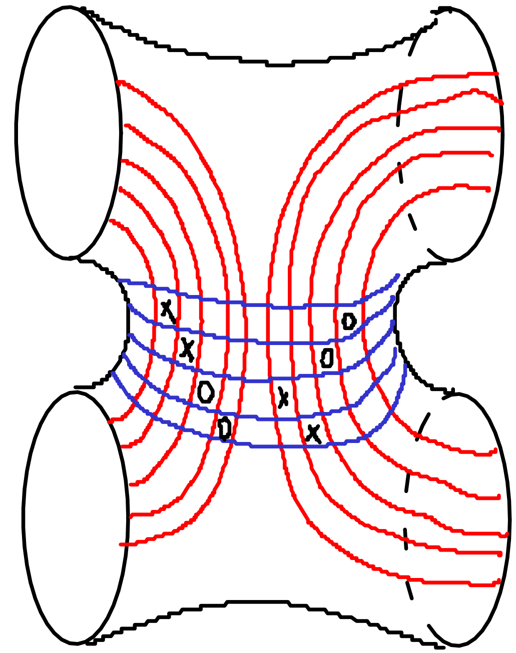
Tangle Floer Homology



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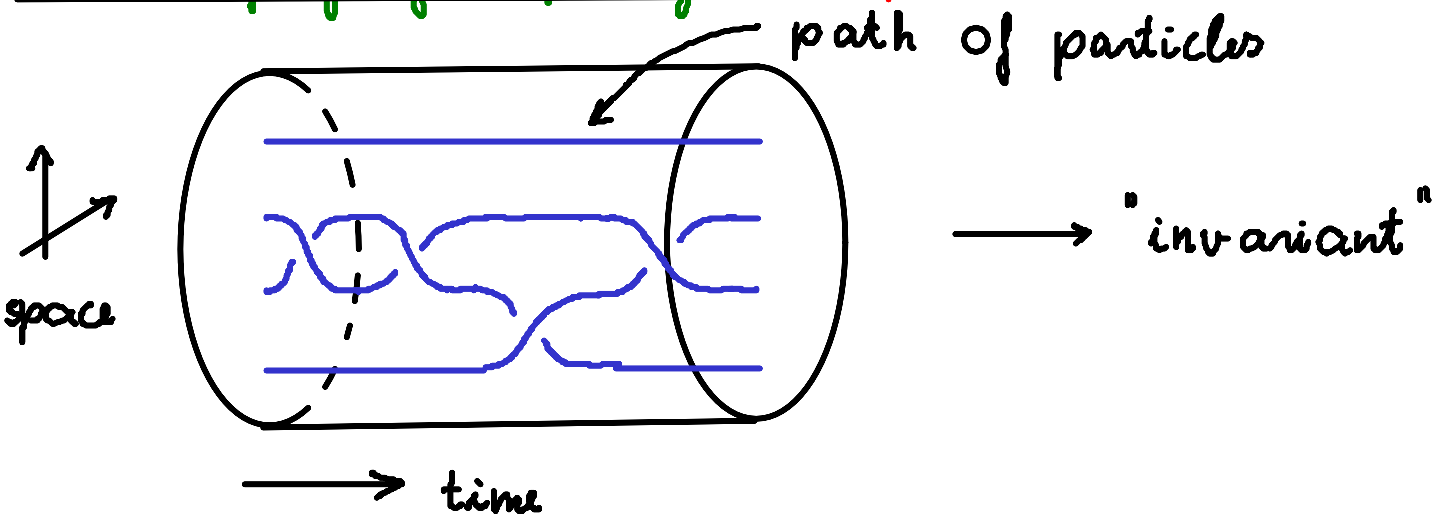
Joint with:

Ina Petkova

TQFT

Goal: Define a 0+1 dimensional embedded TQFT that recovers knot Floer homology (HFK)

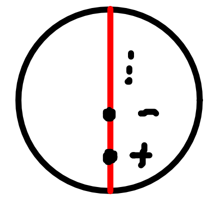
Philosophy of Topological Quantum Field Theories



Would like to compute the invariant as time progresses..

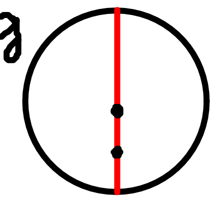
Formulated by Turaev in 1990

Category of Tangles

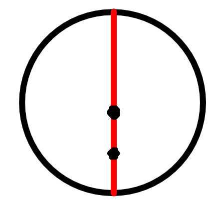
Objects: sequence of +/-'s : $\sigma \in \{+,-\}^n \longleftrightarrow$ 

σ

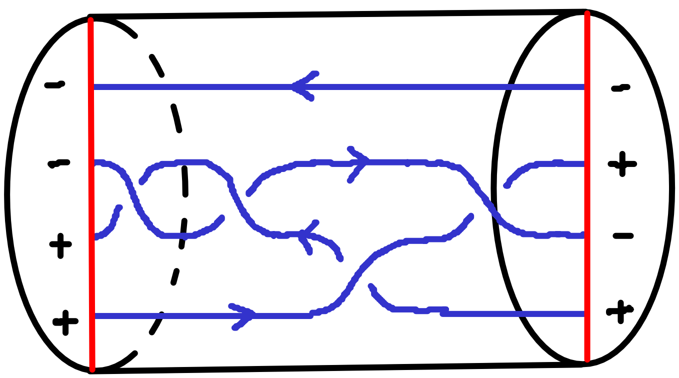
Morfisms: $\hom(\sigma_0,\sigma_1) = \sqcup I \sqcup \sqcup S^1 \xhookrightarrow{\iota} D^2 \times I$

- ι is a proper embedding
- $\iota(\sqcup I \sqcup \sqcup S^1) \cap D^2 \times \{0\} =$  σ_0

&

- $\iota(\sqcup I \sqcup \sqcup S^1) \cap D^2 \times \{1\} =$  σ_1

/ rel ∂ isotopy



$\in \hom(++--, +-+-)$

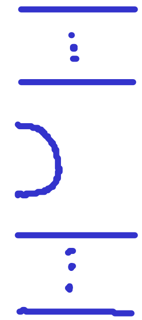
knots & links $\in \hom(\phi, \phi)$

a TQFT is a functor Tangle $\longrightarrow$ some category

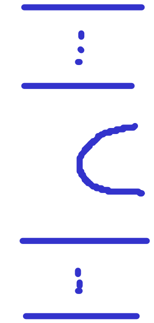
in our case $\mathcal{DGM}$ = differential graded bimodules

Category of Tangles – Generators & Relations

crossings



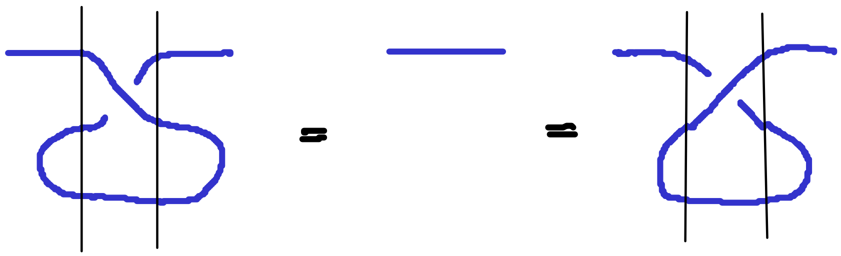
cap



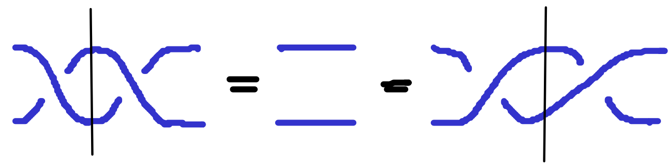
cup

relations:

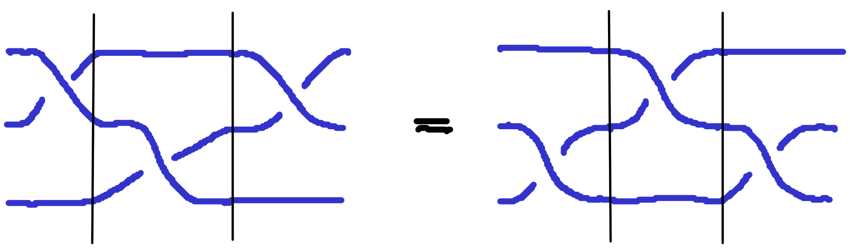
R1:



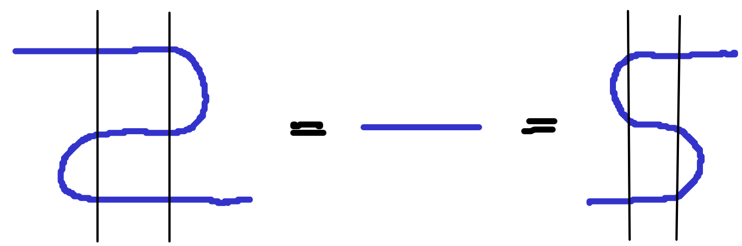
R2:

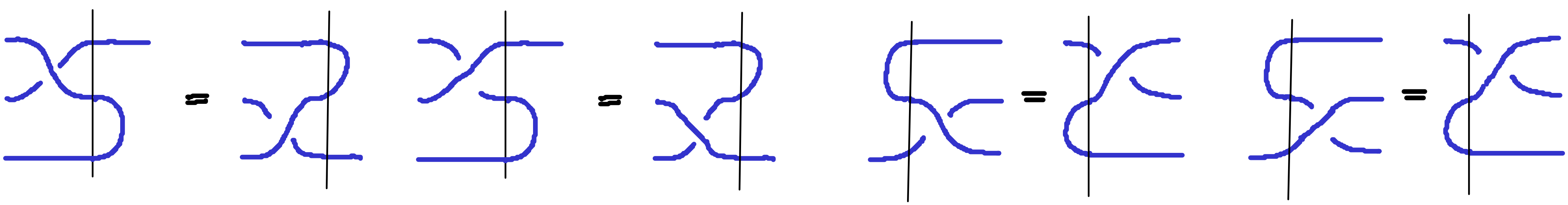


R3:



zig-zag





What is Heegaard Floer Theory?

$\mathbb{F} = \mathbb{Z}$ or $\mathbb{Z}/(2)$

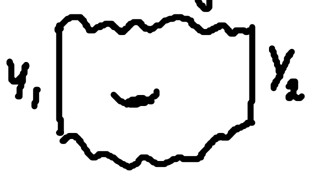
Y 3-manifold $\rightsquigarrow$ $\widehat{HF}(Y)$ graded $\mathbb{F}$-module

$HF^-(Y)$ $\mathbb{F}[U]$-module

$HF^\infty(Y)$ $\mathbb{F}[U,U^{-1}]$-module (determined by $H^*(Y)$)

$HF^+(Y)$ $\mathbb{F}[U^{-1}]$-module

W cobordism btwn 3-manifolds $\rightsquigarrow$ $\hat{F}_W^{+,-,\infty} : \widehat{HF}^{+,-,\infty}(Y_1) \to \widehat{HF}^{+,-,\infty}(Y_2)$



X 4-manifold $\rightsquigarrow$ a number for every spin^c-structure

$K \subset Y$ knot $\rightsquigarrow$ $\widehat{HFK}(Y,K)$ bigraded $\mathbb{F}$-module

$HFK^-(Y,K)$ $\mathbb{F}[U]$-module

Why is Heegaard Floer Theory Useful?

Geometric content

- Ozsváth - Szabó: Detects smooth structures on 4-manifolds
- Ozsváth - Szabó, Ni: Detects the genus of knots
  Thurston norm of 3-manifolds
- Ozsváth - Szabó, Ghiggini, Ni, Juhász,...
  Detects fiberedness of knots and 3-manifolds
- Ozsváth - Szabó,... Bounds the slice genus
  minimal class representatives of homology classes

Computability

- defined using a PDE but sometimes can be combinatorial:
- Manolescu - Ozsváth - Sarkar: $\widehat{HFK}^-$ for knots
- Sarkar - Wang, Ozsváth - Stipsicz - Szabó: $\widehat{HF}(Y)$, easier version of HF^-
- Manolescu - Ozsváth - Thurston: $\widehat{HF}^{\pm,\infty}$, 4-manifold invariant

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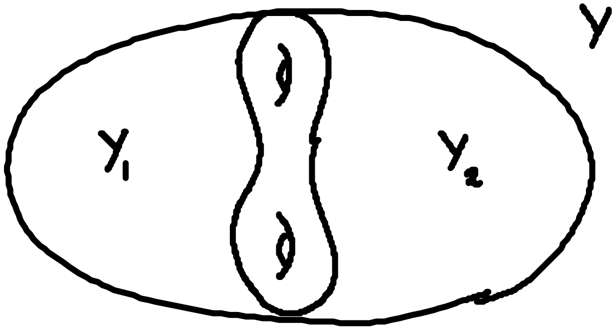
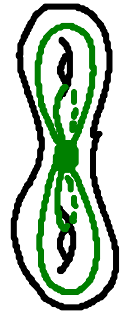
- Manolescu-Ozsváth-Thurston: $\widehat{HF}^{\pm,\infty}$, 4-manifold invariant
- still HARD to compute in practice

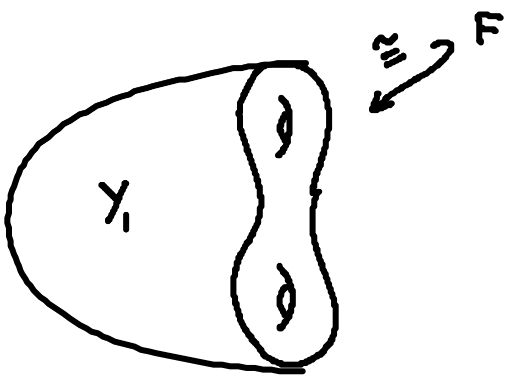
New Approach - Bordered Floer Homology (Lipshitz - Ozsváth - Thurston)



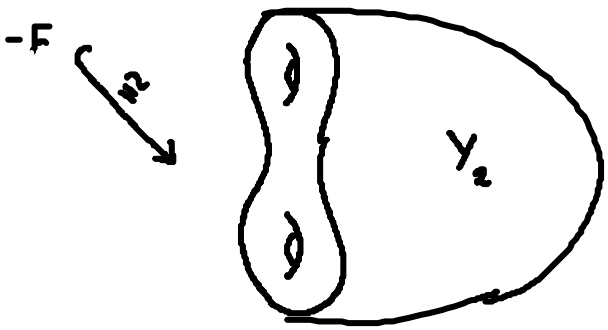


model surface w/ a given handle-decomposition

$\mathcal{F} \rightsquigarrow \mathcal{A}(\mathcal{F})$ - Differential Graded Algebra (DGA)



$\widehat{CFA}_{\mathcal{A}(F)}(Y_1)$ - right $\mathcal{A}_\infty$ - module over $\mathcal{A}(F)$



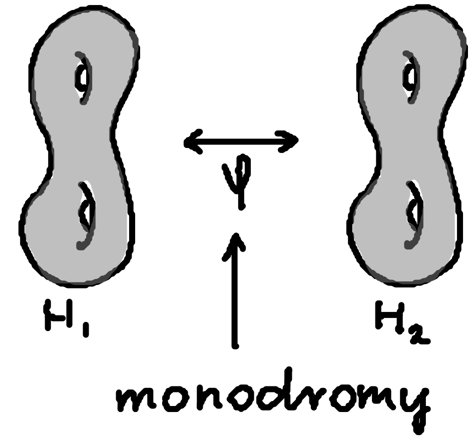
${}^{\mathcal{A}(F)}\widehat{CFD}$ - left DG - module over $\mathcal{A}(-F)$

Pairing Thm:

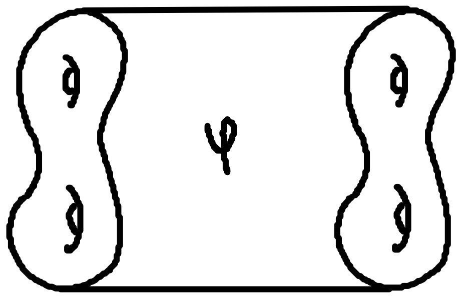
$$\widehat{CFA}(Y_1) \,\tilde{\otimes}\, \widehat{CFD}(Y_2) \cong \widehat{CF}(Y_1 \cup_F Y_2)$$

Computation - Strategy

- take a Heegaard decomposition of Y



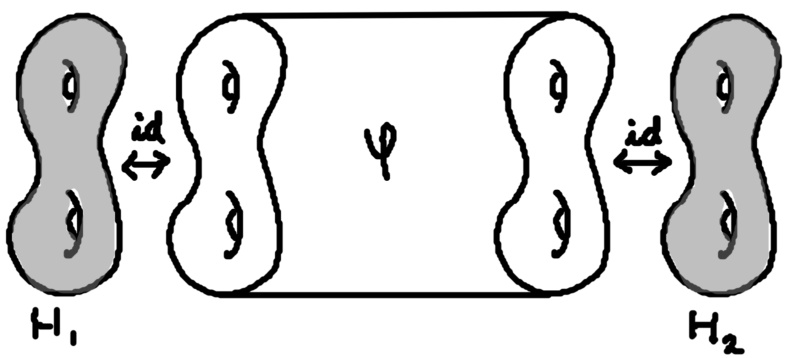
- define



$F \times I$

$\mathcal{A}(F)$ $\widehat{CFAD}_{\mathcal{A}(F)}(\varphi)$

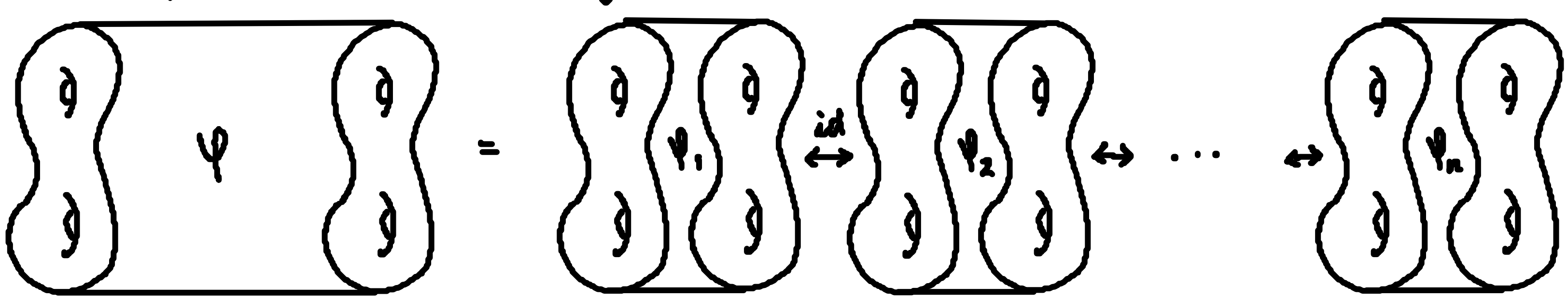
bimodule



$\rightsquigarrow$ $\widehat{CF}(Y) \cong \widehat{CFA}(H) \otimes \widehat{CFAD}(\varphi) \otimes \widehat{CFD}(H)$

simple   simple

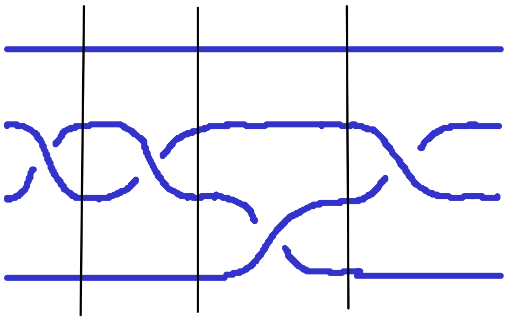
- "cut" φ into elementary pieces



$\widehat{CFAD}(\varphi) \cong \widehat{CFAD}(\varphi_1) \,\tilde{\otimes}\, \widehat{CFAD}(\varphi_2) \,\tilde{\otimes} \quad \tilde{\otimes}\, \widehat{CFAD}(\varphi_n)$

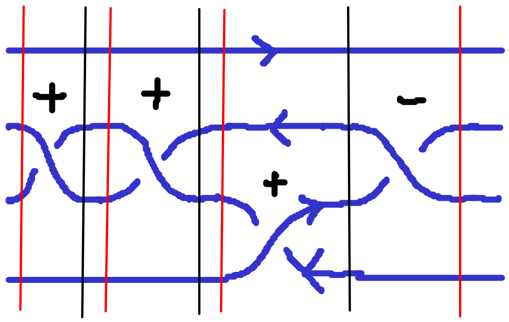
simple

TANGLE FLOER HOMOLOGY



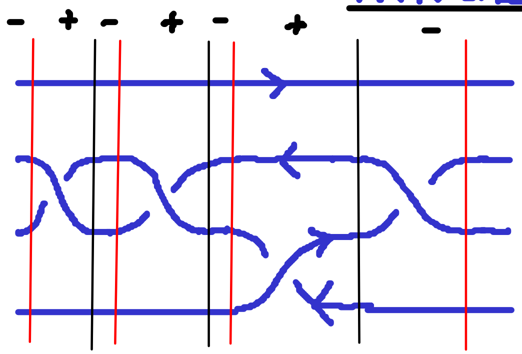
- cut it into product of generators

TANGLE FLOER HOMOLOGY

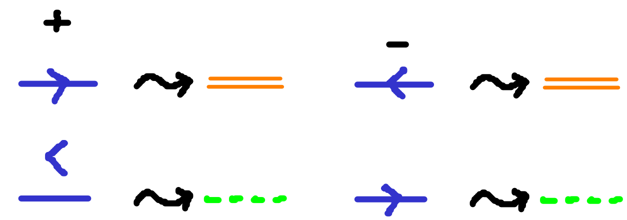


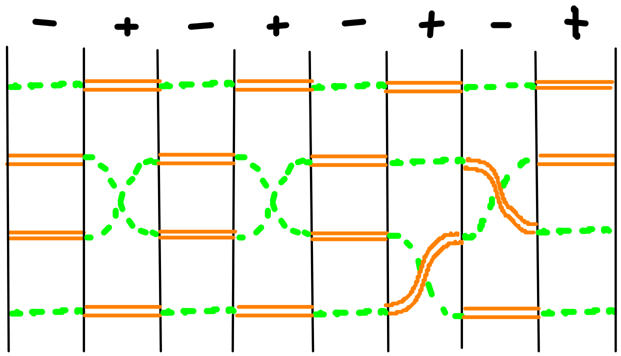
- cut it into product of generators
- by possibly adding trivial sections make sure that + crossings are on even places & - crossings are on odd places

Tangle Floer Homology

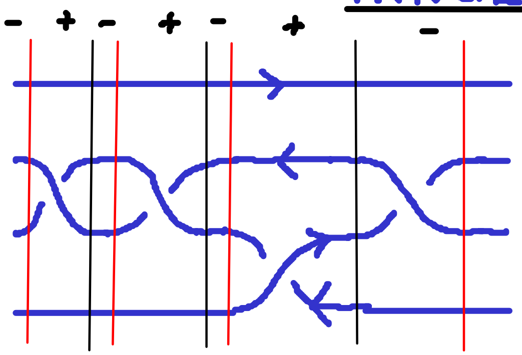


- cut it into product of generators
- by possibly adding trivial sections make sure that + crossings are on even places & − crossings are on odd places
- forget the crossings & orientations & color:





Tangle Floer Homology



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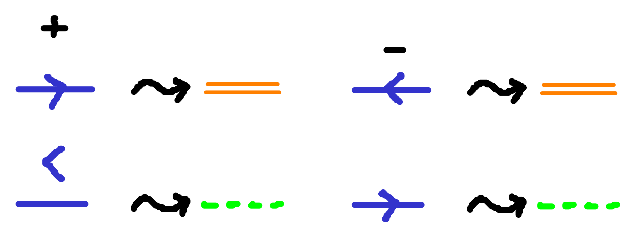
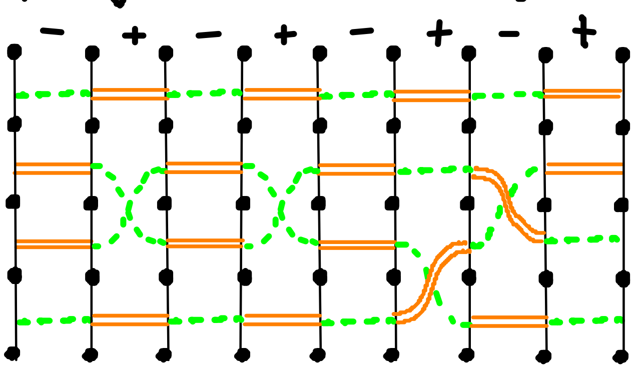
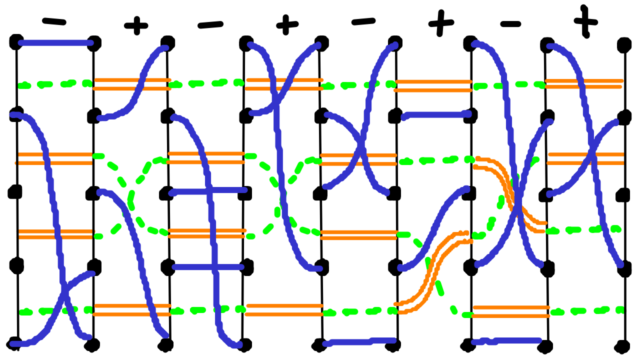




- insert dots on the separating lines between the intersections with the tangle
- connect all pts on the two separating lines bordering a tangle

⇝ bipartite graph $K_{5,5,5,5,5,5,5,5}$

Tangle Floer Homology - Generators & ∂



$m_1^* \quad m_2 \quad m_3^* \quad m_4 \quad m_5^* \quad m_6 \quad m_7^* \quad m_8$

$\updownarrow$

$m_1^* \vee m_2 \vee m_3^* \vee m_4 \vee m_5^* \vee m_6 \vee m_7^* \vee m_8$

<u>generators</u> are partial matchings

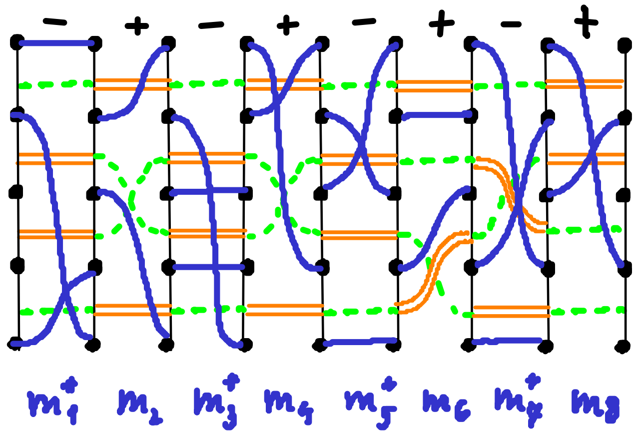
s.t : - pts on the border have $\leqslant 1$ edge

- pts in the interior have $= 1$ edge

<u>boundary map</u> $\partial(m_1^* \vee m_2 \vee m_3^* \vee m_4) =$

$$\partial^* m_1^* \vee m_2 \vee m_3^* \vee m_4 + m_1^* \vee \partial m_2 \vee m_3 \vee m_4 + m_1^* \vee m_2 \vee \partial^* m_3^* \vee m_4 + m_1^* \vee m_2 \vee m_3^* \vee \partial m_4$$

Tangle Floer Homology - Generators & ∂



$\updownarrow$

$m_1^* \vee m_2 \vee m_3^* \vee m_4 \vee m_5^* \vee m_6 \vee m_7^* \vee m_8$

generators are partial matchings

s.t.:
- pts on the border have ≤ 1 edge
- pts in the interior have $= 1$ edge

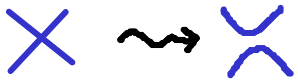
boundary map $\partial(m_1^* \vee m_2 \vee m_3^* \vee m_4) =$

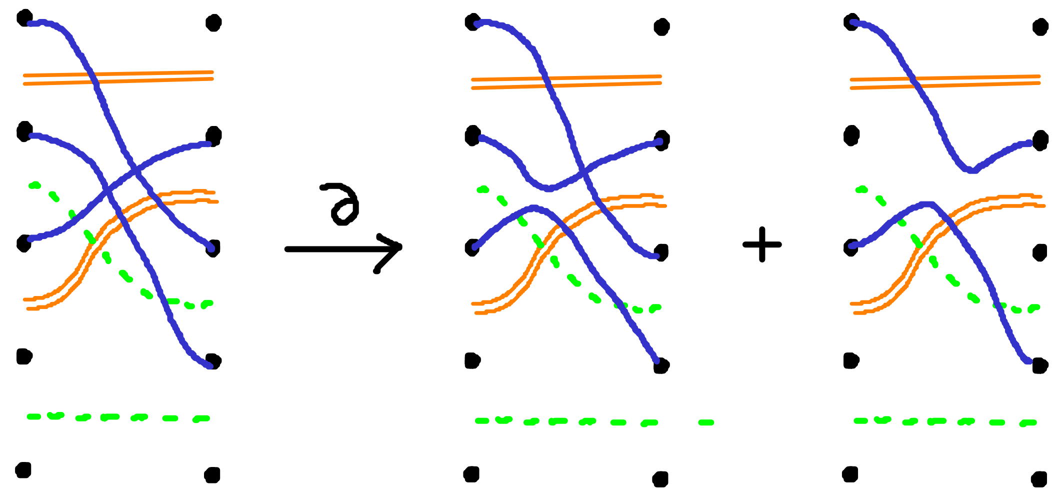
$$\partial^* m_1^* \vee m_2 \vee m_3^* \vee m_4 + m_1^* \vee \partial m_2 \vee m_3 \vee m_4 + m_1^* \vee m_2 \vee \partial^* m_3^* \vee m_4 + m_1^* \vee m_2 \vee m_3^* \vee \partial m_4$$

$$+ \partial_{mixed}(m_1^* \vee m_2) \vee m_3^* \vee m_4 + m_1^* \vee \partial^*_{mixed}(m_2 \vee m_3^*) \vee m_4 + m_1^* \vee m_2 \vee \partial_{mixed}(m_3^* \vee m_4)$$

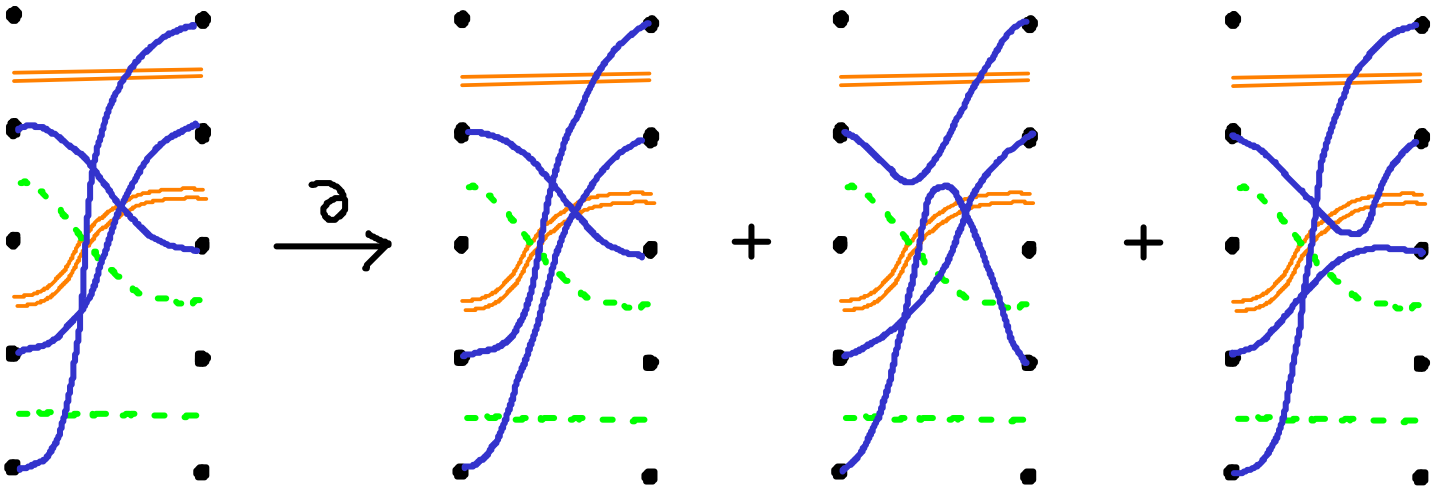
Boundary map

on $+$: resolving intersections





or:



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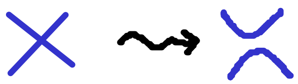

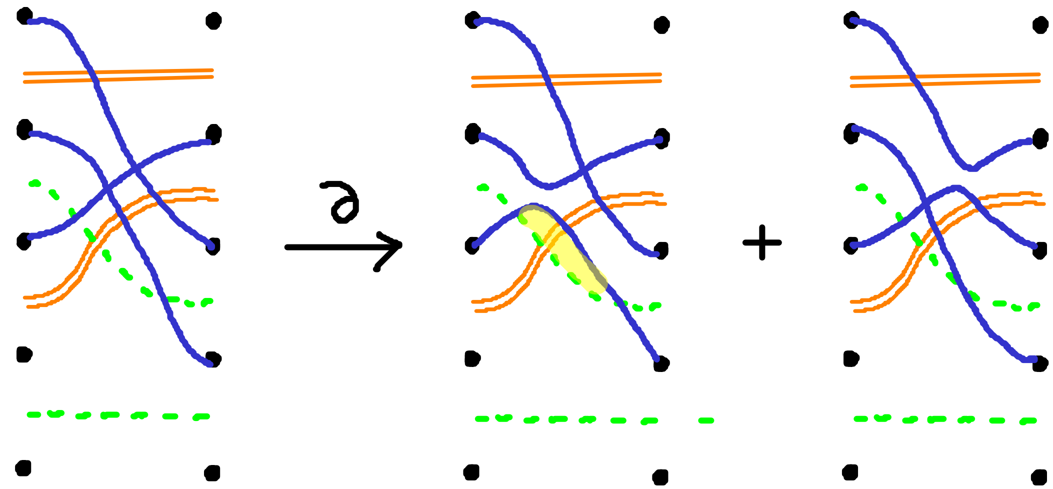

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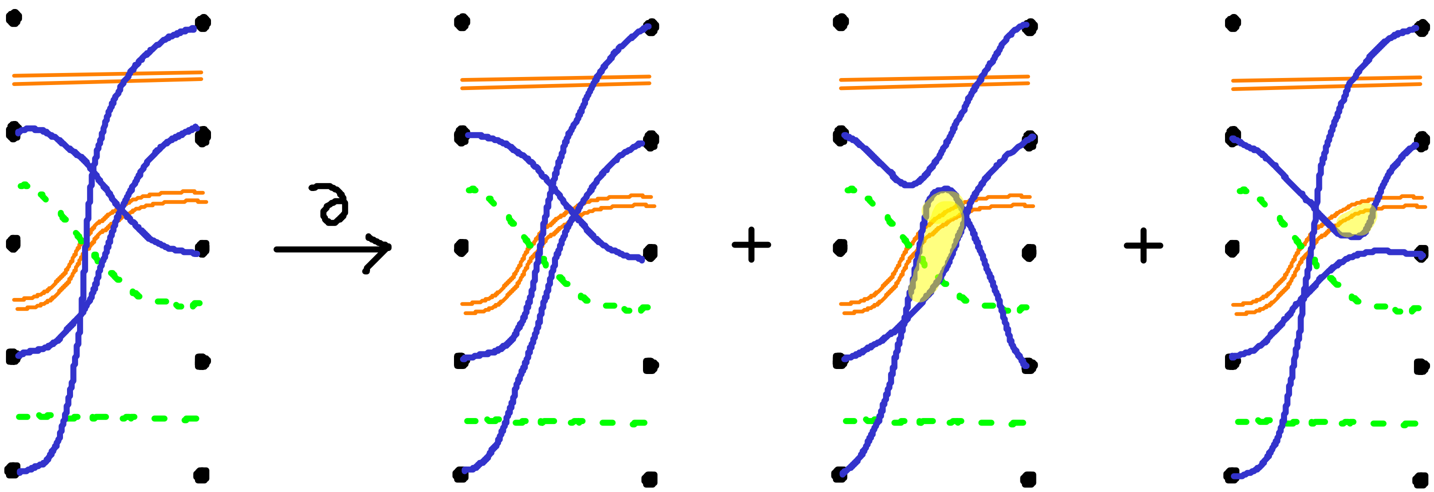
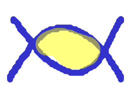
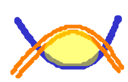
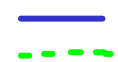
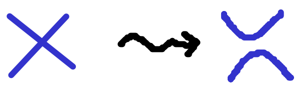
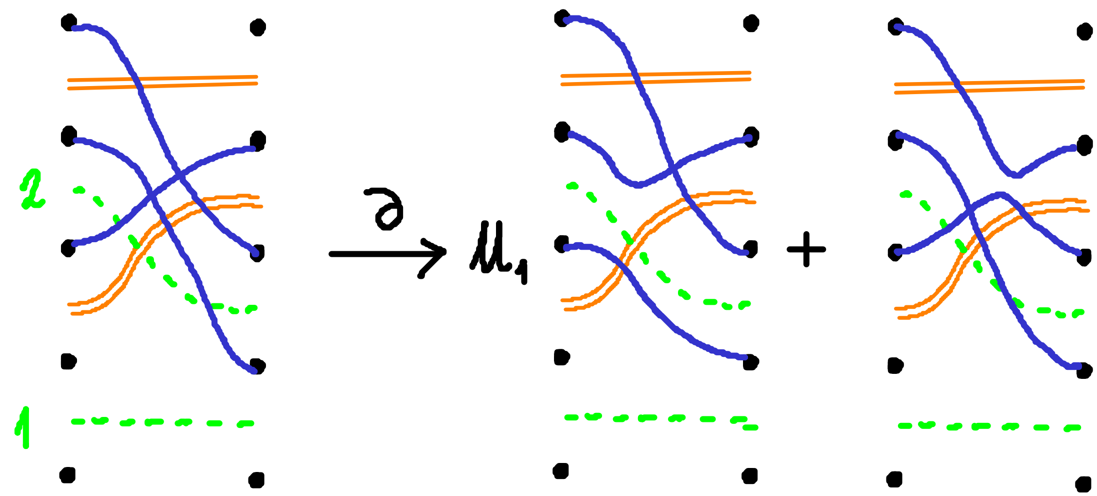


with relations:  $= 0$  $= 0$  $= u_i$ 

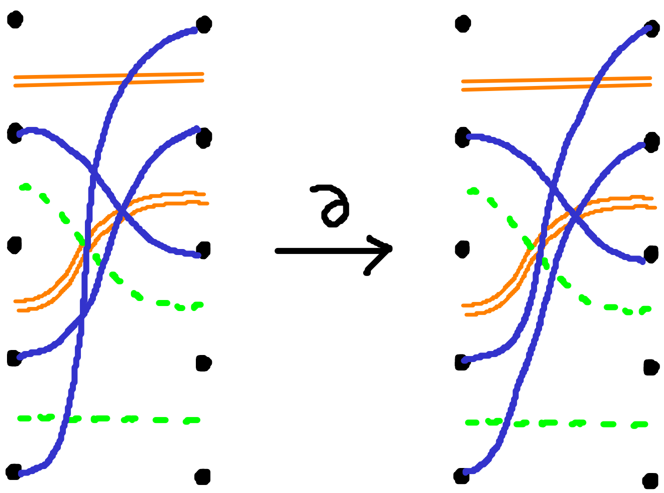
BOUNDARY MAP

on +: resolving intersections

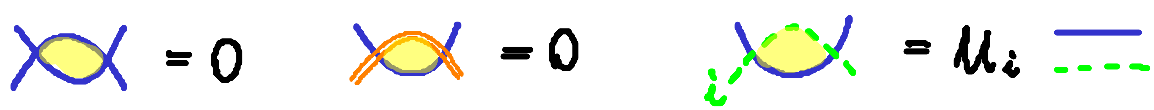




Or:

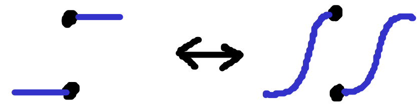


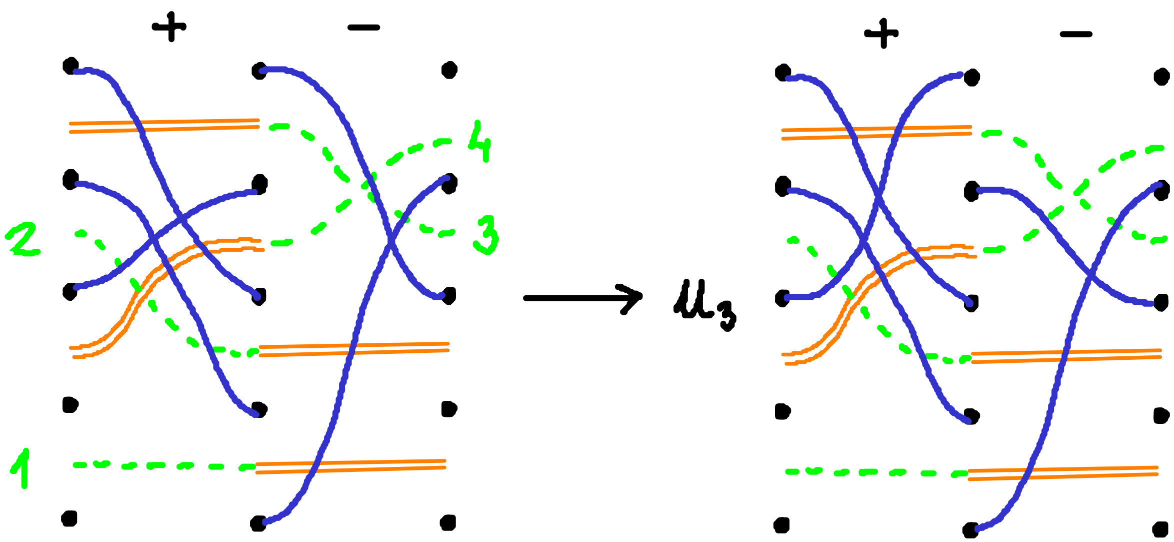
with relations:



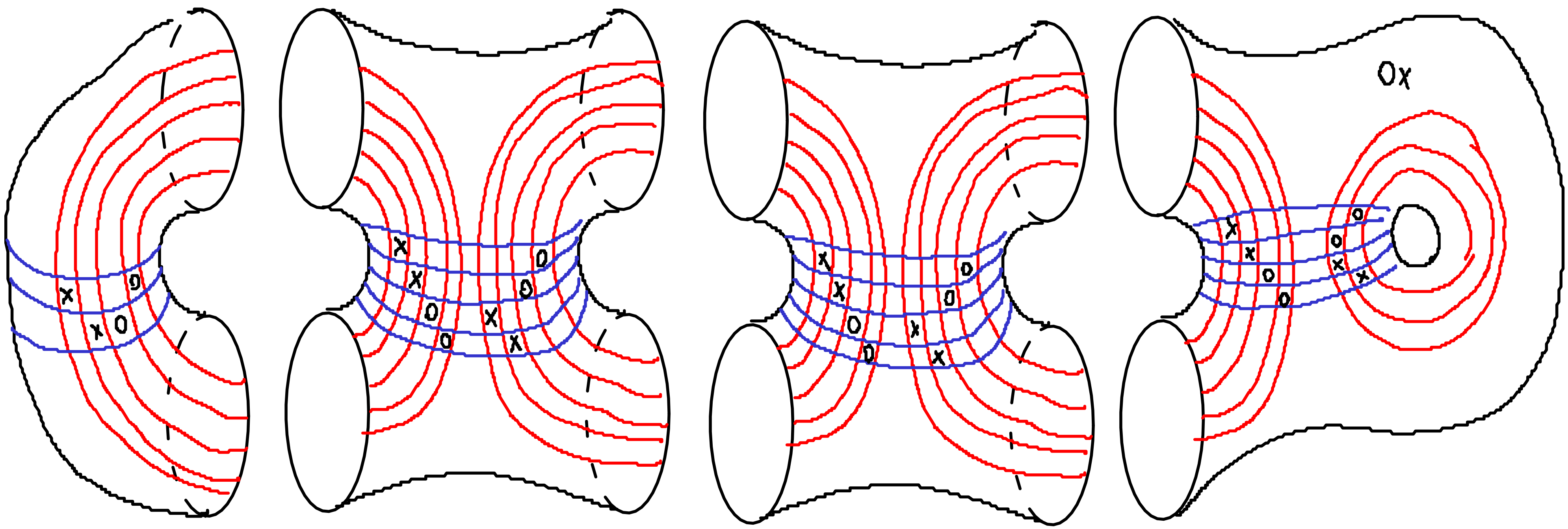
Boundary map

on $-$: introducing intersections

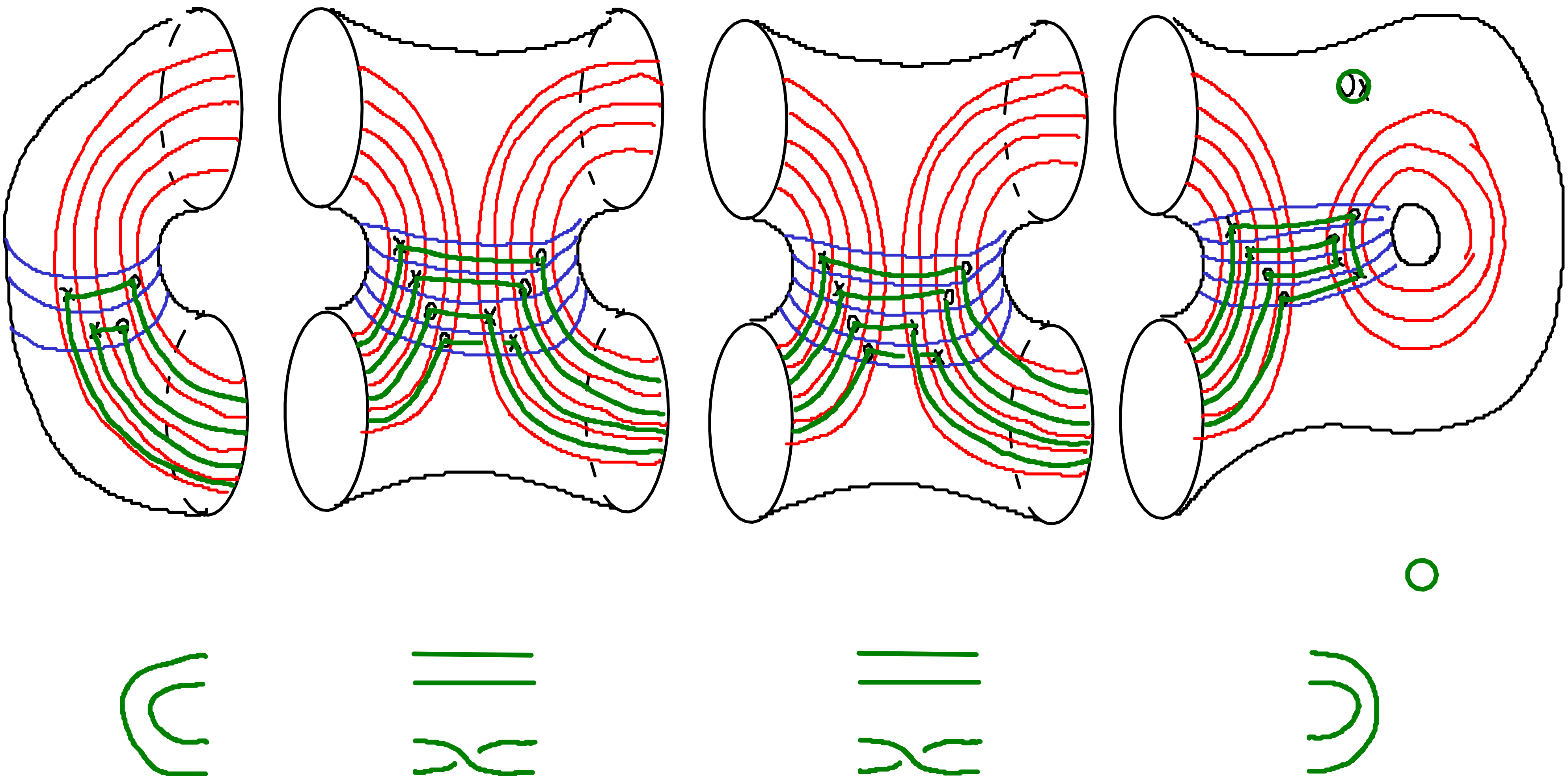
∂_{mixed}: exchanging endpoints:  w/ some relations



Relation with HFK



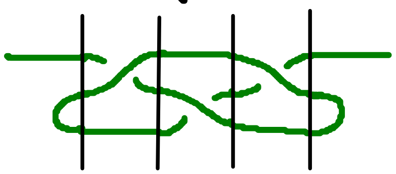
Relation with HFK



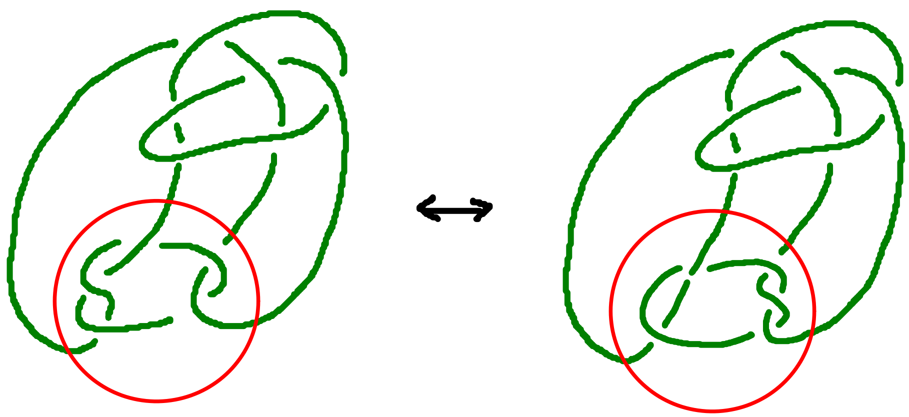
right handed trefoil

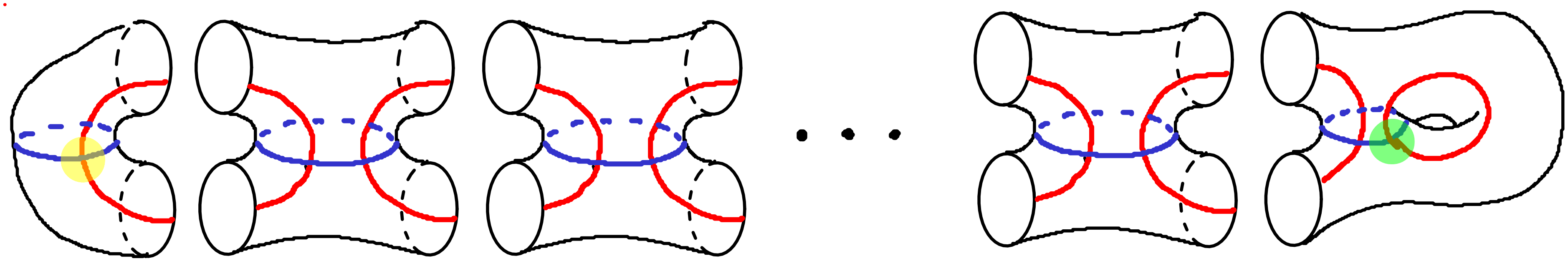
Applications, Generalisations

- can define bimodules for $\supset$ & $\subset$, thus can deal with general tangles



- another combinatorial definition for HFK
- localisation of questions:
    - another proof that HFK categorifies Δ_K
    - copy-paste arguments
    - mutation





GRATULÁLOK

ANDRÁS!