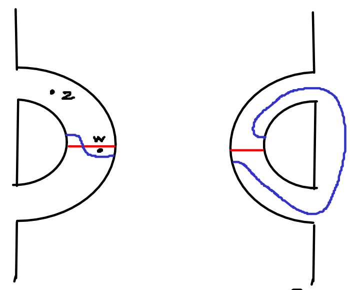
and stabilize it



here T is a transverse braid that intersects the pages in 1 pt & its Legendrian approximation L lies on S^3

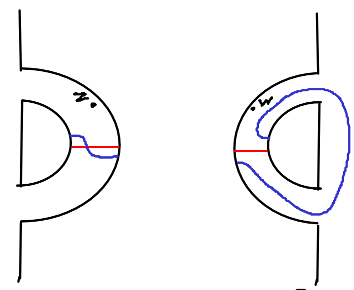
thus we can draw Heegaard diagrams corresponding to both invariants:

for $\mathcal{L}^-(L)$ (or $\vartheta^-(T)$)



$S_{\frac{1}{2}}$ $S_{-\frac{1}{2}}$

for $t^-(T)$

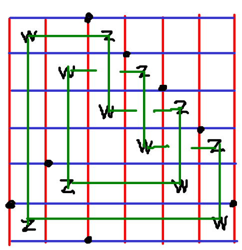


$S_{\frac{1}{2}}$ $S_{-\frac{1}{2}}$ $\square$ T

these are the same $\square$

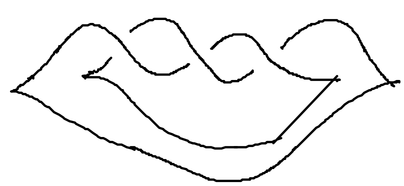
Ozsváth - Szabó - Thurston in (S^3, ξ_{st})

grid diagram



$\rightsquigarrow$

Legendrian knot



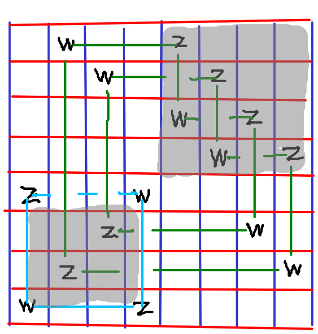
there is a grid diagram for every Legendrian knot

$\underline{x}$:= upper left corner of the z's

Thm: $[\underline{x}] \in \widehat{HFK}^-(K)$ is independent of the grid representation of the Legendrian knot. Thus we get a Legendrian invariant $\hat{\lambda}^-(L)$

Moreover $\hat{\lambda}^-(L)$ is unchanged under negative stabilization. $\rightsquigarrow \hat{\vartheta}^-(T)$

Let B be a transverse braid in (D^2, id). Take a grid diagram for a Legendrian approximation of T that contains the braid axis:



the filtration of a generator with respect to the axis can be computed as the winding # of the axis.

Thus the bottommost generators must be in the shaded region. Each part is generated by 1 element. Thus:

Thm (Baldwin - Vela-Vick - V) $\mathcal{L}^-(T) = t^-(T) = \lambda^-(T)$