Spectra of large non-self-adjoint Toeplitz matrices subject to small random perturbations

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Outline

Non-self-adjoint operators and spectral instability

Non-self-adjoint operators appear naturally in many areas, e.g.:

- In the theory of linear PDEs given by non-normal operators
  - solvability theory
  - evolution equations given by non-normal operators
  - the Kramers-Fokker-Planck type operators
  - the damped wave equation
  - ...
- In mathematical physics when studying scattering poles (resonances).

Bad resolvent control: For non-normal operators $P : \mathcal{H} \to \mathcal{H}$ the norm of the resolvent may be very large even far away from the spectrum $\sigma(P)$:

$$\|(P - z)^{-1}\| \gg \frac{1}{\mathrm{dist}(z, \sigma(P))}.$$

Consequence:

- The spectrum can be very unstable under small perturbations of the operator.

Pseudospectrum

A way to quantify this zone of spectral instability is given by the ε-pseudospectrum [Trefethen-Embree '05], defined by

$$\sigma_\varepsilon(P) := \sigma(P) \cup \left\{ z \in \rho(P) : \ \|(P-z)^{-1}\| > \varepsilon^{-1} \right\};$$

or equivalently

$$\sigma_\varepsilon(P) := \bigcup_{\substack{Q \in \mathcal{B}(\mathcal{H}) \\ \|Q\| < \varepsilon}} \sigma(P+Q).$$

- Renewed interest has started in numerical analysis with the works of [Trefethen '97] (and [Trefethen-Embree '05]);
- Active subject in the field of PDE: Davies, Zworski, Sjöstrand, Bulton, Pravda-Starov, ... ;
- It is natural to add small random perturbations; Hager '06, Hager-Sjöstrand '08, Bordeaux-Montrieux '08, Davies-Hager '09, Sjöstrand '08-'15, Zworski-Christiansen '10.

Laurent and Toeplitz operators

Laurent operator: For $p \in L^\infty(S^1)$, $L(p) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$L(p)u = \hat{p} * u, \quad (L(p)u)(n) = \frac{1}{2\pi} \int_{S^1} p(\xi)\hat{u}(\xi)\mathrm{e}^{in\xi} d\xi.$$

Toeplitz operator: For $p \in L^\infty(S^1)$, $T(p) : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$T(p) := 1_{\mathbb{N}} L(p) 1_{\mathbb{N}}.$$

Identifying $\mathbb{C}^N$ with $\ell^2([1, N])$, we define a $N \times N$ **Toeplitz matrix** by

$$T_N(p) = 1_{[1,N]} L(p) 1_{[1,N]},$$

or by its matrix representation

$$T_N(p) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{1-N} \\ a_1 & a_0 & a_{-1} & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & & & \vdots \\ a_{N-1} & \cdots & & \cdots & a_0 \end{pmatrix}, \quad a_j = \hat{a}(j) \in \mathbb{C}$$

Spectra and Pseudospectra

The spectral theory of these operators were extensively studied, cf. Böttcher-Silbermann (also Trefethen-Embree), Widom, Goldsheid-Khoruzenko, Hatano-Nelson, Ghoberg, ...

We assume for simplicity that the symbol p is a trigonometric polynomial (i.e. the corresponding operators are banded).

i) Laurent operator: is normal, so $\sigma(L(p)) = p(S^1)$.

ii) Toeplitz operator: by truncating we may loose normality, and by [Gohberg '52]

$$\sigma_{ess}(T(p)) = p(S^1) \text{ and } \sigma(T(p)) = p(S^1) \cup \{z \in \mathbb{C};\ wind(p,z) \neq 0\}$$

iii) Toeplitz matrix:

- for non-normal $T_N(p)$, in general $\lim_{N\to\infty} \sigma(T_N(p)) \neq \lim_{N\to\infty} \sigma(T(p))$.
- Set $p_r(z) = p(rz)$, then by [Schmidt-Spitzer '60]

$$\lim_{N\to\infty} \sigma(T_N(p)) = \bigcap_{r>0} \sigma(T(p_r)),$$

$$\sigma(T_N(p)) \subset \text{a finite union of analytic connected arcs}\ .$$

Pseudospectra are well behaved !

i) $\lim_{N\to\infty} \sigma_\varepsilon(T_N(p)) = \sigma_\varepsilon(T(p))$ [Landau '75, Reichel-Trefethen '92, Böttcher '96]

ii) For every $\varepsilon > 0$, there exists N_0 s.t. for all $N \geq N_0$, $\sigma(T(p)) \subset \sigma_\varepsilon(T_N(p))$

Example: $p(z) = 2z^{-3} - z^{-2} + 2iz^{-1} - 4z^2 - 2iz^3$

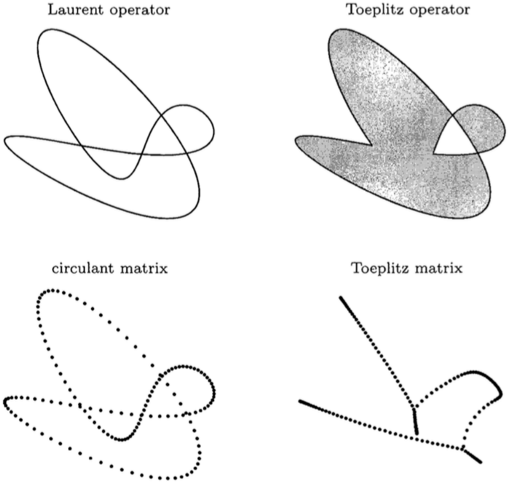
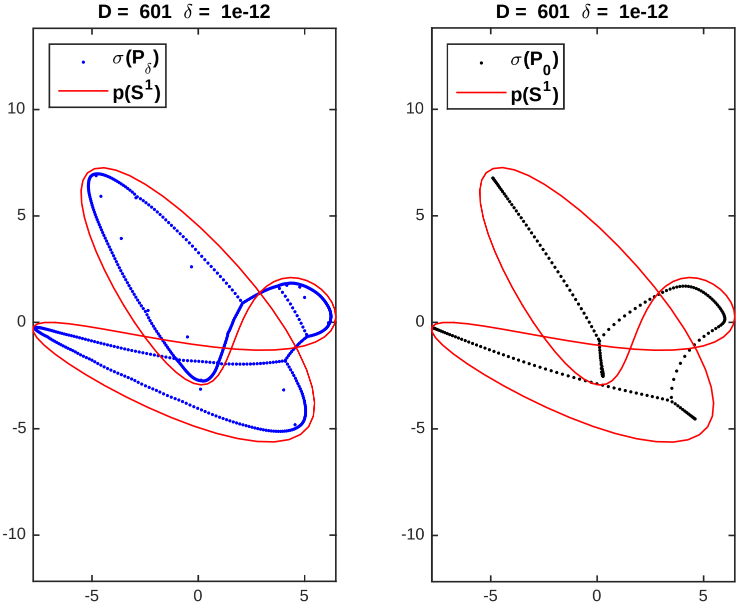
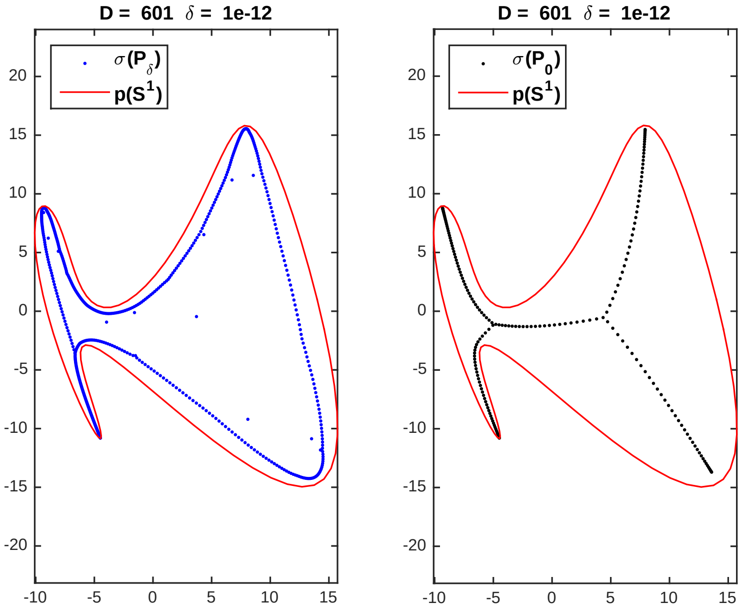


Figure: Picture taken from [Trefethen-Embree '05], represents the spectra of the Laurent, Toeplitz operators and Toeplitz and circulant matrix corresponding to the symbol p.

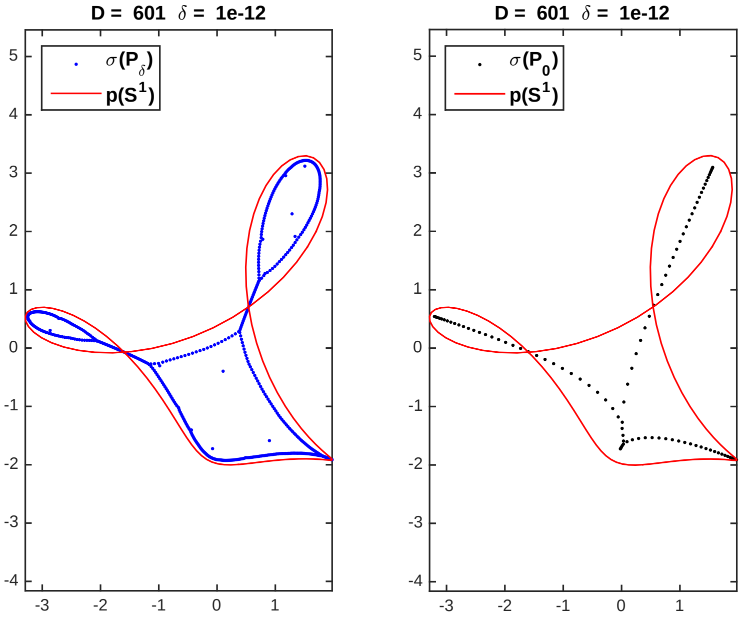
Small random perturbations of large Toeplitz matrices



$$p(\xi) = 2e^{-i3\xi} - e^{-i2\xi} + 2ie^{-i\xi} - 4e^{i2\xi} - 2ie^{i3\xi}$$



$$p(z) = -z^{-4} - (3+2i)z^{-3} + iz^{-2} + z^{-1} + 10z + (3+i)z^2 + 4z^3 + iz^4$$



$$p(z) = 2iz^{-1} + z^2 + \frac{7}{10}z^3$$

Small random perturbations of large Toeplitz matrices

We are interested in small random perturbations of $P_N^0 : \mathbb{C}^N \to \mathbb{C}^N$, a non-normal Toeplitz matrix for $N \gg 1$, of the form:

$$P^{\delta,\omega} := P_N^0 + \delta Q_\omega , \quad 0 < \delta \ll 1$$

where

$$Q_\omega = (q_{j,k}(\omega))_{1 \le j,k \le N} \quad \text{with } q_{j,k} \sim \mathcal{N}_{\mathbb{C}}(0,1) \ \text{(iid)}.$$

- If $C_1 > 0$ is large enough, then

$$(q \in B_{\mathbb{C}^{N^2}}(0, C_1 N) \ \Leftrightarrow \ ) \quad \|Q\|_{\mathrm{HS}} \le C_1 N, \text{ with probability } \ge 1 - e^{-N^2}.$$

We study the following two cases of P_N^0:

Case 1: Large Jordan block matrices

Case 2: Large bi-diagonal matrices

Perturbations of large Jordan blocks

We are interested in the spectrum of a random perturbation of the large Jordan block A_0 :

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} : \mathbb{C}^N \to \mathbb{C}^N.$$

- The spectrum of A_0 is $\sigma(A_0) = \{0\}$;
- $D(0,1)$ is a **region of spectral instability**;
- in $\mathbb{C}\backslash\overline{D(0,1)}$ we have **spectral stability**, i.e. a good resolvent estimate.

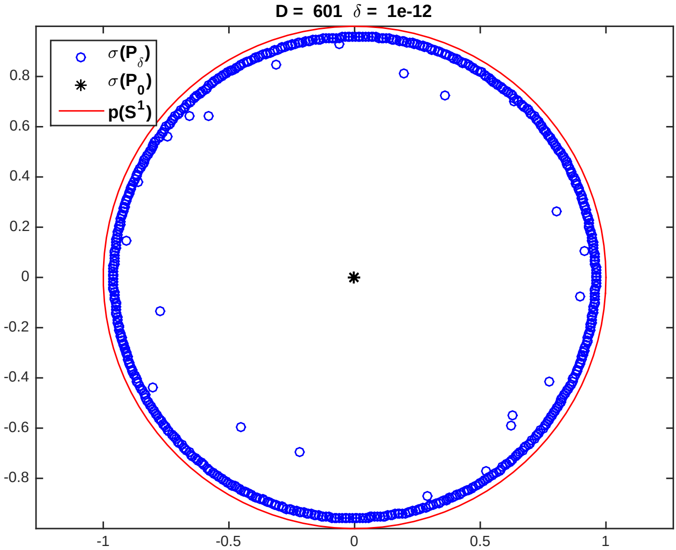
For a small $(0 < \delta \ll 1)$ (random) perturbation

$$A_\delta = A_0 + \delta Q_\omega, \quad Q_\omega = (q_{j,k}(\omega))_{1\leq j,k\leq N}, \quad q_{j,k}(\omega) \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ (iid)}$$

we expect the spectrum to move in a small neighborhood of $\overline{D(0,1)}$.

Numerical simulation

Numerical simulation for the eigenvalues of A_δ, a complex Gaussian random perturbation of A_0, with $p(z) = z$



Previous results

Theorem (Davies-Hager '09)

If $0 < \delta \le N^{-7}$, $R = \delta^{1/N}$, $\sigma > 0$, then with probability $\ge 1 - 2N^{-2}$, we have $\sigma(A_\delta) \subset D(0, RN^{3/N})$ and

$$\#(\sigma(A_\delta) \cap D(0, Re^{-\sigma})) \le \frac{2}{\sigma} + \frac{4}{\sigma} \ln N.$$

- With probability close to 1, **most eigenvalues** are close to a circle, contained in $D(0, RN^{3/N}) \backslash D(0, Re^{-\sigma})$.
- At most $\mathcal{O}(\ln N)$ eigenvalues inside $D(0, R\mathrm{e}^{-\sigma})$.
- Sjöstrand improved on this result by giving a probabilistic angular Weyl law for the eigenvalues close to the S^1

Theorem (Guionnet-Matched-Wood-Zeitouni '14)

Assume that $N^{-1-\kappa'} \le \delta \le N^{-1-\kappa}$ for some $0 < \kappa < \kappa'$, then

$$\frac{1}{N} \sum_{\mu \in \sigma(A_\delta)} \delta(z - \mu) \to \text{the uniform measure on } S^1,$$

weakly in probability as $N \to \infty$.

Interior density of eigenvalues

To obtain more information in the interior, we consider the random point process (related works on the zeros of random polynomials by Shiffman and Zelditch, Sodin, Hough-Krishnapur-Peres-Virág)

$$\Xi = \sum_{z \in \sigma(A_\delta)} \delta_z,$$

Study for $\varphi \in \mathcal{C}_0(D(0,r))$ the first intensity measure of Ξ:

$$\mathbb{E}[\Xi(\varphi) \mathbb{1}_{B(0,C_1 N)}] = \int \varphi(z) d\nu(z) \quad \text{(recall: } \|Q\|_{\mathrm{HS}} \leq C_1 N).$$

Theorem (Sjöstrand-V '14)

Let $e^{-N/C} \leq \delta \ll N^{-3}$ and $N \gg 1$. Let r_0 belong to a parameter range,

$$\frac{1}{C} \leq r_0 \leq 1 - \frac{1}{N}, \quad \text{s.t.} \quad \frac{r_0^{N-1} N}{\delta}(1-r_0)^2 + \delta N^3 \ll 1.$$

Then, for all $\varphi \in \mathcal{C}_0(D(0, r_0 - 1/N))$

$$\mathbb{E}\left[\mathbb{1}_{B_{\mathbb{C}^{N^2}}(0,C_1 N)}(q) \sum_{\lambda \in \sigma(A_\delta)} \varphi(\lambda)\right] = \frac{1}{2\pi} \int \varphi(z) \xi(z) L(dz),$$

where

$$\xi(z) = \frac{4}{(1-|z|^2)^2}\left(1 + \mathcal{O}\left(\frac{|z|^{N-1} N}{\delta}(1-|z|)^2 + \delta N^3\right)\right).$$

Numerical simulation

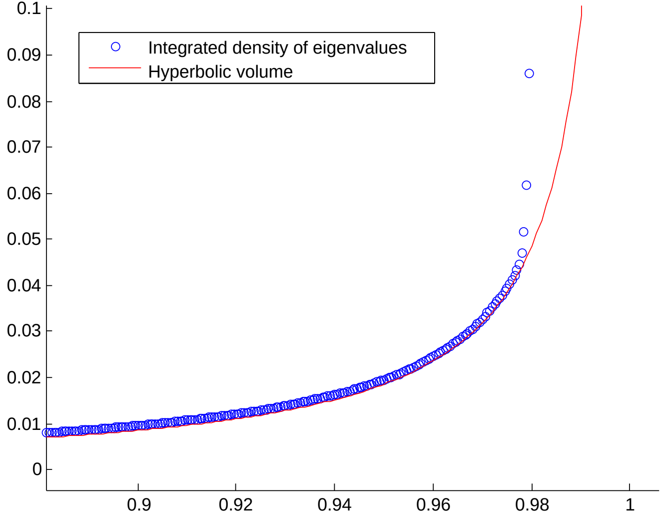


Figure: The experimental integrated density of eigenvalues (averaged over 500 realizations), as a function of the radius, of a 1001 $\times$ 1001-Jordan block matrix perturbed with a random complex Gaussian matrix and with coupling $\delta = 2 \cdot 10^{-\mathbf{10}}$. The red line is the hyperbolic volume on the unit disk as a function of the radius.

Large bi-diagonal matrices - first results

We now consider the following two cases:

$$P_{\mathrm{I}} = \begin{pmatrix} 0 & a & 0 & .. & .. & 0 \\ b & 0 & a & .. & .. & 0 \\ 0 & b & 0 & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & 0 & a \\ 0 & 0 & .. & .. & b & 0 \end{pmatrix} \quad \text{and} \quad P_{\mathrm{II}} = \begin{pmatrix} 0 & a & b & 0 & .. & .. & 0 \\ 0 & 0 & a & b & .. & .. & 0 \\ 0 & 0 & 0 & a & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & a & b \\ 0 & .. & .. & .. & .. & 0 & a \end{pmatrix}.$$

- Here $a, b \in \mathbb{C} \setminus \{0\}$ and $N \gg 1$.
- Identifying $\mathbb{C}^N$ with $\ell^2([1, N])$, $[1, N] = \{1, 2, .., N\}$ and also with $\ell^2_{[1,N]}(\mathbb{Z})$ (the space of all $u \in \ell^2(\mathbb{Z})$ with support in $[1, N]$), we have:

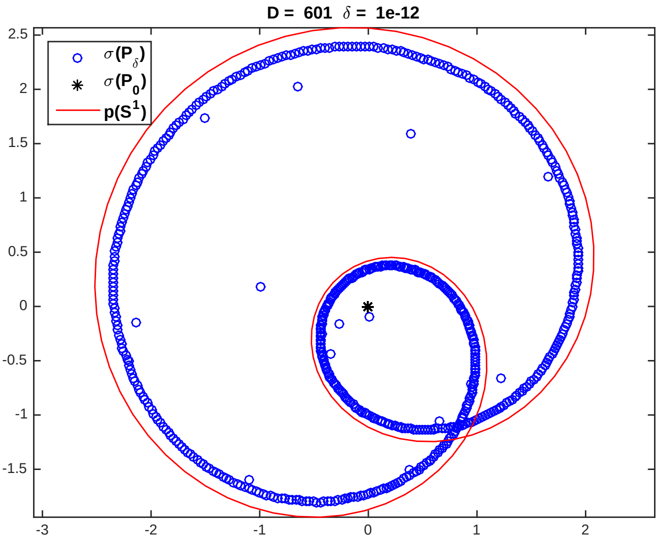
$$P_{\mathrm{I}} = 1_{[1,N]}(a\mathrm{e}^{iD_x} + b\mathrm{e}^{-iD_x}),$$

$$P_{\mathrm{II}} = 1_{[1,N]}(a\mathrm{e}^{iD_x} + b\mathrm{e}^{2iD_x}).$$

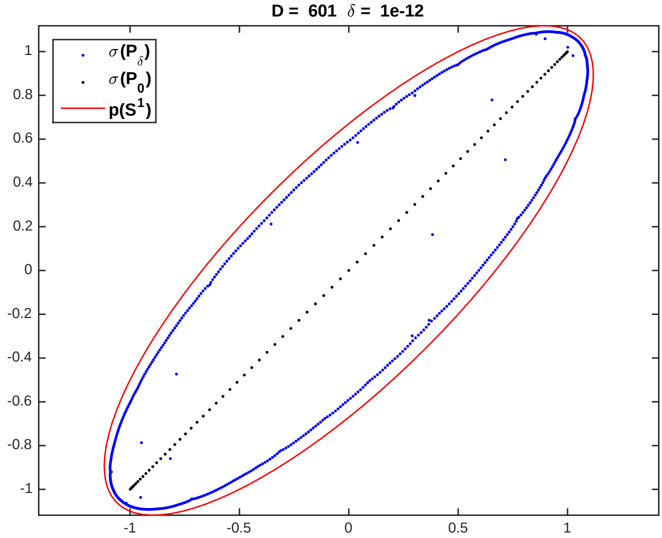
- The symbols of these operators are,

$$p_{\mathrm{I}}(\xi) = a\mathrm{e}^{i\xi} + b\mathrm{e}^{-i\xi}, \;\; p_{\mathrm{II}}(\xi) = a\mathrm{e}^{i\xi} + b\mathrm{e}^{2i\xi}.$$

Numerical simulation for P_{II}



Numerical simulation for P_I



Theorem (Sjöstrand-V '15)

Let $P = P_{\mathrm{I}}$ where $a, b \in \mathbb{C}$ satisfy $0 < |b| < |a|$. Let $P_\delta = P + \delta Q_\omega$. Choose $\delta \asymp N^{-\kappa}$, $\kappa > 5/2$ and consider the limit of large N. Let γ be a segment of the ellipse $E_1 = P_{\mathrm{I}}(S^1)$ and let $\Gamma = \Gamma(r,\gamma) = \{z \in \mathbb{C}; \mathrm{dist}(z, E_1) = \mathrm{dist}(z,\gamma) < r\}$ with $(\ln N)/N \ll r \ll 1$. Let δ_0 be small and fixed.
Then with probability

$$\geq 1 - \mathcal{O}(1)\left(\frac{1}{r} + \ln N\right) N^{2\kappa} \mathrm{e}^{-2N^{\delta_0}}, \tag{1}$$

we have

$$\left| \#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{2\pi}\mathrm{vol}_{]0,N]\times S^1} p_{\mathrm{I}}^{-1}(\Gamma) \right| \leq \mathcal{O}(1) N^{\delta_0}\left(\frac{1}{r} + \ln N\right). \tag{2}$$

- If we choose $\gamma = E^1$, we have
$$\frac{1}{2\pi}\mathrm{vol}_{]0,N]\times S^1} p_{\mathrm{I}}^{-1}(\Gamma) = N$$
(= total number of eigenvalues of P_δ), so the number of eigenvalues outside of Γ is bounded be the right hand side of (2).
- With $r > 0$ fixed but arbitrarily small we get

Corollary

Let Γ be any fixed neighborhood of E_1. Then with probability as in (1), we have

$$|\#\left(\sigma(P_\delta) \cap (\mathbb{C} \setminus \Gamma)\right)| \leq \mathcal{O}(1) N^{\delta_0} \ln N.$$

Outlook

1. Consider general non-normal Toeplitz matrices.
2. Density in the interior of the pseudospectrum
3. Correlation functions, universality
4. limiting point-process

Thank you for your attention !