SERRE'S MODULARITY CONJECTURE (II)

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to Jean-Pierre Serre

Abstract. We provide proofs of Theorems 4.1 and 5.1 of [32].

Contents

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1. Introduction

We fix a representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ with $\mathbb{F}$ a finite field of characteristic p, that is of S-type (odd and absolutely irreducible), $2 \leq k(\bar{\rho}) \leq p+1$ if $p > 2$. We assume that $\bar{\rho}$ has non-solvable image when $p = 2$, and $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is absolutely irreducible when $p > 2$. For a number field F we set $\bar{\rho}_F := \bar{\rho}|_{G_F}$.

In this part we provide proofs of the technical results Theorems 4.1 and 5.1 stated in [32]. We adapt the methods of Wiles, Taylor-Wiles and Kisin (see [62],[60], [33]) to prove the needed modularity lifting results (see Proposition 9.2 and Theorem 9.7 below). We also need to generalise slightly Taylor's potential modularity lifting results in [54] and [55] (see Theorem 6.1 below) to have it in a form suited to our needs.

Modularity lifting results proved here lead to the proof of Theorem 4.1 of [32]. Modularity lifting results when combined with presentation results for deformation rings due to Böckle [4] (see Proposition 4.5 below), and Taylor's potential version of Serre's conjecture, lead by the method of [31] and [30] to the existence of p-adic lifts asserted in Theorem 5.1 of [32] (see Corollary 4.7 below). These lifts are made part of compatible systems using arguments of Taylor (see 5.3.3 of [59]) and Dieulefait (see [21], [63]).

1.1. **Some features of our work.** We remark on the arguments in the paper which differ from the main references we use:

– The definition of local deformation rings follows [33], but there are some novelties in the formalism we use and the calculations we make. We also follow Kisin's suggestion of working with framed deformations at finite and infinite place

– We overcome non-neatness problems encountered in proving properties of spaces of modular forms by the arguments used in [5] (see its appendix).

– We give a different proof of the lower bounds on dimensions of global deformation rings (see Proposition 4.5), that are needed in our method for producing characteristic 0 lifts of global mod p Galois representations with prescribed ramification behaviour, than in the literature. The proof is more consistently relative to the structure of these global deformation rings as algebras over certain local deformation rings.

– For results about presentations of deformation rings we fix the determinants of the deformations we consider. The reason we succeed in terms of the numerics in the Wiles' formula (see (2) of Section 4.2 below) producing a positive lower bound on the dimension of these rings is that the reduced tangent spaces of the deformation rings we consider, that parametrise deformations that in particular have fixed determinants, are isomorphic to the images of certain cohomology groups with $\mathrm{Ad}^0(\bar{\rho})$ coefficients in related cohomology groups with $\mathrm{Ad}(\bar{\rho})$ coefficients.

– We arrive at what has turned out to be the main innovation of the paper: the patching argument to prove modularity of 2-adic lifts. For $p = 2$ when doing the patching of deformation and Hecke rings in §9.1, we fix determinants locally at places in the set S for which the local deformation rings are put in the coefficients (see §9.1). As in [20], we do not fix determinant locally at auxiliary primes. We do this because, as the referee pointed out to us, for $p = 2$, the usual argument for getting auxiliary primes fixing the determinant does not work. We then need to prove equality of dimensions of rings obtained by patching deformation rings and Hecke rings, fixing the determinant. To be sure that fixing the determinant give the right number of relations, we use twists. We need to impose a further condition on the auxiliary primes Q_n in the Taylor-Wiles patching argument, to make certain class groups grow (which allows us to us the formalism that we introduce in §2.4, 2.5 and 2.6), which is one of the novel features of the work here (cf. §5.5). We find it miraculous that the method of Wiles and Taylor-Wiles in [62] and [60] can be made to work with modifications *in extremis*. (For a sketch of the proof, the reader might look at [64].)

Throughout the paper the fact that we can prove modularity of Galois representations after solvable base change is exploited extensively following the work of Skinner-Wiles in [52].

The debt that this paper owes to the work of Wiles, Taylor-Wiles, Skinner-Wiles, Diamond, Fujiwara and Kisin (see [62],[60],[52],[16], [33]) on modularity lifting theorems, and the work of Taylor on the potential version of Serre's conjecture (see [54],[55]) will be readily visible to readers.

1.2. **Notation.** We let p be a rational prime. Let E be a finite extension of $\mathbb{Q}_p$ and call $\mathcal{O}$ the ring of integers of E. Let π be a uniformizer of $\mathcal{O}$ and let $\mathbb{F}$ be the residue field.

For F a number field, $\mathbb{Q} \subset F \subset \overline{\mathbb{Q}}$, we write G_F for the Galois group of $\overline{\mathbb{Q}}/F$. For v a prime/place of F, we mean by D_v (resp., I_v when v is a finite place) a decomposition (resp., inertia) subgroup of G_F at v. We denote by $\mathbb{N}(v)$ the cardinality of the residue field k_v at v. We denote by F_v a completion of F at v and denote by $\mathcal{O}_{F_v}$ the ring of integers of F_v, and sometimes suppress F from the notation. We denote $\mathcal{O}_{F_p} = \Pi_v \mathcal{O}_{F_v}$ with the product over places v of F above a prime p of $\mathbb{Q}$. For each place p of $\mathbb{Q}$, we fix embeddings ι_p of $\overline{\mathbb{Q}}$ in its completions $\overline{\mathbb{Q}_p}$.

Denote by χ_p the p-adic cyclotomic character, and ω_p the Teichmüller lift of the mod p cyclotomic character $\overline{\chi}_p$ (the latter being the reduction mod p of χ_p). By abuse of notation we also denote by ω_p the ℓ-adic character $\iota_\ell \iota_p^{-1}(\omega_p)$ for any prime ℓ: this should not cause confusion as from the context it will be clear where the character is valued. We also denote by $\omega_{p,2}$ a fundamental character of level 2 (valued in $\mathbb{F}_{p^2}^*$) of I_p: it factors through the quotient of I_p that is isomorphic to $\mathbb{F}_{p^2}^*$. We denote by the same symbol its Teichmüller lift, and also all its ℓ-adic incarnations $\iota_\ell \iota_p^{-1}(\omega_{p,2})$. For a number field F we denote the restriction of a character of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to G_F by the same symbol. We denote by $\mathbb{A}_F$ the adeles of F.

Consider a totally real number field F. Recall that in [57], 2-dimensional p-adic representations ρ_π of G_F are associated to cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ that are discrete series at infinity of weight $(k, \cdots, k)$, $k \geq 2$. We say that such forms are of (parallel) weight k. We say that $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$, with $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_p$, is modular if it is isomorphic to (an integral model of) such a ρ_π. For a place v above p we say that the local component π_v at v of π is ordinary if the corresponding eigenvalue of the Hecke operator (T_v or U_v) acting on the representation space of π_v is a unit (with respect to the chosen embedding ι_p). If π_v is ordinary, so is $\rho_\pi|_{D_v}$ in the sense of Definition 3.4 below.

A compatible system of 2-dimensional representations of G_F is said to be modular if one member of the system is modular ; then all members are also modular. We say that $\bar{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F})$, with $\mathbb{F}$ a finite field of characteristic p, is modular if either it is irreducible and isomorphic to the reduction of (an integral model of) such a ρ_π modulo the maximal ideal of $\mathcal{O}$, or it is reducible and totally odd (i.e., $\det(\bar{\rho}(c)) = -1$ for all complex conjugations $c \in G_F$). We denote by $\mathrm{Ad}^0(\bar{\rho})$ the trace zero matrices of $\mathrm{Ad}(\bar{\rho}) = M_2(\mathbb{F})$ and regard it as a G_F-module via the composition of $\bar{\rho}$ with the conjugation action of $\mathrm{GL}_2(\mathbb{F})$ on $M_2(\mathbb{F})$. We oftentimes suppress $\bar{\rho}$ from the notation, as we work with a fixed one, and write $\mathrm{Ad}^0(\bar\rho)$ or $\mathrm{Ad}(\bar\rho)$ as Ad^0 and Ad.

For a local field F we denote by W_F the Weil group of F and normalise the isomorphism $F^* \simeq W_F$ of local class field theory by demanding that a uniformiser is sent to an arithmetic Frobenius.

For a number field we recall the isomorphism of global class field theory

$$\mathbb{A}_F^*/\overline{F^*(F_\infty^*)^0} \simeq G_F^{\mathrm{ab}}$$

that is compatible with the isomorphism of local class field theory.

If $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$, we will consider *arithmetic characters* $\psi : \mathbb{A}_F^*/\overline{F^*(F_\infty^*)^0} \to \mathcal{O}^*$ such that for an open compact subgroup U of $(\mathbb{A}_F^\infty)^*$, $\psi(zu) = \mathbb{N}(u_p)^t\psi(z)$ for $z \in \mathbb{A}_F^*$ and $u \in U$ where u_p is the projection to the places above p of u and $\mathbb{N}$ is the norm map (the product of the local norms), and t an integer. We fix such a character ψ. These give rise to a Galois representation $\rho_\psi : G_F \to \mathcal{O}^*$ that is of the form $\chi_p^{-t}\epsilon$ with ϵ a finite order character. When $\overline{F^*(F_\infty^*)}$ lies in the kernel of ψ, we consider ψ as a character $\psi : F^*\backslash(\mathbb{A}_F^\infty)^* \to \mathcal{O}^*$, and the corresponding ρ_ψ is then totally even.

If F'/F is a finite extension and $\mathbb{N}_{F'/F}$ is the corresponding norm we will sometimes denote ψ and its composition with $\mathbb{N}_{F'/F}$ by the same symbol, or sometimes by $\psi_{F'}$. We will also allow ourselves to use the isomorphism of local and global class field theory to identify characters of G_F and of the idele class group in the global case and characters of the Weil group W_F and of F^* in the local case.

2. Deformation rings: the general framework

References for this section are [40] and §2 of [33].

Let p, E, $\mathcal{O}$, π and $\mathbb{F}$ be like in 1.2.

2.1. $\mathrm{CNL}_{\mathcal{O}}$**-algebras.** Denote by $\mathrm{CNL}_{\mathcal{O}}$ the category whose objects are complete, Noetherian, local $\mathcal{O}$-algebras, with a fixed isomorphism of the residue field to $\mathbb{F}$, and whose maps are local homomorphisms that are compatible with the fixed isomorphism of residue fields. Given $A \in \mathrm{CNL}_{\mathcal{O}}$ we denote by m_A, or sometimes simply by m, its maximal ideal.

Let A be a $\mathrm{CNL}_{\mathcal{O}}$-algebra. We denote by Sp_A the functor of points defined by A, *i.e.* for any $\mathrm{CNL}_{\mathcal{O}}$-algebra B, $\mathrm{Sp}_A(B)$ is the set $\mathrm{Hom}_{\mathrm{CNL}_{\mathcal{O}}}(A, B)$. In particular, the morphism $A \to \mathbb{F}$ defines the unique point in $\mathrm{Sp}_A(\mathbb{F})$; we note it $\xi_{A,\mathbb{F}}$. If one associates to A the functor Sp_A, one gets a contravariant equivalence of the category of $\mathrm{CNL}_{\mathcal{O}}$-algebras into the sub-category of functors from $\mathrm{CNL}_{\mathcal{O}}$-algebras to sets which are representable. If X is a functor from $\mathrm{CNL}_{\mathcal{O}}$-algebras to sets which is representable, we note $A(X)$ the $\mathrm{CNL}_{\mathcal{O}}$-algebra that represents X *i.e.* is such that $X = \mathrm{Sp}_{A(X)}$. We shall say that X has property P if $A(X)$ has. For example, we shall say that $X \to Y$ is smooth if $A(Y) \to A(X)$ is (formally) smooth.

Let A be a $\mathrm{CNL}_{\mathcal{O}}$-algebra. We denote by t_A the (relative) tangent space. Let $\mathbb{F}[\epsilon]$ be the dual number algebra ($\epsilon^2 = 0$). As a set, t_A identifies to $\mathrm{Sp}_A(\mathbb{F}[\epsilon])$. There is a natural bijection of t_A to $\mathrm{Hom}_{\mathbb{F}}(\mathsf{m}_A/(\mathsf{m}_A^2 + \pi A), \mathbb{F})$, and t_A inherits a structure of $\mathbb{F}$-vector space. The 0 of t_A corresponds to the compositum of $\xi_{A,\mathbb{F}}$ with the unique $\mathrm{CNL}_{\mathcal{O}}$-morphism $\mathbb{F} \to \mathbb{F}[\epsilon]$. We write t_A^* the cotangent space *i.e.* the dual vector space $\mathrm{Hom}_{\mathbb{F}}(t_A, \mathbb{F})$. We can also identify t_A to the $\mathbb{F}$-vector space of $\mathcal{O}$-derivations of A with values in $\mathbb{F}$.

As A is noetherian, the tangent and cotangent spaces are finite dimensional. If (x_i) are finitely many elements in m_A such that their images in t_A^* generate t_A^*, (x_i) generates A as a topological $\mathcal{O}$-algebra, and A is the quotient of $\mathcal{O}[[X_i]]$ by the morphism that sends X_i to x_i.

Let $f : A \to B$ be a morphism of $\mathrm{CNL}_{\mathcal{O}}$-algebras. The morphism f induces morphisms of $\mathbb{F}$-vector spaces $t(f) : t_B \to t_A$ and $t^*(f) : t_A^* \to t_B^*$. The relative tangent space $t_{B/A}$ is the kernel of $t(f)$. The relative cotangent space is the cokernel of $t^*(f)$. We see that $t_{B/A}$ identifies to the A-derivations of B with values in $\mathbb{F}$.

Let A, B_1 and B_2 be $\mathrm{CNL}_{\mathcal{O}}$-algebras with morphisms $A \to B_1$ and $A \to B_2$. The completed tensor product $C := B_1 \hat{\otimes}_A B_2$ is a $\mathrm{CNL}_{\mathcal{O}}$-algebra. To see this, one remarks that its quotient by $\mathsf{m} := \mathsf{m}_{B_1} \otimes_A B_2 + B_1 \otimes \mathsf{m}_{B_2}$ is naturally isomorphic to $\mathbb{F}$. As $\mathbb{F}^*$ embeds in C and C is complete for its m-adic topology, every element of C which is not in m is invertible and C is local. It is by definition complete. As C is a quotient of $\mathcal{O}[[X_1, \ldots, X_{d_1+d_2}]]$, $d_i = \dim_{\mathbb{F}}(t_{B_i})$, C is noetherian. For D a $\mathrm{CNL}_{\mathcal{O}}$-algebra, one has a natural bijection of $\mathrm{Sp}_C(D)$ to $\mathrm{Sp}_{B_1}(D) \times_{\mathrm{Sp}_A(D)} \mathrm{Sp}_{B_2}(D)$. To give oneself a A-derivation d_C of C with values in $\mathbb{F}$ is the same as to give oneself A-derivations d_{B_1} and d_{B_2} with values in $\mathbb{F}$ (by the formula $d_C(b_1 \otimes b_2) = b_2 d_{b_1}(b_1) + b_1 d_{b_2}(b_2)$). It follows that $t_{C/A}$ identifies to the direct sum $t_{B_1/A} \oplus t_{B_2/A}$.

A morphism $f : B \to C$ of $\mathrm{CNL}_{\mathcal{O}}$-algebras is called *a closed immersion* if it is surjective. If f is a closed immersion, it is clear that for every $\mathrm{CNL}_{\mathcal{O}}$-algebra A, the map $\mathrm{Sp}_C(A) \to \mathrm{Sp}_B(A)$ is injective. The converse is also true, as one sees easily using the surjectivity of $\mathrm{Sp}_C(A) \to \mathrm{Sp}_B(A)$ for $A = \mathbb{F}[\epsilon]$ ([40]).

2.2. **Lifts and deformations of representations of profinite groups.** Let G be a profinite group satisfying the p-finiteness property 1.1. of [40], *i.e.* for any open subgroup G' of G, there are only finitely many continuous morphisms from G' to $\mathbb{Z}/p\mathbb{Z}$. Let $d \geq 1$ be an integer and $\bar{\rho} : G \to \mathrm{GL}_d(\mathbb{F})$ be a continuous representation. In this paragraph, we compare lifts and deformations of $\bar{\rho}$. This paragraph is not used in our proof of modularity results, but the link between the two points of view seems to us interesting to be spelled out.

If A is a $\mathrm{CNL}_{\mathcal{O}}$ algebra, a *lift* ρ of $\bar{\rho}$ with values in A is a continuous morphism $G \to \mathrm{GL}_d(A)$ such that its reduction $G \to \mathrm{GL}_d(\mathbb{F})$ is $\bar{\rho}$. One defines in an obvious way a functor $\mathcal{D}^{\square} : \mathrm{CNL}_{\mathcal{O}} \to \mathrm{SETS}$ such that $\mathcal{D}^{\square}(A)$

is the set of lifts ρ of $\bar{\rho}$ to $\mathrm{GL}_d(A)$. By a theorem of Grothendieck (18 of [40]), the functor $\mathcal{D}^{\square}$ is representable by a $\mathrm{CNL}_{\mathcal{O}}$-algebra that we note $R^{\square}$. We note $\rho^{\square}_{\mathrm{univ}} : G \to \mathrm{GL}_d(R^{\square})$ the universal lift. The relative tangent space is the $\mathbb{F}$-vector space of 1-cocyles $Z^1(G, \mathrm{Ad})$ where Ad is the adjoint representation of $\bar{\rho}$. For each $g \in G$, the entries of the matrix $\rho^{\square}_{\mathrm{univ}}(g)$ are functions in $R^{\square}$: these functions topologically generate $R^{\square}$ over $\mathcal{O}$ as follows from the description of the tangent space.

A *deformation* of $\bar{\rho}$ is an equivalence set of lifts, two lifts being equivalent if they are conjugate by a matrix of the kernel $\mathrm{GL}_d(A)_1$ of the morphism $\mathrm{GL}_d(A) \to \mathrm{GL}_d(\mathbb{F})$. We note $A \mapsto \mathcal{D}(A)$ the functor of deformations of $\bar{\rho}$. We also call a lift a *framed deformation*. One has a natural morphism of functors $\mathcal{D}^{\square} \to \mathcal{D}$ and a natural action of $(\mathrm{GL}_d)_1$ on $\mathcal{D}^{\square}$. For each A, $\mathcal{D}(A)$ is identified with the set of orbits of $\mathrm{GL}_d(A)_1$ in $\mathcal{D}^{\square}(A)$.

The functor $\mathcal{D}$ has a hull that is unique up to isomorphism. More precisely, there is a $\mathrm{CNL}_{\mathcal{O}}$ algebra R and a versal deformation $\rho_{\mathrm{vers}} : G_F \to \mathrm{GL}_d(R)$. Let, as in the preceding section, Sp_R be the functor represented by R. One has a natural morphism of functors $\mathrm{Sp}_R \to \mathcal{D}$ which is smooth. For each A and each deformation $\rho \in \mathcal{D}(A)$, one has a point $\xi \in \mathrm{Sp}_R(A)$ such that ρ is equivalent to the compositum of $\rho_{\mathrm{vers}} : G \to \mathrm{GL}_d(R)$ with the morphism $\mathrm{GL}_d(R) \to \mathrm{GL}_d(A)$ induced by ξ. Furthermore, if $A = \mathbb{F}[\varepsilon]/\varepsilon^2$ is the dual numbers algebra, then $\mathcal{D}(A)$ is isomorphic as an $\mathbb{F}$-vector space to the tangent space t_R of R. We express this by saying that R is universal for lifts to dual numbers. The tangent space t_R is naturally isomorphic to the $\mathbb{F}$-vector space $H^1(G, \mathrm{Ad})$.

Let us apply the versal property to the deformation defined by the universal lift $\rho^{\square}_{\mathrm{univ}}$. One gets a point $\xi \in \mathrm{Sp}_R(R^{\square})$. It defines a morphism of functors $f_\xi : \mathcal{D}^{\square} \to \mathrm{Sp}_R$ such that the image of $\mathrm{id}_{R^{\square}}$ in $\mathrm{Sp}_R(R^{\square})$ is ξ. As $\mathrm{id}_{R^{\square}}$ is the point defined by $\rho^{\square}_{\mathrm{univ}}$, one sees that the compositum of f_ξ with the natural functor $\mathrm{Sp}_R \to \mathcal{D}$ is the natural functor $\mathcal{D}^{\square} \to \mathcal{D}$.

Proposition 2.1. *The morphism of functors $f_\xi : \mathcal{D}^{\square} \to \mathrm{Sp}_R$ is smooth. The morphism $\xi : R \to R^{\square}$ is formally smooth of dimension $d^2 - \dim(H^0(G, \mathrm{Ad}))$.*

Proof. We have to prove that if A is Artinian object in $\mathrm{CNL}_{\mathcal{O}}$ which is a small extension of $\overline{A}$, then:

$$\mathcal{D}^{\square}(A) \to \mathrm{Sp}_R(A) \times_{\mathrm{Sp}_R(\overline{A})} \mathcal{D}^{\square}(\overline{A})$$

is surjective. Recall that $A \to \overline{A}$ being small means that the morphism of $\mathrm{CNL}_{\mathcal{O}}$-algebras is surjective with principal kernel I such that $\mathsf{m}_A I = (0)$.

Let us first prove that $\mathcal{D}^{\square}(A) \to \mathrm{Sp}_R(A)$ is surjective. Let us choose a representative for ρ_{vers}. It defines a point in $\mathcal{D}^{\square}(R)$ *i.e.* a morphism $R^{\square} \to R$. Let $Z^1(G, \mathrm{Ad})$ be the $\mathbb{F}$-vector space of 1-cocycles. The morphism $R^{\square} \to R$ induces on tangent spaces a $\mathbb{F}$-linear morphism $H^1(G, \mathrm{Ad}) \to Z^1(G, \mathrm{Ad})$ which is a section of the natural morphism. The morphism ξ induces on tangent spaces the natural projection $Z^1(G, \mathrm{Ad}) \to H^1(G, \mathrm{Ad})$.

So one sees that the compositum $R \to R^{\square} \to R$ induces isomorphisms on tangent and cotangent spaces. It induces a surjective endomorphism of R/m^n for each integer n, hence an automorphism of R/m^n, and hence it is an automorphism of R that we note a. If we compose $R^{\square} \to R$ with a^{-1}, we obtain a section of ξ. The existence of this section implies that $\mathcal{D}^{\square}(A) \to \mathrm{Sp}_R(A)$ is surjective.

Let $x_A \in \mathrm{Sp}_R(A)$ and $x_{\overline{A}}^{\square} \in \mathcal{D}^{\square}(\overline{A})$ having the same image in $\mathrm{Sp}_R(\overline{A})$. Let us prove that they come from a $z \in \mathcal{D}^{\square}(A)$. We just proved that there exists $y \in \mathcal{D}^{\square}(A)$ lifting x_A. The image $\overline{y}$ of y in $\mathcal{D}^{\square}(\overline{A})$ and $x_{\overline{A}}^{\square}$ have the same images in $\mathcal{D}(\overline{A})$. So there exists $\overline{g} \in \mathrm{GL}_d(\overline{A})_1$ such that $x_{\overline{A}}^{\square} = \overline{g}\ \overline{y}$. Lifting $\overline{g}$ to $g \in \mathrm{GL}_d(A)_1$, one gets $\tilde{z} = gy \in \mathcal{D}^{\square}(A)$ which is a lift of $x_{\overline{A}}^{\square}$. As x_A and $f_\xi(\tilde{z})$ have the same image in $\mathrm{Sp}_R(\overline{A})$, there exists $\delta \in I \otimes_{\mathbb{F}} H^1(G, \mathrm{Ad})$ such that $f_\xi(\tilde{z}) = x_A + \delta$. Let $\hat{\delta}$ be a lift of δ in $I \otimes_{\mathbb{F}} Z^1(G, \mathrm{Ad})$. One has $f_\xi(\tilde{z} - \hat{\delta}) = x_A$ and the image of $z := \tilde{z} - \hat{\delta}$ in $\mathcal{D}^{\square}(\overline{A})$ is $x_{\overline{A}}^{\square}$. This proves the smoothness.

As ξ is formally smooth, the relative dimension of ξ is the dimension of the relative tangent space, *i.e.* the dimension of the 1-coboundaries $B^1(G, \mathrm{Ad})$, which is $d^2 - \dim(H^0(G, \mathrm{Ad}))$. This proves the proposition. □

Let X be a deformation condition as in 18 of [40]. The deformations that satisfy this condition define a subfunctor $\mathcal{D}_X \subset \mathcal{D}$ which is relatively representable. Let $\mathrm{Sp}_{R,X}$ be the subfunctor of Sp_R defined by $\mathrm{Sp}_{R,X} = \mathrm{Sp}_R \times_{\mathcal{D}} \mathcal{D}_X$. For each A, $\mathrm{Sp}_{R,X}(A)$ is the inverse image of $\mathcal{D}_X(A)$ in $\mathrm{Sp}_R(A)$. The functor $\mathrm{Sp}_{R,X}$ is represented by a quotient $\overline{R}_X$ of R. In the same way, let $\mathcal{D}_X^{\square}$ be the subfunctor of framed deformations with condition X *i.e.* $\mathcal{D}_X^{\square}(A)$ is the inverse image of $\mathcal{D}_X(A)$ in $\mathcal{D}^{\square}(A)$. One sees that $\mathcal{D}_X^{\square}$ is represented by a quotient $\overline{R}_X^{\square}$ of $R^{\square}$. By restriction to $\mathrm{Sp}_{R,X}$, the proposition implies that $\overline{R}_X \to \overline{R}_X^{\square}$ is also formally smooth of dimension $d^2 - \dim(H^0(G, \mathrm{Ad}))$.

2.3. **Points and tensor products of $\mathrm{CNL}_{\mathcal{O}}$ algebras.** We will need the following well-known proposition (lemma 3.4.12. of [33]).

Proposition 2.2. *i) Let R be a flat $\mathrm{CNL}_{\mathcal{O}}$-algebra. Then, there exist a finite extension E' of E such that R has a point with values in the ring of integers of E'. Every maximal ideal of $R[1/p]$ is the image of the generic point of a local morphism : $\mathrm{Spec}(\mathcal{O}') \to \mathrm{Spec}(R)$ over $\mathrm{Spec}(\mathcal{O})$.*

ii) Let I be a finite set and R_i, $i \in I$, be $\mathrm{CNL}_{\mathcal{O}}$-algebras which are flat, have a point with values in $\mathcal{O}$, are domains, and are such that the $R_i[1/p]$ are regular. Then the completed tensor product of the R_i satisfies the same properties.

Proof. Let us prove i).

Let d be the dimension of the special fiber $R/\pi R$ of R. By flatness, the absolute dimension of R is $d+1$. Let $\bar{x_1}, \ldots, \bar{x_d}$ be a system of parameters of $R/\pi R$ and let be $x_1, \ldots, x_d$ be elements of R which reduces to $\bar{x_1}, \ldots, \bar{x_d}$.

The elements $\pi, x_1, \dots, x_d$ form a system of parameters of R. Let R' be $R/(x_1, \dots, x_d)$. It is of dimension 1. Let Q be a minimal prime ideal of R' such that R'/Q is of dimension 1. As $R'/(Q, \pi)$ is of finite length, and R'/Q is separate and complete for the π-adic topology and is of dimension 1, R'/Q is a finitely generated $\mathcal{O}$-module which is not of finite length. It is finite as an $\mathcal{O}$-module and has a non-empty generic fiber. Its normalization is the ring of integers $\mathcal{O}'$ of a finite extension of E. We see that R has a point with values in $\mathcal{O}'$.

Let Q be a maximal ideal of $R[1/p]$ and let $Q_R = Q \cap R$. The $\mathrm{CNL}_{\mathcal{O}}$-algebra R/Q_R is flat. By what we just proved, it has a point ξ with values in $\mathcal{O}'$ for E' a finite extension of E. The image of the generic point of ξ is Q, as Q is maximal. This finishes the proof of i).

Let us prove ii). Let $\hat{R}$ be the complete tensor product of the R_i. We saw in the preceeding section that $\hat{R}$ is a $\mathrm{CNL}_{\mathcal{O}}$. Let, for each i, $\xi_i : R_i \to \mathcal{O}$ a point of R_i with values in $\mathcal{O}$. Let P_i be the kernel of ξ_i. Let ξ and P be the point $\prod_i \xi_i$ and the ideal defining it. Let S be the completion of $(\otimes_i R_i)[1/p]$ at $\xi[1/p]$. As the $R_i[1/p]$ are regular, S is isomorphic to $E[[X_1, \dots, X_d]]$ with $d = \sum_i d_i$, d_i being the relative dimension of R_i. The ring R_i is complete for the P_i topology. It follows from a theorem of Chevalley (cor. 5 of th. 13 chapter 8.5. of [65]) that the P_i topology on R_i is the same as the topology defined by the $\widetilde{P_i^n}$, where $\widetilde{P_i^n} = P_i^n[1/p] \cap R_i$. As $\otimes_i R_i/\widetilde{P_i^n}$ injects in $\otimes_i R_i/P_i^n[1/p]$ for all n, we see that $\hat{R}$ injects in S. This implies that $\hat{R}$ is a domain. By i), the maximal ideals of $\hat{R}[1/p]$ correspond to points of the R_i with values in the ring of integers $\mathcal{O}'$ of a finite extension E' of E. The completion of $\hat{R}[1/p]$ at such a point is a power series ring $E'[[X_1, \dots, X_d]]$. This proves that $\hat{R}[1/p]$ is regular (prop. 28.M. of [38]). □

The next proposition shows that the points with values in the rings of integers $\mathcal{O}'$ of finite extensions of K determine a flat and reduced quotient of a $\mathrm{CNL}_{\mathcal{O}}$ algebra.

Corollary 2.3. *Let R be a $\mathrm{CNL}_{\mathcal{O}}$-algebra and R' be a quotient of R which is flat and reduced. Let I be the kernel of the map $R \to R'$. Then I is the intersection of the kernels of the local $\mathcal{O}$-algebra morphisms $R \to \mathcal{O}'$ which factor through R'.*

Proof. Let us call f the map $R \to R'$ and $I' \subset R'$ the intersection of kernels of the morphisms $R' \to \mathcal{O}'$. We have $I = f^{-1}(I')$, so we have to prove that $I' = (0)$. As R' is flat over $\mathcal{O}$, this is equivalent to $I'[1/p] = (0)$. As R' is noetherian and p belongs to the radical of R', $R'[1/p]$ is a Jacobson ring (cor. 10.5.8. of EGA 4 part 3). As $R'[1/p]$ is reduced, it then follows from the i) of the proposition that $I'[1/p] = (0)$.

□

Definition 2.4. *Let R be a $\mathrm{CNL}_{\mathcal{O}}$-algebra, and X a set of local $\mathcal{O}$-algebra morphisms $R \to \mathcal{O}'$ where $\mathcal{O}'$ runs through the ring of integers of all finite extensions of the fraction field of $\mathcal{O}$. We say that a flat and reduced* *quotient $R' = R/I$ of R classifies the morphisms in X if the set of local $\mathcal{O}$-algebra morphisms $R' \to \mathcal{O}'$ is identified with X. By the corollary above I is identified with the intersection of the kernels of elements in X.*

2.4. **Quotients by group actions for functors represented by** $\mathrm{CNL}_{\mathcal{O}}$ **algebras.** Let G be a functor from $\mathrm{CNL}_{\mathcal{O}}$-algebras to sets which is representable by the $\mathrm{CNL}_{\mathcal{O}}$-algebra $A(G)$. A group structure on G is the data for every $\mathrm{CNL}_{\mathcal{O}}$-algebra A, of a group structure on $G(A)$ such that for $A \to A'$, the map $G(A) \to G(A')$ is a morphism of groups. Such a structure is defined by morphisms $A(G)\hat{\otimes}_{\mathcal{O}}A(G) \to A(G)$, $A(G) \to A(G)$ and $A(G) \to \mathcal{O}$, corresponding to the product in G, the map $g \mapsto g^{-1}$, and the neutral element, satisfying the usual compatibilities.

Let X be a representable functor from $\mathrm{CNL}_{\mathcal{O}}$-algebras to sets. We define in a similar way an action of G on X. We say that the action of G on X is *free* if the map $G \times X \to X \times X$ corresponding to $(g, x) \mapsto (x, gx)$ is a closed immersion *i.e.* for every A, $G(A)$ acts on $X(A)$ without fixed points.

Let O be the functor of orbits from $\mathrm{CNL}_{\mathcal{O}}$-algebras to set which associates to A the set of orbits of $G(A)$ acting on $X(A)$. We call $A(X)_0$ the sub-algebra of $A(X)$ of elements a such that $\gamma(a) = 1 \otimes a$, where γ is the morphism $A(X) \to A(G)\hat{\otimes}_{\mathcal{O}}A(X)$ defined by the morphism $(g, x) \mapsto gx$ from $G \times X$ to X. The functions $a \in A(X)_0$ are the functions on X that are constant on orbits, *i.e.* such that for every $\mathrm{CNL}_{\mathcal{O}}$-algebra A, $g \in G(A)$ and $x \in X(A)$, one has $a(gx) = a(x)$.

The map $G \times G \to G$ defining the product in G defines a morphism of $\mathbb{F}$-vector spaces $t_G \oplus t_G \to t_G$. In the identification of t_G with the group $G(\mathbb{F}[\epsilon])$, this morphism corresponds with the addition in t_G. To see it, first, the unity in $G(\mathbb{F}[\epsilon])$ is the point obtained by composing $1_G \in G(\mathcal{O})$ with $\mathcal{O} \to \mathbb{F} \to \mathbb{F}[\epsilon]$. It follows that it coincides with the 0 in t_G. Then, for $l \in t_G$, $(0, l)$ and $(l, 0)$ go to l as $1 \times g = g \times 1 = g$.

In the same way, the map $G \times X \to X$ defines a morphism $t_G \oplus t_X \to t_X$. For $x \in t_X$, the image of $(0, x)$ is x, as $(1_G, x) \mapsto x$. Let o be the morphism $t_G \to t_X$ which associates to l the image of $(l, 0)$ in t_X by the map $t_G \oplus t_X \to t_X$. The image of (l, x) by the same map is $o(l) + x$. The map o identifies with the map which associates to $l \in G(\mathbb{F}[\epsilon])$ the point $l\xi_{X,\mathbb{F}[\epsilon]} \in X(\mathbb{F}[\epsilon])$, where $\xi_{X,\mathbb{F}[\epsilon]}$ is obtained by composing the unique point $\xi_{X,\mathbb{F}} \in X(\mathbb{F})$ with the morphism of $\mathrm{CNL}_{\mathcal{O}}$-algebras $\mathbb{F} \hookrightarrow \mathbb{F}[\epsilon]$. It follows that if the action of G on X is free, the map o is injective, so we get an embedding $t_G \hookrightarrow t_X$. The quotient t_X/t_G is the biggest quotient of t_X on which t_G acts trivially.

Proposition 2.5. *Let G be a group acting on X as above. One supposes that G is smooth and that the action of G on X is free. Then, $A(X)_0$ is a $\mathrm{CNL}_{\mathcal{O}}$-algebra and the functor of orbits is represented by $A(X)_0$. The map $X \to O$ is smooth. The tangent space t_O is naturally isomorphic to t_X/t_G so that the map $t_X \to t_O$ identifies with the quotient map $t_X \to t_X/t_G$. Let $G_O = O \times G$. The map $G_O \times_O X \to X$ makes X a torsor over O of group*

G_O (the natural map $G_O \times_O X \to X \times_O X$ is an isomorphism). This torsor is trivial and there exists an isomorphism of X with $O \times G$.

Proof. Using Schlessinger theorem (th. 2.11 of [50]), to prove (pro)-representability of the functor O by a $\mathrm{CNL}_{\mathcal{O}}$-algebra, it suffices to check the two following conditions :

- 1) if B_1, B_2 and A are $\mathrm{CNL}_{\mathcal{O}}$-algebras which are Artinian with morphisms $B_1 \to A$ and $B_2 \to A$, $B_1 \to A$ being surjective, the natural map

$$O(B_1 \times_A B_2) \to O(B_1) \times_{O(A)} O(B_2)$$

is bijective ;

- 2) 1) being established, one has a natural struture of $\mathbb{F}$-vector space on $O(\mathbb{F}[\epsilon])$ (lemma 2.10 of [50]); the condition 2) is that this $\mathbb{F}$-vector space is finite dimensional.

Let us prove 1) and first the surjectivity of the above map. Let $\overline{x_1} \in O(B_1)$ and $\overline{x_2} \in O(B_2)$ which have the same image $\overline{x}_A$ in $O(A)$. Let $x_i \in X(B_i)$ in the orbits $O(B_i)$. There exists $g_A \in G(A)$ such that, if $x_{i,A}$ are the images of x_i in $X(A)$, we have $g_A x_{1,A} = x_{2,A}$. As G is smooth and $B_1 \to A$ is surjective, there exists $g_1 \in G(B_1)$ whose image in $G(A)$ is g_A. Let $x'_1 = g_1 x_1$. Then x'_1 and x_2 have the same images in $X(A)$ and the couple (x'_1, x_2) defines an element in $X(B_1 \times_A B_2)$ whose orbit maps to $(\overline{x_1}, \overline{x_2})$. This proves the surjectivity.

Lest us prove the injectivity. Let (x_1, x_2) and (x'_1, x'_2) in $X(B_1 \times_A B_2)$ which have the same images in $O(B_1) \times_{O(A)} O(B_2)$. There exist $g_i \in G(B_i)$ such that $x'_i = g_i x_i$. If we write $*_A$ for the images by the maps induced by $B_i \to A$, we have $x_{1,A} = x_{2,A}$ and $x'_{1,A} = x'_{2,A}$. If we note $x_{i,A} = x_A$ and $x'_{i,A} = x'_A$, we see that $x'_A = g_{1,A} x_A = g_{2,A} x_A$. As the action is free, this implies that $g_{1,A} = g_{2,A}$. We see that (g_1, g_2) is an element g of $G(B_1 \times_A B_2)$. We have $(x'_1, x'_2) = g(x_1, x_2)$ and (x_1, x_2) and (x'_1, x'_2) have the same image in $O(B_1 \times_A B_2)$. This proves injectivity.

Let us prove 2). The map $X(\mathbb{F}[\epsilon]) \to O(\mathbb{F}[\epsilon])$ is surjective (by definition of O) and is a morphism of $\mathbb{F}$-vector spaces (as $X \to O$ is a morphism of functors). In other words, $t_X \to t_O$ is a surjective morphism of $\mathbb{F}$-vector spaces. It follows that t_O is finite dimensional. The representability of O is proved.

Let us prove that $X \to O$ is smooth. Let $B \to A$ be a surjective map of Artinian $\mathrm{CNL}_{\mathcal{O}}$-algebras. Let $x_A \in X(A)$ and $o_B \in O(B)$ which have the same images in $O(A)$. We have to prove that there exists $x_B \in X(B)$ which maps to x_A and o_B in the maps $X(B) \to X(A)$ and $X(B) \to O(B)$ respectively. Let $\widetilde{o_B}$ be an element of $X(B)$ in the orbit o_B. If $\widetilde{o_B}_A$ is the image of $\widetilde{o_B}$ in $O(A)$, there exists $g_A \in G(A)$ such that $x_A = g_A \widetilde{o_B}_A$. As G is smooth, there exists $g_B \in G(B)$ whose image in $G(A)$ is g_A. We can take $x_B = g_B \widetilde{o_B}$. This proves the smoothness of $X \to O$.

It is clear that the orbits of t_G acting on t_X identifies to the quotient t_X/t_G. This proves the description of the tangent spaces.

As $X \to O$ is smooth, it has a section s. This allows to define a map $f : X \to G$ by sending, for any Artinian A, $x_A \in X(A)$ to the element $g_A \in G(A)$ such that $x_A = g_A s(\overline{x_A})$ where $\overline{x_A}$ is the image of x_A in $O(A)$. The morphism $X \to G \times O$ which associates to x_A the couple $(f(x_A), \overline{x_A})$ is obviously an isomorphism. This proves that X is a torsor on O with group G_O and that this torsor is trivial.

We are left to prove that the affine algebra $A(O)$ identifies to $A(X)_0$. We have seen that X with its G action is isomorphic to $G_O = G \times O$. We have to prove that if a is a function on G_O which is such that $a(g'g) = a(g)$ for any $A(O)$-algebra B which is $\mathrm{CNL}_{\mathcal{O}}$ and Artinian, each g and g' in $G(B)$, then a is a constant of $G_O \to O$ *i.e.* belongs to $A(O)$. But this is clear. This ends the proof of the proposition. □

2.5. **Diagonalizable groups.** Let $\mathfrak{a}$ be an abelian finitely generated group whose torsion is of order a power of p. We define the diagonalizable group with character group $\mathfrak{a}$ in the $\mathrm{CNL}_{\mathcal{O}}$ category as the completion of the usual diagonalizable group ([14] exp.8) at the neutral element of the special fiber. We note it $(\mathfrak{a})^*$. If we have an action of $(\mathfrak{a})^*$ on a representable set X in the $\mathrm{CNL}_{\mathcal{O}}$ category, we define as in the previous section the algebra $A(X)_0$ of functions on X that are constant in the orbits of $(\mathfrak{a})^*$.

Proposition 2.6. *Let X be a representable set in $\mathrm{CNL}_{\mathcal{O}}$ and $D = (\mathfrak{a})^*$ be a diagonalizable group in $\mathrm{CNL}_{\mathcal{O}}$ acting freely on X. Then :*

1) A quotient $D\backslash X$ exists in $\mathrm{CNL}_{\mathcal{O}}$ (i.e. the morphism $X \to D\backslash X$ is universal for D-morphisms $X \to Y$ with the action of D on Y trivial). The morphism $X \to D\backslash X$ makes X a torsor under $D_{D\backslash X}$. The affine algebra of $D\backslash X$ is $A(X)_0$.

2) If $\mathfrak{a}' \to \mathfrak{a}$ is a surjective morphism, so $D' = (\mathfrak{a}')^*$ is a closed subgroup of D, $D'\backslash X$ has a natural free action of D/D' and $D\backslash X$ is naturally isomorphic to the quotient of $D'\backslash X$ by the action of D/D'.

Proof. In case $\mathfrak{a}$ is without torsion, 1) is a special case of proposition 2.5. In case $\mathfrak{a}$ is torsion, it is finite and of order a power of p. The affine algebra of the diagonalizable group of character group $\mathfrak{a}$ in the category of group schemes over $\mathcal{O}$ is a $\mathrm{CNL}_{\mathcal{O}}$-algebra. It follows that it coincides with the affine algebra $A((\mathfrak{a})^*)$. The case where $\mathfrak{a}$ is finite then follows from th. 5.1. of [14] exp.8. Then, the proofs of general case of 1) and the proof of 2) is a "dévissage de routine". □

2.6. **Truncations and chunks.** Let m be an integer ≥ 1. Let $\mathrm{CNL}_{\mathcal{O}}^{[m]}$ be the full subcategory of $\mathrm{CNL}_{\mathcal{O}}$-algebras whose objects are $\mathrm{CNL}_{\mathcal{O}}$-algebras A such that $\mathsf{m}_A^m = (0)$. Such algebras are Artinian. For A a $\mathrm{CNL}_{\mathcal{O}}$-algebra, we note $A^{[m]}$ the $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-algebra A/m_A^m. We see that $A \mapsto A^{[m]}$ defines a functor from $\mathrm{CNL}_{\mathcal{O}}$-algebras to $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-algebras. We have an isomorphism of functors from the restrictions of the functors of points Sp_A and $\mathrm{Sp}_{A^{[m]}}$ to the category $\mathrm{CNL}_{\mathcal{O}}^{[m]}$. If A represents the functor X we denote by $X^{[m]}$ the functor represented by $A^{[m]}$. We say that $A^{[m]}$ is obtained by truncation of level m from A. If $X = X^{[m]}$, any $\mathrm{CNL}_{\mathcal{O}}$ map $X \to Y$ factors through $Y^{[m]}$. A map $X \to Y$ is a closed immersion if and only if the maps $X^{[m]} \to Y^{[m]}$ are for all m : this follows from the fact that $A(Y) \to A(X)$ is surjective if and only if the $A(Y)^{[m]} \to A(X)^{[m]}$ are.

One has to be careful that if A_1 and A_2 are $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-algebras, $A_1 \otimes_{\mathcal{O}} A_2$ is not necessarily a $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-algebra. The restriction of the functor $\mathrm{Sp}_{A_1} \times \mathrm{Sp}_{A_2}$ to $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-algebras is represented by $(A_1 \otimes_{\mathcal{O}} A_2)^{[m]}$.

This leads to the definition of a group chunk of level m. It is given by a $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-algebra $A(G)$ and a morphism $(G \times G)^{[m]} \to G$, a map "$g \mapsto g^{-1}$" from G to G and a "neutral element" in $G(O^{[m]})$ which satisfy commutative diagrams which are similar to the usual commutative diagrams of the definition of a group scheme. Such a group chunk defines a functor of groups on $\mathrm{CNL}_{\mathcal{O}}^{[m]}$.

Let G be a representable group chunk of level m and X be a representable set of level m. We define in a similar way a group action chunk as being a map $(G \times X)^{[m]} \to X$ satisfying analogue of usual compatibilities. The map $(g, x) \mapsto (gx, x)$ defines a map $(G \times X)^{[m]} \to X \times X$ that factors through $(X \times X)^{[m]}$. One says that the group chunk action is free if the map $(G \times X)^{[m]} \to (X \times X)^{[m]}$ is a closed immersion.

If G is a representable group in $\mathrm{CNL}_{\mathcal{O}}$, we get by reduction a group chunk $G^{[m]}$ of level m. Idem if G acts on a representable X.

Let G be a group acting on a set X which are representable in $\mathrm{CNL}_{\mathcal{O}}$. The data of the truncated group actions of level m, for all m, determines G, X and the action of G on X. The algebras $A(G)$ and $A(X)$ are the projective limits of the $A(G^{[m]})$ and $A(X^{[m]})$. The action is free if and only if the group action chunks are. This follows from the fact that $G \times X \to X \times X$ is a closed immersion if and only if the $(G \times X)^{[m]} \to (X \times X)^{[m]}$ are.

We shall need two technical results. The first one allows to construct group actions as inductive limit of group action chunks. The second one allows to construct actions of tori.

Let G be a representable group in $\mathrm{CNL}_{\mathcal{O}}$. Let be for each m a $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-algebra A_m, with morphisms $\pi_m : A_{m+1} \to A_m$. Call $X_m := \mathrm{Sp}_{A_m}$. One supposes that one has a $\mathrm{CNL}_{\mathcal{O}}$-algebra C and surjective morphisms $C \to A_m$ that are compatible with the morphisms π_m. Let A_∞ be the projective limit of the A_m, so that A_∞ is a $\mathrm{CNL}_{\mathcal{O}}$-algebra. If X_∞ is the functor Sp_{A_∞}, one has natural immersions $X_m \to X_\infty^{[m]}$.

Let for each m be a group action chunk of level m of $G^{[m]}$ on X_m, with compatibility with the map $G^{[m]} \to G^{[m+1]}$ and the map $X_m \to X_{m+1}$ induced by π_m.

Proposition 2.7. *Then we have a unique group action of G on X_∞ such that for each m the group action chunk of level m of $G^{[m]}$ on $X_\infty^{[m]}$ is compatible with the given group action chunk of $G^{[m]}$ on X_m via the immersion*

$X_m \to X_\infty^{[m]}$. If the group action chunks of $G^{[m]}$ on X_m are free, so is the action of G on X_∞.

Proof. It follows easily from the facts :

- for each m there exists an n such that $X_\infty^{[m]} \hookrightarrow X_n$; this allows to replace the X_m by the $X_\infty^{[m]}$;

- we have immersions $G^{[m]} \times X_\infty^{[m]} \to (G^{[2m]} \times X_\infty^{[2m]})^{[2m]}$ that allow to define the map $(G \times X_\infty) \to X_\infty$. □

Proposition 2.8. *Let $\mathfrak{T} = (\mathbb{Z}^t)^*$ be the (split) torus of dimension t. Let $m \geq 0$ be an integer and let $\mathfrak{T}_{p^m} \hookrightarrow \mathfrak{T}$ be the closed subgroup $((\mathbb{Z}/p^m\mathbb{Z})^t)^*$. Then, the closed immersion $\mathfrak{T}_{p^m} \hookrightarrow \mathfrak{T}$ induces an isomorphism of truncated group schemes $\mathfrak{T}_{p^m}^{[m+1]} \simeq \mathfrak{T}^{[m+1]}$.*

Proof. It follows from the fact that the polynomial $(1+X)^{p^m}-1$ that defines μ_{p^m} as a closed subscheme of $\hat{\mathbb{G}}_{\rm m} := (\mathbb{Z})^*$ belongs to the ideal $(p, X)^{m+1}$. □

2.7. **Inertia-rigid deformations.** The present paragraph will be applied to deformations of representations of local Galois groups with finite and fixed restriction to inertia.

Let G be a profinite group with $I \subset G$ a finite normal subgroup such that the quotient G/I is isomorphic to the free rank one profinite group. Let $F \in G$ be such that the image of F in G/I is a generator of G/I.

Let $\bar{\rho} : G \to \mathrm{GL}_d(\mathbb{F})$ be a continuous representation. We fix a lift $\rho_0 : G \to \mathrm{GL}_d(\mathcal{O})$ of $\bar{\rho}$. Let $\phi : G \to \mathcal{O}^*$ be the determinant of ρ_0.

Let $\mathcal{M}_\phi$ be the affine $\mathcal{O}$-scheme of finite type whose points with values in a $\mathcal{O}$-algebra A are the following data:

- DATA : a morphism ρ_I of I to $\mathrm{GL}_d(A)$, an element $f \in \mathrm{GL}_d(A)$ which normalizes $\rho_I(I)$ and such that $\mathrm{int}(f)(\rho_I(\tau)) = \rho_I(\mathrm{int}(F)(\tau))$ for $\tau \in I$, $\det(\rho_I) = \phi_{|I}$ and $\det(f) = \phi(F)$, where $\mathrm{int}(f)$ and $\mathrm{int}(F)$ are conjugation by f and F respectively.

Let $|I|$ be the cardinality of I. The map from $\mathcal{M}_\phi$ to $(\mathrm{GL}_d)^{|I|+1}$ which associates to such a DATA the $\rho_I(\tau)$ with $\tau \in I$ and f is a closed immersion. The equations are given by the multiplication table of I, the action of $\mathrm{int}(F)$ on I, and the condition that the determinant is ϕ.

Let $\mathcal{M}_{\phi,0}$ be the closed subscheme of $\mathcal{M}_\phi$ given by imposing the equality of the characteristic polynomials $P_{\rho_I(\tau)} = P_{\rho_0(\tau)}$ for all $\tau \in I$. Let us denote by A_* the affine algebra of $\mathcal{M}_*$. We have a universal DATA with values in $A_{\phi,0}$. Let $\mathcal{M}_{\phi,0,\mathrm{fl}}$ be the closed subscheme of $\mathcal{M}_{\phi,0}$ whose affine algebra is the quotient of $A_{\phi,0}$ by its p-torsion.

Lemma 2.9. *Let $\xi_{\bar{\rho}}$ be the point of $\mathcal{M}_{\phi,0}$ defined by $\bar{\rho}$. Then, $\xi_{\bar{\rho}}$ is a point of $\mathcal{M}_{\phi,0,\mathrm{fl}}$.*

Proof. The representation ρ_0 defines a point $\xi_0 \in \mathcal{M}_{\phi,0}(\mathcal{O})$. This point factorizes through $\mathcal{M}_{\phi,0,\mathrm{fl}}$. The image of the closed point of $\mathrm{Spec}(\mathcal{O})$ is $\xi_{\bar{\rho}}$ and lies in $\mathcal{M}_{\phi,0,\mathrm{fl}}$. □

Let $\overline{R}^{\square}_{\phi,0,\mathrm{fl}}$ be the completion of $A_{\phi,0,\mathrm{fl}}$ relative to the maximal ideal defined by $\xi_{\bar\rho}$. It is a faithfully flat local $\mathcal{O}$-algebra ; we still denote by $\xi_{\bar\rho}$ its closed point. The residue field of $\xi_{\bar\rho}$ is $\mathbb{F}$. So we see that $\overline{R}^{\square}_{\phi,0,\mathrm{fl}}$ is an objet of $\mathrm{CNL}_{\mathcal{O}}$.

Proposition 2.10. *Each irreducible component of* $\mathrm{Spec}(\overline{R}^{\square}_{\phi,0,\mathrm{fl}})$ *is faithfully flat of absolute dimension d^2 ; $\overline{R}^{\square}_{\phi,0,\mathrm{fl}}[1/p]$ is regular.*

Proof. Let us prove first that the generic fiber $\mathcal{M}_{\phi,0}[1/p]$ is smooth over E of dimension d^2-1. Let $\mathcal{C}$ be the commutant of $\rho_{0|I}$ in $M_d(\mathcal{O})$. Let C^* be the mutiplicative group of $E\otimes_{\mathcal{O}}\mathcal{C}$ and C_1^* be the subgroup of C^* of elements of determinant 1. We also view C^* and C_1^* as algebraic groups over E. Let M_I be the scheme over E that parametrizes the morphisms of I to $(\mathrm{GL}_d)_E$ that are conjugate to $\rho_{0|I}$. It is isomorphic to $(\mathrm{GL}_d)_E/C^*$. It is smooth of dimension $d^2-\dim(C^*)$. If we forget F, we get a map of $\mathcal{M}_{\phi,0}[1/p]$ to M_I; this map makes $\mathcal{M}_{\phi,0}[1/p]$ a M_I torsor under C_1^*. It follows that $\mathcal{M}_{\phi,0}[1/p]$ is smooth of dimension $d^2-\dim(C^*)+\dim(C_1^*)$. As the homotheties are in C^*, we have $\dim(C^*)=\dim(C_1^*)+1$, and the relative dimension of $\mathcal{M}_{\phi,0}[1/p]$ is d^2-1.

The affine algebra of $\mathcal{M}_{\phi,0}$ is finitely generated over $\mathcal{O}$. It follows that it is excellent (paragraph 34 of [38]). As then, by Grothendieck, the completion morphism is regular (th. 79 of [38]) and $\mathcal{M}_{\phi,0}[1/p]$ is smooth over E, $\overline{R}^{\square}_{\phi,0,\mathrm{fl}}[1/p]$ is regular. As $\mathcal{M}_{\phi,0}[1/p]$ is smooth of relative dimension d^2-1, each of its irreducible components has dimension d^2-1. By faithful flatness, it follows that each irreducible component of the localization of $A_{\phi,0,\mathrm{fl}}$ at $\xi_{\bar\rho}$ has absolute dimension d^2. It follows from th. 31.6. of [39] that $\overline{R}^{\square}_{\phi,0,\mathrm{fl}}$ is equidimensional, and each of its irreducible components is of absolute dimension d^2. □

The tautological DATA with values in $A_{\phi,0,\mathrm{fl}}$ extends to a morphism of G to $\mathrm{GL}_d(\overline{R}^{\square}_{\phi,0,\mathrm{fl}})$ as $\overline{R}^{\square}_{\phi,0,\mathrm{fl}}$ is a projective limit of artinian $\mathcal{O}$-algebras. It is a lift of $\bar\rho$. We call it ρ_X.

Let $\mathcal{O}'$ be the ring of integers of a finite extension E' of the field of fractions E of $\mathcal{O}$, and let ξ be a local morphism of $\overline{R}^{\square}_{\phi,0,\mathrm{fl}}$ to $\mathcal{O}'$. Composing ρ_X with the morphism from $\mathrm{GL}_d(\overline{R}^{\square}_{\phi,0,\mathrm{fl}})$ to $\mathrm{GL}_d(\mathcal{O}')$, we get a lift ρ_ξ of $\bar\rho\otimes\mathbb{F}'$ with values in $\mathrm{GL}_d(\mathcal{O}')$.

Proposition 2.11. *The lifts ρ_ξ are exactly the lifts ρ of $\bar\rho$ with values in $\mathrm{GL}_d(\mathcal{O}')$ which have determinant ϕ and are such that the restriction of $\rho\otimes E'$ to I is conjugate to $(\rho_0)_{|I}\otimes E'$.*

Proof. The proposition follows from the fact that the isomorphism classes of representations of the finite group I with values in $\mathrm{GL}_d(E')$ are determined by their characters. □

Remark. By corollary 2.3, the proposition characterises $\overline{R}^{\square}_{\phi,0,\mathrm{fl}}$ as a quotient of the universal ring $\overline{R}^{\square}$ for lifts of $\bar{\rho}$.

2.8. **Resolutions of framed deformations.** Let G, $\bar{\rho}$ and $R^{\square}$ be as in 2.2. Let $\overline{R}^{\square}_X$ be a non-trivial quotient of $R^{\square}$ by an ideal which is stable by the action of $(\mathrm{GL}_d)_1$ on $R^{\square}$ induced by the conjugaison. As in the last paragraph of 2.2, this defines the subfunctor $\mathcal{D}^{\square}_X$ of framed deformations satisfying the condition X.

We will call a *smooth resolution* of $\mathcal{D}^{\square}_X$ the data of a flat $\mathcal{O}$-scheme $\mathcal{R}$, with an $\mathcal{O}$-morphism $f : \mathcal{R} \to \mathrm{Spec}(\overline{R}^{\square}_X)$ such that :

(1) f is proper surjective and with injective structural morphism : $\mathcal{O}_{\mathrm{Spec}(\overline{R}^{\square}_X)} \to f_*(\mathcal{O}_{\mathcal{R}})$;
(2) $\mathcal{R}[1/p] \to \mathrm{Spec}(R^{\square})[1/p]$ is a closed immersion;
(3) the inverse image $\mathcal{Y} \subset \mathcal{R}$ of the closed point of $\mathrm{Spec}(\overline{R}^{\square}_X)$ is geometrically connected;
(4) there is a smooth algebraization of $\mathcal{R} \to \mathcal{O}$.

By a smooth algebraization, we mean a $\mathcal{O}$-scheme $\mathcal{R}_0$ which is smooth of finite type, a closed subscheme $\mathcal{Y}_0$ of $\mathcal{R}_0$, a $\mathcal{O}$-morphism $\mathcal{R} \to \mathcal{R}_0$ which sends $\mathcal{Y}$ to $\mathcal{Y}_0$ and induces an isomorphism of formal schemes between the completions of $\mathcal{R}$ and $\mathcal{R}_0$ along $\mathcal{Y}$ and $\mathcal{Y}_0$ respectively. The smooth algebraization is part of the data defining the smooth resolution.

Remark : The property (1) implies that $\mathrm{Spec}(\overline{R}^{\square}_X)$ is the scheme theoretical closure of $\mathcal{R}$ in $\mathrm{Spec}(R^{\square})$. This implies that $\mathrm{Spec}(\overline{R}^{\square}_X[1/p])$ is the scheme theoretical closure of $\mathcal{R}[1/p]$ in $\mathrm{Spec}(R^{\square}[1/p])$, and, by (2), the morphism f induces an isomorphism of $\mathcal{R}[1/p]$ to $\mathrm{Spec}(\overline{R}^{\square}_X[1/p])$.

As $\mathcal{R}$ is flat over $\mathcal{O}$, we see that $\overline{R}^{\square}_X$ is flat over $\mathcal{O}$. The functor $\mathcal{D}^{\square}_X$ and its resolution are determined by the morphism $f : \mathcal{R} \to \mathrm{Spec}(R^{\square})$. The last part of the following proposition shows that one may think $\overline{\mathcal{O}}$-points of $\mathcal{R}$ that specialize in $\mathcal{Y}$ as data that define lifts of $\bar{\rho}$ and that the lifts that comes from $\mathcal{R}$ in this way are the lifts that satisfy the condition X.

Proposition 2.12. *Let X be as above a condition on deformations of $\bar{\rho}$ and and let $\mathcal{R}$ be a smooth resolution of $\mathcal{D}^{\square}_X$. Then $\overline{R}^{\square}_X$ is a domain, $\overline{R}^{\square}_X[1/p]$ is regular and the relative dimension of $\overline{R}^{\square}_X$ over $\mathcal{O}$ is the same as the relative dimension of $\mathcal{R}$ over $\mathcal{O}$. Let $\overline{\mathcal{O}}$ be the ring of integers of an algebraic closure of $\mathcal{O}$, and let $\mathcal{R}(\overline{\mathcal{O}})_c$ be the points that send $\mathcal{Y}$ to the closed point of $\overline{\mathcal{O}}$. Then, the set of framed deformations $\mathcal{D}^{\square}_X(\overline{\mathcal{O}})$ of $\bar{\rho}$ with values in $\overline{\mathcal{O}}$ that satisfy the condition X is the image of $\mathcal{R}(\overline{\mathcal{O}})_c$ in $\mathcal{D}^{\square}(\overline{\mathcal{O}})$.*

Proof. Let us first prove that $\overline{R}^{\square}_X$ is a domain. Let $\mathcal{R} \to \mathrm{Spec}(\Gamma(\mathcal{R}, \mathcal{O}_{\mathcal{R}})) \to \mathrm{Spec}(\overline{R}^{\square}_X)$ be the Stein factorization of f. The morphism $\overline{R}^{\square}_X \to \Gamma(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$

is finite. It is injective as f is injective for structural sheaves. It follows that $\Gamma(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$ is semi-local. As $\overline{R}_X^{\square}$ is complete, $\Gamma(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$ is also complete. The maximal ideals of $\Gamma(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$ are in the image of the inverse image $\mathcal{Y}$ of the closed point of $\mathrm{Spec}(\overline{R}_X^{\square})$ in $\mathcal{R}$. As $\mathcal{Y}$ is connected, we see that $\Gamma(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$ is local. Besides, by the theorem of formal functions, it is the ring of global sections of the completion $\widehat{\mathcal{R}}$ of $\mathcal{R}$ along $\mathcal{Y}$. By (2), this formal scheme is isomorphic to the completion of the smooth $\mathcal{O}$-scheme $\mathcal{R}_0$ along $\mathcal{Y}_0$. It follows that $\Gamma(\mathcal{R}, \mathcal{O}_{\mathcal{R}}) = \Gamma(\widehat{\mathcal{R}}, \mathcal{O}_{\widehat{\mathcal{R}}})$ is normal. As it is a local ring, it is a domain. As $\overline{R}_X^{\square}$ injects in it, $\overline{R}_X^{\square}$ is a domain.

Let us prove that the relative dimension of $\overline{R}_X^{\square}$ over $\mathcal{O}$ is the same as the relative dimension of $\mathcal{R}$ over $\mathcal{O}$. The morphism f induces an isomorphism of $\mathcal{R}[1/p]$ to $\mathrm{Spec}(\overline{R}_X^{\square}[1/p])$. As $\overline{R}_X^{\square}$ is flat over $\mathcal{O}$, the relative dimension of $\overline{R}_X^{\square}$ over $\mathcal{O}$ is the same as the dimension of $\overline{R}_X^{\square}[1/p]$. It is the dimension of $\mathcal{R}$ over $\mathcal{O}$.

Let us prove that $\overline{R}_X^{\square}[1/p]$ is regular. Let $\wp \in \mathcal{R}$. Let $V(\wp)$ be its closure in $\mathcal{R}$. By the proper map f, $V(\wp)$ maps onto a closed subset of $\mathrm{Spec}(\overline{R}_X^{\square})$. As $\overline{R}_X^{\square}$ is local, $f(V(\wp))$ contains the closed point, and $V(\wp)$ non trivially intersects $\mathcal{Y}$. Let $Q \in \mathcal{Y} \cap V(\wp)$, and let $\mathcal{U} \subset \mathcal{R}$ be an affine open set containing Q. We see that $\wp \in \mathcal{U}$. Furthermore, as $Q \in V(\wp) \cap \mathcal{U} \cap \mathcal{Y}$, $\wp$ belongs to the image of the map from the spectrum of the completion $\Gamma(\mathcal{U} \cap \mathcal{Y}, \mathcal{O}_{\widehat{\mathcal{R}}})$ of $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{R}})$ along $\mathcal{U} \cap \mathcal{Y}$ to the spectrum of $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{R}})$ (24.B of [38]). Let $\hat{\wp}$ in the spectrum of $\Gamma(\mathcal{U} \cap \mathcal{Y}, \mathcal{O}_{\widehat{\mathcal{R}}})$ mapping to $\wp$. As $\mathcal{R}_0$ is smooth over $\mathcal{O}$, $\Gamma(\mathcal{U} \cap \mathcal{Y}, \mathcal{O}_{\widehat{\mathcal{R}}})$ is regular over $\mathcal{O}$. As the map from the local ring of $\mathcal{R}$ at $\wp$ to the local ring of $\widehat{\mathcal{R}}$ at $\hat{\wp}$ is faithfully flat, it follows that the localization of $\overline{R}_X^{\square}[1/p]$ at $\wp$ is regular (lemma 33.B of [38]). As this is true for all $\wp \in \mathrm{Spec}(\overline{R}_X^{\square}[1/p])$, we have proved that $\overline{R}_X^{\square}[1/p]$ is regular.

Let us prove the description of the points of $\overline{R}_X^{\square}$ given in the statement of the proposition. It is clear that the image of $\mathcal{R}(\overline{\mathcal{O}})_c$ in $\mathcal{D}^{\square}(\overline{\mathcal{O}})$ is included in $\mathcal{D}_X^{\square}(\overline{\mathcal{O}})$. Let us prove the opposite inclusion. Let $\mathcal{O}'$ be the ring of integers of a finite extension E' of the field of fractions E of $\mathcal{O}$. Let $\xi : \mathrm{Spec}(\mathcal{O}') \to \mathrm{Spec}(\overline{R}_X^{\square})$ be a local homomorphism. Let ξ_η be the generic fiber of ξ. As f induces an isomorphism of $\mathcal{R}[1/p]$ to $\mathrm{Spec}(\overline{R}_X^{\square}[1/p])$, we can view ξ_η as a point y_η of $\mathcal{R}$ with values in E'. By properness, y_η extends to a point of $\mathcal{R}$ with values in $\mathcal{O}'$. The closed point of $\mathrm{Spec}(\mathcal{O}')$ has image in $\mathcal{Y}$. This ends the proof of the proposition. $\square$

3. Structure of certain local deformation rings

In this section, $\mathbb{F}$, $\mathcal{O}$, E and π are as in the previous one. In particular, $\mathbb{F}$ is a finite field of characteristic p. Let q be a prime. Consider a local field F_v, finite extension of $\mathbb{Q}_q$, with $D_v = \mathrm{Gal}(\overline{\mathbb{Q}_q}/F_v)$, and a continuous representation $\bar{\rho}_v : D_v \to \mathrm{GL}_2(\mathbb{F})$.

In results below about presentations of deformation rings (see Section 4) and $R = \mathbb{T}$ theorems (see Propositions 9.2 and 9.3 in Section 9.1), we need information about certain local deformation rings $\overline{R}_v^{\square,\psi}$ which are quotients of $R_v^{\square,\psi}$. These classify, in the sense of Definition 2.4, a set of morphisms such that corresponding p-adic Galois representations satisfy prescribed conditions X_v, including fixed determinant $\phi = \psi\chi_p$. In the theorem below, we state the needed information for $\bar{R}_v^{\square,\psi}$ that arises from prescribed conditions that we refer to by a name, and which is explained when we treat the different cases. Thus the morphisms $\bar{R}_v^{\square,\psi} \to \mathcal{O}'$, with $\mathcal{O}'$ the ring of integers of a finite extension E' of E, correspond to liftings which satisfy the prescribed conditions.

Theorem 3.1. *We make the assumption that, when v is above p, F_v is unramified over $\mathbb{Q}_p$ and, if furthermore $\bar{\rho}_v$ is irreducible, F_v is $\mathbb{Q}_p$. We also suppose that in case 3.3.3 of v not above p, $F_v = \mathbb{Q}_q$.*

The rings $\bar{R}_v^{\square,\psi}$ have the following properties:

- *$v = \infty$, odd deformations : $\bar{R}_v^{\square,\psi}$ is a domain, flat over $\mathcal{O}$ of relative dimension 2 , and $\bar{R}_v^{\square,\psi}[\frac{1}{p}]$ is regular.*
- *v above p, low weight crystalline deformations, semistable weight 2 deformations, weight 2 deformations crystalline over $\mathbb{Q}_p^{nr}(\mu_p)$: $\bar{R}_v^{\square,\psi}$ is a domain (see remark below), flat over $\mathcal{O}$ of relative dimension $3 + [F_v : \mathbb{Q}_p]$, and $\bar{R}_v^{\square,\psi}[\frac{1}{p}]$ is regular.*
- *v a finite place not above p and semistable deformations: $\bar{R}_v^{\square,\psi}$ is a domain (see remark below), flat over $\mathcal{O}$, of relative dimension 3, and $\bar{R}_v^{\square,\psi}[\frac{1}{p}]$ is regular.*
- *v a finite place not above p, inertia-rigid deformations: $\bar{R}_v^{\square,\psi}$ is flat over $\mathcal{O}$, with each component of relative dimension 3, and $\bar{R}_v^{\square,\psi}[\frac{1}{p}]$ is regular.*

Thus in particular by Proposition 2.2 we know in each of the cases that $\bar{R}_v^{\square,\psi}$ has points with values in the ring of integers of a finite extension of $\mathbb{Q}_p$.

Remarks : The hypothesis $F_v = \mathbb{Q}_q$ in case 3.3.3 has no really conceptual reason : it is because we are lazy to treat cases that we do not need.

In fact in all cases considered in the theorem, the local deformation rings turn out to be domains. As the calculations to prove this are more elaborate, and we do not need this finer information, we content ourselves with proving the theorem. We stress that in the definitions below, we have to make some choices to guarantee that $\bar{R}_v^{\square,\psi}$ is a domain. When $k(\bar{\rho}_v) = p$ and $\bar{\rho}_v$ is unramified non scalar, we choose one of the two characters and consider ordinary lifts with unramified quotient reducing to the eigenspace for this eigenvalue (3.6). When $p = 2$ and we are in the v above p semistable weight

2 case (3.2.6) or in the v not above p twist of semistable case (3.3.4), we have to choose the character γ_v. See also the inertia rigid case 3.3.2.

The proof of the theorem will take up the rest of the section.

We will need the following proposition.

Proposition 3.2. *Let us suppose that the conditions X_v are one of those of Theorem 3.1. After possibly replacing $\mathcal{O}$ by the ring of integers of a finite extension of E, we have :*

- (i) the completed tensor product $\bar{R}^{\square,\mathrm{loc},\psi} := \hat{\otimes}_{v\in S}\bar{R}_v^{\square,\psi}$ is flat over $\mathcal{O}$, each of its component is of relative dimension $3|S|$, and $\bar{R}^{\square,\mathrm{loc},\psi}[1/p]$ is regular ;

- (ii) if further for finite places in S not above p the corresponding deformation problem considered is of semistable type then it is also a domain.

Proof. The proposition follows from the proposition of section 2.3 and from Theorem 3.1, noticing that the part of the tensor product coming from infinite places contributes $2[F:\mathbb{Q}]$ to the relative dimension and the part above p contributes $3|S_p|+[F:\mathbb{Q}]$, where S_p is the number of places of F above p. □

In the next paragraphs, we will denote by V a free $\mathcal{O}$-module of rank d and for A an $\mathcal{O}$-algebra, we write V_A for $A\otimes_{\mathcal{O}} V$; V_* will be the underlying space of the lifts of $\bar{\rho}_v$. We will call $\overline{e_1},\overline{e_2}$ a basis of $V_{\mathbb{F}}$, and e_1,e_2 a lift of this basis.

3.1. **The case $v=\infty$.** We recall that our notation for $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is D_∞ and that c is the complex conjugation. We give ourselves $\bar{\rho}_\infty : D_\infty \to \mathrm{GL}_V = \mathrm{GL}_2(\mathbb{F})$. We suppose that $\bar{\rho}$ is odd *i.e.* $\det(\bar{\rho}(c)) = -1$.

Proposition 3.3. *There is a flat and reduced $\mathrm{CNL}_{\mathcal{O}}$ algebra $\overline{R}_\infty^{\square,\psi}$ which classifies (in the sense of Definition 2.4) the odd lifts of $\bar{\rho}_\infty$. If $\bar{\rho}(c)\neq \mathrm{id}$ (it is always the case if $p\neq 2$), $\overline{R}_\infty^{\square,\psi}$ is formally smooth of relative dimension 2. If $p=2$ and $\bar{\rho}(c)=\mathrm{id}$, $\overline{R}_\infty^{\square,\psi}$ is a domain of relative dimension 2 with regular generic fiber.*

Proof. We will see that the ring $\overline{R}_\infty^{\square,\psi}$ is the completion of the ring of functions of the affine scheme of 2×2 matrices of characteristic polynomial X^2-1 at the point defined by $\bar{\rho}_\infty$. We surely could give a shorter proof of the proposition using this. We give a proof using resolutions (Section 2.8) to show how it works. Let $\overline{M}=\bar{\rho}(c)\in \mathrm{GL}_2(\mathbb{F})$.

If $p\neq 2$, let $M\in \mathrm{GL}_2(\mathcal{O})$ be a lift of $\overline{M}$ with characteristic polynomial X^2-1. We have a decomposition : $V=L_1\oplus L_2$ with L_1 and L_2 lines that are the eigenspaces for M for eigenvalues 1 and -1 respectively. Let D^* be the $\mathcal{O}$-scheme of diagonal matrices relatively to this decomposition. The quotient GL_2/D^* is isomorphic to the open subset of $\mathbb{P}_1\times\mathbb{P}_1$ which is the complement of the diagonal. It is smooth of relative dimension 2. The matrix $\overline{M}$ defines a closed point $\xi_{\overline{M}}$ of GL_2/D^*. The $\mathcal{O}$-algebra $\overline{R}_\infty^{\square,\psi}$ is the completion of the local ring of GL_2/D^* at this point.

Let $p=2$. Let us construct a smooth resolution which is an isomorphism if $\overline{M}$ is not the identity.

Let $\mathcal{M}_2$ be the $\mathcal{O}$-scheme which represents linear automorphisms of V whose square is id and $\mathcal{M}(X^2-1)$ be its closed subscheme which represents those whose characteristic polynomial is X^2-1. The matrix $\overline{M}$ defines a point $\xi_{\overline{M}} \in \mathcal{M}(X^2-1)(\mathbb{F})$. We can choose the basis $\overline{e_1}, \overline{e_2}$ of $V_{\mathbb{F}}$ such that $\overline{M}$ is either id or :

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The universal ring for framed deformations $R_\infty^\square$ is the completion of the local ring of $\mathcal{M}_2$ at $\xi_{\overline{M}}$.

Let $\mathcal{R}_0$ be the closed subscheme of $\mathcal{M}(X^2-1) \times_{\mathcal{O}} (\mathbb{P}_1)_{\mathcal{O}}$ which represents pairs (M,L), where $M \in \mathcal{M}(X^2-1)$ and L a submodule of V such that V/L is a locally free module of rank 1 ; we furthermore ask that $M(L)=L$ and that M acts as id on L.

The first projection f_1 is projective. Let us prove that it induces an isomorphism of the open subschemes of $\mathcal{R}_0$ and $\mathcal{M}(X^2-1)$ where $M \neq \mathrm{id}$. We have to see that, if A a local $\mathcal{O}$-algebra with residue field $k(A)$, and if $M_A \in \mathcal{M}(X^2-1)(A)$ is such that the image $M_{k(A)}$ of M_A in $\mathcal{M}(X^2-1)(k(A))$ is not the identity, there exists a unique line L in V_A such that M_A acts as identity on L. This follows by elementary linear algebra from $\det(M_A-\mathrm{id}) = 0$ and $\mathrm{rank}(M_{k(A)}-\mathrm{id})=1$.

The only point of $\mathcal{M}(X^2-1)$ such that $M=\mathrm{id}$ is the identity in the special fiber ($M=\mathrm{id}$ implies that $\det(M)=1=-1$). We see that f_1 induces an isomorphism of the generic fibers and above the complement of identity in the special fiber. It contracts $\mathrm{id} \times \mathbb{P}_1$ in the special fiber to the point $\mathrm{id} \in \mathcal{M}(X^2-1)(\mathbb{F})$. We call f_0 the morphism from $\mathcal{R}_0$ to $\mathcal{M}_2$ which is the compositum of the first projection f_1 and the closed immersion of $\mathcal{M}(X^2-1)$ in $\mathcal{M}_2$. The second projection makes $\mathcal{R}_0$ a torsor on $\mathbb{P}_1$ under $\mathrm{Hom}(V/L,L)$. We see that $\mathcal{R}_0$ is smooth over $\mathcal{O}$ of relative dimension 2.

We claim that the scheme theoretical image of f_0 is $\mathcal{M}(X^2-1)$. This is because f_1 is surjective being proper with dense image and $\mathcal{M}(X^2-1)$ is integral (it is the quadric of equation $X_{11}^2+X_{12}X_{21}=1$, X_{ij} being the entries of M). We define $\overline{R}_\infty^{\square,\psi}$ as the completion of the local ring of $\mathcal{M}(X^2-1)$ at $\xi_{\overline{M}}$. We define $\mathcal{R}$ and f as the base change by $\mathrm{Spec}(\overline{R}_\infty^{\square,\psi}) \to \mathcal{M}_2$ of $\mathcal{R}_0$ and f_0.

If $\overline{M} \neq \mathrm{id}$, $\xi_{\overline{M}}$ belongs to the open subscheme of $\mathcal{M}(X^2-1)$ above which f_0 is an isomorphism and $\overline{R}_\infty^{\square,\psi}$ is formally smooth of dimension 2 over $\mathcal{O}$.

Let us suppose that $\overline{M} = \mathrm{id}$ and let us check that we get a smooth resolution as defined in section 2.8. First, $\mathrm{Spec}(\overline{R}_\infty^{\square,\psi}) \to R^\square$ is obviously GL_2-equivariant. The condition (1) is satisfied as the analogous condition is satisfied by $\mathcal{R}_0$, f_1 and $\mathcal{M}(X^2-1)$, and by flatness of the completion. We saw that $f_0[1/2]$ is a closed immersion. This also holds for f and the hypothesis (2) is satisfied. The inverse image $\mathcal{Y}$ of $\xi_{\overline{M}}$ in $\mathcal{R}$ is $(\mathrm{id}_V, \mathbb{P}_1)$. It is geometrically connected. We already saw that $\mathcal{R}_0$ is smooth over $\mathcal{O}$. Thus, (3) and (4) follow by construction of $\mathcal{R}_0$ and $\mathcal{R}$.

It follows that the conclusions of prop. 2.12 are satisfied : $\overline{R}_\infty^{\square,\psi}$ is a domain, faithfully flat over $\mathcal{O}$ of relative dimension 2, with regular generic fiber.

Let us come back to the general hypotheses of the proposition. Let $\overline{\mathcal{O}}$ be the ring of integers of an algebraic closure $\overline{E}$ of E. To finish the proof of the proposition, we have to prove that the points of $\mathrm{Sp}(\overline{R}_\infty^{\square,\psi})(\overline{\mathcal{O}})$, correspond to odd lifts of $\bar{\rho}$. By proposition 2.12, these points correspond to matrices M of $\mathrm{GL}_2(\overline{\mathcal{O}})$ that lift $\overline{M}$, have characteristic polynomial X^2-1 and are such that there exists a line L of $V_{\overline{\mathcal{O}}}$ which is a direct factor and on which M acts as identity. For M with characteristic polynomial X^2-1, the line L which is the intersection of the eigenspace for eigenvalue 1 in $V_{\overline{E}}$ with $V_{\overline{\mathcal{O}}}$ satisfy these conditions. That finishes the proof. □

Remark. If $p = 2$ and $\overline{M} = \mathrm{Id}$, it is not difficult to see that $\overline{R}_\infty^{\square,\psi}$ is isomorphic to $\mathcal{O}[[X_1, X_2, X_3]]/(X_1^2 + X_2X_3 + 2X_1)$; it is a relative complete intersection.

3.2. **The case of v above p.**

3.2.1. *Local behaviour at p of p-adic Galois representations.* It is convenient to make the following definition.

Definition 3.4. *1. Suppose V is a 2-dimensional continuous representation with coefficients in E of G_F with E, F finite extensions of $\mathbb{Q}_p$. We say that V is of weight k if for all embeddings $\iota : E \hookrightarrow \mathbb{C}_p$, $V \otimes_E \mathbb{C}_p = \mathbb{C}_p \oplus \mathbb{C}_p(k-1)$ as G_F-modules.*

2. Suppose V is a continuous representation, with V a free rank 2 module over a $\mathrm{CNL}_{\mathcal{O}}$-algebra R, of G_F with F a finite extension of $\mathbb{Q}_p$. We say that V is ordinary if there is a free, rank one submodule W of V that is G_F stable, such that V/W is free of rank one over R with trivial action of the inertia I_F of G_F and the action of an open subgroup of I_F on W is by χ_p^a, for a rational integer $a \geq 0$.

We have the following lemma ; recall that we have fixed $\bar{\rho}$.

Lemma 3.5. *Let F be an unramified extension of $\mathbb{Q}_p$, V a 2-dimensional vector space over a p-adic field E and $\rho : G_F \to \mathrm{Aut}(V)$ a continuous representation that lifts $\bar{\rho}$. Then:*

(i) if V is crystalline of weight k such that $2 \leq k \leq p$, V is ordinary if residually it is ordinary. The same is true for $2 \leq k \leq p+1$ if $F = \mathbb{Q}_p$.

(ii) if V is semistable non crystalline of weight 2, then V is ordinary.

(iii) if V is of weight 2 and crystalline over $\mathbb{Q}_p^{\mathrm{nr}}(\mu_p)$, then V is ordinary, if residually it is ordinary.

Proof. We sketch a proof of the first part of (i). Let $D(V)$ be the filtered Dieudonné module associated to V by Fontaine and Laffaille ([24]). Let M be a strongly divisible lattice of $D(V)$ which is stable by the action of $\mathcal{O}$ and let $\overline{M}$ its reduction, with its structure of object of the category MF of filtered Dieudonné module of finite length. Let $\overline{M}_{\overline{k_F}}$ be the filtered Dieudonné module obtained from $\overline{M}$ by extending scalars from the residual field k_F of F to its algebraic closure $\overline{k_F}$. It is not difficult to see that :

- Either $\overline{M}_{\overline{k_F}}$ is a direct sum of simple objects (in MF) isomorphic to the filtered Dieudonné module associated by Fontaine-Laffaille to the sequence $(0, k-1)$ in 0.7 of [24] and the restriction of $\bar{\rho}$ to inertia is $\Psi^{k-1} \oplus \Psi^{p(k-1)}$ where Ψ is the fundamental character of level 2 (th. 0.8 of [24]). It follows that $\bar{\rho}$ is not ordinary. By definition of ordinarity ([41] : the simple subquotients of $\overline{M}_{\overline{k_F}}$ are 1-dimensional), M is not ordinary and it follows that V is not ordinary ([41]).

- Either $\overline{M}$, thus $D(V)$, are ordinary ([41]). $D(V)$ is an extension of two filtered Dieudonné modules whose underlying spaces are free $E \otimes_{\mathbb{Q}_p} F$ modules of rank 1, the quotient D' of weight $k-1$ ($\mathrm{gr}_{k-1}(D') = D'$), and the sub-object of weight 0. It follows that V is ordinary. $\overline{M}$ is an extension of two filtered Dieudonné modules whose underlying spaces are free $\mathbb{F} \otimes_{\mathbb{F}_p} k_F$ modules of rank 1, the quotient of weight $k-1$ and the sub-object of weight 0. It follows that the Galois representation $V(\overline{M})$ associated by Fontaine-Laffaille to $\overline{M}$ is an extension of an unramified representation of dimension 1 (over $\mathbb{F}$) by a 1-dimensional representation on which inertia acts by $\overline{\chi}_p^{k-1}$. If $k \neq p$, we can take M to be associated to the lattice chosen to define $\bar{\rho}$ and we see that $\bar{\rho}$ is ordinary. If $k = p$, we conclude by saying that $\bar{\rho}$ is an extension of two 1-dimensional unramified representations and $\bar{\rho}$ is ordinary.

In each of the above cases, the first part of (i) is proved. The second part is a result of [3]. Part (ii) is an easy exercise on filtered Dieudonné modules. Part (iii) is proved in [12] (lemma 2.1.2.). □

3.2.2. *Types of deformations.* In this section, we consider a representation $\bar{\rho}_v : D_v \to \mathrm{GL}_2(\mathbb{F})$, where D_v is the Galois group of a finite unramified extension F_v of $\mathbb{Q}_p$ if $\bar{\rho}_v$ is reducible, and $F_v = \mathbb{Q}_p$ if $\bar{\rho}_v$ is irreducible. We also assume $F_v = \mathbb{Q}_p$ if $k(\bar{\rho}_v) = p+1$ when we consider crystalline deformations. If $F_v = \mathbb{Q}_p$, we set $v = p$. We denote by I_v the inertia subgroup (or I_p if $F_v = \mathbb{Q}_p$). Furthermore, we impose that, if $p \neq 2$, the Serre weight $k(\bar{\rho}_v)$ satisfies $2 \leq k(\bar{\rho}_v) \leq p+1$. If $\bar{\rho}_v$ is reducible and F_v is not $\mathbb{Q}_p$, we mean by this that $(\bar{\rho}_v)_{|I_v}$ is ordinary (see previous paragraph) with $\det((\bar{\rho}_v)_{|I_v})$ a power of the cyclotomic character.

There are two types of deformation rings we consider in this paragraph which are denote by $\bar{R}_v^{\square,\psi}$, and in the two cases the $\mathcal{O}'$ valued morphisms (which $\bar{R}_v^{\square,\psi}$ classifies) give rise to lifts ρ_v of $\bar{\rho}_v$ that are of the following kind :

(i) Weight 2 deformations. For $p \neq 2$, the lifts ρ_v are potentially semistable of weight 2, have fixed determinant $\phi = \psi\chi_p$, and with inertial Weil-Deligne parameter $(\omega_p^{k(\bar{\rho})-2} \oplus 1, 0)$ if $k(\bar{\rho}_v) \neq p+1$ and (id, N) , with N a non-zero nilpotent matrix, if $k(\bar{\rho}_v) = p+1$. For $p = 2$, we consider lifts that are crystalline of weight 2 (equivalently Barsotti-Tate with $\det(\rho_v)|_{I_v} = \chi_p$) if $k(\bar{\rho}_v) = 2$, and semistable of weight 2 if $k(\bar{\rho}_v) = 4$, and with fixed determinant of the form $\psi\chi_2$.

Note that by Lemma 3.5 of Section 3.2.1, if $\bar{\rho}_v|_{I_v}$ is of the form

$$\begin{pmatrix} \overline{\chi}_p^{k-1} & * \\ 0 & 1 \end{pmatrix},$$

with $2 \leq k \leq p+1$, then such a lift ρ_v is of the form

$$\begin{pmatrix} \omega_p^{k-2}\chi_p & * \\ 0 & 1 \end{pmatrix},$$

with the further condition that when $\bar{\rho}_v$ is finite flat at v (which can occur only when $k = 2$), $\rho_v|_{I_v}$ is crystalline of weight 2.

(ii) Low weight crystalline deformations : We assume $k(\bar{\rho}_v) = 2$ if $p = 2$. Note that if $k(\bar{\rho}_v) = p+1$, then $\bar{\rho}_v|_{I_v}$ is of the form

$$\begin{pmatrix} \overline{\chi}_p & * \\ 0 & 1 \end{pmatrix}.$$

The lifts ρ_v of $\bar{\rho}_v$ have fixed determinant $\phi = \psi\chi_p$ and such that ρ_v is:

(a) crystalline of weight $k(\bar{\rho}_v)$ if $k(\bar{\rho}_v) \leq p$ (*i.e.* comes from a Fontaine-Laffaille module, [24]).

(b) if $k(\bar{\rho}_v) = p+1$, then $\rho_v|_{I_v}$ is of the form

$$\begin{pmatrix} \chi_p^p & * \\ 0 & 1 \end{pmatrix}.$$

If $\bar{\rho}_v$ is unramified (this implies that $k(\bar{\rho}) = p$) and there are exactly two lines stabilised by $\bar{\rho}_v$, and thus $\bar{\rho}_v = \bar{\eta}_1 \oplus \bar{\eta}_2$ with $\bar{\eta}_i$ distinct unramified characters, we choose one of these characters and consider only lifts on whose unramified quotient the action of D_v reduces to the chosen character.

Note that whenever $\bar{\rho}_v$ is reducible the lifts that we consider are ordinary. The conditions that we impose to ρ_v determine the character $\det(\rho_v|_{I_v})$ and we impose to ψ to be such that the restriction of $\chi_p\psi$ to I_v coincides with this character.

Now we prove properties of the corresponding deformation rings.

3.2.3. *$\bar\rho_v$ irreducible, low weight crystalline case.* We suppose that the representation $\bar\rho_p$ of $D_v = D_p = G_{\mathbb{Q}_p}$ is irreducible of weight $k \leq p$ and we consider lifts that are crystalline of weight k. The deformation ring is smooth over $\mathcal{O}$ of dimension 1 : this follows from Fontaine-Laffaille theory as in [44] (or [19]). There the case $k(\bar\rho_p) = p$ and the case $p = 2$ is excluded, but as $\bar\rho_p$ is irreducible the argument extends. The argument relies on the fact that the filtered Dieudonné module has the following description. It has a basis v_1, v_2 with v_2 generating Fil^{k-1} and the matrix of ϕ is :

$$\begin{pmatrix} \lambda & p^{k-1} \\ \alpha & 0 \end{pmatrix}$$

with α a unit determined by the determinant and λ any element of the maximal ideal of the coefficient ring.

It follows from Proposition 2.1 that the framed deformation ring is smooth of dimension 4 over $\mathcal{O}$. In this case, the endomorphism ring of $\bar\rho_p$ is $\mathbb{F}$, and in fact the proposition 2.1 is obvious as the universal framed deformation scheme is the completion of a torsor above the universal deformation scheme under PGL_2.

3.2.4. *$\bar\rho_v$ irreducible, weight 2 deformations.* If $p = 2$, we have $k(\bar\rho_v) = 2$ and we are in the crystalline case, that we just handled. We suppose $p \neq 2$.

Let R_v^ψ be the universal deformation ring with fixed determinant and $\overline{R}_v^\psi$ be its quotient by the intersection of the prime ideals $\wp$ kernel of the morphisms $R \to \mathcal{O}'$, $\mathcal{O}'$ ring of integers of a finite extension E' of E, corresponding to deformations that are of the type required. Savitt proved that, provided that $\mathcal{O}$ is sufficiently big, this ring is isomorphic to $\mathcal{O}[[T_1, T_2]]/(T_1T_2 - p)$ (3 of th. 6.22 of [49]). Furthermore, for every morphism of $\overline{R}_v^\psi$ to the ring $\mathcal{O}'$ of integers of a finite extension E' of E, the corresponding deformation is of the type required (th. 6.24). By the (obvious case of) Proposition 2.1, the corresponding framed deformation ring $\bar{R}^{\square,\psi}$ is isomorphic to $\mathcal{O}[[T_1, T_2, T_3, T_4, T_5]]/(T_1T_2 - p)$.

3.2.5. *$\bar\rho_v$ ordinary with $k(\bar\rho_v) \leq p$ and low weight crystalline or weight 2 potentially Barsotti-Tate lifts.* We remind the reader that $D_v = G_{F_v}$ with F_v an unramified extension of $\mathbb{Q}_p$. We suppose that $k(\bar\rho_v) \leq p$ dealing with the $k(\bar\rho_v) = p+1$ case in 3.2.6 and 3.2.7. We recall that if $\bar\rho_v$ is unramified, we have $k(\bar\rho_v) = p$.

Proposition 3.6. *There are flat and reduced $\mathrm{CNL}_{\mathcal{O}}$-algebras $\overline{R}_v^{\square,\psi}$ which classify weight 2 and low-weight crystalline framed deformations (3.2.2). $\overline{R}_v^{\square,\psi}$ is a domain, of relative dimension $3 + [F : \mathbb{Q}_p]$, with regular generic fiber, and if either $\bar\rho_v$ is ramified or $\bar\rho_v$ is isomorphic to $\overline{\eta_1} \oplus \overline{\eta_2}$ with $\overline{\eta_1}$ and $\overline{\eta_2}$ two distinct unramified characters, $\overline{R}_v^{\square,\psi}$ is formally smooth.*

Proof. We write the fixed determinant $\phi = \psi\chi_p$ as $\chi_1\eta$ where χ_1 is a character of the Galois group of the cyclotomic extension and η is unramified. The character χ_1 is $\chi_p^{k(\bar\rho)-1}$ in the crystalline case, and $\chi_p\omega_p^{k(\bar\rho)-2}$ in the weight 2 case. One supposes that $\bar\rho_v$ is of the form :

$$\begin{pmatrix} \overline{\chi_1}\,\overline{\eta_1} & * \\ 0 & \overline{\eta_2} \end{pmatrix},$$

with $\overline{\eta_1}$ and $\overline{\eta_2}$ unramified. The conditions impose that the lifts that we consider are of the form:

$$\begin{pmatrix} \chi_1\,\eta_1 & * \\ 0 & \eta_2 \end{pmatrix},$$

with η_1 and η_2 unramified lifts of $\overline{\eta}_1$ and $\overline{\eta}_2$, and $\eta_1\eta_2 = \eta$.

Let $A = \mathcal{O}[[T]]$ and let $[\overline{\eta_1}]$ be the Teichmüller lift of $\overline{\eta_1}$. Let $\eta_T : D_v \to (1+T\mathcal{O}[[T]])^*$ be the unramified character which factors through the $\mathbb{Z}_p$ unramified extension of F_v and sends the Frobenius to $1+T$. Let $\eta_1 = [\overline{\eta_1}]\eta_T$ and let η_2 be the unramified character defined by $\eta = \eta_1\eta_2$. We see A as the affine algebra of deformations of the character $\overline{\eta_1\chi_1}$ whose restriction to the inertia I_v coincides χ_1 ; $\eta_1\chi_1$ is the universal character. We write Ξ for $\chi_1\eta_1\eta_2^{-1}$.

Let B be a $\mathrm{CNL}_{\mathcal{O}}$ A-algebra and N be a finitely generated B-module. Let $M = N(\Xi)$. Let $Z^1(M)$ be the B-module of continuous 1-cocycles of D_v with values in M. If $k(\bar\rho_v) = 2$ (and so $\chi_1 = \chi_p$) let us denote by $Z^1_f(M)$ be the submodule of finite cocyles. To explain what we mean by finite, let F_{nr} be the maximal unramified extension of F_v. The restriction of Ξ to I_v is equal to the cyclotomic character χ_p. One has a morphism that we call v_Z of B-modules :

$$v_Z \;:\; Z^1(M) \to H^1_{\mathrm{cont}}(D_v, M) \to H^1_{\mathrm{cont}}(I_v, N(\chi_p)) \to N,$$

where the last arrow is the map given by Kummer theory :

$$H^1_{\mathrm{cont}}(I_v, N(\chi_p)) \simeq (F^*_{\mathrm{nr}} \otimes N)\widehat{\ },$$

where $\widehat{\ }$ is the $\mathfrak{m}_B$-adic completion, composed with the map $(F^*_{\mathrm{nr}} \otimes N)\widehat{\ } \to N$ which is the completion of the map $v \otimes \mathrm{id}$ with v the valuation of F^*_{nr} normalized by $v(p) = 1$. A finite cocyle is a cocycle whose image in N is trivial. If $k(\bar\rho_v) > 2$, we set $Z^1_f(M) = Z^1(M)$.

Lemma 3.7. *The B-module $Z^1_f(B(\Xi))$ is free of rank $1+[F_v : \mathbb{Q}_p]$; morever $Z^1_f(N(\Xi)) = Z^1_f(B(\Xi)) \otimes_B N$.*

Proof. Since $Z^1_f(N(\Xi)) = \varprojlim_n Z^1_f((N/\mathfrak{m}^n_B N)(\Xi))$, we reduce the lemma to the case where B is of finite length.

We then view $N \mapsto Z^1_f(D_v, N(\Xi))$ as a functor from the category of finite B-modules to itself. We call it $\mathcal{F}$. We claim that the lemma follows from the assertion :

$$|\mathcal{F}(N)| = |N|^{1+[F:\mathbb{Q}_p]},$$

for each finite A-module N.

Indeed, then, as $\mathcal{F}$ is left exact, it is exact, and for each N the natural morphism $N \otimes_B \mathcal{F}(B) \to \mathcal{F}(N)$ is surjective . By Nakayama, $\mathcal{F}(B)$ is finite free of the correct rank, the morphisms $N \otimes_B \mathcal{F}(B) \to \mathcal{F}(N)$ are isomorphisms and the lemma is proved assuming the assertion.

We prove the assertion. We have, with the notation $M = N(\Xi)$:

$$|Z^1(D_v, M)| = |H^1(D_v, M)|\ |M|\ |H^0(D_v, M)|^{-1}.$$

It follows using Euler characteristic and duality :

$$(*)\ |Z^1(D_v, M)| = |M|^{1+[F:\mathbb{Q}_p]}\ |H^0(D_v, M^*)|.$$

If $N^\vee = \mathrm{Hom}_{\mathcal{O}}(N, F/\mathcal{O})$, we have $M^* = N^\vee(\chi_p \Xi^{-1})$ and $\chi_p \Xi^{-1} = \chi_p \chi_1^{-1} \eta_1^{-1} \eta_2$.

For $k(\bar\rho_v) \neq 2$ (thus $p \neq 2$), the restriction of $\chi_p \chi_1^{-1} \eta_1^{-1} \eta_2$ to the inertia I_v has non trivial reduction and the group $H^0(D_v, M^*)$ is trivial. The lemma follows in this case.

Suppose that $k(\bar\rho_v) = 2$. The map :

$$H^1(D_v, M) \to \left(H^1(I_v, N(\chi_p))(\eta_1 \eta_2^{-1})\right)^{D_v}$$

is surjective, as $H^2(\mathrm{Gal}(F_{\mathrm{nr}}/F_v), N(\Xi))$ is trivial, since $N(\Xi)$ is torsion. Let us note $\tilde\eta = \eta_1 \eta_2^{-1}$. Kummer theory gives an identification of the right hand side :

$$\left(H^1(I_v, N(\chi_p))(\tilde\eta)\right)^{D_v} \simeq ((F_{\mathrm{nr}}^* \otimes N)(\tilde\eta))^{D_v}.$$

As Galois module, F_{nr}^* is isomorphic to $\mathbb{Z} \times U$, U units of F_{nr}. Thus the map:

$$((F_{\mathrm{nr}}^* \otimes N)(\tilde\eta))^{D_v} \to (N(\tilde\eta))^{D_v}$$

is surjective. Finally, we see that the map :

$$Z^1(D_v, N(\Xi)) \to (N(\tilde\eta))^{D_v}$$

is surjective. This implies that :

$$|Z^1_f(D_v, N(\Xi))| = |Z^1(D_v, N(\Xi))|\ |\,(N(\tilde\eta))^{D_v}\,|^{-1}.$$

With formula (*), we see that :

$$|Z^1_f(D_v, M)| = |N|^{1+[F:\mathbb{Q}_p]},$$

and the lemma is proved. □

We now complete the proof of Proposition 3.6. Recall that $R_v^{\square,\psi}$ is the universal ring for framed deformations of $\bar\rho_v$ with fixed determinant $\phi = \psi\chi_p = \chi_1\eta$. Let us call $\rho_v^{\mathrm{univ}} : D_v \to \mathrm{GL}_2(R_v^{\square,\psi})$ the corresponding framed deformation. We recall that we have fixed a free $\mathcal{O}$-module V of rank 2 which is the underlying space of the Galois representations that we consider: by a representation with values in $\mathrm{GL}_2(B)$ for a $\mathcal{O}$-algebra B we mean a representation in $\mathrm{Aut}_B(B \otimes_{\mathcal{O}} V)$.

Recall that $A = \mathcal{O}[[T]]$ and we have continuous characters $\eta_1\chi_1$ and η_2 of D_v with values in A^*. Let $\mathcal{Z}^1_f$ be the vector bundle over $\mathrm{Spec}(A)$ whose sections are the elements of the free A-module $Z^1_f(A(\Xi))$. Thus for any A-algebra A' and any point $z \in \mathcal{Z}^1_f(A')$, we get a 1-cocycle $a_z \in Z^1(A'(\Xi))$. Let $\mathcal{BA}$ be the scheme over $\mathrm{Spec}(\mathcal{O})$ of bases (u_1, u_2) of V. The map which sends (u_1, u_2) to the line L_1 generated by u_1 makes $\mathcal{BA}$ a torsor over $(\mathbb{P}_1)_{\mathcal{O}}$ of group the Borel group $\mathcal{B}$ of upper triangular matrices in GL_2. For A' and z as above and a basis $B = (u_1, u_2)$ of $A' \otimes_{\mathcal{O}} V$, we have a representation $\rho_{z,B} : D_v \to \mathrm{Aut}_{A'}(A' \otimes_{\mathcal{O}} V)$ which has matrix :

$$\begin{pmatrix} \chi_1\eta_1 & \eta_2 a_z \\ 0 & \eta_2 \end{pmatrix}.$$

We have a natural action of the Borel $\mathcal{B}$ on $\mathcal{Z}^1_f$ by affine automorphisms such that if $b \in \mathcal{B}(A')$, one has $\rho_{bz,bB} = \rho_{z,B}$. It is given by the formula : for $g = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathcal{B}(A')$, one has $a_{gz} = \beta\alpha^{-1}(\Xi - 1) + \delta\alpha^{-1}a_z$. Taking the quotient of $\mathcal{Z}^1_f \times_{\mathrm{Spec}(A)} \mathcal{BA}$ by the diagonal action of $\mathcal{B}$, we get an affine bundle $\mathcal{E}_f$ over $(\mathbb{P}_1)_A$. For A' an A-algebra and $e \in \mathcal{E}_f(A')$, we define $\rho_e : D_v \to \mathrm{GL}_2(A')$ as the representation $\rho_{z,B}$, for any basis $B = (u_1, u_2)$ of $A' \otimes_{\mathcal{O}} V$ such that u_1 is a generator of the image of e in $\mathbb{P}_1(A')$, and (z, B) being the point $\in (\mathcal{Z}^1_f \times_{\mathrm{Spec}(A)} \mathcal{BA})(A')$ defined by e and B.

We have the following description of the vector bundle of translations of the affine bundle $\mathcal{E}_f$. Let $\mathcal{L}$ the tautological line bundle on $(\mathbb{P}_1)_{\mathcal{O}}$. On $\mathrm{Spec}(A) \times_{\mathrm{Spec}(\mathcal{O})} \mathcal{BA}$, the line bundle $\mathrm{Hom}(V/\mathcal{L}, \mathcal{L})$ is naturally isomorphic to the trivial line bundle. It allows to define the bundle $\mathcal{Z}^1_f(D_v, \mathrm{Hom}(V/\mathcal{L}, \mathcal{L}))$. It clearly descends to a vector bundle on $(\mathbb{P}_1)_A$ that we note $\tilde{\mathcal{Z}}^1_f$; it is the vector bundle of translations of the affine bundle $\mathcal{E}_f$.

Let $\mathcal{R}'$ be the closed subscheme of $\mathcal{E}_f \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(R_v^{\square,\psi})$ defined by the equality of Galois representations : $\rho_e = \rho_v^{\mathrm{univ}}$.

We claim that the morphism $\mathcal{R}' \to (\mathbb{P}_1)_A \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(R_v^{\square,\psi})$ is still a closed immersion. To see that, let U be one of the two open affines of the standard open covering of $(\mathbb{P}_1)_A$, U' the inverse image of U in $\mathcal{E}_f$; U' is affine. Let x be a point of $(U' \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(R_v^{\square,\psi})) \cap \mathcal{R}'$ with values in $\mathrm{Spec}(A')$, for A' an $\mathcal{O}$-algebra. Let $A(U)$ and $A(U')$ the affine algebras. One has to prove that $A(U) \otimes_{\mathcal{O}} R_v^{\square,\psi}$ and $A(U') \otimes_{\mathcal{O}} R_v^{\square,\psi}$ have the same images in A'. This is because the Galois representation $\rho_v^{\mathrm{univ}} \otimes_{R_v^{\square,\psi}} A'$ and the line of $A' \otimes_{\mathcal{O}} V$ defined by x determine the cocycle defined by the morphism $\mathrm{Spec}(A') \to \mathcal{E}_f$.

Unless $\bar{\rho}_v$ is isomorphic to a direct sum of two non isomorphic unramified representations of dimension 1, we let $\mathcal{R} = \mathcal{R}'$ and we define $\overline{R}_v^{\square,\psi}$ as the affine algebra of the scheme theoretical image of $\mathcal{R}$ in $\mathrm{Spec}(R_v^{\square,\psi})$. The projection $\mathcal{R} \to \mathrm{Spec}(\overline{R}_v^{\square,\psi})$ is proper and surjective.

Let us first suppose that either $k(\bar{\rho}_v) \neq p$ or $k(\bar{\rho}_v) = p$ and $\bar{\rho}_v$ is ramified.

We claim that in these cases, the projection $\mathcal{R} \to \mathrm{Spec}(\overline{R}_v^{\square,\psi})$ is an isomorphism. One has to prove that, if S is any $\overline{R}_v^{\square,\psi}$-algebra, and if $\rho_{v,S}$ is the Galois representation $\rho_v^{\mathrm{univ}} \otimes_{R_v^{\square,\psi}} S$, there is a unique line L in the underlying space $S \otimes_{\mathcal{O}} V$ of $\rho_{v,S}$ on which the inertia subgroup I_v acts by the character χ_1 and the projective coordonates of L can be expressed by functions in S. To see this, let $\sigma \in I_v$ such that :

- $\overline{\chi_1}(\sigma) \neq 1$ if $k(\bar{\rho}_v) \neq p$, where $\overline{\chi_1}$ is the reduction of χ_1 ;
- $\bar{\rho}_v(\sigma) \neq \mathrm{id}$ if $k(\bar{\rho}_v) = p$.

The matrix $M(\sigma) = (m_{i,j}(\sigma))$ of $\rho_{v,S}(\sigma) - \chi_1(\sigma)\mathrm{id}$ in the fixed basis (e_1, e_2) of the underlying space V has determinant 0 and its reduction has rank 1. We for example see that the line L above the open subscheme of $\mathrm{Spec}(S)$ defined by $m_{1,1}(\sigma) \neq 0$ is the line defined by the vector $(-m_{1,2}(\sigma), m_{1,1}(\sigma))$. This proves the claim.

It follows that $\bar{\rho}$ defines a point $\bar{e}$ of $\mathcal{E}_f$ and $\mathcal{R}$ identifies to the completion of $\mathcal{E}_f$ at $\bar{e}$. More precisely, let $\overline{L}$ the unique line of $V_{\mathbb{F}}$ on which I_v act by the character $\overline{\chi_1}$, Let $\widehat{\mathcal{L}}$ be the formal scheme of lines that reduce to $\overline{L}$: it is the completion of $(\mathbb{P}_1)_A$ at the point $\xi_{\overline{L}}$ of the special fiber that corresponds to $\overline{L}$. It is formally smooth over $\mathrm{Specf}(\mathcal{O})$ of dimension 1. Let $\mathcal{X} = \mathrm{Specf}(A)$. We get a map from $\mathrm{Specf}(\overline{R}_v^{\square,\psi})$ to $\mathcal{X}$ which maps (ρ_v, L) to the character giving the action of D_v on L (recall that we see A as the affine algebra of deformations of characters of D_v). The map $\mathrm{Specf}(\overline{R}_v^{\square,\psi}) \to \widehat{\mathcal{L}} \hat{\times}_{\mathcal{O}} \mathcal{X}$ is a torsor under the completion of $\tilde{\mathcal{Z}}_f^1$ along the zero section of its fiber at $\xi_{\overline{L}}$. As Z_f^1 is free over A of rank $1 + [F : \mathbb{Q}_p]$, we see that $\overline{R}_v^{\square,\psi}$ is formally smooth of relative dimension $3 + [F : \mathbb{Q}_p]$ over $\mathcal{O}$.

Let us now consider the case $k(\bar{\rho}) = p$ and $\bar{\rho}_v \simeq \overline{\eta_1} \oplus \overline{\eta_2}$, with $\overline{\eta_1}$ and $\overline{\eta_2}$ two distinct unramified characters. Recall that we have chosen one of them, say $\overline{\eta_1}$. One sees that $\mathcal{R}'$ has two closed points, $(\bar{\rho}_v, \overline{L_1})$ and $(\bar{\rho}_v, \overline{L_2})$ where $\overline{L_1}$ and $\overline{L_2}$ are the eigenspaces of $V_{\mathbb{F}}$ corresponding respectively to $\overline{\eta_1}$ and $\overline{\eta_2}$. We see that $\mathcal{R}'$ is the spectrum of a semilocal ring. We call $\mathcal{R}$ the spectrum of the local ring of $\mathcal{R}'$ at the closed point corresponding to $\overline{L_1}$. We define $\overline{R}_v^{\square,\psi}$ as the affine algebra of the scheme theoretical image of $\mathcal{R}$ in $\mathrm{Spec}(R_v^{\square,\psi})$. We claim that the natural map $\mathcal{R} \to \mathrm{Spec}(\overline{R}_v^{\square,\psi})$ is an isomorphism. To see this, let as above S be a $\overline{R}_v^{\square,\psi}$ algebra and $\rho_{v,S} = \rho_v^{\mathrm{univ}} \otimes_{R_v^{\square,\psi}} S$. If $F \in D_v$ projects to the Frobenius in the unramified quotient of D_v, the characteristic polynomial of $\rho_{\mathrm{univ}}(F)$ has one unique root λ_1 in the complete local ring $\overline{R}_v^{\square,\psi}$ which reduces to $\overline{\eta_1}(F)$. We define the line L as the eigenspace of $\rho_{v,S}(F)$ for the eigenvalue λ_1. Le line L reduces to $\overline{L_1}$ and we prove the claim as in the previous case using the matrix of $\rho_{v,S}(F) - \lambda_1\mathrm{id}$. Finally, thanks to Lemma 3.7, we have that $\mathcal{R}$ is formally smooth over $\mathcal{O}$ of relative dimension $3 + [F : \mathbb{Q}_p]$.

Let us now suppose that $\bar{\rho}_v$ is unramified and $\bar{\rho}_v$ acts by homotheties on the semisimplification of $\bar{\rho}_v$. Let us check that $\mathcal{R}$ is a smooth resolution (2.8).

We already noticed that $f : \mathcal{R} \to \mathrm{Spec}(\overline{R}_v^{\square,\psi})$ is proper and surjective. By the definition of scheme theoretical closure, f has an injective structural morphism and we get property (1) of the definition of smooth resolution.

There exists $\sigma \in I_v$ such that $\chi_1(\sigma) - 1$ is invertible in E. It follows that if A' is a E-algebra, and (ρ_v, L) is a point of $\mathcal{R}(A')$ with values in A, the line L is determined by $\rho_v(\sigma)$. It follows as above that the projection $\mathcal{R}[1/p] \to \mathrm{Spec}(R_v^{\square,\psi}[1/p])$ is a closed immersion. We get (2).

The inverse image $\mathcal{Y}$ of the closed point of $\mathrm{Spec}(R_v^{\square,\psi})$ in $\mathcal{R}$ is isomorphic to $(\mathbb{P}_1)_{\mathbb{F}}$ if D_v acts by homotheties, and is a point corresponding to the unique line stabilized by $\bar{\rho}_v$ if D_v does not act by homotheties. In either case, it is connected and we get (3).

It remains to produce a smooth algebraization. Let $A_0 = \mathcal{O}[T]$ and let Z_0 be a free A_0-module of rank $1 + [F_v, \mathbb{Q}_p]$. By the lemma below, one can find an element m_0 of Z_0 with an isomorphism of $A \otimes_{A_0} Z_0$ with $Z^1_f(A(\Xi))$ sending m_0 to the cocycle $\Xi - 1$. We define $\mathcal{R}_0$ as the affine bundle $\mathcal{E}_0$ over $(\mathbb{P}_1)_{A_0}$ which is defined by gluing $A_0[\lambda_1] \otimes_{A_0} Z_0$ and $A_0[\lambda_2] \otimes_{A_0} Z_0$, λ_1 being the usual coordonate on $\mathbb{A}_1$ and λ_2 the usual coordonate on $(\mathbb{P}_1) - \{\infty\}$, by the formula for changing charts : $z_2 = \lambda_1 m_0 - \lambda_1^2 z_1$. It is not difficult to get an isomorphism from $\mathcal{E}_f$ to the affine bundle which is obtained by the same way gluing $A[\lambda_1] \otimes_A Z^1_f(A(\Xi))$ and $A[\lambda_2] \otimes_A Z^1_f(A(\Xi))$ by the formula $z_2 = \lambda_1(\Xi - 1) - \lambda_1^2 z_1$. Thus, one gets an isomorphism :

$$\mathcal{R}_0 \times_{(\mathbb{P}_1)_{A_0}} (\mathbb{P}_1)_A \simeq \mathcal{E}_f$$

that sends m_0 to $\Xi - 1$.

We define the closed subscheme $\mathcal{Y}_0$:

- if D_v does not act as homotheties in $\bar{\rho}$, $\mathcal{Y}_0$ is the closed point of the special fiber of $\mathcal{R}_0$ which is defined by $T \mapsto 0$, the Galois stable line $\overline{L} \subset V_{\mathbb{F}}$ and, for a basis (u_1, u_2) of $V_{\mathbb{F}}$ with $u_1 \in \overline{L}$, by the cocycle defining $\bar{\rho}$;

- if D_v acts by homotheties in $\bar{\rho}$, let $(\mathbb{P}_1)_{\mathbb{F}} \hookrightarrow (\mathbb{P}_1)_A$ be defined by $A \to \mathbb{F}$ sending π and T to 0. As $\Xi - 1 \in \mathsf{m}_A Z^1_f(A(\Xi))$, $m_0 \in (\pi, T)Z_0$ and one sees that the restriction of $\mathcal{R}_0$ to $(\mathbb{P}_1)_{\mathbb{F}}$ is a vector bundle. The subscheme $\mathcal{Y}_0$ is the zero section of this vector bundle.

We define the map $\mathcal{R} \to \mathcal{R}_0$ as the compositum of the projection $\mathcal{R} \to \mathcal{E}_f$ with the map $\mathcal{E}_f \to \mathcal{R}_0$. We have a natural isomorphism between the completions of $\mathcal{R}_0$ and $\mathcal{R}$ along respectively $\mathcal{Y}_0$ and $\mathcal{Y}$. This proves the proposition, granted the promised lemma : □

Lemma 3.8. *Let M_0 be a free A_0-module of finite type. Let $M = A \otimes_{A_0} M$ and $m \in M$. Then, there exists an $m_0 \in M_0$ and an isomorphism $A \otimes_{A_0} M_0 \simeq M$ sending m_0 to m.*

Proof. Let $(e_1, \ldots, e_r)$ a basis of M_0 and let $m = \sum_{i=1}^r a_i e_i$. Obviously, the lemma reduces to the case where some a_i, say a_1, is not divisible by the uniformizer π of $\mathcal{O}$. Let K be the fraction field of A. Let B be the matrix of the automorphism of $K \otimes_A M$ which sends m to $\sum_{i=1}^r a'_i e_i$ and fixes e_i for $i \geq 2$. One has $B(e_1) = a'_1 a_1^{-1} e_1 + \sum_{i=2}^r (a'_i - a_i) a_1^{-1} e_i$. By the Weierstrass preparation and division theorems, we can find $a'_i \in A_0$ such that $a'_1 a_1^{-1} \in A^*$ and $(a'_i - a_i) a_1^{-1} \in A$. Then $B \in \mathrm{GL}_r(A)$. This proves the lemma. □

3.2.6. *$\bar{\rho}_v$ reducible and semistable weight 2 deformations.* Suppose $\bar{\rho}_v$ is of the form

$$\begin{pmatrix} \bar{\gamma}_v\bar{\chi}_p & * \\ 0 & \bar{\gamma}_v \end{pmatrix},$$

with $\bar{\gamma}_v$ an unramified character, then $\rho|_{F_v}$ is of the form

$$\begin{pmatrix} \gamma_v\chi_p & * \\ 0 & \gamma_v \end{pmatrix},$$

where γ_v is a fixed unramified character of D_v that lifts $\bar{\gamma}_v$ and such that $\gamma_v^2\chi_p = \phi$.

The argument is as in the last paragraph. For an $\mathcal{O}$-algebra B and a B-module N which are finite, the cocycles group $Z^1(D_v, N(\chi_p))$ is by (*) of the proof of the lemma of the last paragraph of cardinality $|N|^{2+[F:\mathbb{Q}_p]}$. We get a smooth resolution $\mathcal{R}$.

Unless $p = 2$ and D_v acts by homotheties, the morphism from $\mathcal{R}$ to $\mathrm{Spec}(\overline{R}^{\square,\psi})$ is an isomorphism ; $\mathcal{R}$ is a torsor over the completion of $(\mathbb{P}_1)_{\mathbb{F}}$ along the point corresponding to the unique stable line of $\bar{\rho}$ with structural group the completion of a free module of rank $2 + [F : \mathbb{Q}_p]$ along the zero section of its special fiber. The ring $\overline{R}_v^{\square,\psi}$ is formally smooth over $\mathcal{O}$ of relative dimension $3 + [F : \mathbb{Q}_p]$.

When $p = 2$ and D_v acts by homotheties, $\mathcal{R}$ is torsor over the completion of $(\mathbb{P}_1)_{\mathcal{O}}$ along its special fiber with structural group the completion of a free module of rank $2 + [F : \mathbb{Q}_p]$ along the zero section of its special fiber. By proposition 2.12, the ring $\overline{R}_v^{\square,\psi}$ is a domain, faithfully flat over $\mathcal{O}$ of relative dimension $3 + [F : \mathbb{Q}_p]$, and with regular generic fiber.

3.2.7. *The case $k(\bar{\rho}) = p + 1$, $p \neq 2$; crystalline lifts of weight $p + 1$.* By [3], we know that such lifts are ordinary (Lemma 3.5, recall that we are supposing $F_v = \mathbb{Q}_p$). Ordinary lifts of weight $p + 1$ are the lifts that are extensions of an unramified free rank one representation by a free rank one representation with action of I_v by χ_p^p. Let us analyse the deformation ring of such lifts (and now we suppose that the ground field is F_v a finite unramified extension of $\mathbb{Q}_p$). Let us prove that the corresponding framed deformation ring $\overline{R}_v^{\square,\psi}$ is formally smooth over $\mathcal{O}$ of relative dimension $3 + [F_v : \mathbb{Q}_p]$ *i.e.* is isomorphic to $\mathcal{O}[[T_1, \ldots, T_{3+[F_v:\mathbb{Q}_p]}]]$.

First we prove that the relative tangent space is of dimension $3+[F_v : \mathbb{Q}_p]$. For that, we note that, for A a CNL$_{\mathcal{O}}$-algebra with $\pi A = (0)$, the framed deformations with values in A that we consider are the same as we were considering in the previous paragraph. This proves the claim by the same calculation as in the last paragraph. As $\bar{\chi}_p^p \equiv \bar{\chi}_p$ mod.π, one has to prove that if :

$$\begin{pmatrix} \eta_1\bar{\chi}_p & \eta_2 a \\ 0 & \eta_2 \end{pmatrix},$$

is a D_v-representation with values in A with $\pi A = (0)$ and η_1 and η_2 unramified, then $\eta_1 = \eta_2$. If F is an element of D_v that maps to the Frobenius, and $v_Z : Z^1(A(\Xi)) \to A$ is the map defined in 3.2.5, with $\Xi = \bar{\chi}_p \eta_1 \eta_2^{-1}$, one has $v_Z(a) = v_Z(a \circ \mathrm{int}(F)) = \eta_1(F)\eta_2(F)^{-1} v_Z(a)$. As the cocycle a has a très ramifiée reduction in $\mathbb{F}$, $v_Z(A)$ is a unit in A and we get $\eta_1(F) = \eta_2(F)$.

To prove the formal smoothness of $\overline{R}_v^{\square,\psi}$ over $\mathcal{O}$, we have to extend a lift ρ_A of the required type with values in $\mathrm{GL}_2(A)$, for A a finite $\mathrm{CNL}_{\mathcal{O}}$-algebra, to a lift $\rho_{A'}$ of the required type with values in $\mathrm{GL}_2(A')$, for A' a small extension of A. One first lifts the line L_A stabilized by D_v to a line $L_{A'}$. If the characters giving the action on L_A and $(A \otimes_{\mathcal{O}} V)/L_A$ are respectively $\chi_p^p \eta_1$ and η_2, we claim that there exists a lift of $\eta_1\eta_2^{-1}$ such that the cocycle giving the extension lifts ; that proves the existence of $\rho_{A'}$.

We prove the existence of the lift of $\eta_1\eta_2^{-1}$. In a convenient basis, ρ_A is, after a suitable twist, of the form :

$$\begin{pmatrix} \delta & a \\ 0 & 1 \end{pmatrix},$$

where $\delta : D_v \to A^*$ is a character whose reduction modulo m_A is $\overline{\chi_p}$ and whose restriction to I_v is the compositum of $\chi_p^p : D_v \to \mathcal{O}^*$ and $\mathcal{O}^* \to A^*$. Let $\widehat{\delta}$ be a lift of δ to $(A')^*$ whose restriction to I_v is in the same way the compositum of $\chi_p^p : D_v \to \mathcal{O}^*$ and $\mathcal{O}^* \to (A')^*$. Let $\gamma : D_v \to \mathbb{F}$ be an unramified character. We define the character $\delta_\gamma : D_v \to (A')^*$ by : $\delta_\gamma(\sigma) = 1 + e\gamma(\sigma)$ where e is a generator of the ideal $\mathrm{Ker}(A' \to A)$. Write f_γ for the connecting homomorphism : $H^1(A(\delta)) \to H^2(\mathbb{F}(\overline{\chi_p}))$ for the exact sequence :

$$(0) \to \mathbb{F}(\overline{\chi_p}) \to A'(\widehat{\delta}\delta_\gamma) \to A(\delta) \to (0)$$

(the cohomology is for the group D_v). As $H^2(\mathbb{F}(\overline{\chi_p}))$ is of dimension 1, f_γ is either 0 or is surjective. In the first case we get the existence of $\rho_{A'}$. Let us suppose that f_γ is surjective. We have the exact sequence :

$$(0) \to H^1(\mathbb{F}(\overline{\chi_p})) \to H^1(A'(\widehat{\delta}\delta_\gamma)) \to H^1(A(\delta)) \to H^2(\mathbb{F}(\overline{\chi_p})) \to (0).$$

Let us still denote by a the class in $H^1(\mathbb{F}(\overline{\chi_p}))$ of the cocycle a. A direct calculation gives that $f_\gamma(a) - f_0(a)$ is the cup product of $\gamma \in H^1(\mathbb{F})$ and the reduction $\overline{a} \in H^1(\mathbb{F}(\overline{\chi_p}))$ of a. As $\overline{a}$ is "très ramifié", class field theory implies that, if γ is non zero, this cup product is non zero. We find a (unique) γ_1 such that the $f_{\gamma_1}(a) = 0$. This implies that a lifts to an $\widehat{a} \in H^1(A'(\widehat{\delta}\delta_{\gamma_1}))$. This proves the claim.

Remark: It is not difficult to see using the proof above that the map $\mathrm{Specf}(\overline{R}_v^{\square,\psi}) \to \mathcal{X}$ which corresponds to associate to ρ the character describing the action of D_v on the stable line is not formally smooth.

3.3. **The case of a finite place v not above p.** Let q be the residue characteristic of v. We fix the determinant ϕ. In 3.3.1. to 3.3.3., after possibly enlarging $\mathcal{O}$, we construct a lift $\rho_0 : D_v \to \mathrm{GL}_2(\mathcal{O})$ of $\bar{\rho}_v$ with $\rho_0(I_v)$ finite and with determinant ϕ. We consider inertia-rigid lifts (2.7), *i.e.* lifts whose restriction of inertia is conjugate to the restriction of ρ_0 to inertia and whose determinant is ϕ. The corresponding affine $\mathrm{CNL}_{\mathcal{O}}$-algebra has the required properties by propositions 2.10 and 2.11.

3.3.1. *Inertially finite deformations: minimally ramified lifts.* We consider minimal lifts as we define below (see also [17] for the case $p \neq 2$). We exclude the case that projectively $\bar{\rho}_v(I_v)$ is cyclic of order p as that is treated in 3.3.4. For $p \neq 2$, or $p = 2$ and the projective image of I_v is not dihedral, minimal lifts ρ satisfy $\bar{\rho}_v(I_v) = \rho(I_v)$. In all cases, the restriction to I_v of the determinant of a minimal lift is the Teichmüller lift and the conductor of a minimal lift equals the conductor of $\bar{\rho}_v$.

We construct the required lift ρ_0 of $\bar{\rho}_v$. We distinguish 2 cases.

- the projective image of I_v has order prime to p. As $\bar{\rho}_v(I_v)$ has order prime to p, there is a lift ρ_I of $(\bar{\rho}_v)_{|I_v}$ in $\mathrm{GL}_2(\mathcal{O})$ such that $\bar{\rho}_v(I_v)$ is isomorphic to $\rho_I(I_v)$. The lift ρ_I is unique up to conjugation. Let $F \in D_v$ be a lift of the Frobenius. The representation $\bar{\rho}_v \circ \mathrm{int}(F)$ is isomorphic to $\bar{\rho}_v$. It follows that ρ_I is isomorphic to $\rho_I \circ \mathrm{int}(F)$. Let g be such an isomorphism. Let $\bar{g}$ be the reduction of g ; $\bar{g}^{-1}\bar{\rho}(F)$ is in the centralizer of $(\bar{\rho}_v)_{|I_v}$. As $(\bar{\rho}_v)_{|I_v}$ is semisimple, one easily sees that the centralizer of ρ_I surjects to the centralizer of $(\bar{\rho}_v)_{|I_v}$. So, we can choose g such that it reduces to $\bar{\rho}_v(F)$. We extend ρ_I to D_v by sending F to g, and twist by an unramified character so that the determinant of ρ_0 coincide with ϕ.

- the projective image G of I_v has order divisible by p and is non-cyclic. As G is non-cyclic, the image of the wild inertia in G is non-trivial, hence also the center C of the image of wild inertia.

Suppose first that C is cyclic. Then C has exactly two fixed points in $\mathbb{P}_1(\overline{\mathbb{F}_p})$, defining two distinct lines $\overline{L_1}$ and $\overline{L_2}$ of $V_{\mathbb{F}}$. As G normalizes C, G stabilizes the set of these two lines ; as G is non cyclic it does not fix these two lines. The group G is a dihedral group of order $2d$, with d prime to $p = 2$ and q divides d. The representation $\bar{\rho}_v$ is isomorphic to the induced representation $\mathrm{Ind}_L^{F_v}(\gamma)$, for a character γ of G_L with values in $\overline{\mathbb{F}_2}^*$, L a ramified quadratic extension of F_v. Let δ be a ramified character of G_L of order 2 with values in $\mathbb{Q}_2^*$. Let us call ϵ the character of $D_v = G_{F_v}$ defined by L. Let us define ρ_0' as $\mathrm{Ind}_L^{F_v}(\hat{\gamma}\delta)$, where $\hat{\gamma}$ is the Teichmüller lift of γ. We have

$$\det(\rho_0') = \epsilon \times (\hat{\gamma} \circ t) \times (\delta \circ t),$$

where t is the transfer from D_v^{ab} to G_L^{ab}. As the restrictions to I_v of $\delta \circ t$ and ϵ coincide, we see that

$$\det(\rho_0')_{|I_v} = (\hat{\gamma} \circ t)_{|I_v}.$$

It is the Teichmüller lift of $\det(\bar{\rho}_v)_{|I_v}$. We define ρ_0 as an unramified twist of ρ_0' whose determinant is ϕ. As the restriction of δ to inertia is unique, we see that the restriction of ρ_0 to inertia does not depend of the choice of δ. As γ is wildly ramified and δ is tamely ramified, the conductor of ρ_0' equals the conductor of $\bar{\rho}_v$.

Suppose now C is non-cyclic. Let $c \in C$ be a non trivial element, and $\overline{L_1}$ and $\overline{L_2}$ be the two eigenspaces for c ; C stabilizes the set of these two lines. As C is abelian non cyclic, it is of order 4 and is conjugate to the projective image of the group of matrices :

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

The residual characteristic q of F_v is 2. The normalizer of C in $\mathrm{PGL}_2(\overline{\mathbb{F}_p})$ is isomorphic to the symmetric group S_4, $p = 3$ and G is isomorphic to A_4. The projective image of $\bar{\rho}_v$ is contained in the normalizer of C, *i.e.* in S_4. We have an isomorphic lift of S_4 : it is given by the normalizer of the projective image of the above matrices in $\mathrm{PGL}_2(\overline{\mathbb{Z}_3})$. We define $(\rho_0)_{\mathrm{proj}}$ as the lift given by this lift of S_4. As $p \neq 2$, there is a unique lift ρ_0 of $\bar{\rho}_v$ such that the projective representation defined by ρ_0 coincide with $(\rho_0)_{\mathrm{proj}}$ and whose determinant is ϕ.

3.3.2. *Inertially finite deformations: abelian lifts with fixed inertial character.* We enlarge $\mathbb{F}$ so that it contains all the $q-1$st roots of 1 of $\overline{\mathbb{F}}$.

Suppose that, in the basis $\overline{e_1}, \overline{e_2}$ of $V_{\mathbb{F}}$, $\bar{\rho}_v$ is upper triangular and $\bar{\rho}|_{I_v}$ is

$$\begin{pmatrix} \overline{\chi} & * \\ 0 & 1 \end{pmatrix},$$

where $\overline{\chi}$ arises from a mod. p character of $\mathrm{Gal}(F_v(\mu_q)/F_v)$. Let χ be its Teichmüller lift. This is a power of the character $\iota_p\iota_q^{-1}(\omega_q)$ which we recall that by our conventions is again denoted by ω_q.

Consider a lift χ' of χ that is some non trivial power of ω_q. We see that, if p^r is the exact p power which divides $q-1$, we have, for some integer a : $\chi' = \chi\omega_q^{\frac{a(q-1)}{p^r}}$. We suppose that the restrictions of χ' and ϕ to I_v coincide.

We consider lifts such that $\rho \otimes E$ is the sum of two spaces of dimension 1 on which I_v acts by the characters 1 and χ'. In the case $\overline{\chi}$ unramified and either $\bar{\rho}_v$ ramified or $\bar{\rho}_v$ is a direct sum of two unramified distinct characters (case we call particular), we furthermore impose to ρ to be, in a basis e_1, e_2 lifting the basis $\overline{e_1}, \overline{e_2}$ of the form :

$$\begin{pmatrix} \chi' & * \\ 0 & 1 \end{pmatrix}.$$

As χ' is invariant by conjugation by Frobenius, the action of D_v in ρ is abelian and factorizes through the Galois group G of the compositum L of the maximal unramified extension of F_v with $F_v(\mu_q)$. Unless we are in the particular case, we are considering inertia-rigid lifts. In the particular case, the lifts that we consider are irreducible components of the inertia-rigid lifts.

One has to prove that there exists such a lift with value in the ring of integers of a finite extension of E. Let σ be a generator of the inertia subgroup of G and let us still denote by F the Frobenius of the extension $L/F(\mu_q)$. Let us write $\epsilon = \chi'(\sigma)$. One has to find lifts $\underline{F}$ and $\underline{\sigma}$ of $\bar{\rho}(F)$ and

$\bar{\rho}(\sigma)$ respectively such that $\underline{F}$ and $\underline{\sigma}$ commute, the characteristic polynomial of $\underline{\sigma}$ is $(X-1)(X-\epsilon)$, and $\det(\underline{F}) = \phi(F)$. Granted the first two conditions, we can realize the third by an unramified twist.

We find $\underline{F}$ and $\underline{\sigma}$ satisfying the first two conditions. If $\bar{\rho}(\sigma)$ or $\bar{\rho}(F)$ is semi-simple and not an homothety, both are semi-simple and the existence of the lift is clear. Otherwise, it follows from the lemma :

Lemma 3.9. *Let :*

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a' \\ 0 & 1 \end{pmatrix},$$

be two matrices with coefficient in $\mathbb{F}$. Then, possibly after enlarging $\mathcal{O}$, there exist lifts of these matrices that commute and the first one has characteristic polynomial $(X-1)(X-\epsilon)$.

Proof. One uses the following elementary fact. Let us write v for the valuation of $\mathcal{O}$ such that $v(\pi) = 1$. Let L_1 and L_2 be lines of the free $\mathcal{O}$-module V which are direct factors. Let us note $v(L_1, L_2)$ the least integer v such that the images of L_1 and L_2 in V/π^{v+1} are distinct. Let α be a unit in $\mathcal{O}$ with $v(\alpha - 1) > 0$. Let $g \in \mathrm{GL}_2(E)$ which has eigenspaces L_1 and L_2 with eigenvalues 1 and α. Then one has $g(V) = V$ if and only if $v(L_1, L_2) \leq v(\alpha - 1)$. If $v(L_1, L_2) < v(\alpha - 1)$, the reduction of g is the identity ; if $v(L_1, L_2) = v(\alpha - 1)$, the reduction $\overline{g}$ of g is unipotent $\neq$ id. In the last case, the only line fixed by $\overline{g}$ is the common reduction of L_1 and L_2. Furthermore, one obtains all possible such unipotent matrices, either if α is fixed, by varying L_1 and L_2 such that $v(L_1, L_2) = v(\alpha - 1)$, either if L_1 and L_2 are fixed, by varying α with this condition.

Let us prove the lemma. One lifts the first matrix by a matrix which has eigenvectors e_1 and $e_1 + \lambda e_2$ with eigenvalues 1 and ϵ with $0 < v(\lambda) < v(\epsilon - 1)$ if $a = 0$ and $v(\lambda) = v(\epsilon - 1)$ if $a \neq 0$. One lifts the second matrix by a matrix which has the same eigenvectors and the eigenvalues 1 and α with $v(\lambda) < v(\alpha - 1)$ if $a' = 0$ and $v(\lambda) = v(\alpha - 1)$ if $a' \neq 0$. This proves the lemma. □

3.3.3. *Inertially finite deformations: non-abelian liftings with fixed non-trivial inertial character.* We assume $F_v = \mathbb{Q}_q$. Assume that $p^r | q + 1$ for an integer $r > 0$, and $\bar{\rho}|_{D_q}$ is up to unramified twist of the form

$$\begin{pmatrix} \overline{\chi}_p & * \\ 0 & 1 \end{pmatrix},$$

in the basis $\overline{e_1}, \overline{e_2}$.

We enlarge $\mathbb{F}$ so that it contains all the $q^2 - 1$st roots of 1 of $\overline{\mathbb{F}}$. We consider $\mathcal{O}$ whose residue field is $\mathbb{F}$ and which contains all the $q^2 - 1$st roots of 1 of $\overline{\mathbb{Q}_p}$.

Let χ' be some character of level 2 of I_q (it factors through $(\mathbb{F}_{q^2})^*$ and not through $(\mathbb{F}_q)^*$) which is of order a power of p.

We construct a lift $\rho : D_q \to \mathrm{GL}_2(\mathcal{O}')$ of $\bar{\rho}$ of fixed determinant $\phi = \psi\chi_p$, such that $\rho|_{I_q}$ is of the form

$$\begin{pmatrix} \chi' & * \\ 0 & \chi'^q \end{pmatrix},$$

in a basis lifting the basis $\overline{e_1}, \overline{e_2}$.

As χ' is of level 2, χ' and χ'^q are distinct, and we see that for any lift ρ considered, the restriction to I_q of $\rho \otimes E$ is isomorphic to a direct sum of two representations of dimension 1 with characters χ' and χ'^q. It follows that the lifts considered are inertia-rigid. We have to prove the existence of a lift, the condition on the determinant then being satisfied by an unramified twist (of course, ψ is such that the restriction of ϕ to I_q is $(\chi')^{1+q}$).

Let $F \in D_q$ mapping to the Frobenius, let σ be a generator of tame inertia and let us note $\epsilon = \chi'(\sigma)$.

If $p \neq 2$, as $\chi_p(F) \equiv -1$ mod.p and we can choose the basis of $V_{\mathbb{F}}$ such that the matrix of $\bar{\rho}_v(F)$ is:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix for $\bar{\rho}(\sigma)$ is of the form:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

We choose for $\rho(F)$ the lift with the same matrix as $\bar{\rho}(F)$. We choose for $\rho(\sigma)$ the matrix which has eigenspaces the lines generated by $e_1 + \lambda e_2$ and $e_1 - \lambda e_2$ with eigenvalues ϵ and $\epsilon^{-1} = \epsilon^q$. Note that $\rho(F)$ permutes the two eigenspaces, so that we have $\mathrm{int}(\rho(F))(\rho(\sigma)) = \rho(\sigma)^q$. This matrix $\rho(\sigma)$ is :

$$\begin{pmatrix} (\epsilon + \epsilon^{-1})/2 & (\epsilon - \epsilon^{-1})/2\lambda \\ \lambda(\epsilon - \epsilon^{-1})/2 & (\epsilon + \epsilon^{-1})/2 \end{pmatrix}.$$

We choose λ such that $0 < v(\lambda) < v(\epsilon - \epsilon^{-1})$ if $a = 0$, and if $a \neq 0$, $v(\lambda) = v(\epsilon - \epsilon^{-1})$ and $(\epsilon - \epsilon^{-1})/2\lambda \equiv a$ mod.π.

Let us suppose $p = 2$. Note that $\epsilon \neq -1$ as χ' is of level 2. We choose $\rho(F)$ of the form :

$$\begin{pmatrix} -1 & z \\ 0 & 1 \end{pmatrix},$$

with $v(z) > 0$ if $\bar{\rho}(F)$ is the identity matrix, and $v(z) = 0$ if $\bar{\rho}(F)$ is unipotent not equal to identity. We choose z such that $v(z) < v(\epsilon^2 - 1)$. We choose $\rho(\sigma)$ with eigenspaces $e_1 + \lambda e_2$ and $\rho(F)(e_1 + \lambda e_2)$ with eigenvalues ϵ and ϵ^{-1} respectively. The matrix of $\rho(\sigma)$ in this basis is:

$$\begin{pmatrix} \epsilon^{-1} + \frac{(\epsilon-\epsilon^{-1})}{(2-\lambda z)} & \frac{(\epsilon-\epsilon^{-1})(1-\lambda z)}{\lambda(2-\lambda z)} \\ \frac{\lambda(\epsilon-\epsilon^{-1})}{(2-\lambda z)} & \epsilon - \frac{(\epsilon-\epsilon^{-1})}{(2-\lambda z)} \end{pmatrix}.$$

If $\bar{\rho}(\sigma) = \mathrm{id}$, we choose λ with $v(\lambda) > 0$, $2v(\lambda) + v(z) < v(\epsilon^2 - 1)$. This is possible as we have choosen $v(z) < v(\epsilon^2 - 1)$. As then we have

$v(\lambda)+v(z)<v(2)$, we have $v(2-\lambda z)=v(\lambda)+v(z)$, we see that the above matrix has reduction id.

If $\bar{\rho}(\sigma)\neq$ id, we choose λ with $2v(\lambda)+v(z)=v(\epsilon^2-1)$. We have $v(\lambda)>0$ as we have choosen $v(z)<v(\epsilon^2-1)$. It follows that $v(\lambda z)<v(2)$ and $v(2-\lambda z)=v(\lambda)+v(z)$. The reduction of the above matrix is upper triangular unipotent, not equal to identity. If we note, for an element $x\in K^*$, $r(x)$ the reduction of $x\pi^{-v(x)}$, the reduction of the upper right term $\frac{(\epsilon-\epsilon^{-1})(1-\lambda z)}{\lambda(2-\lambda z)}$ of the above matrix is $r(\epsilon-\epsilon^{-1})/(r(z)r(\lambda)^2)$: we see that we can choose λ such that the reduction of the above matrix is $\bar{\rho}(\sigma)$.

3.3.4. *Twist of semistable deformations.* We use the formalism of Section 2.8. Suppose $\bar{\rho}|_{D_v}$ is of the form

$$\begin{pmatrix} \bar{\gamma}_v\bar{\chi}_p & * \\ 0 & \bar{\gamma}_v \end{pmatrix}.$$

We consider liftings ρ of the form

$$\begin{pmatrix} \gamma_v\chi_p & * \\ 0 & \gamma_v \end{pmatrix},$$

where γ_v is a fixed character of D_v that lifts $\bar{\gamma}_v$ such that its restriction to I_v is the Teichmüller lift and $\gamma_v^2\chi_p=\phi$.

The situation is analogous to the semistable case considered in 3.2.6 except the cocycles form an $\mathcal{O}$-module of rank 2. All we have to note is that for A a finite CNL$_{\mathcal{O}}$-algebra instead of the formula (*) of 3.2.5, we use

$$|Z^1(G_F,A(\chi_p))|=|A|\ |H^0(G_F,A)|=|A|^2.$$

Furthermore, if $\bar{\rho}_v$ is ramified, the conductor of such a lift equals the conductor of $\bar{\rho}_v$.

4. Global deformation rings: basics and presentations

References for this section are [4] and 3.1 of [34].

Given F,F' number fields with $\mathbb{Q}\subset F\subset F'\subset\overline{\mathbb{Q}}$, and a finite set of places S of F, we denote by $G_{F,S}$ (resp. $G_{F',S}$) the Galois group of the maximal extension of F (resp. F') in $\overline{\mathbb{Q}}$ unramified outside S (resp. places of F' above S).

Let F be a totally real number field and let $\bar{\rho}_F:G_F\to \mathrm{GL}_2(\mathbb{F})=\mathrm{GL}(V_{\mathbb{F}})$ be absolutely irreducible and (totally) odd. We assume that $\bar{\rho}$ has non-solvable image when $p=2$, and $\bar{\rho}|_{F(\mu_p)}$ is absolutely irreducible when $p>2$. We also write $\bar{\rho}$ for $\bar{\rho}_F$. We suppose that F is unramified at places above p, and even split at p if $\bar{\rho}\ _{|D_v}$ is irreducible for v a place of F over p. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ as before.

Let $\phi=\psi\chi_p:G_F\to\mathcal{O}^*$ be a totally odd character that lifts $\det\bar{\rho}_F$. In all cases that we will consider, ψ will be arithmetic (1.2). We will consider deformations of $\bar{\rho}$ to CNL$_{\mathcal{O}}$-algebras with fixed determinant $\phi=\psi\chi_p$, or deformations where we fix the determinant to be ϕ locally at places belonging to a finite set S as below. As we fix $\bar{\rho}$ we denote $\mathrm{Ad}(\bar{\rho})$ and $\mathrm{Ad}^0(\bar{\rho})$ by Ad and Ad^0 as usual.

4.1. **Basics.**

4.1.1. *Various deformation rings.* Let $W = S \cup V$ be a finite set of places of F, with S and V disjoint, such that $\bar{\rho}$ and ϕ are unramified outside the places in W, such that all infinite places are in S, and all places above p are in S. Thus we may consider $\bar{\rho}$ as a representation of $G_W = G_{F,W}$ the Galois group of the maximal extension of F in $\overline{F}$ unramified outside W.

For $v \in S$ consider the $\mathrm{CNL}_{\mathcal{O}}$-algebra $R_v^{\square,\psi}$ which represents the functor obtained by assigning to a $\mathrm{CNL}_{\mathcal{O}}$-algebra A, the isomorphism classes of lifts of $\bar{\rho}_{|D_v}$ in $\mathrm{GL}_2(A)$ having determinant $\psi\chi_p$. We can also say that $R_v^{\square,\psi}$ represents the functor of pairs $(V_A, \beta_{v,A})$ where V_A is a deformation of the D_v-representation $V_{\mathbb{F}}$ to A , having determinant $\psi\chi_p$, and $\beta_{v,A}$ is a lift of the chosen basis of $V_{\mathbb{F}}$.

Call $R_S^{\square,\mathrm{loc},\psi}$ the completed tensor product $\hat{\otimes}_{v\in S} R_v^{\square,\psi}$. Consider the $\mathrm{CNL}_{\mathcal{O}}$-algebra $R_{S\cup V}^{\square,\psi}$ (resp., $R_{S\cup V}^{\square}$) which represents the functor obtained by assigning to A, the isomorphism classes of tuples $(V_A, \{\beta_{v,A}\}_{v\in S})$ where V_A is a deformation of the G_W-representation $\bar{\rho}$ to A having determinant $\psi\chi_p$ (resp., $V_A|_{D_v}$ has determinant $\psi\chi_p$ for $v \in S$), and for $v \in S$, $\beta_{v,A}$ is a lift of the chosen basis of $V_{\mathbb{F}}$. Then in a natural way $R_{S\cup V}^{\square,\psi}$ (resp., $R_{S\cup V}^{\square}$) is a $R_S^{\square,\mathrm{loc},\psi}$-algebra.

For $v \in S$ consider one of the quotients $\bar{R}_v^{\square,\psi}$ defined in the earlier section that classifies certain lifts that satisfy a prescribed condition X_v. Let us write : $\bar{R}_S^{\square,\mathrm{loc},\psi} := \hat{\otimes}_{v\in S} \bar{R}_v^{\square,\psi}$.

We consider the $\mathrm{CNL}_{\mathcal{O}}$-algebra $\bar{R}_{S\cup V}^{\square,\psi} = R_{S\cup V}^{\square,\psi} \hat{\otimes}_{R_S^{\square,\mathrm{loc},\psi}} \bar{R}_S^{\square,\mathrm{loc},\psi}$. Analogously we consider $\bar{R}_{S\cup V}^{\square} = R_{S\cup V}^{\square} \hat{\otimes}_{R_S^{\square,\mathrm{loc},\psi}} \bar{R}_S^{\square,\mathrm{loc},\psi}$.

If A is a $\mathrm{CNL}_{\mathcal{O}}$-algebra, then a morphism $R_{S\cup V}^{\square,\psi} \to A$ of $\mathrm{CNL}_{\mathcal{O}}$-algebras factorises through $\bar{R}_{S\cup V}^{\square,\psi}$ if and only if the corresponding local representations for $v \in S$ factorise through the specialisation $R_v^{\square,\psi} \to \bar{R}_v^{\square,\psi}$.

4.1.2. *Relative dimension of framed/unframed deformation rings.*

Proposition 4.1. *Let $\bar{R}_{S\cup V}^{\psi}$ (resp., $\bar{R}_{S\cup V}$) be the subring of $\bar{R}_{S\cup V}^{\square,\psi}$ (resp., $\bar{R}_{S\cup V}^{\square}$) generated by the traces of the corresponding universal deformation. This is the same as the image of the usual (unframed) universal deformation ring $R_{S\cup V}^{\psi}$ (resp., $R_{S\cup V}$) in $\bar{R}_{S\cup V}^{\square,\psi}$ (resp., $\bar{R}_{S\cup V}^{\square}$) . Then* $\mathrm{Specf}(\bar{R}_{S\cup V}^{\square,\psi})$ *(resp.,* $\mathrm{Specf}(\bar{R}_{S\cup V}^{\square})$*) is a* $\mathrm{Specf}(\bar{R}_{S\cup V}^{\psi})$*-torsor (resp.,* $\mathrm{Specf}(\bar{R}_{S\cup V})$*-torsor) under* $(\prod_{v\in S}(\mathrm{GL}_2)_1)/\mathbb{G}_{\mathrm{m}}$ *and $\bar{R}_{S\cup V}^{\square,\psi}$ (resp., $\bar{R}_{S\cup V}^{\square}$) is a power series ring over $\bar{R}_{S\cup V}^{\psi}$ (resp., $\bar{R}_{S\cup V}$) in $4|S| - 1$ variables.*

Proof. We will prove the statements only for deformations with fixed determinants. The proof in the other case is similar. Recall that we denote by

$\mathrm{GL}_2(A)_1$ the kernel of $\mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbb{F})$. Note that $R^{\square,\psi}_{S\cup V}$ represents the functor that associates to A equivalence classes of tuples $(\rho,(\rho_v)_{v\in S},(g_v)_{v\in S})$ where ρ (resp. ρ_v) is a lift of $\bar\rho$ (resp. $\bar\rho_{|D_v}$) to $\mathrm{GL}_2(A)$ of determinant ϕ, and for each $v \in S$, g_v is an element of $\mathrm{GL}_2(A)_1$ such that $\rho_v = \mathrm{int}(g_v)(\rho_{|D_v})$. Two tuples $(\rho,(\rho_v)_{v\in S},(g_v)_{v\in S})$ and $(\rho',(\rho'_v)_{v\in S},(g'_v)_{v\in S})$ are said to be equivalent if they are conjugate under the action of $\mathrm{GL}_2(A)_1$ defined by $(\rho,(\rho_v),(g_v)) \mapsto (\mathrm{int}(g)(\rho),(\rho_v),(g_v g^{-1}))$.

Let us call $\tilde R^{\square,\psi}_{S\cup V}$ the $\mathrm{CNL}_{\mathcal{O}}$-algebra which classifies such tuples (without going modulo the equivalence relation). Note that $\mathrm{Specf}(\tilde R^{\square,\psi}_{S\cup V})$ is a $\mathrm{Specf}(R^{\square,\psi}_{S\cup V})$-torsor under $(\mathrm{GL}_2)_1$. The action of $(\mathrm{GL}_2)_1$ commutes with the action of $(h_v) \in \prod_{v\in S}(\mathrm{GL}_2)_1$ defined by $(\rho,(\rho_v),(g_v)) \mapsto (\rho,\mathrm{int}(h_v)(\rho_v),h_v g_v)$. The action of homotheties embedded diagonally in $(\mathrm{GL}_2)_1 \times \prod_{v\in S}(\mathrm{GL}_2)_1$ is trivial and $\mathrm{Specf}(\tilde R^{\square,\psi}_{S\cup V})$ is a torsor on $\mathrm{Specf}(R^{\psi}_{S\cup V})$ with group $(\mathrm{GL}_2)_1 \times \prod_{v\in S}(\mathrm{GL}_2)_1/\mathbb{G}_{\mathrm{m}}$. We denote $\widetilde{\bar R^{\square,\psi}_{S\cup V}} = \tilde R^{\square,\psi}_{S\cup V}\hat\otimes_{R^{\square,\mathrm{loc},\psi}_S}\bar R^{\square,\mathrm{loc},\psi}_S$. Then the proposition follows from the fact that $\mathrm{Specf}(\widetilde{\bar R^{\square,\psi}_{S\cup V}})$ is a torsor on $\mathrm{Specf}(\bar R^{\psi}_{S\cup V})$ under $(\mathrm{GL}_2)_1 \times \prod_{v\in S}(\mathrm{GL}_2)_1/\mathbb{G}_{\mathrm{m}}$. $\square$

4.1.3. *Dimension of $\bar R^{\square,\mathrm{loc},\psi}_S$.* Recall from Theorem 3.1 that we have the following properties of $\bar R^{\square,\psi}_v$ for $v \in S$:

- $\bar R^{\square,\psi}_v$ is flat over $\mathcal{O}$,
- The relative to $\mathcal{O}$ dimension of each component of $\overline{R}^{\square,\psi}_v$ is :
  - 3 if $\ell \neq p$;
  - $3 + [F_v : \mathbb{Q}_p]$ if $\ell = p$;
  - 2 if v is an infinite place.
- $\bar R^{\square,\psi}_v[\frac{1}{p}]$ is regular.

The completed tensor product $\bar R^{\square,\mathrm{loc},\psi}_S$ is thus flat over $\mathcal{O}$ (see Proposition 2.2), with each component of relative dimension $3|S|$, and $\bar R^{\square,\mathrm{loc},\psi}_S[\frac{1}{p}]$ is regular (see Proposition 3.2).

4.1.4. *Tangent spaces.* For a discrete G_F module M on which the action is unramified outside the finite set of places W, we denote by $H^*(W,M)$ the cohomology group $H^*(G_{F,W},M)$. If for each $v \in W$ we are given a subspace L_v of $H^*(D_v,M)$, we denote by $H^*_{\{L_v\}}(W,M)$ the preimage of the subspace $\Pi_{v\in W}L_v \subset \Pi_{v\in W}H^*(D_v,M)$ under the restriction map $H^*(W,M) \to \Pi_{v\in W}H^*(D_v,M)$.

We consider the following two situations:

- Case 1. $* = 1$ and $M = \mathrm{Ad}^0(\bar\rho)$, for any place $v \in S$, L_v is the image of $H^0(D_v,\mathrm{Ad}/\mathrm{Ad}^0)$ in $H^1(D_v,\mathrm{Ad}^0)$. It is either 0 or 1-dimensional over $\mathbb{F}$, with the latter possibility only when $p = 2$.For $v \in V$ we take L_v to be all of $H^1(D_v,\mathrm{Ad}^0)$.

- Case 2. $* = 1$ and $M = \mathrm{Ad}(\bar{\rho})$, for any place $v \in S$, $L_v = 0$. For $v \in V$ we take L_v to be all of $H^1(D_v, \mathrm{Ad})$.

Note that $(\mathrm{Ad}^0)^* = \mathrm{Hom}_{\mathbb{F}}(\mathrm{Ad}^0, \mathbb{F})$ is isomorphic to Ad/Z as a G_F-module, where Z are the scalar matrices in $M_2(\mathbb{F})$. For $p \neq 2$, we have $\mathrm{Ad} = \mathrm{Ad}^0 \oplus Z$. The G_F-module Ad/Z is isomorphic to Ad^0 when $p > 2$, and need not be so when $p = 2$. (Note that in many references for example [13] what we call $(\mathrm{Ad}^0)^*(1)$ is denoted by $(\mathrm{Ad}^0)^*$.) We have G_F-equivariant perfect pairings $\mathrm{Ad}^0 \times (\mathrm{Ad}^0)^*(1) \to \mathbb{F}(1)$ and $\mathrm{Ad} \times \mathrm{Ad}(1) \to \mathbb{F}(1)$.

We define $L_v^{\perp}$ to the annihilator of L_v under the perfect pairing given by local Tate duality (see Theorem 2.17 of [13] for instance)

$$H^1(D_v, \mathrm{Ad}^0) \times H^1(D_v, (\mathrm{Ad}^0)^*(1)) \to \mathbb{F},$$

and

$$H^1(D_v, \mathrm{Ad}) \times H^1(D_v, (\mathrm{Ad})(1)) \to \mathbb{F}$$

respectively.

We denote by the superscript η the image of the corresponding cohomology with $\mathrm{Ad}^0(\bar{\rho})$-coefficients in the cohomology with $\mathrm{Ad}(\bar{\rho})$-coefficients. Thus the images of the maps $H^1(W, \mathrm{Ad}^0) \to H^1(W, \mathrm{Ad})$ and $H^1_{\{L_v\}}(W, \mathrm{Ad}^0) \to H^1(W, \mathrm{Ad})$ are denoted by $H^1(W, \mathrm{Ad}^0)^\eta$ and $(H^1_{\{L_v\}}(W, \mathrm{Ad}^0))^\eta$ respectively. (Note that this differs slightly from the notation of [4]: for instance what we call $H^1(W, \mathrm{Ad}^0)^\eta$ is denoted using the conventions there by $H^1(W, \mathrm{Ad})^\eta$.)

In Case 2 above we denote the corresponding $H^1_{\{L_v\}}(S \cup V, \mathrm{Ad})$ by $H^1_{S-split}(V, \mathrm{Ad})$. As notation we denote the dimension over $\mathbb{F}$ of a cohomology group considered above by substituting h for H.

Definition 4.2. *For a prime p, define $\delta_p = 0$ if $p > 2$ and $\delta_2 = 1$.*

Then we have the following result.

Lemma 4.3. *1. Consider the exact sequence*

$$0 \to H^0(W, \mathrm{Ad}^0) \to H^0(W, \mathrm{Ad}) \to H^0(W, \mathrm{Ad}/\mathrm{Ad}^0)(= \mathbb{F})$$
$$\to H^1(W, \mathrm{Ad}^0) \to H^1(W, \mathrm{Ad}).$$

The dimension of the kernel of the surjective maps $H^1(W, \mathrm{Ad}^0) \to H^1(W, \mathrm{Ad}^0)^\eta$ and $H^1_{\{L_v\}}(W, \mathrm{Ad}^0) \to (H^1_{\{L_v\}}(W, \mathrm{Ad}^0))^\eta$ is δ_p.

2. (i) For $p > 2$ we have that $H^0(G_F, \mathrm{Ad}^0) = H^0(G_F, (\mathrm{Ad}^0)^*(1)) = 0$.

(ii) For $p = 2$ we have that the projective image of $\bar{\rho}$, denoted by proj.im.$(\bar{\rho})$, *is $G := \mathrm{SL}_2(\mathbb{F}_{2^r})$ for $r > 1$. Further $H^0(G, \mathrm{Ad}) = H^0(G, \mathrm{Ad}^0) = Z$ is 1-dimensional over $\mathbb{F}$, and $H^0(G, \mathrm{Ad}/Z) = H^0(G, (\mathrm{Ad}^0)^*) = 0$.*

3. We have

$$\frac{|H^0(G_F, \mathrm{Ad}^0)|}{|H^0(G_F, (\mathrm{Ad}^0)^*(1))|} = |\mathbb{F}|^{\delta_p}.$$

4. We have an injection $H^1(G_F, Z) \hookrightarrow H^1(G_F, \mathrm{Ad})$.

5. For $p = 2$, we have from 2 (ii) that the projective image of $\bar{\rho}$ is $G := \mathrm{SL}_2(\mathbb{F}_{2^r})$ for $r > 1$.

(i) We have that $H^1(G,\mathrm{Ad})=0$.
(ii) The only G*-submodules of* Ad *are* $0, Z, \mathrm{Ad}^0, \mathrm{Ad}$.

Proof. 1. As $\bar{\rho}$ is absolutely irreducible, we have $H^0(G_F,\mathrm{Ad})=\mathbb{F}.\mathrm{id}=Z$. Also note that $\mathbb{F}.\mathrm{id}$ is not inside Ad^0 if $p\neq 2$ and $\mathbb{F}.\mathrm{id}\subset \mathrm{Ad}^0$ if $p=2$. (1) follows. (See also the exact sequences (7) and (8) of [4].)

2. (i)For $p>2$, as $\bar{\rho}|_{F(\mu_p)}$ is absolutely irreducible, both $H^0(G_F,\mathrm{Ad}^0)$ and $H^0(G_F,(\mathrm{Ad}^0)^*(1))$ are 0 (see proof of Corollary 2.43 of [13]).

(ii) In the case of $p=2$, by our assumption that $\bar{\rho}$ has non-solvable image and Dickson's theorem the projective image of $\bar{\rho}$ is $\mathrm{PGL}_2(\mathbb{F}')\simeq \mathrm{SL}_2(\mathbb{F}')$ for some $\mathbb{F}'\subset\mathbb{F}$, and $|\mathbb{F}'|=2^r$ with $r>1$.

As $\bar{\rho}$ is absolutely irreducible, $H^0(G_F,\mathrm{Ad}^0)$ and $H^0(G_F,\mathrm{Ad})$ are one-dimensional over $\mathbb{F}$ generated by id.

Let us prove that $H^0(G_F,(\mathrm{Ad}^0)^*)=H^0(G_F,(\mathrm{Ad}/Z))$ is trivial. If not, let $l\in\mathrm{Ad}$ whose image in Ad/Z is a non trivial element in $H^0(G_F,(\mathrm{Ad}/Z))$. Let g be an element of G_F whose image in $\mathrm{PGL}_2(\mathbb{F})$ is a non trivial semisimple element. Let $\overline{D}_1$ and $\overline{D}_2$ be the two eigenspaces of $\bar{\rho}(g)$. As the restriction of $\mathrm{Ad}(\bar{\rho})(g)$ to the plane P generated by id and l is unipotent and 1 is not an eigenvalue of $\mathrm{Ad}(\bar{\rho})(g)$ acting on Ad/P, we see that P is the subspace of diagonal matrices relatively to the decomposition $\overline{D}_1\oplus\overline{D}_2$ of the underlying space of $\bar{\rho}$. As P is stable under G_F, the set of the two lines $\overline{D}_1$ and $\overline{D}_2$ is stable by G_F and $\bar{\rho}(G_F)$ is solvable. This is not the case and (2) is proved. (See also Lemma 42 of [20].)

3. This follows from (2).

4. This is clear for $p>2$ and in the case $p=2$ follows from the fact that $H^0(G_F,\mathrm{Ad}/Z)=0$ and the long exact sequence of cohomology.

5. (i) This is Lemma 42 of [20]: in loc. cit. it is proved that given finite fields $\mathbb{F}'\subset\mathbb{F}$ with $\mathbb{F}'=\mathbb{F}_{2^r}$ for some $r>1$, then $H^1(\mathrm{SL}_2(\mathbb{F}'),M_2(\mathbb{F}))=0$.

(ii) To see this note that any 1-dimensional module for $G=\mathrm{SL}_2(\mathbb{F}_{2^r})$ ($r>1$) is trivial as G is perfect. Note also that the only G-submodules of Ad^0 are $0, Z, \mathrm{Ad}^0$. For if V was another G-submodule, then as $H^0(G_F,\mathrm{Ad}^0)=Z$, we have $\dim(V)=2$, and this would contradict that $H^0(G,(\mathrm{Ad}^0)^*)=0$.

If V is now a G-submodule of Ad other than the asserted ones, then $\dim(V)$ is either 2 or 3. If $\dim(V)=2$, then the G-submodule $V\cap\mathrm{Ad}^0$ is non-zero and we see from the list of G-submodules of Ad^0 that it can only be Z. But then V/Z is a one-dimensional G submodule of Ad/Z which contradicts $H^0(G,\mathrm{Ad}/Z)=0$. If $\dim(V)=3$, then $V\cap\mathrm{Ad}^0$ is at least 2-dimensional, and then we see that $V=\mathrm{Ad}^0$ by inspection of the list of G-submodules of Ad^0.

□

4.1.5. *Number of generators of global/local deformation rings.* By arguments as in the proof of Lemma 3.2.2 of [33] we have:

Lemma 4.4. *1. The minimal number of generators of* $R^{\square,\psi}_{S\cup V}$ *(resp.,* $\bar{R}^{\square,\psi}_{S\cup V}$*) over* $R^{\square,\mathrm{loc},\psi}_{S}$ *(resp.,* $\bar{R}^{\square,\mathrm{loc},\psi}_{S}$*) is* $\dim_{\mathbb{F}}(H^1_{\{L_v\}}(S\cup V,\mathrm{Ad}^0))-\delta_p+\Sigma_{v\in S}\dim_{\mathbb{F}}(H^0(D_v,\mathrm{Ad}))-\dim_{\mathbb{F}}(H^0(G_F,\mathrm{Ad}))$.

2. The minimal number of generators of $R^{\square}_{S\cup V}$ *(resp.,* $\bar{R}^{\square}_{S\cup V}$*) over* $R^{\square,\mathrm{loc},\psi}_{S}$ *(resp.,* $\bar{R}^{\square,\mathrm{loc},\psi}_{S}$*) is* $\dim_{\mathbb{F}}(H^1_{S-split}(S\cup V,\mathrm{Ad}))+\Sigma_{v\in S}\dim_{\mathbb{F}}(H^0(D_v,\mathrm{Ad}))-\dim_{\mathbb{F}}(H^0(G_F,\mathrm{Ad}))$.

Proof. We prove only 1. as 2. is similar.

Let m_{gl} (resp. $\mathsf{m}_{\mathrm{loc}}$) the maximal ideal of $R^{\square,\psi}_{S\cup V}$ (resp. $R^{\square,\mathrm{loc},\psi}_{S}$). Let us prove that the dimension of the relative cotangent space $(\mathsf{m}_{\mathrm{gl}}/\mathsf{m}_{\mathrm{gl}}^2)\otimes_{R^{\square,\mathrm{loc},\psi}_{S}}\mathbb{F}$, or equivalently that of the relative tangent space $\mathrm{Hom}_{\mathcal{O}}((\mathsf{m}_{\mathrm{gl}}/\mathsf{m}_{\mathrm{gl}}^2)\otimes_{R^{\square,\mathrm{loc},\psi}_{S}}\mathbb{F},\mathbb{F})$, is $\dim_{\mathbb{F}}(H^1_{\{L_v\}}(S\cup V,\mathrm{Ad}^0))-\delta_p+\Sigma_{v\in S}\dim_{\mathbb{F}}(H^0(D_v,\mathrm{Ad}))-\dim_{\mathbb{F}}(H^0(G_F,\mathrm{Ad}))$.

The relative tangent space T of the previous sentence corresponds to the set of deformations of $V_{\mathbb{F}}$ to finite free, rank 2, dual algebra $\mathbb{F}[\varepsilon]$-modules ($\varepsilon^2=0$), $V_{\mathbb{F}[\varepsilon]}$, with prescribed determinant ϕ, together with a collection of bases $\{\beta_v\}_{v\in S}$ lifting the chosen basis of $V_{\mathbb{F}}$, such that for each $v\in S$, the pair $(V_{\mathbb{F}[\varepsilon]}|_{D_v},\beta_v)$ is isomorphic to $V_{\mathbb{F}}\otimes_{\mathbb{F}}\mathbb{F}[\varepsilon]$ equipped with the action of D_v induced by the action on $V_{\mathbb{F}}$ and the basis induced by the chosen basis of $V_{\mathbb{F}}$. The space of such deformations $V_{\mathbb{F}[\varepsilon]}$ is given by $(H^1_{\{L_v\}}(W,\mathrm{Ad}^0))^{\eta}$ whose $\mathbb{F}$-dimension by Lemma 4.3 is $\dim_{\mathbb{F}}(H^1_{\{L_v\}}(S\cup V,\mathrm{Ad}^0))-\delta_p$. After this the proof is completed just as in [33] by observing that two sets of choices $\{\beta_v\}_{v\in S}$ and $\{\beta'_v\}_{v\in S}$ are equivalent if there is an automorphism of $V_{\mathbb{F}[\varepsilon]}$ respecting the action of D_v, reducing to an homothety on $V_{\mathbb{F}}$, and taking β_v to β'_v. The referee has remarked that this may be expressed as

$$0\to H^0(G_F,\mathrm{Ad})\to\prod_{v\in S}H^0(D_v,\mathrm{Ad})\to T\to H^1_{\{L_v\}}(W,\mathrm{Ad}^0)^{\eta}\to 0.$$

□

4.1.6. *Wiles formula.* Recall the formula of Wiles (see Theorem 2.19 of [13]). Consider a module M as above which is finite, and with $nM=0$ for some $n\in\mathbb{N}$. Let $M^*(1)$ be the Galois module $\mathrm{Hom}(M,\mu_n)$. Then:

(1)
$$\frac{|H^1_{\{L_v\}}(S\cup V,M)|}{|H^1_{\{L_v^{\perp}\}}(S\cup V,M^*(1))|}=\frac{|H^0(G_F,M)|}{|H^0(G_F,M^*(1))|}\prod_{v\in S}\frac{|L_v|}{|H^0(D_v,M)|}\prod_{v\in V}\frac{|L_v|}{|H^0(D_v,M)|}$$

In Cases 1 and 2 above this gives:

(2)
$$\frac{|H^1_{\{L_v\}}(S\cup V,\mathrm{Ad}^0)|}{|H^1_{\{L_v^{\perp}\}}(S\cup V,(\mathrm{Ad}^0)^*(1))|}=\frac{|H^0(G_F,\mathrm{Ad}^0)|}{|H^0(G_F,(\mathrm{Ad}^0)^*(1))|}\prod_{v\in S\cup V}\frac{|L_v|}{|H^0(D_v,\mathrm{Ad}^0)|}$$

(3)
$$\frac{|H^1_{S-split}(V,\mathrm{Ad})|}{|H^1_{V-split}(S,\mathrm{Ad}(1))|} = \frac{|H^0(G_F,\mathrm{Ad})|}{|H^0(G_F,\mathrm{Ad}(1))|}\prod_{v\in S}\frac{1}{|H^0(D_v,\mathrm{Ad})|}\prod_{v\in V}\frac{|H^1(D_v,\mathrm{Ad})|}{|H^0(D_v,\mathrm{Ad})|}$$

4.2. **Presentations.** For the rest of this subsection we assume that V is empty, consider only Ad^0 coefficients, and remove V from the notation.

Proposition 4.5. *The absolute dimension of $\bar{R}^{\psi}_S$ is ≥ 1.*

Proof. Let

$$g := \dim_{\mathbb{F}}(H^1_{\{L_v\}}(S,\mathrm{Ad}^0)) - \delta_p + \Sigma_{v\in S}\dim_{\mathbb{F}}(H^0(D_v,\mathrm{Ad})) - \dim_{\mathbb{F}}(H^0(G_F,\mathrm{Ad}))$$
$$= \dim_{\mathbb{F}}(H^1_{\{L_v^{\perp}\}}(S,(\mathrm{Ad}^0)^*(1)) - \dim_{\mathbb{F}}(H^0(G_F,\mathrm{Ad}))$$
$$+\Sigma_{v\in S}\left(\dim_{\mathbb{F}}(H^0(D_v,\mathrm{Ad})) + \dim_{\mathbb{F}}(L_v) - \dim_{\mathbb{F}}(H^0(D_v,\mathrm{Ad}^0))\right).$$

The second equality follows from Wiles' formula (2), and 3. of Lemma 4.3.

We have the exact sequence :

$$(0) \to H^0(D_v,\mathrm{Ad}^0) \to H^0(D_v,\mathrm{Ad}) \to \mathbb{F} \to L_v \to (0).$$

It follows that, in the preceeding formula, each term of the sum over $v \in S$ is 1 and we find :

$$g = \dim_{\mathbb{F}}(H^1_{\{L_v^{\perp}\}}(S,(\mathrm{Ad}^0)^*(1))) + |S| - 1.$$

By Lemma 4.4, we have a presentation

$$R^{\square,\psi}_S \simeq R^{\square,\mathrm{loc},\psi}_S[[X_1,\cdots,X_g]]/J,$$

which induces an isomorphism on the relative to $R^{\square,\mathrm{loc},\psi}_S$ tangent spaces of $R^{\square,\mathrm{loc},\psi}_S[[X_1,\cdots,X_g]]$ and $R^{\square,\psi}_S$. Let us call $r(J)$ the minimal number of generators of J : if m is the maximal ideal of $R^{\square,\mathrm{loc},\psi}_S[[X_1,\cdots,X_g]]$, we have $r(J) = \dim_{\mathbb{F}}(J/\mathsf{m}J)$.

Lemma 4.6. *We have the inequality :*

$$r(J) \leq \dim_{\mathbb{F}}(H^1_{\{L_v^{\perp}\}}(S,(\mathrm{Ad}^0)^*(1))).$$

Proof. We prove the lemma. We define a $\mathbb{F}$-linear map

$$f:\ \mathrm{Hom}_{\mathbb{F}}(J/\mathsf{m}J,\mathbb{F}) \to H^1_{\{L_v^{\perp}\}}(S,(\mathrm{Ad}^0)^*(1))^*,$$

and we prove that f is injective.

To define f, we have to construct a pairing (,) :

$$H^1_{\{L_v^{\perp}\}}(S,(\mathrm{Ad}^0)^*(1)) \times \mathrm{Hom}_{\mathbb{F}}(J/\mathsf{m}J,\mathbb{F}) \to \mathbb{F}_p,$$

that satisfies $(\lambda *, *) = (*, \lambda *)$ for $\lambda \in \mathbb{F}$.

Let $u \in \mathrm{Hom}_{\mathbb{F}}(J/\mathsf{m}J, \mathbb{F})$ and $[x] \in H^1_{\{L_v^\perp\}}(S, (\mathrm{Ad}^0)^*(1))$. We have the exact sequence :

$$0 \to J/\mathsf{m}J \to R_S^{\square,\mathrm{loc},\psi}[[X_1, \cdots, X_g]]/\mathsf{m}J \to R_S^{\square,\psi} \to 0.$$

We push-forward the exact sequence above by u and we get an exact sequence

$$0 \to I_u \to R_u \to R_S^{\square,\psi} \to 0.$$

Note that $I_u^2 = 0$, and I_u is isomorphic to $\mathbb{F}$ as an R_u-module.

Let $(\rho, (\rho_v)_{v\in S}, (g_v)_{v\in S})$ be tuple as in the proof of proposition 4.1 representing the tautological point of $R_S^{\square,\psi}$. As R_u is a $R_S^{\square,\mathrm{loc}}$-algebra, for all $v \in S$, we get a lift $\tilde{\rho}_v$ of ρ_v with values in $\mathrm{GL}_2(R_u)$. Let us choose lifts $\tilde{g}_v \in \mathrm{GL}_2(R_u)$ of the g_v and let us write $\tilde{\rho}'_v = \mathrm{int}(\tilde{g}_v^{-1})(\tilde{\rho}_v)$.

Consider a set theoretic lift $\tilde{\rho} : G_S \to \mathrm{GL}(V_{R_u})$ of the universal representation $\rho_S^{\square,\mathrm{univ}} : G_F \to \mathrm{GL}_2(R_S^{\square,\mathrm{univ}})$, such that the image of $\tilde{\rho}$ consists of automorphisms of determinant the fixed determinant ϕ. This is possible as SL_2 is smooth. We define the 2-cocycle :

$$c : (G_S)^2 \to \mathrm{Ad}^0,\ c(g_1, g_2) = \tilde{\rho}(g_1)\tilde{\rho}(g_2)\tilde{\rho}(g_1g_2)^{-1} - 1.$$

We define a 1-cochain a_v by the formula :

$$a_v : D_v \to \mathrm{Ad}^0,\ \tilde{\rho}(g) = (1 + a_v(g))\tilde{\rho}'_v(g).$$

We see that $c_{|D_v} = \delta(a_v)$. We define (x, u) by the formula :

$$\sum_{v\in S} \mathrm{inv}((x_v \cup a_v) + z_v),$$

where x is a 1-cocycle representing $[x]$, z is a 2-cochain of G_S with values in O_S^* such that $\delta(z) = (x \cup c)$ (see proof of th. 8.6.7 of [43]). We have written $(\ \cup\ )$ for the cup-product followed by the map on cochains defined by the pairing $(\mathrm{Ad}^0)^*(1) \times \mathrm{Ad}^0 \to \mathbb{F}(1) \to \mathbb{F}_p(1)$, where the map $\mathbb{F}(1) \to \mathbb{F}_p(1)$ is induced by the trace $\mathbb{F} \to \mathbb{F}_p$.

The formula is meaningful as $\delta((x_v \cup a_v) + z_v) = 0$. By the product formula, it does not depend on the choice of z. It does not depend on the choice of the representative x of $[x]$. Indeed, if we choose $x + \delta(y)$ as a representative, we can replace z by $z + (y \cup c)$, and in the formula defining the pairing we have to add :

$$\sum_{v\in S} \mathrm{inv}((\delta(y_v) \cup a_v) + (y_v \cup c_v)).$$

Each term of this last sum is 0 as it is $\mathrm{inv}(\delta((y_v \cup a_v)))$. It does not depend on the choice of $\tilde{\rho}$. If we take as a section $(1 + b)\tilde{\rho}$, we replace a_v by $a_v + b_v$, c by $c + \delta(b)$ and we can replace z by $z - (x \cup b)$, and the formula is changed by :

$$\sum_{v\in S} \mathrm{inv}((x_v \cup b_v) - (x_v \cup b_v)) = 0.$$

It also does not depend on the choice of the $\tilde{g}_v$. If we change $\tilde{g}_v$ by $\tilde{g}_v(1+h_v)$, we change a_v by $g \mapsto a_v(g) - (\mathrm{ad}(g)(h_v) - h_v)$, $(x_v \cup a_v)$ by $(x_v \cup a_v - x_v \cup \delta(h_v))$. As $x_v \cup \delta(h_v) = -\delta(x_v \cup h_v)$, we do not change $\mathrm{inv}((x_v \cup a_v) + z_v)$.

The application $x \mapsto (x,u)$ is obviously $\mathbb{F}_p$-linear. The $\mathbb{F}_p$-linearity of $u \mapsto (x,u)$ follows from the fact that c (resp. a_v) is defined by evaluating at u a cochain $(G_S)^2 \to J/\mathsf{m}J \otimes \mathrm{Ad}^0$ (resp. $D_v \to J/\mathsf{m}J \otimes \mathrm{Ad}^0$). As $(\cup)$ satifies the identity $(\lambda c_1 \cup c_2) = (c_1 \cup \lambda c_2)$ for $\lambda \in \mathbb{F}$, $(\ , \ )$ satisfies the analogous identity, and we get the $\mathbb{F}$-linearity.

Let us prove that the map f defined by this pairing is injective. Suppose $u \neq 0$. We have a part of the Poitou-Tate exact sequence :

$$H^1(S,\mathrm{Ad}^0) \to \oplus_{v \in S} H^1(D_v,\mathrm{Ad}^0)/L_v \to H^1_{\{L_v^\perp\}}(S,(\mathrm{Ad}^0)^*(1))^* \to \text{Ш}^2(S,\mathrm{Ad}^0).$$

If $f(u) = 0$, we have in particular that for all $[x] \in \text{Ш}^1(S,(\mathrm{Ad}^0)^*(1))$, $([x],u) = 0$. For such an $[x]$, it follows from 8.6.8 of [43] that $([x],u) = 0$ coincides with Poitou-Tate product of $[x]$ and the image of the class $[c]$ of c in $\text{Ш}^2(S,\mathrm{Ad}^0)$. As the Poitou-Tate pairing is non-degenerate, we see that if $f(u) = 0$, we have $[c] = 0$. Thus we can suppose that $\tilde{\rho}$ is a Galois representation and that $[z] = 0$. The formula defining $f(u)$ shows that $f(u)$ comes from the image of (a_v) in $\oplus_{v \in S} H^1(D_v,\mathrm{Ad}^0)/L_v$. As $f(u) = 0$, we can find $b \in Z^1(S,\mathrm{Ad}^0)$ and $h_v \in Z^0(D_v,\mathrm{Ad})$ such that $b_v = a_v + \delta(h_v)$ for each $v \in S$. If we replace the Galois representation $\tilde{\rho}$ by $(1-b)\tilde{\rho}$ and $\tilde{g}_v$ by $\tilde{g}_v(1-h_v)$, we obtain a tuple $(\tilde{\rho},(\tilde{\rho}_v),(\tilde{g}_v))$ defining a section of the morphism of $R_S^{\square,\mathrm{loc},\psi}$-algebras : $R_u \to R_S^{\square,\psi}$. This is impossible as $R_u \to R_S^{\square,\psi}$ induces an isomorphism on tangent spaces relative to $R_S^{\square,\mathrm{loc},\psi}$, hence so does the section which is thus an isomorphism. This proves that f is injective and the lemma. $\square$

Using the presentation

$$R_S^{\square,\psi} \simeq R_S^{\square,\mathrm{loc},\psi}[[X_1,\cdots,X_g]]/J$$

and the natural maps $R_S^{\square,\mathrm{loc},\psi} \to \bar{R}_S^{\square,\mathrm{loc},\psi}$ and $R_S^{\square,\psi} \to \bar{R}_S^{\square,\psi}$ we deduce a presentation.

$$\bar{R}_S^{\square,\psi} \simeq \bar{R}_S^{\square,\mathrm{loc},\psi}[[X_1,\cdots,X_g]]/J',$$

and get that J' is generated by $\dim(H^1_{\{L_v^\perp\}}(S,(\mathrm{Ad}^0)^*(1)))$ elements.

Thus we get a lower bound for the (absolute) dimension of $\bar{R}_S^{\square,\psi}$ as $|S| - 1 + \text{abs. dim.}(\bar{R}_S^{\square,\mathrm{loc},\psi})$. We know that $\bar{R}_S^{\square,\mathrm{loc},\psi}$ is flat over $\mathcal{O}$ such that $\text{abs. dim.}(\bar{R}_S^{\square,\mathrm{loc},\psi}) = 1 + 3|S|$. Thus a lower bound for the (absolute) dimension of $\bar{R}_S^{\square,\psi}$ is $3|S| + 1 + |S| - 1 = 4|S|$. Comparing this with another expression for the (absolute) dimension which is $\text{abs. dim.}(\bar{R}_S^{\psi}) + 4|S| - 1$ (see Proposition 4.1) proves the proposition.

$\square$

Remarks: The proof above is along the lines of Lemma 4.1.1 of [34], or Theorem 5.2 of [4]. The map

$$f:\ \mathrm{Hom}(J/\mathsf{m}J,\mathbb{F}) \to H^1_{\{L_v^\perp\}}(S,(\mathrm{Ad}^0)^*(1))^*$$

constructed above, and its injectivity, answers a question of §4 of [34].

Let $r_1,\ldots,r_{r(J)}$ be elements of J which reduce to a basis of $J/\mathsf{m}J$. We shall prove in Theorem 10.1 that $\bar{R}_S^{\psi}$ is $\mathcal{O}$-module of finite type. It follows that $y_1,\cdots,y_{4|S|-1},r_1,\ldots,r_{r(J)},p$ is a system of parameters of $\bar{R}_S^{\square,\mathrm{loc},\psi}[[X_1,\cdots,X_g]]$ where the images of y_i generate $\bar{R}_S^{\square,\psi}$ over $\bar{R}_S^{\psi}$. If the $\mathcal{O}$-algebras $\bar{R}_v^{\square,\psi}$ are Cohen-Macaulay, it is a regular sequence. It follows that $\bar{R}_S^{\psi}$ is flat over $\mathcal{O}$, and Cohen-Macaulay (resp. Gorenstein, resp complete intersection) if the $\bar{R}_v^{\square,\psi}$ are.

Corollary 4.7. *If $\bar{R}_S^{\psi}$ is a finitely generated $\mathbb{Z}_p$-module, then there is a map of $CNL_{\mathcal{O}}$-algebras $\pi:\bar{R}_S^{\psi}\to\mathcal{O}'$ for $\mathcal{O}'$ the ring of integers of a finite extension of $\mathbb{Q}_p$. As $\bar{R}_S^{\square,\psi}$ is smooth over $\bar{R}_S^{\psi}$, we also get a morphism $\bar{R}_S^{\square,\psi}\to\mathcal{O}''$, for $\mathcal{O}''$ like $\mathcal{O}'$.*

Proof. Let $R=\bar{R}_S^{\psi}$. From the hypothesis and the proposition we see that $p\in R$ is not nilpotent and hence there is a prime ideal I of R with $p\notin I$, and thus the fraction field of R/I is a finite extension E' of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}'$. Thus the map $R\to R/I\hookrightarrow\mathcal{O}'(\hookrightarrow E')$ is the required morphism. □

5. Galois cohomology: auxiliary primes and twists

5.1. **Generalities on twists.** We preserve the notation of §4. Thus F is a totally real number field and let $\bar{\rho}_F:G_F\to\mathrm{GL}_2(\mathbb{F})=\mathrm{GL}(V_{\mathbb{F}})$ be absolutely irreducible and (totally) odd. We assume that $\bar{\rho}$ has non-solvable image when $p=2$, and $\bar{\rho}|_{F(\mu_p)}$ is absolutely irreducible when $p>2$.

Consider a set of primes $S\cup V$, and character ψ as in Section 4. Let F_V^S/F denote the maximal abelian extension of F of order a power of p, unramified outside V and split at primes in S. Let $G_V=\mathrm{Gal}(F_V^S/F)$. As S contains all primes above p, G_V is a finite abelian p-group. Let us denote by $(G_V)^*$ the diagonalizable group of group of characters G_V in the category of functors from $\mathrm{CNL}_{\mathcal{O}}$-algebras to sets (2.5). For $A\in\mathrm{CNL}_{\mathcal{O}}$, $(G_V)^*(A)$ is the group $\mathrm{Hom}(G_V,A^*)_1$ of continuous characters that reduce to the trivial character modulo m_A.

We have an action of $(G_V)^*(R^{\square}_{S\cup V})$ on $R^{\square}_{S\cup V}$. To see it, let us note $\rho_{S\cup V}^{\square,\mathrm{univ}}$ the universal representation with values in $R^{\square}_{S\cup V}$. Given a character $\chi\in\mathrm{Hom}(G_V,(R^{\square}_{S\cup V})^*)_1$, for $v\in S$, as χ split at v, the restriction of $\det(\chi\otimes\rho_{S\cup V}^{\square,\mathrm{univ}})$ to D_v is $\psi\chi_p$. This allows to define the automorphism a_χ of $R^{\square}_{S\cup V}$ by $a_\chi\circ\rho_{S\cup V}^{\square,\mathrm{univ}}=\chi\otimes\rho_{S\cup V}^{\square,\mathrm{univ}}$.

If $p = 2$, let $G_{V,2} = G_V/2G_V$ be the maximal quotient of G_V which is a group killed by 2. Then, if $\chi \in (G_{V,2})^*(R^{\square}_{S\cup V})$, we have $\chi^2 = 1$ and $\det(\chi \otimes \rho^{\square,\mathrm{univ}}_{S\cup V}) = \det(\rho^{\square,\mathrm{univ}}_{S\cup V})$. It follows that $(G_{V,2})^*$ acts on $R^{\square,\psi}_{S\cup V}$.

5.2. **Freeness of action by twists.** Suppose that $p = 2$. The following lemma implies that the actions of $(G_V)^*$ (resp. $(G_{V,2})^*$) on $R^{\square}_{S\cup V}$ (resp. $R^{\square,\psi}_{S\cup V}$) are free (2.4) :

Lemma 5.1. *If A is a $\mathrm{CNL}_{\mathcal{O}}$-algebra, then a deformation $\rho_A : G_F \to \mathrm{GL}(V_A)$ of $\bar\rho : G_F \to \mathrm{GL}_2(\mathbb{F})$ is not equivalent to any of its twists by a non-trivial character $\chi \in \mathrm{Hom}(G_V, A^*)_1$.*

Proof. As we have supposed that $\bar\rho$ has non solvable image, the projective image of $\bar\rho$ is isomorphic to $\mathrm{SL}_2(F_0)$ for a finite field F_0 of order $2^* > 2$. If χ is a non-trivial character we may find a $g \in G_F$ such that $\chi(g) \neq 1$ and $\mathrm{tr}(\bar\rho(g)) \neq 0$. Then $\chi(g)\mathrm{tr}(\rho_A(g)) \neq \mathrm{tr}(\rho_A(g))$ as $\mathrm{tr}(\rho_A(g))$ is a unit. $\square$

5.3. **A useful lemma.**

Lemma 5.2. *Let F be a totally real number field that is unramified above p. Let $\bar\rho : G_F \to \mathrm{GL}_2(\mathbb{F})$, as before, be such that $\bar\rho$ is (totally) odd and has non-solvable image when $p = 2$, and $\bar\rho|_{F(\mu_p)}$ is absolutely irreducible when $p > 2$. (Hence $\bar\rho|_{G_{F(\zeta_{p^m})}}$ is irreducible for all non-negative integers m.)*

Let K/F be the extension of F cut out by $\mathrm{Ad}^0(\bar\rho)$ when $p > 2$ and that cut out by $\mathrm{Ad}(\bar\rho)$ when $p = 2$ and set $F_m = F(\zeta_{p^m})$ in both cases. Then:

1. $p > 2$: $H^1(\mathrm{Gal}(KF_m/F), \mathrm{Ad}^0(\bar\rho)^*(1)) = 0$.

2. $p = 2$: $H^1(\mathrm{Gal}(KF_m/F), \mathrm{Ad}(\bar\rho)) = H^1(F_m/F, Z)$, and $H^1(F_m/F, Z)$ has dimension 2 over $\mathbb{F}$ for $m > n_0$, with n_0 the largest value of m for which $F_m^+ \subset F$, with F_m^+ the totally real subfield of F_m (We recall that $Z = \mathbb{F}\mathrm{id} \subset \mathrm{Ad}$ are the homotheties.)

Proof. 1. The case of $p \geq 5$ is dealt with in Lemma 2.5 of [55]. The case of $p = 3$ is dealt with exactly as in proof of Theorem 2.49 of [13] using our assumption that $\bar\rho|_{G(F(\zeta_p))}$ is irreducible. This reference assumes that $F = \mathbb{Q}$, but because of our assumption that F is unramified at p, the arguments there remains valid in our situation.

2. We turn to $p = 2$. By our assumption and Dickson's theorem the projective image of $\bar\rho$ is $\mathrm{PGL}_2(\mathbb{F}') \simeq \mathrm{SL}_2(\mathbb{F}')$ for $|\mathbb{F}'| = 2^m$ with $m > 1$, and thus is a simple group. Hence $\mathrm{Gal}(KF_m/F_m) \simeq \mathrm{SL}_2(\mathbb{F}')$. Using that $H^0(\bar\rho, M_2(\mathbb{F})) = Z$, inflation restriction gives

$$0 \to H^1(F_m/F, \mathbb{F}) \to H^1(KF_m/F, \mathrm{Ad}) \to H^1(\mathrm{Gal}(KF_m/F_m) \simeq \mathrm{SL}_2(\mathbb{F}'), \mathrm{Ad}).$$

By Lemma 4.3 (5), $H^1(\mathrm{SL}_2(\mathbb{F}'), \mathrm{Ad}) = 0$, and thus $H^1(\mathrm{Gal}(KF_m/F), \mathrm{Ad}(\bar\rho)) = H^1(F_m/F, Z)$. The last statement follows from the fact that if n_0 is the largest value of m for which $F_m^+ \subset F$, with F_m^+ the maximal totally real subfield of F_m, then for $m > n_0$ the maximal elementary abelian $(2, \cdots, 2)$ quotient of $\mathrm{Gal}(F_m/F)$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. $\square$

5.4. $p > 2$. For the definition of L_v in the following lemma, see case 1 of 4.1.4.

Lemma 5.3. *Let p be an odd prime. Assume that F is a totally real number field unramified at p. For each positive integer n, there is a set of primes Q_n such that:*

- $|Q_n| = \dim_{\mathbb{F}} H^1_{\{L_v^\perp\}}(S, (\mathrm{Ad}^0)^*(1))$,
- *for $v \in Q_n$, v is unramified in $\bar{\rho}$ and ψ, $\bar{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues α_v, β_v, and $\mathbb{N}(v) = 1$ mod p^n,*
- $H^1_{\{L_v^\perp\}}(S \cup Q_n, (\mathrm{Ad}^0)^*(1)) = 0$ *where $L_v^\perp = 0$ for $v \in Q_n$.*

Remark: The referee pointed out to us that the lemma above may not be true for $p = 2$. Indeed, for $p = 2$, we have $(\mathrm{Ad}^0)^*(1) \simeq \mathrm{Ad}/Z$, the only irreducible subspace of Ad/Z is Ad^0/Z, any $g \in \mathrm{GL}_2(\mathbb{F})$ which has distinct eigenvalues does not have 1 as eigenvalue in Ad^0/Z, thus the argument of Lemma 2.5 of [55] does not apply.

Proof. The condition (1) in the proof of Lemma 2.5 of [55] follows from Lemma 5.2 (1). After this, the lemma follows just as in the proof of Lemma 2.5 of [55] by using the Cebotarev density theorem. □

Lemma 5.4. *For $v \in Q_n$, $\dim_{\mathbb{F}}(H^1(D_v, \mathrm{Ad}^0)) = 2 = 1 + \dim_{\mathbb{F}}(H^0(D_v, \mathrm{Ad}^0))$.*

Proof. By the Euler characteristic formula and Tate duality it is enough to show that

$$\dim_{\mathbb{F}}(H^0(D_v, (\mathrm{Ad}^0)^*(1))) = 1$$

and this follows from the fact that $\mathbb{N}(v)$ is 1 mod p and that $\bar{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues for $v \in Q_n$. □

Proposition 5.5. *The ring $R^{\square,\psi}_{S \cup Q_n}$ is generated over the ring $R^{\square,\mathrm{loc},\psi}_S$ by $|Q_n| + |S| - 1 = h^1_{\{L_v^\perp\}}(S, (\mathrm{Ad}^0)^*(1)) + |S| - 1$ generators.*

Proof. This follows from Lemma 5.4, Lemma 4.4 and Wiles' formula (2). □

5.5. $p = 2$. Recall that F is a totally real number field and $\bar{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F})$ has non-solvable image with $\mathbb{F}$ of residue characteristic 2. Let S be a finite set of places of F containing the archimedean places and the primes above 2 and the primes which are ramified in $\bar{\rho}$.

5.5.1. *Linear disjointness.* We need a result about disjointness of certain field extensions, Proposition 5.6, that is crucial for the existence of auxiliary sets of primes Q_n proved in Lemma 5.10.

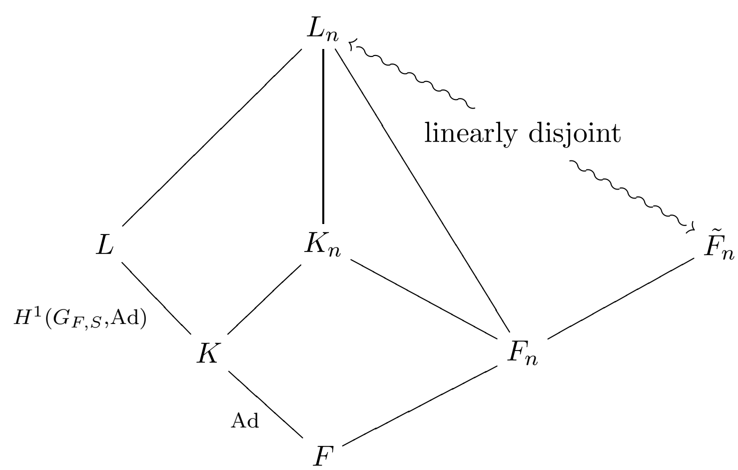
Let $M_n = \mathbb{Q}(\mu_{2^n})$ for $n \geq 1$ and M_n^+ be the the maximal totally real subfield of M_n. Let ϵ_n be roots of unity : $\epsilon_1 = -1$ and $\epsilon_{n+1}^2 = \epsilon_n$ for $n \geq 1$. Let $x_n = \frac{\epsilon_n + \epsilon_n^{-1}}{2}$. We have $M_n = \mathbb{Q}(\epsilon_n)$ and $M_n^+ = \mathbb{Q}(x_n)$; $x_2 = 0$, $x_3 = 1/\sqrt{2}$ where $\sqrt{2}$ is a square root of 2. We have for $n \geq 1$: $2x_{n+1}^2 = x_n + 1$. Let $y_n = \frac{x_n+1}{2}$ so that $y_n = x_{n+1}^2$ and $M_{n+1}^+ = M_n^+(\sqrt{y_n})$.

Let n_0 be the biggest integer $n \geq 2$ such that $M_n^+ \subset F$. For $n > n_0$, let $\bar{y}_n \in F^*/(F^*)^{2^n}$ the image of y_{n_0}, and let κ_n be the image of $\bar{y}_n$ in $H^1(G_F, \mu_{2^n})$ by the Kummer map.

Let:

- $n > n_0$
- K be the splitting field of $\mathrm{Ad}(\bar{\rho})$
- L be the splitting field over K of the image of $H^1(G_{F,S}, \mathrm{Ad}(\bar{\rho}))$ in $\mathrm{Hom}(G_K, \mathrm{Ad}(\bar{\rho}))$
- $F_n = F(\mu_{2^n})$, $K_n = K(\mu_{2^n})$ and $L_n = L(\mu_{2^n})$
- $\tilde{F}_n$ be the extension of F_n corresponding to the kernel of the image of κ_n in $\mathrm{Hom}(G_{F_n}, \mu_{2^n})$.

The inclusions between the fields above may be summarised by the following diagram (which also includes the conclusion of the main result, Proposition 5.6, of the present §5.5.1):



Proposition 5.6. *Let $n > n_0$. Then L_n and $\tilde{F}_n$ are linearly disjoint extensions of F_n.*

Proof. We accomplish the proof via a succession of lemmas. We leave the proof of the following lemma to the reader.

Lemma 5.7. *Let p a prime number and Ω be a field of characteristic $\neq p$. Let n be an integer ≥ 1. We suppose that Ω contains a primitive p^n root of unity. Let $y \in \Omega^*$. Let $1 \leq m \leq n$. If $\Omega(y^{1/p^m})/\Omega$ is of degree p^a, $a \geq 1$ then $\Omega(y^{1/p^n})/\Omega$ is of degree p^{a+n-m}.*

Next we have:

Lemma 5.8. *$F_{n_0}(y_{n_0}^{1/4})$ is a dihedral extension of F of degree 8.*

Proof. Let us prove the lemma. Let $P := F_{n_0}(y_{n_0}^{1/4})$. As $n_0 \geq 2$, the extension P/F is Galois. As $M_{n_0} \subset F$, we have $F(i) = F_{n_0}$ and it is a quadratic CM extension of F. By Lemma 5.7, to prove that P/F is of degree 8, we have to prove that $F(i, \sqrt{y_{n_0}})/F(i)$ is of degree 2. But this follows from the fact that $F(\sqrt{y_{n_0}}) = FM_{n_0+1}$ is a quadratic totally real extension of F and it is linearly disjoint of $F(i)/F$. Once we know that P/F is of degree 8, it is dihedral as $\mathrm{Gal}(F(i)/F)$ acts by $\epsilon \mapsto \epsilon^{-1}$ on μ_4. □

Recall $n > n_0$. The extensions K/F and $\tilde{F}_n/F$ are linearly disjoint as $\mathrm{Gal}(K/F)$ is a simple, non-cyclic group and $\tilde{F}_n/F$ is a solvable Galois extension. In particular, $\tilde{F}_n$ and K_n are linearly disjoint over F_n. So it suffices to prove that $\tilde{F}_n K_n$ and L_n are linearly disjoint over K_n.

We have $F_n(\sqrt{y_{n_0}}) = F_n$ as $n \geq n_0 + 1$. We do not have $F_n(y_{n_0}^{1/4}) = F_n$ as by lemma 5.8 , $F_{n_0}(y_{n_0}^{1/4})$ is not an abelian extension of F. From Lemma 5.7 we now immediately see the first part of the following lemma:

Lemma 5.9. *1. The extension $\tilde{F}_n/F_n$ is cyclic of degree 2^{n-1} and its cyclic sub-extension of degree 2 is $F_n(y_{n_0}^{1/4})$.*

2. The element κ_n is unramified outside S, i.e. is $\in H^1(G_{F,S}, \mu_{2^n})$. The order of κ_n is divisible by 2^{n-1}.

Proof. It remains only to prove 2. It suffices to prove that $y_{n_0} = x_{n_0+1}^2$ is a unit at primes not above 2. Let us prove that the x_n are units at primes not above 2 for $n \geq 3$. Let R_n be the minimal polynomial of x_n over $\mathbb{Q}$. We have $R_2(X) = X$, $R_3(X) = 2X^2 - 1$ and for $n \geq 3$: $R_n(X) = R_{n-1}(2X^2 - 1)$. We see that R_n has integral coefficients, $R_n(X) = 2^{n-2}X^{2^{n-2}} + \ldots$, $R_3(0) = -1$ and for $n \geq 4$:

$$R_n(0) = R_{n-1}(-1) = R_{n-1}(1) = R_2(1) = 1.$$

We have the second equality as for $n \geq 3$, R_n is an even polynomial and third equality by the recurrence formula for the R_n. We see that for $n \geq 3$, x_n^{-1} is integral of norm 2^{n-2} if $n \geq 4$ and -2 for $n = 3$. The restriction of κ_n to G_{F_n} cuts out an extension $\tilde{F}_n/F_n$ which by the first part of the lemma is of degree 2^{n-1}. □

As K_n/F_n and $\tilde{F}_n/F_n$ are linearly disjoint, $\tilde{F}_n K_n/K_n$ is also cyclic of degree 2^{n-1} and its cyclic sub-extension of degree 2 is $K'_n := K_n(y_{n_0}^{1/4})$.

To prove that $\tilde{F}_n K_n$ and L_n are linearly disjoint over K_n, and as the Galois group of L_n/K_n is abelian and of exponent dividing 2, we have to prove that K'_n/K_n is linearly disjoint of L_n/K_n. We have to prove that K'_n is not contained in L_n.

Let us suppose that K'_n is contained in L_n. Then K'_n would be an abelian extension of K. As $\mathrm{Gal}(K'_n/K)$ is isomorphic to $\mathrm{Gal}(F_n(y_{n_0}^{1/4})/F)$ it would imply that $\mathrm{Gal}(F_n(y_{n_0}^{1/4})/F)$ is abelian, which is not the case by lemma 5.8. □

5.5.2. *Auxiliary primes in even characteristic.* Here is the analog of Lemma 5.3 for $p = 2$.

Lemma 5.10. *For each $n >> 0$ ($n > n_0$ with n_0 as above suffices), there is a finite set of primes Q_n with the following properties:*

- *(a) Q_n is of constant cardinality $h^1(S, \mathrm{Ad}) - 2$;*
- *(b) for $v \in Q_n$, $\mathbb{N}(v)$ is 1 mod 2^n, $\bar{\rho}$ is unramified at v, and $\bar{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues α_v, β_v;*
- *(c) $v \in Q_n$ splits in the Kummer extension $\tilde{F}_n/F$, $h^0(D_v, \mathrm{Ad}) = 2$, and $h^1_{Q_n-split}(S, \mathrm{Ad}) = 2$;*
- *(d) $h^1_{S-split}(Q_n, \mathrm{Ad}) = 2 - \sum_{v \in S} h^0(D_v, \mathrm{Ad}) + \sum_{v \in Q_n} h^0(D_v, Ad) = 2 - \sum_{v \in S} h^0(D_v, \mathrm{Ad}) + 2|Q_n|$;*
- *(e) The minimal number of generators of $R^{\square}_{S \cup Q_n}$ over $R_S^{\square, \mathrm{loc}, \psi}$ is $= 2 + 2|Q_n| - 1$;*
- *(f) Consider $F^S_{Q_n}$, the maximal abelian extension of F of degree a power of 2 which is unramified outside Q_n and is split at primes in S. Define $G_n := \mathrm{Gal}(F^S_{Q_n}/F)$. Then we have $G_n/2^{n-2}G_n = (\mathbb{Z}/2^{n-2}\mathbb{Z})^t$ where $t = h^1_{S-split}(Q_n, \mathbb{F})$;*
- *(g) We have $t = 2 - |S| + |Q_n|$;*
- *(h) Thus $[h^1_{S-split}(Q_n, \mathrm{Ad}) - h^1_{S-split}(Q_n, \mathbb{F})] + [\sum_{v \in S} h^0(D_v, \mathrm{Ad}) - h^0(G_F, \mathrm{Ad})] = |Q_n| + |S| - 1$.*

Proof. We begin with the following observations:

- The representations $\bar{\rho}$ and $\bar{\rho}|_{G_{\tilde{F}_n}}$ have the same projective image $G \simeq \mathrm{SL}_2(\mathbb{F}_{2^r})$ for some $r > 1$. We know from Lemma 4.3 (5) that the only non-zero, irreducible G- submodule V of Ad is Z.
- Thus we see that for any non-zero, irreducible G-submodule V of Ad, there is an element σ in G such that $\mathrm{Ad}(\sigma)$ has an eigenvalue $\neq 1$, while $\mathrm{Ad}(\sigma)|V$ has 1 as an eigenvalue. (Any element $\sigma \in G$ with distinct eigenvalues works.)

We deduce:

(1) Given a $\psi \in H^1(G_F, \mathrm{Ad})$ that has non-trivial restriction to $H^1(\tilde{F}_n, \mathrm{Ad})$, and hence also to $H^1(K\tilde{F}_n, \mathrm{Ad})$ as $H^1(G, \mathrm{Ad}) = 0$, arguing as in proof of Lemma 2.5 of [55], we can find a place $v \notin S$ of F such that

- v splits in $\tilde{F}_n/F$
- $\bar{\rho}$ is unramified at v and $\bar{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues α_v, β_v
- ψ maps non-trivially to $H^1(\mathbb{F}_v, \mathrm{Ad}) \subset H^1(F_v, \mathrm{Ad})$.

This is also proved in Lemma 40 of [20].

(We give the argument of Lemma 2.5 of [55]. By the Cebotarev density theorem it is enough to find an element $\sigma' \in G_{\tilde{F}_n}$ such that $\bar{\rho}(\sigma')$ has distinct eigenvalues, and $\psi(\sigma') \notin (\sigma' - 1)\mathrm{Ad}$. (The set of such σ' is then a non-empty open subset of $G_{\tilde{F}_n}$.) As $H^1(K\tilde{F}_n/\tilde{F}_n, \mathrm{Ad}) = 0$, we see that $\psi(G_{K\tilde{F}_n})$ is a non-trivial G-submodule of Ad, and thus contains non-zero irreducible submodule V (necessarily $= Z$). Consider the element σ above that we regard as an element of $\mathrm{Gal}(K\tilde{F}_n/\tilde{F}_n)$ ($\simeq G$): it has the property that $\psi(G_{K\tilde{F}_n})$ is not contained in $(\sigma - 1)\mathrm{Ad}$. Denote by σ again (an arbitrarily chosen) lift of $\sigma \in \mathrm{Gal}(K\tilde{F}_n/\tilde{F}_n)$ to $G_{\tilde{F}_n}$. For $\tau \in G_{K\tilde{F}_n}$ we have $\psi(\tau\sigma) = \psi(\tau) + \psi(\sigma)$, $(\tau\sigma - 1)\mathrm{Ad} = (\sigma - 1)\mathrm{Ad}$, and $\bar{\rho}(\tau\sigma)$ has distinct eigenvalues. If σ has the property that $\psi(\sigma) \notin (\sigma - 1)\mathrm{Ad}$ then set σ' to be σ. Otherwise using the fact that $\psi(G_{K\tilde{F}_n})$ is not contained in $(\sigma - 1)\mathrm{Ad}$, we may find a $\tau \in G_{K\tilde{F}_n}$ such that $\psi(\tau\sigma) \notin (\tau\sigma - 1)\mathrm{Ad}$, and set σ' to be $\tau\sigma$.)

(2) Further from Proposition 5.6, and $H^1(\mathrm{proj.im}(\bar{\rho}), \mathrm{Ad}(\bar{\rho})) = 0$ (Lemma 4.3(5)) we see that

$$\ker(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{\tilde{F}_n}, \mathrm{Ad})) = \ker(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{F_n}, \mathrm{Ad}))$$

$$= H^1(F_n/F, \mathrm{Ad}^{\mathrm{im}(\bar{\rho})}) = H^1(F_n/F, \mathbb{F}),$$

and the latter for $n > n_0$ has $\mathbb{F}$-dimension 2. This is also the kernel of the restriction map $H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{KF_n,S}, \mathrm{Ad}))$ by Lemma 5.2.

Applying (1) to elements in $H^1(G_{F,S}, \mathrm{Ad})$ that are in a chosen $\mathbb{F}$-subspace of $H^1(G_{F,S}, \mathrm{Ad})$ that is complementary to the kernel of the restriction map $H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{\tilde{F}_n}, \mathrm{Ad})$ we get that for each $n > n_0$:

– there is a set Q_n of cardinality $h^1(S, \mathrm{Ad}) - 2$, such that $v \in Q_n$ splits in $\tilde{F}_n/F$,

– $\bar{\rho}$ is unramified at v, and $\bar{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues for $v \in Q_n$ and

– $H^1_{Q_n-split}(S, \mathrm{Ad}) = \ker(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{\tilde{F}_n}, \mathrm{Ad})) = H^1(F_n/F, (\mathrm{Ad})^{\mathrm{im}(\bar{\rho})}) = H^1(F_n/F, Z)$, where the second equality uses (2) above.

As the last vector space has dimension 2 we get that $h^1_{Q_n-split}(S, \mathrm{Ad}) = 2$. For $v \in Q_n$, as $\bar{\rho}$ is unramified and Frob_v has distinct eigenvalues in $\bar{\rho}$, we have $h^0(D_v, \mathrm{Ad}) = 2$. This proves (a),(b),(c) of the Proposition. Part (d) follows using Wiles' formula (3) of §4.1.6, and noting that for $v \in Q_n$

$$h^1(D_v, \mathrm{Ad}) - h^0(D_v, \mathrm{Ad}) = h^2(D_v, \mathrm{Ad}) = h^0(D_v, \mathrm{Ad}).$$

The first equality follows from the Euler-Poincaré characteristic formula, and the second by Tate duality.

Part (e) follows from part (d) using Lemma 4.4. The computation of $t = h^1_{S-triv}(Q_n, \mathbb{F}) (= h^1_{S-triv}(Q_n, \mathbb{Z}/2\mathbb{Z}))$ as $2 + |Q_n| - |S|$ follows from noting that:

– $H^1(G_F, \mathbb{F}) \hookrightarrow H^1(G_F, \mathrm{Ad})$ (by Lemma 4.3(4)), $H^1_{Q_n-split}(S, \mathrm{Ad})$ is the image of $H^1(F_n/F, Z = \mathbb{F})$ under inflation, and thus $H^1_{Q_n-split}(S, \mathbb{F})$ is also the image of $H^1(F_n/F, Z = \mathbb{F})$ under inflation.

– Then by Wiles' formula (cf. §4.1.6), applied to $M = \mathbb{F}$, we derive the formula for t.

It remains to prove (f). This time we apply Wiles' formula (cf. §4.1.6) with $M = \mathbb{Z}/2^n\mathbb{Z}$ to the finite $\mathbb{Z}/2^n\mathbb{Z}$-module $H^1(S \cup Q_n, \mathbb{Z}/2^n\mathbb{Z})$ to get

$$(4) \qquad \frac{|H^1_{S-split}(Q_n, \mathbb{Z}/2^n\mathbb{Z})|}{|H^1_{Q_n-split}(S, \mathbb{Z}/2^n\mathbb{Z}(1) \simeq \mu_{2^n})|}$$

$$= \frac{|H^0(G_F,\mathbb{Z}/2^n\mathbb{Z})|}{|H^0(G_F,\mathbb{Z}/2^n\mathbb{Z}(1))|}\prod_{v\in S}\frac{1}{|H^0(D_v,\mathbb{Z}/2^n\mathbb{Z})|}\prod_{v\in Q_n}\frac{|H^1(D_v,\mathbb{Z}/2^n\mathbb{Z})|}{|H^0(D_v,\mathbb{Z}/2^n\mathbb{Z})|}.$$

A simple computation, using that $H^1(D_v,\mathbb{Z}/2^n\mathbb{Z})\simeq(\mathbb{Z}/2^n\mathbb{Z})^2$ as $\mathbb{N}(v)$ is 1 mod 2^n for $v\in Q_n$, then gives that

$$|H^1_{S-split}(Q_n,\mathbb{Z}/2^n\mathbb{Z})| = |H^1_{Q_n-split}(S,\mathbb{Z}/2^n\mathbb{Z}(1))|2^{n-1}2^{n(|Q_n|-|S|)}.$$

Note that κ_n (cf. §5.6) is an element of $H^1(S,\mathbb{Z}/2^n\mathbb{Z}(1))$, and κ_n has order divisible by 2^{n-1} by Lemma 5.9. As places in Q_n split in the Kummer extension $\tilde{F}_n/F$ cut out by κ_n, we get that $\kappa_n\in H^1_{Q_n-split}(S,\mathbb{Z}/2^n\mathbb{Z}(1))$.

1. Thus we get that $2^{n(2+|Q_n|-|S|)}2^{-2}$ divides the order of the finite group $H^1_{S-split}(Q_n,\mathbb{Z}/2^n\mathbb{Z})$ that has exponent dividing 2^n.

2. As $h^1_{S-split}(Q_n,\mathbb{Z}/2\mathbb{Z})=t=2+|Q_n|-|S|$, we deduce that the $\mathbb{Z}/2^n\mathbb{Z}$-module $H^1_{S-split}(Q_n,\mathbb{Z}/2^n\mathbb{Z}))$ is generated by t elements.

From 1. and 2. it easily follows that there is an isomorphism $G_n/2^{n-2}G_n\simeq(Z/2^{n-2}\mathbb{Z})^t$ as desired.

□

5.6. **Action of inertia of auxiliary primes.** The following proposition is standard (see Lemma 2.1 of [55]): p may be odd or even.

Proposition 5.11. *Let $v\in Q_n$. The universal deformation $\rho^{\mathrm{univ},\psi}_{S\cup Q_n}$ corresponding to the ring $\bar{R}^{\psi}_{S\cup Q_n}$ is such that $\rho^{\mathrm{univ},\psi}_{S\cup Q_n}|_{D_v}$, is of the form*

$$\begin{pmatrix}\gamma_{\alpha_v} & 0\\ 0 & \gamma_{\beta_v}\end{pmatrix},$$

where $\gamma_{\alpha_v},\gamma_{\beta_v}$ are characters of D_v such that γ_{α_v} and γ_{β_v} modulo the maximal ideal are unramified and takes Frob_v *to α_v and β_v respectively. Note that $\gamma_v:=\gamma_{\alpha_v}|_{I_v}=\gamma_{\beta_v}^{-1}|_{I_v}$ for a character $\gamma_v: I_v\to\Delta'_v\to(\bar{R}^{\psi}_{S\cup Q_n})^*$ where Δ'_v is the maximal p-quotient of k_v^*. This naturally endows $\bar{R}^{\psi}_{S\cup Q_n}$ (and hence $\bar{R}^{\square,\psi}_{S\cup Q_n}$) with a $\mathcal{O}[\Delta'_{Q_n}]=\otimes_{v\in Q_n}\mathcal{O}[\Delta'_v]$ module structure, and its quotient by the augmentation ideal of $\mathcal{O}[\Delta'_{Q_n}]$ is isomorphic to $\bar{R}^{\psi}_S$ (resp., $\bar{R}^{\square,\psi}_S$).*

Let $p=2$ and let $G_{n,2}$ be the maximal quotient of G_n (5.10) which is a group killed by 2 so that we have an action of $(G_{n,2})^*$ on $\bar{R}^{\psi}_{S\cup Q_n}$. We have the compatibility with the action of Δ'_{Q_n} :

Lemma 5.12. *For $\delta\in\Delta'_{Q_n}$ and $\chi\in(G_{n,2})^*(\bar{R}^{\psi}_{S\cup Q_n})$ we have : $a(\chi)\circ\delta=\chi(\delta)\times(\delta\circ a(\chi))$.*

Proof. The action of δ on $\bar{R}^{\psi}_{S\cup Q_n}$ is by multiplication by $\gamma_v(\delta)$. The lemma follows from the identity : $a(\chi)(\gamma_v(\delta))=\chi(\delta)\gamma_v(\delta)$. □

6. Taylor's potential version of Serre's conjecture

We will need the following variant and extension of Taylor's results on a potential version of Serre's conjecture (see [54], [55]):

Theorem 6.1. *Let $\bar{\rho}$ a $G_{\mathbb{Q}}$ representation of S-type, with $2 \le k(\bar{\rho}) \le p+1$ if $p > 2$. We assume that $\bar{\rho}$ has non-solvable image when $p = 2$, and $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is absolutely irreducible when $p > 2$. Then there is a totally real field F that is Galois over $\mathbb{Q}$ of even degree, F is unramified at p, and even split above p if $\bar{\rho}|_{D_p}$ is irreducible, $\mathrm{im}(\bar{\rho}) = \mathrm{im}(\bar{\rho}|_{G_F})$, $\bar{\rho}|_{F(\mu_p)}$ absolutely irreducible, and such that:*

(i) Assume $k(\bar{\rho}) = 2$ if $p = 2$. Then $\bar{\rho}|_{G_F}$ arises from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ that is discrete series of weight $k(\bar{\rho})$ at the infinite places and unramified at all the places above p. If $\bar{\rho}$ is ordinary at p, then for all places v above p, π_v is ordinary.

(ii) $\bar{\rho}|_{G_F}$ also arises from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ such that π_v, at all places v above p, is of conductor dividing v (and is unramified if $\bar{\rho}$ is finite flat at v), and is of weight 2 at the infinite places. Further π_v is ordinary at all places v above p in the case when $\bar{\rho}$ is ordinary at p.

(iii) Further :

a) In the case $k(\bar{\rho}) = p$ and the representation $\bar{\rho}|_{I_p}$ is trivial, we may choose F so that at places $\wp$ of F above p, $\bar{\rho}|_{D_\wp}$ is trivial.

b) Given finitely many primes $\ell_i \neq p$ and extensions $F_{\ell_i}/\mathbb{Q}_{\ell_i}$, then we may choose F so that for every embedding $F \hookrightarrow \overline{\mathbb{Q}}_{\ell_i}$, the closure of F contains F_{ℓ_i}.

c) In the case that $p > 2$ and weight $k(\bar{\rho}) = p+1$, we may ensure that F is split at p.

d) Given a finite extension L of $\mathbb{Q}$, we can morever impose that F and L are linearly disjoint.

Proof. We give the arguments that one has to add to Taylor's papers [54], [55], and Theorem 2.1 of [30], to get the additional needed statements that are written in italics below, and which are not explicitly in these papers.

In the case when the projective image of $\bar{\rho}$ is dihedral, we ensure that the fields considered below, besides being linearly disjoint from the extension cut out by $\bar{\rho}$, are split at a prime which splits in the field cut out by the projectivisation of $\bar{\rho}$, but which is inert in the quadratic subfield of $\mathbb{Q}(\mu_p)$. This ensures that $\bar{\rho}|_{G_{F(\mu_p)}}$ is irreducible for all the number fields F considered below.

We first dispose of (i) and (ii) in the cases where the image of $\bar{\rho}$ is solvable. Then it is a standard consequence of results of Langlands and Tunnell (see [37] and [61]) that $\bar{\rho}$ arises from $S_k(\Gamma_1(N))$ for some positive integers N and $k \geq 2$ (see pg. 220 of [51] for a hint). Using results of [22] and Propositions 8.13 and 8.18 of [25] it furthermore follows that $\bar{\rho}$ arises from $S_k(\Gamma_1(N))$ for $k = k(\bar{\rho})$ and some integer N prime to p, and also from $S_2(\Gamma_1(Np))$. Thus the hypotheses (α), (β) of §8.2 are satisfied for $\bar{\rho}$. After this the theorem follows from Theorem 8.2 below.

We assume that $\bar{\rho}$ has non-solvable image till further notice. In the proof of Taylor, one has a moduli problem X for Hilbert-Blumenthal abelian varieties A with polarisation and level structures. It is Hilbert-Blumenthal relative to a totally real field that we call M (as in [54] ; in [55] it is called E). One has an embedding $i : O_M \to \mathrm{End}(A)$. The polarisation datum is an isomorphism j of a fixed ordered invertible O_M-module to the ordered invertible O_M-module $\mathcal{P}(A, i)$ of polarisations of (A, i). The level structure is called α. In [54], the level structure is at a prime λ above p and an auxiliary prime that Taylor calls p (in Taylor, the residue characteristic of $\bar{\rho}$ is ℓ). We call this auxiliary prime p_0. In [55], there are two auxiliary primes p_1 and p_2. The level structure at λ is given by $\bar{\rho}$. There is a prime $\wp$ of M above p_0 (resp. p_1) for which the residual representation is irreducible with solvable image. There exists a point x of $X(F)$ giving rise to a data(A, i, j, α). In [54] A is defined over F and a modularity lifting theorem gives the automorphy of the Tate module $V_\wp(A)$, hence of A and of $\bar{\rho}_{|G_F}$. In [55], A is defined over a totally real extension $N'F$ of F which gives rise to an abelian variety B over F (see lemma 4.4.). The modularity of B implies that of $\bar{\rho}_{|G_F}$.

The existence of F and a point of X with values in F follows by a theorem of Moret-Bailly from the existence of points of X with values in the completion of $\mathbb{Q}$ at ∞, p and the auxiliary primes. For $p = 2$ one proves the existence of points with values in $\mathbb{Q}_2$ or $\mathbb{Q}_{p_i}$, p_i auxiliary prime, as for $p \neq 2$.

- *For $p = 2$, there exists a point of $X(\mathbb{R})$.*

Let us first consider the case where $\bar{\rho}_{|D_2}$ is reducible ([54]). The polarization data j is an isomorphism $(O_M^+) \simeq \mathcal{P}(A, i)$ (see erratum page 776 of [55]). In the erratum, Taylor gives a data (A, i, j) over $\mathbb{R}$. The torus $A(\mathbb{C})$ is $\mathbb{C}^{\mathrm{Hom}(M,\mathbb{R})}/L$, where $L = \delta_M^{-1}1 + O_M z$, δ_M is the different of M and $z \in (i\mathbb{R}_{>0})^{\mathrm{Hom}(M,\mathbb{R})}$. For $a \in O_M$, $j(a)$ corresponds to the Riemann form :

$$E(x + yz, u + vz) = \mathrm{tr}_{M/\mathbb{Q}}(a(yu - xv)).$$

The action of the complex conjugation c over $A(\mathbb{C})$ is the natural one on the torus and we see that the action of c on the points of order 2 of A is trivial. It follows that, if $\bar{\rho}(c)$ is trivial, one can define a level structure α such that (A, i, j, α) is a real point of X. If $\bar{\rho}(c)$ is non trivial, let $L' \subset L$ be defined by, for $\beta \in \delta_M$ such that $\delta_{M,2} = \beta O_{M,2}$:

$$L' = \{u + zv \in L \mid v \equiv \beta u \text{ mod. } 2O_M\}.$$

As $c(L') = L'$, L' defines an abelian variety A' over $\mathbb{R}$, which is isogenous to A. It has an action i' of O_M. For $a \in O_M$, $1/2E$ defines a Riemannian form on A' ; this gives a polarisation datum j'. If $\gamma \in \delta_M^{-1} \otimes \mathbb{Z}_2$ is such that $\gamma\beta \equiv 1$mod. 2, the matrix of c acting on $L' \otimes \mathbb{Z}_2$ in the basis $(\gamma + z, 2z)$ is

$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$. This implies that one can find a level structure α' such that (A', i', j', α') defines a point in $X(\mathbb{R})$.

When $\bar{\rho}_{|D_2}$ is irreducible ([55]), the polarization data j is an isomorphism $((\delta_M^{-1})^+) \simeq \mathcal{P}(A,i)$. We can find a point (A,i,j,α) in $X(\mathbb{R})$ by taking $A = E \otimes O_M$, E elliptic curve over $\mathbb{R}$ with 4 or 2 points $\mathbb{R}$-rational of order 2 according whether $\bar{\rho}(c)$ is trivial or not.

- *The case $p = 2$, $k(\bar{\rho}) = 4$.* We have to prove that $\bar{\rho}_{|G_F}$ is associated to a cuspidal automorphic form of $\mathrm{GL}_2(\mathbb{A}_F)$ of parallel weight 2 and which is Steinberg at places v above 2. We do this as Taylor does ([54]) when the character χ_v (p. 130) is such that $\chi_v^2 = 1$. The abelian variety A is chosen to have completely toric reduction at primes above 2 (first case of lemma 1.2. of [54]). This proves that we can take π of weight 2 and of level v for v above 2 .

- *If $k(\bar{\rho}) = 2$, one can ensure π is of weight 2 and unramified at primes above p (including $p = 2$) and ordinary at these places if $\bar{\rho}$ is ordinary.* When the restriction of $\bar{\rho}$ to the decomposition group D_p is irreducible, this follows from the fact that in the lemma 4.3. of [55], we can impose that χ is unramified at p (ℓ in [55]). Then we obtain in 3. of prop. 4.1. of [55] that the Weil-Deligne parameter at p is unramified.

When $\bar{\rho}$ is ordinary, we do as in [31], as follows.

We can ensure that the point of $X(\mathbb{Q}_p)$ given by the theorem of Moret-Bailly (see [42]) defines an abelian variety A_v which has good ordinary reduction. To prove this, first we twist by a character unramified at p to reduce to the case where the restriction of $\det(\bar{\rho})\overline{\chi_p}^{-1}$ to D_p is trivial. Then, $\bar{\rho}_{|D_p}$ has the shape :

$$\begin{pmatrix} \overline{\chi_p}\chi_v^{-1} & * \\ 0 & \chi_v \end{pmatrix}$$

with χ_v unramified. As in p. 131 of [54], we define a lifting $\tilde{\chi}_v$ of χ_v : we choose the first definition even if $\chi_v^2 = 1$, so that $\tilde{\chi}_v$ sends the Frobenius to the chosen Weil number β_v. The field $\tilde{F}_v$ of p. 130 of [54] is $\mathbb{Q}_p$. In Lemma 1.2. of [54], we do not need to do descent. We have, as p. 135 of [54], to lift the class $\overline{x} \in H^1(D_v, O_M/\lambda(\overline{\chi_p}\chi_v^{-2}))$ defining $\bar{\rho}_v$ to an x_λ in $H^1(D_v, O_{M,\lambda}(\chi_p\tilde{\chi}_v^{-2}))$. The obstruction to do this is an element of $H^2(D_v, O_{M,\lambda}(\chi_p\tilde{\chi}_v^{-2}))$. This group is dual to $H^0(D_v, M_\lambda/O_{M,\lambda}(\tilde{\chi}_v^2))$. When χ_v^2 is non trivial, there is no obstruction. Let us prove that, when χ_v^2 is trivial, the obstruction o is trivial. The character $\tilde{\chi}_v^2$ is non trivial as the Weil number β_v^2 is not 1. Let a be the least integer such that $\tilde{\chi}_v^2$ is non trivial modulo p^{a+1} (recall that M is unramified at p). Let us write $\tilde{\chi}_v^2 = 1 + p^a\eta \text{ mod.} p^{a+1}$. Let us denote o_0 the corresponding obstruction for $\eta = 0$. In fact o_0 is trivial by Kummer theory. By comparing the obstructions o and o_0, we prove that the obstruction o is the cup product η with $\overline{x}$. As η is unramified and $\overline{x}$ is finite this obstruction vanishes. This proves that we can find A_v which has good ordinary reduction.

The abelian variety A_v with the polarization and level structures define a point $x_v \in X(\mathbb{Z}_p)$ (take as integral structure on X the normalization of the integral structure for the moduli problem without level structure for primes above p). One considers Ω_v to be points of $X(\mathbb{Q}_p)$ that reduce to x_v. Then, applying Moret-Bailly theorem 1.3. of part 2 of [42] with this Ω_v, we can impose that the point $x \in X(F)$ that we get has the same reduction as x_v. The abelian variety A has ordinary good reduction at primes of F above p.

- *for $p = 3$ and $\bar{\rho}_{|D_p}$ irreducible, adjustment of the weight.* Although $p = 3$ is excluded in (Section 5 of) [55], as explained in Section 2 of [30], Lemma 2.2 of [30] allow one to lift this restriction.

- *$p \neq 2$, $\bar{\rho}_{|D_p}$ reducible, $k(\bar{\rho}) > 2$, adjustment of the weight.* The proof of lemma 1.5. of [54] shows that A is ordinary at v such that the inertial Weil-Deligne parameter at v of A is $(\omega^{k(\bar{\rho})-2} \oplus 1, 0)$ if $k(\bar{\rho}) \neq p+1$ and $(1 \oplus 1, N)$ if $k(\bar{\rho}) = p+1$ with N a non-zero 2×2 nilpotent matrix. (In the case $k(\bar{\rho}) = p$ and $\bar{\rho}_{|D_p}$ is semisimple, the proof of the quoted lemma gives that the inertial Weil-Deligne parameter at v of A is $(\omega^{-1} \oplus 1, 0)$, it does not say to which line of $\bar{\rho}$ reduces the line of ρ on which an open subgroup of I_p acts by the cyclotomic character). Thus as in [54] (and using Appendix B of [12]) one knows that $\bar{\rho}$ arises from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of parallel weight 2, and at places v above p, π_v is ordinary such that the inertial Weil-Deligne parameter of π_v is the same as that of A at v. It follows from this, using Hida theory (see Section 8 of [27], using also Lemma 2.2 of [30] to avoid the neatness hypothesis there) that $\bar{\rho}_{|G_F}$ also comes from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ that is unramified at places above p and of parallel weight $k(\bar{\rho})$.

Now we relax the assumption that $\bar{\rho}$ has non-solvable image.

- *for $k(\bar{\rho}) = p$ and the representation $\bar{\rho}|_{I_p}$ is trivial, one can impose that at places $\wp$ of F above p, $\bar{\rho}|_{G_\wp}$ is trivial* by enlarging F by an extension that is unramified at p (but not split).

- *one can impose that closures of F contain locally given extensions F_{ℓ_i},* $\ell_i \neq 2, p$, by successive applications of Grunwald-Wang theorem.

- (iii) *If $k(\bar{\rho})$ is even and $p > 2$, we can take F split at p (that proves c) of (iii))* . If $\bar{\rho}_{|D_p}$ is irreducible, we apply [55]. Let us suppose $\bar{\rho}_{|D_p}$ reducible. As $p > 2$ and $k(\bar{\rho})$ is even, the restriction to D_p of $\det(\bar{\rho})\overline{\chi_p}^{-1}$ is a square. It follows that there exists a character $\eta_1 : G_{\mathbb{Q}} \to (\overline{\mathbb{F}_p})^*$ that is even and split at p such that $\eta_2 := \det(\bar{\rho})\overline{\chi_p}^{-1}\eta_1^{-1}$ is the square of a character γ_2. We impose to F to contain the field F_0 fixed by the kernel of η_1. We then apply theorem 1.6. of [54] to $\bar{\rho}_{|G_{F_0}}$ twisted by $\gamma_2{}^{-1}_{|G_{F_0}}$ and we get F split at p.

- *given a finite extension L of $\mathbb{Q}$, we can impose that L and F are linearly disjoint* . As in proposition 2.1 of [26], when we apply Moret-Bailly theorem, we furthermore impose that F is split at a finite set of primes that are unramified in L and whose Frobenius generate the Galois group of the Galois closure of L. We choose the set of these primes disjoint from the finite set of primes above p and the auxiliary primes so that we can get these conditions simultaneously.

□

7. p-adic modular forms on definite quaternion algebras

The reference for this section is Sections 2 and 3 of [55]. The modifications used here at many places of the usual arguments to deal with non-neatness problems is an idea of [5].

Let p be any prime, and F a totally real number field of even degree in which p is unramified. Let D denote a quaternion algebra over F that is ramified at all infinite places, and ramified at a finite set Σ of finite places of F.

Fix a maximal order $\mathcal{O}_D$ in D and isomorphisms $(\mathcal{O}_D)_v \simeq M_2(\mathcal{O}_{F_v})$ for all places v at which D is split. Let A be a topological $\mathbb{Z}_p$-algebra which is either an algebraic extension of $\mathbb{Q}_p$, the ring of integers in such an extension or a quotient of such a ring of integers.

For a place v at which D is split denote by

$$U_0(v) = \{g \in \mathrm{GL}_2(\mathcal{O}_{F_v}) : g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mathrm{mod.}(\pi_v)\},$$

and

$$U_1(v) = \{g \in \mathrm{GL}_2(\mathcal{O}_{F_v}) : g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mathrm{mod.}(\pi_v)\}.$$

Let $U = \Pi_v U_v$ be an open subgroup of $(D \otimes_F \mathbb{A}_F^\infty)^*$ that is of the following shape: at places $v \notin \Sigma' \subset \Sigma$, $U_v \subset (\mathcal{O}_D)_v^*$ and is $= (\mathcal{O}_D)_v^*$ for almost all v, and for $v \in \Sigma'$, U_v is D_v^*. Note that in the latter case $[U_v : (\mathcal{O}_D)_v^* F_v^*] = 2$. The case of non-empty Σ' is considered only when $p = 2$, and its consideration may be motivated by Lemma 7.2.

Let $\psi : (\mathbb{A}_F^\infty)^*/F^* \to A^*$ be a continuous character. Let $\tau : U \to \mathrm{Aut}(W_\tau)$ be a continuous representation of U on a finitely generated A-module W_τ. We assume that

$$\tau|_{U \cap (\mathbb{A}_F^\infty)^*} = \psi^{-1}|_{U \cap (\mathbb{A}_F^\infty)^*}.$$

The character ψ and the representation τ will always be such that on an open subgroup of $\mathcal{O}_{F_p}^*$, ψ is an integral power of the norm character. (The norm character $\mathbb{N} : \Pi_{v|p} F_v^* \to \overline{\mathbb{Q}}_p^*$ is defined by taking products of the local norms.)

We regard W_τ as a $U((\mathbb{A}_F^\infty)^*)$-module with U acting via τ and $(\mathbb{A}_F^\infty)^*$ acting via ψ^{-1}. We define $S_{\tau,\psi}(U)$ to be the space of functions

$$f : D^* \backslash (D \otimes_F \mathbb{A}_F^\infty)^* \to W_\tau$$

such that:

$$f(gu) = u^{-1} f(g)$$
$$f(gz) = \psi(z) f(g)$$

for all $g \in (D \otimes_F \mathbb{A}_F^\infty)^*, u \in U, z \in (\mathbb{A}_F^\infty)^*$. We also use the notation $S_{\tau,\psi}(U,A)$ for $S_{\tau,\psi}(U)$ when we want to emphasise the role of the coefficients A.

Fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, and let $E \subset \overline{\mathbb{Q}_p}$ be a sufficiently large finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$, and residue field $\mathbb{F}$. We assume that E contains the images of all embeddings $F \hookrightarrow \overline{\mathbb{Q}_p}$. For all such embeddings we assume $D_v \otimes_{F \hookrightarrow E} E$ is split for all places v above p. Write W_k and $\bar{W}_k$ for $\otimes_{F \hookrightarrow E}\mathrm{Sym}^{k-2}{\mathcal{O}_E}^2$ and $\otimes_{F \hookrightarrow E}\mathrm{Sym}^{k-2}\mathbb{F}^2$ respectively, where $k \geq 2$ is an integer, and $k = 2$ if $p = 2$. These are $\Pi_{v|p}(\mathcal{O}_D)_v^*$-modules using an identification of $\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_E$ with $M_2(\mathcal{O}_E)$. The character $\psi : (\mathbb{A}_F^\infty)^*/F^* \to A^*$ acts on these modules W_k and $\bar{W}_k$ via the natural action of A^* ; ψ restricted to an open subgroup of $(\mathbb{A}_F^\infty)^*$ is $\mathbb{N}^{2-k}$ where $\mathbb{N}$ is the product of the local norms at places above p.

In the cases of U non-compact, and hence Σ' non-empty, that are considered (and thus $p = 2$), denote by $U' = \Pi_v U'_v$ the open compact subgroup of U where for places in $v \in \Sigma'$, U'_v is maximal compact, and for finite places v not in Σ', $U_v = U'_v$, i.e. U' is the maximal compact subgroup of U. The actions of U' on W_2 and $\bar{W}_2$ are trivial. Furthermore, $U(\mathbb{A}_F^\infty)^*/U'$ is abelian and the quotient $U(\mathbb{A}_F^\infty)^*/U'(\mathbb{A}_F^\infty)^*$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\Sigma'}$. It follows that the module W_2 of $U'(\mathbb{A}_F^\infty)^*$ can be extended to one of $U(\mathbb{A}_F^\infty)^*$ in one of $2^{|\Sigma'|}$ possible ways, and we denote these extensions by the same symbol W_2 (as we will fix such an extension). The module $\bar{W}_2$ has a unique extension to one of $U(\mathbb{A}_F^\infty)^*$, and that we again denote by the same symbol $\bar{W}_2$. For $p > 2$, W_k or $\bar{W}_k$ are naturally $U_p = \Pi_{v|p}U_v$ and hence U-modules. Thus in all cases we may regard W_k and $\bar{W}_k$ as $U(\mathbb{A}_F^\infty)^*$-modules.

The modules W_τ below will be of the form $W_k \otimes_{\mathcal{O}} V$ where V is a finite free $\mathcal{O}$-module on which U acts through a finite quotient, or $\bar{W}_k \otimes_{\mathcal{O}} V$ where V is a finite dimensional $\mathbb{F}$-vector space which is a U module (and with $k = 2$ if $p = 2$).

When $W_\tau = W_k, \bar{W}_k$ we also denote $S_{\tau,\psi}(U)$ by $S_{k,\psi}(U,\mathcal{O})$ and $S_{k,\psi}(U,\mathbb{F})$ respectively. (To be consistent we should also use $\overline{\psi}$ in the latter, but this inconsistency should cause no confusion.)

If $(D \otimes_F \mathbb{A}_F^\infty)^* = \amalg_{i \in I} D^* t_i U(\mathbb{A}_F^\infty)^*$ for a finite set I and with $t_i \in (D \otimes \mathbb{A}_F^\infty)^*$, then $S_{\tau,\psi}(U)$ can be identified with

$$\oplus_{i \in I} W_\tau^{(U(\mathbb{A}_F^\infty)^* \cap t_i^{-1} D^* t_i)/F^*} \tag{5}$$

via $f \to (f(t_i))_i$.

Let S be a finite set of places of F containing the places at infinity, Σ, the primes dividing p, and the set of places v of F such that either $U_v \subset D_v^*$ is not maximal compact, or U_v acts on W_τ non-trivially.

For each finite place v of F we fix a uniformiser π_v of F_v. We consider the left action of $g \in (D \otimes_F \mathbb{A}_F^\infty)^*$ by right translation on the W_τ-valued functions f on $(D \otimes \mathbb{A}_F^\infty)^*$ and denote this action by $g.f$ or gf. This induces an action of the double cosets $U\begin{pmatrix}\pi_v & 0\\ 0 & \pi_v\end{pmatrix}U$ and $U\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}U$ on $S_{k,\psi}(U)$ for $v \notin S$: we denote these operators by S_v (which is simply multiplication by $\psi(\pi_v)$) and T_v respectively. They do not depend on the choice of π_v.

We denote by $\mathbb{T}_\psi(U)$ the $\mathcal{O}$-algebra generated by the endomorphisms T_v and S_v acting on $S_{k,\psi}(U,\mathcal{O})$ for $v \notin S$. (Note that we are suppressing the weight k in the notation for the Hecke algebra, but this should not cause any confusion in what follows.)

A maximal ideal m of $\mathbb{T}_\psi(U)$ is said to be Eisenstein if $T_v - 2, S_v - 1 \in \mathsf{m}$ for all but finitely many v that split in a fixed finite abelian extension of F. We will only be interested in non-Eisenstein maximal ideals.

We consider the localisations of the above spaces of modular forms at non-Eisenstein ideals m: $S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$ denotes the localisation at m of $S_{k,\psi}(U,\mathcal{O})$. In the case $k=2$, the functions in $S_{k,\psi}(U,\mathcal{O})$ that factor through the norm die in such non-Eisenstein localisations. These spaces $S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$ for non-Eisenstein m can be identified with a certain space of cusp forms using the Jacquet-Langlands correspondence as in Lemma 1.3 of [55]. From this we deduce that a Hecke eigenform $f \in S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$ gives rise to a representation $\rho_f : G_F \to \mathrm{GL}_2(\mathcal{O})$ as in [10] and [57] which is residually irreducible. The representation ρ_f is characterised by the property that for almost all places v of F, ρ_f is unramified at v and $\rho_f(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 - a_v X + \mathbb{N}(v)\psi(\mathrm{Frob}_v)$ where Frob_v is the arithmetic Frobenius and a_v is the eigenvalue of the Hecke operator T_v acting on f. It is easy to see that non-Eisenstein maximal ideals m of $\mathbb{T}_\psi(U)$ give rise to irreducible Galois representations $\bar{\rho}_{\mathsf{m}} : G_F \to \mathrm{GL}_2(\mathbb{T}_\psi(U)/\mathsf{m})$.

We record a lemma that we use a few times below. It is the analog of a lemma of Ihara and due to Taylor [57].

Lemma 7.1. *Let $U = \Pi_v U_v$ be an open compact subgroup of $(D \otimes_F \mathbb{A}_f^\infty)^*$ and $\bar{W}_\tau$ a $U(\mathbb{A}_F^\infty)^*$ module as before that is a finite dimensional vector space over $\mathbb{F}$ such that $(\mathbb{A}_F^\infty)^*$ acts on it by $\bar{\psi}^{-1}$. Let $w \notin \Sigma$ be a finite place of F such that $U_w = \mathrm{GL}_2(\mathcal{O}_w)$ is maximal compact at w and acts trivially on $\bar{W}_\tau$. Let $U' = \Pi_v U'_v$ be a subgroup of U such that $U_v = U'_v$ for $v \neq w$, and $U'_w = U_0(w)$. Consider the degeneracy map $\alpha_w : S_{\bar{W}_\tau,\bar{\psi}}(U,\mathbb{F})^2 \to S_{\bar{W}_\tau,\bar{\psi}}(U',\mathbb{F})$ given by*

$$(f_1, f_2) \to f_1 + \begin{pmatrix}1 & 0\\ 0 & \pi_w\end{pmatrix} f_2.$$

The maximal ideals of the Hecke algebra $\mathbb{T}_\psi(U)$ (where we assume that S contains w) which acts diagonally on $S_{\bar{W}_\tau,\bar{\psi}}(U,\mathbb{F})^2$ that are in the support of $\ker(\alpha_w)$ *are Eisenstein.*

Proof. By passing to an open subgroup we may assume that the action of U on $\bar{W}_\tau$ is trivial. Now observe that if (f_1, f_2) is an element of the kernel of α_w then f_1 is invariant under $U\mathrm{SL}_2(F_w)$. Thus by strong approximation we see that f_1 is invariant under right translation by $D^1(\mathbb{A}_F^\infty)$, with D^1 the derived subgroup of D, and thus f_1 factors through the norm. Thus it dies in the localisation at any non-Eisenstein maximal ideal.

□

7.1. **Signs of some unramified characters.** We record a lemma which is used in Section 9.1.

Lemma 7.2. *We assume the conventions of the present Section 7. Let* $U = \Pi_v U_v$ *be as before, but we further ask that:*

(i) for all $v \in \Sigma$, $U_v = (\mathcal{O}_D)_v^*$ *for* $p > 2$,

(ii) for all $v \in \Sigma$, *we assume that* $U_v = D_v^*$ *for* $p = 2$ *(i.e. in the earlier notation* $\Sigma' = \Sigma$*).*

Consider $S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$ *for a non-Eisenstein maximal ideal as above, where again by our conventions* $k = 2$ *when* $p = 2$. *Assume that* Σ *is disjoint from* $\{v|p\}$ *if* $k > 2$. *Then for each* $v \in \Sigma$ *there is a fixed unramified character* $\gamma_v : G_{F_v} \to \mathcal{O}^*$ *such that for any Hecke eigenform* $f \in S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$, $\rho_f|_{D_v}$ *is of the form*

$$\begin{pmatrix} \chi_p\gamma_v & * \\ 0 & \gamma_v \end{pmatrix}.$$

Proof. From the Jacquet-Langlands correspondence (and its functoriality at all places including those in Σ), and the compatibility of the local and global Langlands correspondence for the association $f \to \rho_f$ proved in [10] and [57] it follows that $\rho_f|_{D_v}$ is of the form

$$\begin{pmatrix} \chi_p\gamma_{v,f} & * \\ 0 & \gamma_{v,f} \end{pmatrix},$$

with $\gamma_{v,f} : G_{F_v} \to \mathcal{O}^*$ an unramified character such that $\gamma_{v,f}^2 = \psi_v$.

The claim that $\gamma_{v,f}$ is independent of f follows:

(i) in the case $p > 2$ from the fact that the residual representation attached to a $f \in S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$ is independent of f;

(ii) in the case $p = 2$ from the quoted results and the fact that $U_v = D_v^*$ for $v \in \Sigma$. In a little more detail we first deduce that the local component at v of automorphic forms on $(D \otimes \mathbb{A}_F^\infty)^*$ corresponding to the eigenforms $f \in S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$ is independent of f, and thus by the functoriality at places in Σ of the Jacquet-Langlands correspondence we deduce that the corresponding forms on $\mathrm{GL}_2(\mathbb{A}_F)$ have the same property. Then using the results of [10] and [57] we are done. □

Remark: We remark that $\gamma_v(\mathrm{Frob}_v)$ is the inverse of the eigenvalue of a uniformiser of D_v^* acting on such f's.

7.2. **Isotropy groups.** For any $t = \Pi_v t_v \in (D \otimes_F \mathbb{A}_F^\infty)^*$, we have the following exact sequences with U as before but we further assume that U is compact (see [55]):

(6)

$$0 \to UV \cap t^{-1}D^{\det=1}t/\{\pm 1\} \to (U(\mathbb{A}_F^\infty)^* \cap t^{-1}D^*t)/F^* \to (((\mathbb{A}_F^\infty)^*)^2 V \cap F^*)/(F^*)^2$$

with $V = \Pi_{v<\infty}\mathcal{O}^*_{F_v}$, and

$$0 \to \mathcal{O}^*_F/(\mathcal{O}^*_F)^2 \to (((\mathbb{A}^\infty_F)^*)^2V \cap F^*)/(F^*)^2 \to H[2] \to 0 \tag{7}$$

where H denotes the class group of $\mathcal{O}_F$.

It is easy to see that $UV \cap t^{-1}D^{\det=1}t$ is a finite group and the p-part of its order is bounded independently of t and U. For this note that $tUVt^{-1} \cap D^{\det=1}$ is a discrete subgroup of the compact group $tUVD^{\det=1}_\infty t^{-1}$ and maps injectively to $t_wU_wV_wt_w^{-1}$ for a finite place w of F not above p (at which D splits for instance). The latter has a pro-q subgroup whose index is bounded independently of t_w and U_w, with q a prime different from p.

Note also that $(((\mathbb{A}^\infty_F)^*)^2V \cap F^*)/(F^*)^2$ is finite of exponent 2.

Thus in the case U is compact we note (for use in Lemma 7.3) that the exponent of a Sylow p-subgroup of the finite groups $(U(\mathbb{A}^\infty_F)^* \cap t^{-1}D^*t)/F^*$ divides $2N_w$ where N_w is the cardinality of $\mathrm{GL}_2(k_w)$.

In the cases of U non-compact that are considered, denote as before by $U' = \Pi_vU'_v$ its maximal compact subgroup. Then $U'(\mathbb{A}^\infty_F)^*$ is normal of finite index in $U(\mathbb{A}^\infty_F)^*$ and $U(\mathbb{A}^\infty_F)^*/U'(\mathbb{A}^\infty_F)^*$ is of type $(2,\cdots,2)$. We deduce that $(U'(\mathbb{A}^\infty_F)^* \cap t^{-1}D^*t)/F^*$ is normal, of finite index in $(U(\mathbb{A}^\infty_F)^* \cap t^{-1}D^*t)/F^*$, and the corresponding quotient is of type $(2,\cdots,2)$. To see this we may use the obvious fact that if G, H, K are subgroups of a group G', and H is normal and of finite index in G, then $H \cap K$ is normal in $G \cap K$ and $[G \cap K : H \cap K] | [G : H]$.

Thus in the cases considered where U is non-compact we note (for use in Lemma 7.3) that the exponent of a Sylow p-subgroup of the finite groups $(U(\mathbb{A}^\infty_F)^* \cap t^{-1}D^*t)/F^*$ divides $4N_w$ where N_w is the cardinality of $\mathrm{GL}_2(k_w)$.

7.3. **Base change and isotropy groups.** Let $\{v\}$ be a finite set of finite places of F, not above p, at which D is split. Let w be a place of F of residue characteristic different from p at which D is split, and let N_w be the order of $\mathrm{GL}_2(k_w)$.

Let F'/F be any totally real finite extension of F that is completely split at w. Let us denote by $\{v'\}$ the places of F' above the fixed finite set of finite places $\{v\}$ of F.

Let $U_{F'} = \Pi_rU_{F',r}$ be a subgroup of $(D_{F'} \otimes_{F'} \mathbb{A}^\infty_{F'})^*$ as fixed at the beginning of the section (taking the F there to be F', and D to be $D_{F'} = D \otimes F'$). The first part of the following lemma has already been proved in Section 7.2.

Lemma 7.3. *1. The exponent of the Sylow p-subgroup of the isotropy groups $(U_{F'}(\mathbb{A}^\infty_{F'})^* \cap t^{-1}D^*_{F'}t)/F'^*$ divides $4N_w$ for any $t \in (D_{F'} \otimes_{F'} \mathbb{A}^\infty_{F'})^*$.*

2. Let us further assume that for all places $\{v'\}$ of F' above the places $\{v\}$, the order of the p-subgroup of $k^*_{v'}$ is divisible by the p-part of $2p(4N_w)$. Assume that $U_{F'}$ is such that at places $\{v'\}$ it is of the form

$$U_{F',v'} = \{g \in \mathrm{GL}_2(\mathcal{O}_{F'_{v'}}) : g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mathrm{mod.}(\pi_{v'})\}.$$

A character $\chi = \Pi_{v'}\chi_{v'}$ *of* $\Pi_{v'}k_{v'}^*$ *may be regarded as a character of* $\Pi_{v'}U_{F',v'}$*, and hence of* $U_{F'}$*, via the map* $\Pi_{v'}U_{F',v'} \to \Pi_{v'}k_{v'}^*$ *with kernel*

$$\Pi_{v'}\{g \in \mathrm{GL}_2(\mathcal{O}_{F'_{v'}}) : g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mathrm{mod.}(\pi_{v'}), ad^{-1} = 1\}.$$

Thus χ *is trivial on* $U_{F'} \cap (\mathbb{A}_{F'}^\infty)^*$*, and may be extended to a character* χ *of* $U_{F'}(\mathbb{A}_{F'}^\infty)^*$ *by defining it to be trivial on* $(\mathbb{A}_{F'}^\infty)^*$*.*

There is a character $\chi = \Pi_{v'}\chi_{v'}$ *of* $\Pi_{v'}k_{v'}^*$ *of order a power of* p*, with each* $\chi_{v'}$ *non-trivial (and of order divisible by 4 when* $p = 2$*), such that when regarded as a character of* $U_{F'}(\mathbb{A}_{F'}^\infty)^*$ *as above, it annihilates* $(U_{F'}(\mathbb{A}_{F'}^\infty)^* \cap t^{-1}D_{F'}^*t)/F'^*$ *for any* $t \in (D_{F'} \otimes_{F'} \mathbb{A}_{F'}^\infty)^*$*.*

Proof. For the second part we only have to notice that by the hypotheses it follows that there is a character $\chi' = \Pi_{v'}\chi'_{v'}$ of $\Pi_{v'}k_{v'}^*$ of order a power of p, with each $\chi'_{v'}$ of order divisible by the p-part of $2p(4N_w)$. Then set $\chi = \Pi_{v'}\chi_{v'} = \chi'^{4N_w}$. When regarded as characters of $(U_{F'}(\mathbb{A}_{F'}^\infty)^* \cap t^{-1}D_{F'}^*t)/F'^*$, we still have $\chi = \chi'^{4N_w}$. As the exponent of a Sylow p-subgroup of $(U_{F'}(\mathbb{A}_{F'}^\infty)^* \cap t^{-1}D_{F'}^*t)/F'^*$ divides $4N_w$, we get that χ is trivial on $(U_{F'}(\mathbb{A}_{F'}^\infty)^* \cap t^{-1}D_{F'}^*t)/F'^*$. As $\chi = \Pi_{v'}\chi_{v'}$ also has the property that each $\chi_{v'}$ is of order divisible by the p-part of $2p$, we are done. □

7.4. **Δ_Q-freeness in presence of isotropy.** Let N (a power of p) be the least common multiple of the exponent of the Sylow p-subgroups of the finite groups $(U(\mathbb{A}_F^\infty)^* \cap t^{-1}D^*t)/F^*$ for all $t \in (D \otimes_F \mathbb{A}_F^\infty)^*$. The integer N exists by the discussion in Section 7.2.

Consider an integer n and a finite set of places $Q = \{v\}$ of F disjoint from S. We denote by k_v the residue field at v and by $\mathbb{N}(v)$ its cardinality. Assume that $\mathbb{N}(v) = 1$ mod p^n, $n \geq 1$, and let Δ'_v be the pro-p quotient of the cyclic group $k_v^* = (\mathcal{O}_{F_v}/\pi_v)^*$.

Let Δ_v be the quotient of Δ'_v by its N-torsion. Hence any character $\chi : \Delta_v \to \mathcal{O}^*$, when regarded as a character of Δ'_v, is an Nth power. We consider subgroups $U_Q = \Pi_v(U_Q)_v$ and $U_Q^0 = \Pi_v(U_Q^0)_v$ of U which have the same local component as U at places outside Q and for $v \in Q$,

$$(U_Q)_v = \{g \in \mathrm{GL}_2(\mathcal{O}_{F_v}) : g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mathrm{mod.}(\pi_v), ad^{-1} \to 1 \in \Delta_v\},$$

and

$$\left(U_Q^0\right)_v = \{g \in \mathrm{GL}_2(\mathcal{O}_{F_v}) : g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mathrm{mod.}(\pi_v)\}.$$

Then there is a natural isomorphism

$$\frac{U_Q^0}{U_Q} \simeq \Pi_v\Delta_v := \Delta_Q$$

via which characters of Δ_Q may be regarded as characters of U_Q^0.

The space $S_{k,\psi}(U_Q, \mathcal{O})$ carries an action of Δ_Q and of the operators U_{π_v} and S_{π_v} for $v \in Q$. The natural action of $g \in \Delta_v$, denoted by $\langle g \rangle$, arises from the double coset

$$U_Q \begin{pmatrix} \tilde{g} & 0 \\ 0 & 1 \end{pmatrix} U_Q$$

where $\tilde{g}$ is a lift of g to $(\mathcal{O}_F)_v^*$. The operators and S_{π_v} and U_{π_v} for $v \in Q$ are defined just as before by the action of $U_Q \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} U_Q$ and $U_Q \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} U_Q$. By abuse of notation we denote these by S_v (which is just multiplication by $\psi(\pi_v)$) and U_v although they might depend on choice of π_v. We consider the extended (commutative) Hecke algebra $\mathbb{T}_{\psi,Q}(U_Q)$ generated over $\mathbb{T}_\psi(U_Q)$ by these operators and Δ_Q.

A character $\chi : \Delta_Q \to \mathcal{O}^*$ induces a character of U_Q^0, and this character is trivial on $U_Q^0 \cap (\mathbb{A}_F^\infty)^*$. Thus χ may be extended to a character of $U_Q^0(\mathbb{A}_F^\infty)^*$ by declaring it to be trivial on $(\mathbb{A}_F^\infty)^*$. Let $W_k(\chi)$ denote the $U_Q^0(\mathbb{A}_F^\infty)^*$-module which is the tensor product $W_k \otimes_{\mathcal{O}} \mathcal{O}(\chi)$. Thus $S_{W_k(\chi),\psi}(U_Q^0, \mathcal{O})$ denotes the space of continuous functions

$$f : D^* \backslash (D \otimes_F \mathbb{A}_F^\infty)^* \to W_k(\chi)$$

such that:

$$f(gu) = (u)^{-1} f(g)$$
$$f(gz) = \psi(z) f(g)$$

for all $g \in (D \otimes_F \mathbb{A}_F^\infty)^*, u \in U_Q^0, z \in (\mathbb{A}_F^\infty)^*$.

Lemma 7.4. *1. The rank of the $\mathcal{O}$-module $S_{W_k(\chi),\psi}(U_Q^0, \mathcal{O})$ is independent of the character χ of Δ_Q. Further we have a Hecke equivariant isomorphism $S_{W_k(\chi),\psi}(U_Q^0, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \simeq S_{W_k(\chi'),\psi}(U_Q^0, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F}$ for characters χ, χ' of Δ_Q.*

2. $S_{k,\psi}(U_Q, \mathcal{O})$ is a free $\mathcal{O}[\Delta_Q]$-module of rank equal to the rank of $S_{k,\psi}(U_Q^0, \mathcal{O})$ as an $\mathcal{O}$-module.

Proof. We first claim that a character χ of Δ_Q, kills $(U_Q^0(\mathbb{A}_F^\infty)^* \cap t^{-1} D^* t)/F^*$ for all $t \in (D \otimes_F \mathbb{A}_F^\infty)^*$.

To prove the claim we note that the p-power order character χ, regarded as a character of $(U_Q^0(\mathbb{A}_F^\infty)^* \cap t^{-1} D^* t)/F^*$ is an Nth power, and N by definition is divisible by the exponent of the Sylow p-subgroups of $(U_Q^0(\mathbb{A}_F^\infty)^* \cap t^{-1} D^* t)/F^*$.

We next claim that for each $t \in (D \otimes_F \mathbb{A}_F^\infty)^*$, we have:

$$(U_Q^0(\mathbb{A}_F^\infty)^* \cap t^{-1} D^* t)/F^* = (U_Q(\mathbb{A}_F^\infty)^* \cap t^{-1} D^* t)/F^*. \tag{8}$$

To prove this second claim, note that we have a natural isomorphism $U_Q^0(\mathbb{A}_F^\infty)^*/U_Q(\mathbb{A}_F^\infty)^* \simeq \Delta_Q$ and a natural injection $\frac{U_Q^0(\mathbb{A}_F^\infty)^* \cap t^{-1} D^* t}{U_Q(\mathbb{A}_F^\infty)^* \cap t^{-1} D^* t} \hookrightarrow U_Q^0(\mathbb{A}_F^\infty)^*/U_Q(\mathbb{A}_F^\infty)^*$. Thus we get a surjective map from the characters of $(U_Q^0(\mathbb{A}_F^\infty)^* \cap t^{-1} D^* t)/F^*$ induced by Δ_Q, which are as noted in the first claim trivial, to the character group of $\frac{(U_Q^0(\mathbb{A}_F^\infty)^* \cap t^{-1} D^* t)/F^*}{(U_Q(\mathbb{A}_F^\infty)^* \cap t^{-1} D^* t)/F^*}$. This proves the second claim.

Let $\{t_i\}, i \in I_0$ be a set of representatives of the double cosets $D^*\backslash(D \otimes_F \mathbb{A}_F^\infty)^*/U_Q^0$. We get from (5) an isomorphism of $S_{W_k(\chi),\psi}(U_Q^0(\mathbb{A}_F^\infty)^*, \mathcal{O})$ with

$$\oplus_{i \in I_0} W_k(\chi)^{(U_Q^0(\mathbb{A}_F^\infty)^* \cap t_i^{-1} D^* t_i)/F^*}.$$

This $\mathcal{O}$-module and its image in $S_{k,\psi}(U, \mathbb{F})$ does not depend on χ, for χ a character of Δ_Q, as by the first claim we know that such χ kill $(U_Q^0(\mathbb{A}_F^\infty)^* \cap t_i^{-1} D^* t_i)/F^*$.

Namely for characters $\chi, \chi' : \Delta_Q \to \mathcal{O}^*$ we have a commutative diagram

$$\begin{array}{ccc} S_{W_k(\chi),\psi}(U_Q^0(\mathbb{A}_F^\infty)^*, \mathcal{O}) & \longrightarrow & S_{W_k(\chi'),\psi}(U_Q^0(\mathbb{A}_F^\infty)^*, \mathcal{O}) \\ \downarrow & & \downarrow \\ S_{k,\psi}(U, \mathbb{F}) & \xrightarrow{=} & S_{k,\psi}(U, \mathbb{F}) \end{array}$$

where the top arrow is a non-Hecke-equivariant isomorphism, and the other arrows are Hecke equivariant. This proves 1).

For 2), we note that, by the second claim (see (8)), a set of representatives I of the double cosets $D^*\backslash(D \otimes_F \mathbb{A}_F^\infty)^*/U_Q(\mathbb{A}_F^\infty)^*$ is $\{t_i u_j\}$ where $\{u_j\}$ is a set of representative of the elements of the quotient $U_Q^0(\mathbb{A}_F^\infty)^*/U_Q(\mathbb{A}_F^\infty)^* \simeq \Delta_Q$. Then, 2) follows from (5).

□

For the following corollary, consider a non-Eisenstein maximal ideal m of $\mathbb{T}_\psi(U)$ and assume that the eigenvalues of $\bar{\rho}_{\mathsf{m}}(\mathrm{Frob}_v)$, α_v and β_v, for $v \in Q$, are distinct. Let $Q' \subset Q$. By Hensel's lemma the polynomial $X^2 - T_v X + \mathbb{N}(v)\psi(\pi_v) \in \mathbb{T}_\psi(U)_{\mathsf{m}}[X]$ splits as $(X - A_v)(X - B_v)$ where A_v modulo m is α_v and B_v modulo m is β_v. Then we may pull back the maximal ideal m to a maximal ideal of $\mathbb{T}_{\psi,Q'}(U_{Q'})$ or $\mathbb{T}_{\psi,Q'}(U_{Q'}^0)$, denoted again by m, by declaring that $U_v - \tilde{\alpha}_v \in \mathsf{m}$ for $v \in Q'$ with $\tilde{\alpha}_v$ some lift of α_v: that this is possible follows from 2) of Lemma 1.6 of [55].

Consider $v \in Q \backslash Q'$. Since

$$\mathbb{T}'_{\psi,Q'}(U_{Q'}^0)_{\mathsf{m}'} \to \mathbb{T}_{\psi,Q'}(U_{Q'}^0)_{\mathsf{m}}$$

is an isomorphism, where $\mathbb{T}'_{\psi,Q'}(U_{Q'}^0)$ is defined without T_v, and m' is the preimage of m, it follows that

$$S_{k,\psi}(U_{Q'}^0)_{\mathsf{m}'} \to S_{k,\psi}(U_{Q'}^0)_{\mathsf{m}}$$

is an isomorphism. The matrix $\begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix}$ defines a map

$$S_{k,\psi}(U_{Q'}^0)_{\mathsf{m}'} \to S_{k,\psi}(U_{Q' \cup \{v\}}^0)_{\mathsf{m}'}$$

(where abusing notation as before, m' is also the maximal ideal over m' in the Hecke algebra $\mathbb{T}_{\psi,Q'}(U_{Q' \cup \{v\}})$ defined without U_v).

Thus the formula

$$\xi_v(f) = A_v f - \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} f,$$

composed with localisation at m, defines a map $S_{k,\psi}(U^0_{Q'},\mathcal{O})_{\mathsf{m}} \to S_{k,\psi}(U^0_{Q'\cup\{v\}},\mathcal{O})_{\mathsf{m}}$ for $Q' \subset Q$ and $v \in Q\backslash Q'$. These maps are used in Corollary 7.5 below.

Corollary 7.5. *$S_{k,\psi}(U_Q,\mathcal{O})_{\mathsf{m}}$ is a free $\mathcal{O}[\Delta_Q]$-module. The rank of $S_{k,\psi}(U_Q,\mathcal{O})_{\mathsf{m}}$ as an $\mathcal{O}[\Delta_Q]$-module is the rank of $S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$ as an $\mathcal{O}$-module. The Δ_Q covariants of $S_{k,\psi}(U_Q,\mathcal{O})_{\mathsf{m}}$ are isomorphic to $S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$, compatibly with a map $\mathbb{T}_{\psi,Q}(U_Q)_{\mathsf{m}} \to \mathbb{T}_{\psi}(U)_{\mathsf{m}}$ sending T_v to T_v for v not in $S \cup Q$, $\langle g\rangle \to 1$ for $\langle g\rangle \in \Delta_Q$ and $U_v \to A_v$ for $v \in Q$.*

Proof. (see also Lemma 2.2, Lemma 2.3 and Corollary 2.4 of [55]) The first assertion follows from Lemma 7.4 as $S_{k,\psi}(U_Q,\mathcal{O})_{\mathsf{m}}$ is isomorphic to a direct factor, as a module over the local ring $\mathcal{O}[\Delta_Q]$, of $S_{k,\psi}(U_Q,\mathcal{O})$. The other assertions follow from proving $S_{k,\psi}(U^0_Q,\mathcal{O})_{\mathsf{m}} \simeq S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$. The fact that the natural map $S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}} \to S_{k,\psi}(U^0_Q,\mathcal{O})_{\mathsf{m}}$ (given by composing the ξ_v's for $v \in Q$ above) is an isomorphism after inverting p follows as there is no automorphic representation π of $(D \otimes_F \mathbb{A}_F)^*$ which is (a twist of) Steinberg at any place in Q which can give rise to $\bar\rho_{\mathsf{m}}$. This in turn follows the compatibility of the local-global Langlands correspondence proved in [10] and [57] as $\mathbb{N}(v)$ is 1 mod p for $v \in Q$, v is unramified in $\bar\rho_{\mathsf{m}}$ and $\bar\rho_{\mathsf{m}}(\mathrm{Frob}_v)$ has distinct eigenvalues.

As we know that $S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}} \to S_{k,\psi}(U^0_Q,\mathcal{O})_{\mathsf{m}}$ is an injective map of $\mathcal{O}$-modules of the same rank, to prove that it is surjective it is enough to prove that its reduction modulo the maximal ideal of $\mathcal{O}$ is injective. This in turn follows from showing that for any subset Q' of Q and $q \in Q\backslash Q'$, the degeneracy map $S_{k,\psi}(U^0_{Q'},\mathbb{F})^2 \to S_{k,\psi}(U^0_{Q'\cup\{q\}},\mathbb{F})$ has Eisenstein kernel (see Lemma 7.1). □

7.5. **Twists of modular forms for $p = 2$.** Assume $p = 2$, and let $Q := Q_n$ be a set of auxiliary primes as in Lemma 5.10. We also use the notation of §5.6. We assume n is such that $2^n > N$ with N as in §7.4. Consider a character $\chi : G_{n,2} \to \mathcal{O}^*$ of order 2 (recall that $G_{n,2} = G_n/2G_n$). As χ is split at infinite places, we can regard χ also as a character $(\mathbb{A}_F^\infty)^* \to \mathcal{O}^*$. Given $f \in S_{k,\psi}(U_Q,\mathcal{O})$ (so $k = 2$), we can define

$$f_\chi(g) := f(g)\chi(\mathrm{Nm}(g)).$$

We check that f_χ is in $S_{k,\psi}(U_Q,\mathcal{O})$ again. We have to prove that for $u \in U_Q$ and $z \in (\mathbb{A}_F^\infty)^*$ we have $\chi(\mathrm{Nm}(u)) = 1$ and $\chi(\mathrm{Nm}(z)) = 1$. The second equality follows from the fact that χ is of order 2. For the first one, we have $\chi(\mathrm{Nm}(u)) = \prod_{v\in Q}\chi(\mathrm{Nm}(u_v))$ as χ is split at places in S. For $v \in Q$ and $u_v \in (U_Q)_v$, $\det(u_v) = \mathrm{Nm}(u_v)$ is a square in k_v^*, hence $\chi(\mathrm{Nm}(u_v)) = 1$.

The character χ may also be regarded as a character $\chi : \Delta_Q \to \mathcal{O}^*$, by considering the map $\Delta_Q \to G_{n,2}$ which maps a generator of Δ_v (which by our assumption $2^n > N$ is a non-trivial cyclic group of order a power of 2) to a generator of an inertia group in $G_{n,2}$ above v (for $v \in Q$) (which is of order dividing 2). This gives a meaning to the 2. of the following proposition.

Proposition 7.6. *1. For $T_v \in \mathbb{T}_{\psi,Q}(U_Q)$, with v a place not above S, and π_v a uniformiser at v, we have:*
(i)

$$f_\chi|T_v = \chi(\pi_v)(f|T_v)_\chi$$

for $v \notin Q$;
(ii)

$$f_\chi|U_v = \chi(\pi_v)(f|U_v)_\chi$$

for $v \in Q$.

(Note that this is well-defined as for $v \in Q$, U_v depends on choice of uniformiser π_v.)

2. For $h \in \Delta_Q$ we have

$$(f|\langle h\rangle)_\chi = \chi^{-1}(h) f_\chi|\langle h\rangle.$$

Proof. The proposition follows easily from the definition of f_χ. For example, 1 (ii) follows from the formula $(f|U_v)(g) = \sum_i f(gu_i)$ where the u_i belongs to the double coset $U_Q \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} U_Q$, hence $\chi(\mathrm{Nm}(u_i)) = \chi(\pi_v)$. □

It follows from the proposition that we define an action of $\mathrm{Hom}(G_{n,2}, \mathcal{O}^*)$ on $\mathbb{T}_{\psi,Q}(U_Q)$ by sending T_v to $\chi(\pi_v)T_v$, U_v to $\chi(\pi_v)U_v$, h to $\chi(h)h$ and S_v to itself. This action is compatible with the action of $\mathrm{Hom}(G_{n,2}, \mathcal{O}^*)$ on $S_{k,\psi}(U_Q, \mathcal{O})$. As $\chi(\sigma) \equiv 1 \mathrm{mod}\ \mathsf{m}$ for every $\sigma \in G_{n,2}$, it follows from the proposition that the action of $\mathrm{Hom}(G_{n,2}, \mathcal{O}^*)$ on $\mathbb{T}_{\psi,Q}(U_Q)$ preserves its maximal ideal m (defined in §7.4) . It follows that we get actions of $\mathrm{Hom}(G_{n,2}, \mathcal{O}^*)$ on $\mathbb{T}_{\psi,Q}(U_Q)_\mathsf{m}$ and $S_{k,\psi}(U_Q, \mathcal{O})_\mathsf{m}$ and that satisfy the compatibility conditions of Proposition 7.6.

7.6. **A few more preliminaries.**

7.6.1. *Local behaviour at p of automorphic p-adic Galois representations.* The following result is the corollary in the introduction to [36] which extends to some more cases the results of [48].

Lemma 7.7. *Let F be a totally real number field that is unramified at p and π a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ that is discrete series of (parallel) weight $k \geq 2$ at the infinite places. Consider the Galois representation $\rho_\pi : G_F \to \mathrm{GL}_2(E)$ associated to π, and assume that residually it is absolutely irreducible. Then the association $\pi \to \rho_\pi$ is compatible with the Langlands-Fontaine correspondence.*

This has the following more explicit corollary.

Corollary 7.8. *Let v be a place of F above p.*

(i) If π_v is unramified at v, then $\rho_\pi|_{D_v}$ is crystalline of weight k. Further if π_v is ordinary then $\rho_\pi|_{I_v}$ is of the form

$$\begin{pmatrix} \chi_p^{k-1} & * \\ 0 & 1 \end{pmatrix}.$$

(ii) ($k = 2$) If $\pi_v^{U_1(v)}$ is non-trivial, but $\pi_v^{U_0(v)}$ is trivial, and the corresponding character of k_v^* factors through the norm to $\mathbb{F}_p^*$, then $\rho_\pi|_{D_v}$ is of weight 2 and crystalline over $\mathbb{Q}_p^{\mathrm{nr}}(\mu_p)$. Further if π_v is ordinary then $\rho_\pi|_{I_v}$ is of the form

$$\begin{pmatrix} \omega_p^{k-2}\chi_p & * \\ 0 & 1 \end{pmatrix}.$$

If $\pi_v^{U_0(v)}$ is non-trivial, but π_v has no invariants under $\mathrm{GL}_2(\mathcal{O}_v)$, (hence π_v is (unramified twist of) Steinberg) then $\rho_\pi|_{D_v}$ is semistable, non-crystalline of weight 2, i.e. of the form

$$\begin{pmatrix} \chi_p\gamma_v & * \\ 0 & \gamma_v \end{pmatrix},$$

with γ_v an unramified character of D_v.

We will sometimes call unramified twists of Steinberg representations of $\mathrm{GL}_2(F_v)$ again Steinberg.

7.6.2. *A definition.* Let $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ as before be a continuous, absolutely irreducible, totally odd representation.

Let F be a totally real number field such that F is unramified at p and split at p if $\bar{\rho}|_{D_p}$ is irreducible. We assume that $\bar{\rho}|_F$ has non-solvable image when $p = 2$, and $\bar{\rho}|_{F(\mu_p)}$ is absolutely irreducible when $p > 2$. We make a useful definition:

Definition 7.9. *A totally real solvable extension F'/F, that is of even degree, unramified at places above p, and split at places above p if $\bar{\rho}|_{D_p}$ is irreducible, such that $\mathrm{im}(\bar{\rho}) = \mathrm{im}(\bar{\rho}|_{F'})$ and $\bar{\rho}|_{F'(\mu_p)}$ absolutely irreducible is said to be an allowable base change.*

In many of the considerations below, the statements of the results will permit allowable base change, primarily because of Langlands theory of base change (see [37]).

A totally real extension F'/F has the property that $\mathrm{im}(\bar{\rho}) = \mathrm{im}(\bar{\rho}|_{F'})$ and $\bar{\rho}|_{F'(\mu_p)}$ is absolutely irreducible if (i) it is linearly disjoint from the fixed field of the kernel of $\bar{\rho}|_F$; and (ii) in the case that the projective image of $\bar{\rho}|_F$ is dihedral, if F'/F has the property that it is split at a prime of F split in the fixed field of the kernel of the projective image of $\bar{\rho}|_F$, but inert in $F(\mu_p)$.

In the constructions below this property can easily be ensured and will not be explictly commented upon.

7.6.3. *Determinants.* We will need the following lemma later to ensure that certain lifts we construct (after twisting and allowable base change which also splits at finitely many specified primes) have a certain prescribed determinant character.

Lemma 7.10. *Suppose $\psi, \psi' : F^* \backslash (\mathbb{A}_F^\infty)^* \to \mathcal{O}^*$ are characters that have the same reduction. Assume that the restrictions of ψ, ψ' to an open subgroup of $\mathcal{O}_{F_p}^*$ are equal. Assume we are given a finite set of finite places $\{v\}$ of F, at which the restrictions of ψ, ψ' to $(\mathcal{O}_F)_v^*$ are equal. Then after enlarging $\mathcal{O}$ if necessary there is a finite order character $\zeta : F^* \backslash (\mathbb{A}_F^\infty)^* \to \mathcal{O}^*$ of order a power of p and unramified at $\{v\}$, and a totally real solvable extension F'/F that can be made disjoint from any given finite extension of F, and that is split at all places in $\{v\}$, such that the characters $\zeta|_{F'}^2 \psi_{F'}, \psi'_{F'}$: $F'^* \backslash (\mathbb{A}_{F'}^\infty)^* \to \mathcal{O}^*$ are equal.*

Proof. Let L/F be a finite Galois extension and choose a finite set of places $\{w\}$ of F, that are unramified in L/F, and also unramified for ψ, ψ', such that the Frob_w exhaust the conjugacy classes of $\mathrm{Gal}(L/F)$.

Our assumptions imply that $\psi\psi'^{-1}$ is a finite order character, of order a power of p which if viewed as character of G_F via the class field theory isomorphism is totally even. For $p > 2$ the lemma is trivial (and we may take $F' = F$).

For $p = 2$ we use the Grunwald-Wang theorem, see Theorem 5 of Chapter 10 of [1], to find ζ of order a power of 2 such that the characters $\zeta^2\psi, \psi'$ have the same restriction to F_v^* for the finite set of places $\{v\} \cup \{w\}$. It follows that there is a finite totally real solvable (and even cyclic) extension F'/F that is split at all places in $\{v\} \cup \{w\}$, and thus linearly disjoint from L/F, such that $\zeta|_{F'}^2 \psi_{F'}, \psi'_{F'} : F'^* \backslash (\mathbb{A}_{F'}^\infty)^* \to \mathcal{O}^*$ are equal.

□

8. Modular lifts with prescribed local properties

While proving modularity lifting theorems by the Wiles, Taylor-Wiles, Diamond, Fujiwara and Kisin patching method (see Propositions 9.2 and 9.3 below) we need to produce modular liftings of a modular residual $\bar{\rho}$ that factor through the quotient of the deformation ring being considered. The purpose of this section is to produce such liftings. As we work with deformations of fixed determinant we also take care to produce modular lifts with the given determinant. This we cannot always do without performing allowable base change (also ensuring splitting behaviour at finitely many specified primes). This is harmless for our applications.

Theorem 8.4 produces modular lifts, up to allowable base change, with some prescribed local conditions (these are always semistable outside primes above p). A crucial input for this is Theorem 8.2 which produces minimal lifts (after allowable base change).

Consider the fixed S-type representation $\bar{\rho}: G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$, with $2 \leq k(\bar{\rho}) \leq p+1$ if $p > 2$, and $\bar{\rho}$ has non-solvable image if $p = 2$ and $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is irreducible if $p > 2$.

8.1. **Fixing determinants.** The proof of the following lemma is easy and is omitted.

Lemma 8.1. *There is a totally real field F such that:*

- *$F/\mathbb{Q}$ is solvable*
- *F is totally real, $[F : \mathbb{Q}]$ is even, unramified at p, and is even split at p if $\bar{\rho}|_{D_p}$ is irreducible, or if $k(\bar{\rho}) = p+1$ (and hence $p > 2$).*
- *$\bar{\rho}_F := \bar{\rho}|_{G_F}$ has non-solvable image if $p = 2$, and $\bar{\rho}|_{G_{F(\mu_p)}}$ is irreducible if $p > 2$.*
- *$\bar{\rho}_F$ is unramified at places that are not above p*
- *if $\bar{\rho}|_{D_p}$ is unramified then for all places $\wp$ of F above p, $\bar{\rho}|_{D_\wp}$ is trivial.*

We use the notations of 1.2. We furthermore suppose given an arithmetic character $\psi : F^* \backslash (\mathbb{A}_F^\infty)^* \to \mathcal{O}^*$, unramified outside the places above p, such that the corresponding Galois representation $\chi_p \rho_\psi : G_F \to \mathcal{O}^*$ (which is totally odd), lifts the determinant of $\bar{\rho}$ and such that the restriction of ψ to $\mathcal{O}_{F_p}^*$ is one of the following kind:

- (i) restricted to $\mathcal{O}_{F_p}^*$ of the form $\mathbb{N}(u)^{2-k(\bar{\rho})}$,
- (ii) restricted to $\mathcal{O}_{F_p}^*$ corresponds to $\omega_p^{k(\bar{\rho})-2}$, or
- (iii) when $k(\bar{\rho}) = 2$, restricted to $\mathcal{O}_{F_p}^*$ it is of the form $\mathbb{N}(u)^{1-p}$.

Remark: As the referee has remarked (i), (ii) and (iii) are not mutually exclusive: (i) and (ii) coincide if $k(\bar{\rho}) = 2$. If $p = 2$ we use only (ii) in what ensues.

We fix such a F and ψ for the rest of this section. When referring to properties of determinant characters we will use the numbering of this section.

8.2. **Minimal at p modular lifts and level-lowering.** Consider the following hypotheses:

(α) $\bar{\rho}|_{G_F}$ arises from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$, such that π_v is unramified for all $v|p$, and is discrete series of weight $k(\bar{\rho})$ at the infinite places.

(β) $\bar{\rho}|_{G_F}$ arises from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$, such that π_v, at all places v above p, is of conductor dividing v, and is of weight 2 at the infinite places.

Using the Jacquet-Langlands correspondence, we may transfer π to inner forms of GL_2 and will call it π again.

When results of this section are used later, we will verify that for $p > 2$ the assumptions (α) and (β) are satisfied. For $p = 2$, condition (α) will be satisfied if $k(\bar{\rho}) = 2$, and (β) will be satisfied for $k(\bar{\rho}) = 2$ *and* 4.

For instance quoting the results of [22] and Propositions 8.13 and 8.18 of [25], we will verify later that these assumptions are satisfied if $\bar{\rho}$ is modular. In cases when $\bar{\rho}$ itself is not supposed to be modular, but we find a F such that $\bar{\rho}_F$ is modular by using Theorem 6.1, that very theorem verifies these hypotheses for us.

A key ingredient in the proof of Theorem 8.4 is the following result (which can be regarded as a level-lowering result) which we prove following the idea of Skinner-Wiles in [52], except that our proof avoids using duality. It strengthens the hypotheses (α) and (β).

Theorem 8.2. *Let $\bar{\rho}$, F, ψ and π as above. In particular, we assume if $p > 2$ that (α) and (β) are satisfied, and if $p = 2$ that (α) is satisfied if $k(\bar{\rho}) = 2$ and (β) in general.*

We suppose that π is as in:

– ($p = 2$) (α) when $k(\bar{\rho}) = p = 2$, or as in (β)

– ($p > 2$) (α) or (β) if we are in case (a) below, and (β) if we are in cases (b) and (c) below.

Let Σ be a subset of the finite places at which π is (unramified twist of) Steinberg. (Note that Σ contains no place above p when π is as in (α).)

Then there is an allowable base change F''/F, that is split at p if $p > 2$, and a cuspidal automorphic representation π'' of $\mathrm{GL}_2(\mathbb{A}_{F''})$ that is discrete series at infinity such that

- *$\rho_{\pi''}$ is a lift of $\bar{\rho}_{F''}$*
- *π'' is unramified (spherical) at all finite places not above $\Sigma \cup \{p\}$, and π'' is Steinberg at places above Σ.*
- *($p = 2$) π'' is of weight 2, if $k(\bar{\rho}) = 2$ π''_v is unramified at all places v dividing 2, if $k(\bar{\rho}) = 4$ π''_v is Steinberg at all places v dividing 2, and π'' has central character $\psi_{F''}$ with ψ supposed as in (i) when $k(\bar{\rho}) = 2$, and ψ supposed as in (ii) when $k(\bar{\rho}) = 4$. For every place v of F above 2, let ψ'_v be a choice of an unramified square-root of the unramified character ψ_v. Then when $k(\bar{\rho}) = 4$ we may further ensure that the Hecke operator $U_{v'}$ at places v' of F'' above v acts on π'' by $\psi'_v \circ N_{F''_{v'}/F_v}$.*
- *($p > 2$) π'' can be chosen so that it satisfies :*

  (a) Suppose ψ as in (i) above. Then π'' is unramified at places v above p, $v \notin \Sigma$, and of parallel weight $k(\bar{\rho})$ with central character given by $\psi_{F''}$.

  When $k(\bar{\rho}) = 2$, and we assume that Σ contains no places above p, and ψ is as in (iii), π'' can also be chosen to be of weight $p+1$ with central character given by $\psi_{F''}$, and unramified at all places above p.

  (b) Suppose ψ as in (ii), and assume $k(\bar{\rho}) < p + 1$. Then π''_v has fixed vectors under $U_1(v)$ for all $v|p$, and the associated character of k_v^* factors through the norm to $\mathbb{F}_p^*$, is of parallel weight 2, with central character given by $\psi_{F''}$.

(c) (considered only when $k(\bar{\rho}) = p+1$) Suppose ψ as in (ii). Then π''_v has fixed vectors under $U_0(v)$ for all $v|p$, is of parallel weight 2, with central character given by $\psi_{F''}$.

Remark: The statement in Theorem 8.2 (and hence Lemma 8.3), corresponding to $p > 2$ case (a) ($k(\bar{\rho}) = 2$ and π'' of weight $p+1$), is not used in the present paper.

Proof. Recall from the statement that we consider π as in:
– ($p = 2$) (α) when $k(\bar{\rho}) = p = 2$, or as in (β)
– ($p > 2$) (α) or (β) in case (a), and (β) in cases (b) and (c).

After an allowable base change that is split at places in Σ, we may assume that at places v not above p such that π_v is ramified, it is Steinberg of conductor v. Denote this set of places, deprived of the places in Σ, by S.

Choose a place $w \notin \Sigma \cup S \cup \{v|p\}$. Using Lemma 2.2 of [56], there is an allowable base change F'/F of even degree, that is split at $\Sigma \cup \{v|p\} \cup \{w\}$, such that for all places $\{v'\}$ of F' above the places $\{v\} = S$ of F, the order of the p-subgroup of $k^*_{v'}$ is divisible by the p-part of $2p(4N_w)$, where N_w is defined in 7.2. As we are permitted allowable base changes in the statement of theorem, we may reinitialise and set $F = F'$. Note that thus $F/\mathbb{Q}$ is of even degree and $|\Sigma|$ is even.

Let $\psi' = \det(\rho_\pi)\chi_p^{-1}$, and consider D the definite quaternion algebra over F ramified at exactly the infinite places and Σ. Then by the JL-correspondence ρ_π arises from an eigenform in $S_{k,\psi'}(U', \mathcal{O})$ where $k = k(\bar{\rho})$ in case (a), and $k = 2$ otherwise. Here $U' = \Pi_v U'_v \subset (D \otimes_F \mathbb{A}_F^\infty)^*$ is an open compact subgroup such that:
– at places above p, U'_v is maximal compact in case (a) and when $p = k(\bar{\rho}) = 2$, and is otherwise $U_1(v)$
– for the places not in S and not above p, U'_v is maximal compact
– for $v \in S$, $U'_v = U_0(v)$.

There is a maximal ideal m of $\mathbb{T}_\psi(U') \subset \mathrm{End}(S_{k,\psi'}(U', \mathcal{O}))$ such that $\bar{\rho}_{\mathsf{m}} \simeq \bar{\rho}_F$.

By Lemma 7.3 for all places v in S, there is a character $\chi = \Pi_v \chi_v$ of $\Pi_v k_v^*$ of order a power of p, with each χ_v non-trivial (and of order divisible by 4 if $p = 2$) with the following property:
– If we regard χ as a character of U' via maps $U'_v \to k_v^*$ with kernel

$$\{g \in U'_v : g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mathrm{mod.}(\pi_v), ad^{-1} = 1\},$$

then χ is trivial on $(U'(\mathbb{A}_F^\infty)^* \cap t^{-1}D^*t)/F^*$ for any $t \in (D \otimes \mathbb{A}_F^\infty)^*$.

Then as in Lemma 7.4 we have the isomorphism $S_{k,\psi'}(U', \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \simeq S_{W_k(\chi),\psi'}(U', \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F}$. We deduce there is a maximal ideal m' of the Hecke algebra $\subset \mathrm{End}(S_{W_k(\chi),\psi'}(U', \mathcal{O}))$ such that $\bar{\rho}_{\mathsf{m}'} \simeq \bar{\rho}_F$. As $\chi = \Pi_{v \in S}\chi_v$ and each χ_v is non-trivial (and of order divisible by 4 if $p = 2$), each (irreducible, cuspidal) automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_F)$ that contributes to $S_{W_k(\chi),\psi'}(U', \mathcal{O})_{\mathsf{m}'}$ via the JL correspondence is a (ramified) principal series at the places $v \in S$. Thus after another allowable base change F''/F that is split at places above p and Σ, we deduce, using also Lemma 7.7, that there is an automorphic representation π'' of $\mathrm{GL}_2(\mathbb{A}_{F''})$ that gives rise to $\bar{\rho}_{F''}$ such that:

- ($p = 2$) π'' is of parallel weight 2, unramified at all primes above 2 if $k(\bar{\rho}) = 2$, or Steinberg at all places above 2 if $k(\bar{\rho}) = 4$
- ($p > 2$) corresponding to the cases above (a) π'' is of parallel weight $k(\bar{\rho})$ and unramified at places above p, $v \notin \Sigma$, and in the case when $k(\bar{\rho}) = 2$, $\Sigma \cap \{v|p\} = \phi$, we may also choose π'' to be unramified at places above p and of weight $p+1$, or (b) of parallel weight 2 and at places above p the base change of the corresonding WD parameter to $\mathbb{Q}_p^{\mathrm{nr}}(\mu_p)$ is unramified, or (c) π'' of parallel weight 2 and Steinberg at places above p when $k(\bar{\rho}) = p + 1$.
- For all p, π'' is unramified at all finite places not above p and Σ and at places $v \in \Sigma$, π''_v is Steinberg.

The last part of case (a) (which is considered only for $p > 2$) is handled by Lemma 8.3. Note that as $\bar{\rho}|_{F''(\mu_p)}$ is absolutely irreducible for $p > 2$, we may assume for our purposes by Lemma 2.2 of [30], the U of Lemma 8.3 satisfies the conclusion of Lemma 1.1 of [55] (which ensures the surjectivity of $S_{p+1,\psi}(U, \mathcal{O}) \to S_{p+1,\psi}(U, \mathbb{F})$).

For $p > 2$ up to replacing π'' by a twist (using the trivial $p \neq 2$ case of Lemma 7.10) we obtain the desired cuspidal automorphic representation π'' of $\mathrm{GL}_2(\mathbb{A}_{F''})$ with central character $\psi_{F''}$.

In the case of $p = 2$, after twisting and invoking another allowable base change that is also split at places above p and Σ, using Lemma 7.10, we may ensure that the central character of π'' is given by $\psi_{F''}$ thus obtaining the desired cuspidal automorphic representation π'' of $\mathrm{GL}_2(\mathbb{A}_{F''})$. The claim for $p = 2, k(\bar{\rho}) = 4$ about the eigenvalue of $U_{v'}$ acting on $\pi''_{v'}$, for v' place of F'' above v, may be ensured by a further allowable base change.

□

Lemma 8.3. *Consider an open compact subgroup $U = \Pi_v U_v$ of $(D_{F''} \otimes \mathbb{A}^{\infty}_{F''})^*$, with $D_{F''}$ the definite quaternion algebra over F'' unramified at all finite places not above Σ, with $\Sigma \cap \{v|p\} = \phi$, and with $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ for places v above p. Let $\psi : (\mathbb{A}^{\infty}_{F''})^* \to \mathbb{F}^*$ be a continuous character such that $\psi|_{U \cap (\mathbb{A}^{\infty}_{F''})^*} = 1$. Assume $\bar{\rho}_{F''}$ arises from a maximal ideal of the Hecke algebra (outside p) acting on $S_{2,\psi}(U, \mathbb{F})$. Then it also arises from a maximal ideal of the Hecke algebra (outside p) acting on $S_{p+1,\psi}(U, \mathbb{F})$.*

Proof. This follows by the group-cohomological arguments in the proof of Proposition 1 of Section 4 of [23]. Although only the case of p inert in F'' is considered in [23], the argument there can be iterated to remove this restriction. We spell this out a little more.

Let $\{w_1, \cdots, w_r\}$ be places of F'' above p, and for each $1 \leq i \leq r$, let $W_i = \{w_1, \cdots, w_i\}$ and W_0 be the empty set. Let E be a large enough unramified extension of $\mathbb{Q}_p$. Let W_{τ_i} be the U_p-module $\otimes_{\iota:F''\hookrightarrow E,\iota\in J_i}\mathrm{Symm}^{p-1}(\mathbb{F})$ with J_i the subset of the embeddings corresponding to W_i. Note that $W_{\tau_i}|_{U\cap(\mathbb{A}^{\infty}_{F''})^*}$ is trivial.

Assume that for an i, $0 \leq i < r$, there is a maximal ideal m of the Hecke algebra (outside p) acting on $S_{\tau_i,\psi}(U,\mathbb{F})$ which gives rise to $\bar{\rho}$, and thus $S_{\tau_i,\psi}(U,\mathbb{F})_{\mathsf{m}} \neq 0$. This assumption for $i=0$ is part of the hypothesis of the lemma.

Let $U'' = \Pi_v U''_v$ be the subgroup of U such that $U_v = U''_v$ for $v \neq w := w_{i+1}$, and $U''_w = U_0(w)$. Then the kernel of the standard degeneracy map

$$S_{\tau_i,\psi}(U,\mathbb{F})^2 \to S_{\tau_i,\psi}(U'',\mathbb{F}),$$

$(f_1,f_2) \to f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \pi_w \end{pmatrix} f_2$, is Eisenstein (see Lemma 7.1).

Next observe that $S_{\tau_i,\psi}(U'',\mathbb{F}) \simeq S_{\tau_i,\psi}(U,\mathbb{F}) \oplus S_{\tau\otimes V,\psi}(U,\mathbb{F})$, where V is the $\mathrm{GL}_2(k_w)$-module $\otimes_{\iota:F\hookrightarrow E,\iota\in J_w}\mathrm{Symm}^{p-1}(\mathbb{F})$ where $J_{w_{i+1}}$ this time consists of embeddings of F'' in E corresponding to $w = w_{i+1}$. Here (see [23]) we use the fact that $\mathbb{F}[\mathbb{P}_1(k_w)]$ is isomorphic as a $\mathrm{GL}_2(k_w)$-module, using the natural action of $\mathrm{GL}_2(k_w)$ on $\mathbb{P}_1(k_w)$, to $\mathrm{id}\oplus V$. The map $f \to \begin{pmatrix} 1 & 0 \\ 0 & \pi_w \end{pmatrix} f$, that sends $S_{\tau_i,\psi}(U,\mathbb{F})_{\mathsf{m}}$ to $S_{\tau_i,\psi}(U'',\mathbb{F})_{\mathsf{m}}$, when composed with the projection $S_{\tau_i,\psi}(U'',\mathbb{F})_{\mathsf{m}} \to S_{\tau_{i+1},\psi}(U,\mathbb{F})_{\mathsf{m}}$, induces a map $S_{\tau_i,\psi}(U,\mathbb{F})_{\mathsf{m}} \to S_{\tau_{i+1},\psi}(U,\mathbb{F})_{\mathsf{m}}$. This last map is seen to be injective by Lemma 7.1. Thus we see at the end (the case $i=r$) that $\bar{\rho}|_{F''}$ also arises from a maximal ideal of the Hecke algebra acting on $S_{p+1,\psi}(U,\mathbb{F})$.

□

8.3. **Lifting data.** We will need to construct automorphic lifts of $\bar{\rho}_{|G_F}$ satisfying various properties that are described by *lifting data* which consists of imposing the determinant and some local conditions at a finite set S of places of F including the infinite places and the places above p (the lift has to be unramified outside S). The condition at infinite places is to be odd, and it is implied by the determinant.

8.3.1. *Determinant condition of lifting data.* We fix determinant of the lifts to be $\psi\chi_p$ as chosen in 8.1.

8.3.2. *Lifting data away from p.* For finitely many places $\{v\}$ of F not above p, which are called the ramified places of the lifting data, we are given local lifts $\tilde{\rho}_v$ of $\bar{\rho}|_{D_v}$, such that $\tilde{\rho}_v$ is ramified and of the form

$$\begin{pmatrix} \gamma_v\chi_p & * \\ 0 & \gamma_v \end{pmatrix},$$

with γ_v a given unramified character and $\gamma_v^2 = \psi_v$.

At all other places v not above p, the lifting data specified is that the lift be unramified and of determinant $\psi_v\chi_p$.

8.3.3. *Lifting data at p.* Suppose also that for all places v above p we are given a lift $\tilde{\rho}_v$ of $\bar{\rho}|_{D_v}$ such that $\det(\tilde{\rho}_v) = \psi_v \chi_p$ and such that

- ($p = 2$) $\tilde{\rho}_v$ is crystalline of weight 2 at all primes above 2 when $k(\bar{\rho}) = 2$, and when $k(\bar{\rho}) = 4$ $\tilde{\rho}_v$ is semistable and non-crystalline of weight 2 at v and of the form

$$\begin{pmatrix} \gamma_v \chi_2 & * \\ 0 & \gamma_v \end{pmatrix},$$

  where χ_2 is the 2-adic cyclotomic character, and γ_v a given unramified character such that $\gamma_v^2 = \psi_v$.
- ($p > 2$) $\tilde{\rho}_v$ is either (simultaneously at all places v above p)
  (A) crystalline of weight k, such that $2 \leq k \leq p+1$, with the case $k = p+1$ considered only when F is split at p and $k(\bar{\rho}) = p+1$, or
  (B) crystalline of weight 2 over $\mathbb{Q}_p^{\mathrm{nr}}(\mu_p)$ of Weil-Deligne parameter $(\omega_p^{k-2} \oplus 1, 0)$ for a fixed k in all embeddings, or
  (C) semistable, non-crystalline of weight 2 and of the form

$$\begin{pmatrix} \gamma_v \chi_p & * \\ 0 & \gamma_v \end{pmatrix},$$

  where χ_p is the p-adic cyclotomic character, and γ_v is a given unramified character such that $\gamma_v^2 = \psi_v$.

To make the conditions uniform with p, notice that in the case of $p = 2$ we only consider lifts of the type considered in (A) of weight 2, and we do not consider the case (B) and we consider (C) only when the residual representation has weight 4.

In (A) the character ψ has to be of the form (i), in case (B) of the form (ii), and in case (C) it has to be of the form (ii).

We fix lifting data as above for the rest of the section. A lift $\rho_F : G_F \rightarrow \mathrm{GL}_2(\mathcal{O})$ fits the lifting data if $\rho_F|_{D_v}$ is of type $\tilde{\rho}_v$ at all places which are in S, is unramified at the other places and ψ is $\det(\rho_F)\chi_p^{-1}$.

8.4. **Liftings with prescribed local properties: Theorem 8.4.** The following theorem proves that, under the hypothesis that $\bar{\rho}$ is modular as in 8.2, we can find after an allowable base change a modular lift ρ of $\bar{\rho}$ which fits the lifting data that we have chosen in the last paragraph.

Theorem 8.4. *Assume if $p > 2$, (α) and (β) of Section 8.2, and if $p = 2$ that (α) is satisfied when $k(\bar{\rho}) = 2$ and (β) is satisfied. There is an allowable base change F'/F, and a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_{F'})$ that is discrete series at infinity of parallel weight such that*

- *$\rho_{\pi'}$ is a lift of $\bar{\rho}_{F'}$*
- *($p = 2$) is crystalline of weight 2 at all primes above 2 when $k(\bar{\rho}) = 2$, and when $k(\bar{\rho}) = 4$ it is semistable of weight 2 of the form prescribed above.*

- *($p > 2$) at all places above p of F' either crystalline of weight k, such that $2 \leq k \leq p+1$ (and when the case $p+1$ is considered F' is split at p and $k(\bar{\rho}) = p+1$), or of weight 2 and crystalline over $\mathbb{Q}_p^{\rm nr}(\mu_p)$ of the prescribed inertial WD parameter as in (B) above , or as in case (C) semistable and non-crystalline of weight 2, and then for $v|p$, $\rho_F|_{D_v}$ is of the form*

$$\left(\begin{array}{cc} \gamma_v\chi_p & * \\ 0 & \gamma_v \end{array} \right),$$

  corresponding to the cases (A), (B), (C) above and where we are denoting the restriction of the character γ_v by the same symbol
- *$\rho_{\pi'}$ is unramified at places where the lifting data is unramified*
- *at all places not above p at which the lifting data is ramified, $\rho_{\pi'}|_{D_v}$ is of the form*

$$\left(\begin{array}{cc} \gamma_v\chi_p & * \\ 0 & \gamma_v \end{array} \right),$$

  with γ_v unramified.
- $\det\rho_{\pi'} = \psi_{F'}\chi_p$.

For $p = 2$, when $k(\bar{\rho}) = 2$, π' is unramified at places v above 2, and when $k(\bar{\rho}) = 4$ is Steinberg at places v above 2.

For $p > 2$, in (A) we may ensure that π' is unramified at places v above p, in (B) that π'_v has fixed vectors under $U_1(v)$, and in (C) that π'_v has fixed vectors under $U_0(v)$.

Proof. After a suggestion of Fred Diamond, and the referee, we give a proof that works uniformly for all p. This is unlike what was done in a previous version of the paper.

It is enough to prove the Theorem 8.4 after base changing to the F'' of Theorem 8.2 with the cases (a), (b), (c) of the latter corresponding to (A), (B), (C) of the former (except that in (a) we do not consider weight $p+1$ liftings unless $k(\bar{\rho}) = p+1$ and F split at p) : we reinitialise and take F to be F''.

Theorem 8.4 follows from the existence of π'' of Theorem 8.2 (assuming the Σ of the statement of Theorem 8.2 to be the empty set) using the method of proof of Corollary 3.1.11 (this is Ribet's method of raising levels using Ihara's lemma) and Lemma 3.5.3 (use of base change and Jacquet-Langlands) of [33]. Note that in the proof of Corollary 3.1.11 of [33] we may allow Σ (in the notation there) to contain the places above p if we are in case (C) and $k(\bar{\rho}) = 2$, and the restriction to weight $k = 2$ there can be replaced by $2 \leq k \leq p+1$.

We give the details. By Theorem 8.2 we know that $\bar{\rho}_F$ arises from a cuspidal automorphic representation π'' of $\mathrm{GL}_2(\mathbb{A}_F)$, of central character ψ, that is unramified at all places not above p, is of weight $k(\bar{\rho})$ when in case (A), and otherwise of weight 2, and such that in case (A) (or $p = 2, k(\bar{\rho}) = 2$)

at places v above p is unramified, and otherwise in cases (B) and (C)(or $p = 2, k(\bar{\rho}) = 4$) has fixed vectors under $U_1(v)$.

We take $\Sigma = \{v_1, \cdots, v_r\}$ to consist of places that are not above p at which the lifting data is ramified, and we also include the places above p if we are in case (C) and $k(\bar{\rho}) = 2$. After an allowable base change split at p we may assume that $|\Sigma|$ is even.

Then as in Lemma 3.5.3 of [33], there is a tower of totally real fields $F = F_0 \subset F_1 \subset \cdots F_r := F'$ such that for $i = 1, \cdots, r$, F_i/F_{i-1} is a quadratic extension such that for $j \in \{1, \cdots, r\}$ any prime w of F_{i-1} over v_j is inert in F_i if $i \neq j$ and splits in F_i if $i = j$, and F_r/F is an allowable base change, unramified at places above p, and even split at places above p if we are not in case (C) with $k(\bar{\rho}) = 2$.

We further ensure that $\bar{\rho}|_{F'}$ has non-solvable image when $p = 2$, and otherwise $\bar{\rho}|_{F'(\mu_p)}$ is absolutely irreducible when $p > 2$.

Choose any prime r_0 of F which is not in Σ and does not lie over p. Inductively as in the proof of Lemma 3.5.3 of [33], using Corollary 3.1.11 of [33] repeatedly, we ensure for each $0 \leq i \leq r$, starting for $i = 0$ with the π'' of Theorem 8.2, the following situation:

- there is a definite quaternion algebra D_i over F_i with center F_i that is ramified exactly at all the infinite places and the places above $\{v_1, \cdots, v_i\}$ (note that the latter has cardinality $2i$),
- there is an open compact subgroup $U_i = \Pi_v (U_i)_v$ of $(D_i \otimes \mathbb{A}_{F_i}^{\infty})^*$ such that $\bar{\rho}_{F_i}$ arises from $S_{k,\psi_{F_i}}(U_i, \mathcal{O})$ such that
  – $(U_i)_{r'}$ is $U(r')$ for r' lying over r_0(this ensures that U has the neatness property descibed in Lemma 1.1 of [55] and hence the corresponding space of modular forms has the usual perfect pairings)
  – $(U_i)_v$ is maximal compact at all finite places v at which D_i is ramified, and all v not above p and r_0,
  – for the places above p at which D_i is not ramified, $(U_i)_v$ is maximal compact if we are in case (A) (or case (C) if $k(\bar{\rho}) = 2$), and otherwise $(U_i)_v$ is $U_1(v)$.

We also define the integer k to be $k(\bar{\rho})$ when we are in case (A), and $k = 2$ in cases (B) and (C). Assume we have proven the statement for some i such that $0 \leq i < r$. Consider the definite quaternion algebra D_i and the subgroup U_i and the unique place w_{i+1} of F_i above v_{i+1}. Let $U_i' = \Pi_v (U_i')_v$ be the subgroup of $U_i = \Pi_v (U_i)_v$ such that $(U_i')_v = (U_i)_v$ for $v \neq w_{i+1}$ and $(U_i')_{w_{i+1}} = U_0(w_{i+1})$. Consider the degeneracy map $S_{k,\psi_{F_i}}(U_i, \mathcal{O})^2 \to S_{k,\psi_{F_i}}(U_i', \mathcal{O})$. By Lemma 7.1 (note that when w_{i+1} is a place above p, then $k = 2$ and the hypotheses of Lemma 7.1 are thus fulfilled), we deduce easily that the kernel of the reduction of this map modulo the maximal ideal of $\mathcal{O}$, and hence the p-torsion of the cokernel of the characteristic 0 map, has only Eisenstein maximal ideals in its support. Then from Corollary 3.1.11 of [33], the Jacquet-Langlands correspondence and the compatibility of the local and global Langlands correspondence proved in [10] and [57], we see that there is an automorphic representation π_i of $\mathrm{GL}_2(\mathbb{A}_{F_i})$ that has non-zero invariants under U_i' and such that ρ_{π_i} lifts $\bar{\rho}_{F_i}$, and π_i is Steinberg at all places of F_i above $\{v_1, \cdots, v_{i+1}\}$. Base changing π_i to F_{i+1} we see by the Jacquet-Langlands correspondence that the conditions over F_{i+1} are ensured.

For $i = r$, invoke Theorem 8.2 (with the Σ there taken to be all places lying above the $\Sigma = \{v_1, \cdots, v_r\}$ here) to get rid of possible ramification at primes above r_0. Then another use of the Jacquet-Langlands correspondence, together with Lemma 7.7, gives that there is a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_{F'})$ such that $\rho_{\pi'}$ lifts $\bar{\rho}_{F'}$ and $\rho_{\pi'}$ gives rise to the lifting data at least up to unramified order 2 twist at places above Σ and above p (the latter considered only when $p > 2, k(\bar{\rho}) = 2$ and we are in case (C) or $p = 2, k(\bar{\rho}) = 4$). After an allowable base change, we may in fact ensure that that there is a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_{F'})$ such that $\rho_{\pi'}$ lifts $\bar{\rho}_{F'}$ and $\rho_{\pi'}$ gives rise to the lifting data.

Note that F' need not be split at p if we are in case $p = 2, k(\bar{\rho}) = 4$, or $p > 2, k(\bar{\rho}) = 2$ and we are in case (C), while otherwise we may arrange it to be split at p.

□

9. $R = \mathbb{T}$ THEOREMS

Throughout this section we consider $\bar{\rho}$ as in 8, $\bar{\rho}_F := \bar{\rho}_{|G_F}$, ψ (see 8.1) and the lifting data (see 8.3) as in Section 8 and assume that $\bar{\rho}_F$ satisfies the assumptions $(\alpha), (\beta)$ if $p \neq 2$, and (α) when $p = k(\bar{\rho}) = 2$ and (β) if $p = 2$ (see 8.2).

After possibly an allowable base change (7.6.2), Theorem 8.4 ensures that there is a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ such that ρ_π fits the prescribed lifting data. Further when we need to consider weight $p+1$ liftings, which we consider only when $k(\bar{\rho}) = p+1$, we may assume by Theorem 6.1 and 8.4 that F is split at p.

9.1. **Taylor-Wiles systems.** For $p > 2$ we reproduce in our context Kisin's modification of the original Taylor-Wiles systems of [60] as later modified by Fujiwara and Diamond (see [16]). In case $p = 2$, we patch deformation rings for non-fixed determinants and we have to take into accounts twists (9.1.3).

We use Proposition 3.3.1 of [33], Section 1.3 of [34], Proposition 1.3 and Corollary 1.4 of [35] as the principal references.

9.1.1. *The map* $R \to \mathbb{T}$. We denote by Σ the set which consists of the places at which the lifting data is ramified that are not above p, and all the places above p if our lifting data is in case (C) or when $p = 2$ and $k(\bar{\rho}) = 4$. We denote by S the union of Σ, the infinite places of F and the places of F above p.

By an allowable base change we may assume that the number of places in Σ above F is even, and $[F:\mathbb{Q}]$ is even.

Consider D the definite quaternion algebra over F that is ramified at exactly the places in Σ and all the infinite places. The existence of π gives rise to a maximal ideal m of the Hecke algebra $\mathbb{T}_\psi(U)$ which acts on $S_{k,\psi}(U,\mathcal{O})$. Here $U := \Pi_v U_v \subset (D \otimes_F \mathbb{A}_F^\infty)^*$ is such that U_v is described as follows:

- U_v is maximal compact at all places v not in S,
- U_v is the group $(\mathcal{O}_D)_v^*$ (resp., D_v^* in case $p=2$) for $v \in \Sigma$,
- For v above p, if we are in case (C), including the case $p=2, k(\bar\rho)=4$, then $v \in \Sigma$ and we are already covered. If this is not the case, then at places v above p, U_v is either maximal compact, or $U_1(v)$, according to whether we are in case (A) (including $p=k(\bar\rho)=2$) or (B).

Here when $p=2$, and hence $k=2$, and we denote by U' the maximal compact subgroup of U as defined in Section 7, it is understood that we have extended the module W_2 of $U'(\mathbb{A}_F^\infty)^*$ to one of $U(\mathbb{A}_F^\infty)^*$, denoted again by W_2, in the unique way that allows the existence of an eigenform in $S_{k,\psi}(U,\mathcal{O})$ that has the same Hecke eigenvalues at places not in S as those arising from π (see Section 7.2).

Consider the deformation ring $\bar{R}_S^{\square,\psi}$ where the corresponding local deformation rings $\bar{R}_v^{\square,\psi}$ for $v \in S$ parametrise the liftings as in the lifting data. Thus these liftings are:

- semistable liftings for places $\in S$, and not above p and infinity with a fixed choice of unramified character γ_v (cd. §3.3.4);
- at infinite places the odd liftings (cf. §3.1);
- At places above p the lifts are uniformly of any of the type as in 8.3:
  - (A) (including $p=2$ when the weight k is 2) : low weight crystalline ((ii) of §3.2.2) ;
  - (B) ($p>2$ and $k(\bar\rho) \le p$) : weight 2 lifts ((i) of §3.2.2) ;
  - (C) (including the case $p=2$ when $k(\bar\rho)=4$) weight 2 semistable lifts ((i) of §3.2.2).

Recall that the $\bar{R}_v^{\square,\psi}$ for $v \in S$ that we consider have the following properties:

- $\bar{R}_v^{\square,\psi}$ is a domain flat over $\mathcal{O}$
- The relative to $\mathcal{O}$ dimension of $\bar{R}_v^{\square,\psi}$ is :
  - 3 if $\ell \neq p$;
  - $3+[F_v:\mathbb{Q}_p]$ if $\ell = p$.
  - 2 if v is an infinite place.
- $\bar{R}_v^{\square,\psi}[\frac{1}{p}]$ is regular.

When $k(\bar\rho)=p$ and $\bar\rho$ is unramified at p, note, for the fact that $\bar{R}_v^{\square,\psi}$ is a domain, that by lemma 8.1, $(\bar\rho_F)_{|D_v}$ is trivial (3.2.5).

The completed tensor product $\bar{R}_S^{\square,\mathrm{loc},\psi}$ is thus flat over $\mathcal{O}$, a domain, and of relative dimension $3|S|$, and $\bar{R}_S^{\square,\mathrm{loc},\psi}[\frac{1}{p}]$ is regular (see Theorem 3.1 and Proposition 3.2).

As in Section 1 of [55], using existence of Galois representations attached to Hilbert modular eigenforms and the Jacquet-Langlands correspondence we get a continuous representation

$$G_F \to \mathrm{GL}_2(\mathbb{T}_\psi(U)_{\mathsf{m}} \otimes_{\mathcal{O}} E).$$

This together with Théorème 2 of [11], and the fact that the traces of the representation are contained in $\mathbb{T}_\psi(U)_{\mathsf{m}}$ yields that the representation has a model

$$\rho_{\mathsf{m}} : G_F \to \mathrm{GL}_2(\mathbb{T}_\psi(U)_{\mathsf{m}}).$$

The representation ρ_{m} is characterised by the property that for $v \notin S$ the Eichler-Shimura relation is satisfied, i.e., the characteristic polynomial of $\rho_{\mathsf{m}}(\mathrm{Frob}_v)$ is $X^2 - T_v X + \mathbb{N}(v)\psi(\pi_v)$. Here Frob_v denotes arithmetic Frobenius at v and $\mathbb{N}(v)$ denotes the order of the residue field at v. We denote by $\bar{\rho}_{\mathsf{m}} : G_F \to \mathrm{GL}_2(\mathbb{F})$ the representation obtained by reducing ρ_{m} modulo m: this is isomorphic to $\bar{\rho}_F$. Thus there is a unique map $\pi' : R_S^\psi \to \mathbb{T}_\psi(U)_{\mathsf{m}}$ which takes the universal representation ρ_S^{univ} to ρ_{m}. Recall that the $\mathcal{O}$-algebra $\bar{R}_S^{\square,\psi}$ has a natural structure of a smooth $\bar{R}_S^\psi$-algebra.

Lemma 9.1. *The map* $\pi' : R_S^\psi \to \mathbb{T}_\psi(U)_{\mathsf{m}}$ *induces a surjective map* $\pi : \bar{R}_S^\psi \to \mathbb{T}_\psi(U)_{\mathsf{m}}$*, that takes the universal representation* $\bar{\rho}_S^{\mathrm{univ}}$ *to* ρ_{m}*. Let* $\mathbb{T}_\psi^\square(U)_{\mathsf{m}} = \mathbb{T}_\psi(U)_{\mathsf{m}} \otimes_{\bar{R}_S^\psi} \bar{R}_S^{\square,\psi}$*. The map* π *also induces a surjective map* $\bar{R}_S^{\square,\psi} \to \mathbb{T}_\psi^\square(U)_{\mathsf{m}}$ *that we again denote by* π*.*

We pull back the maximal ideal m *of* $\mathbb{T}_\psi(U)$ *to a maximal ideal of the Hecke algebra* $\mathbb{T}_{\psi,Q_n}(U_{Q_n})$ *that acts on* $S_{k,\psi}(U_{Q_n},\mathcal{O})$ *as prescribed in Section 7.4, and again denote it by* m*. (Recall from Section 7.4 that for* $\mathsf{m} \subset \mathbb{T}_{\psi,Q_n}(U_{Q_n})$*,* $v \in Q_n$*,* $U_v - \tilde{\alpha}_v \in \mathsf{m}$ *with* $\tilde{\alpha}_v$ *a lift of one of the two distinct eigenvalues of* $\bar{\rho}_{\mathsf{m}}(\mathrm{Frob}_v)$ *that we have fixed.) Then we have a map* $\bar{R}_{S\cup Q_n}^\psi \to \mathbb{T}_{\psi,Q_n}(U_{Q_n})_{\mathsf{m}}$*, compatible with the* $\mathcal{O}[\Delta'_{Q_n}]$*-action, characterised by the property that* $\mathrm{tr}(\bar{\rho}_{S\cup Q_n}^{\mathrm{univ}}(\mathrm{Frob}_v))$ *maps to* T_v *for almost all places* v*.*

Note that as $\bar{R}_S^{\square,\psi}$ is formally smooth over $\bar{R}_S^\psi$, and $\mathbb{T}_\psi(U)_{\mathsf{m}}$ is flat over $\mathcal{O}$ and reduced we get that $\mathbb{T}_\psi^\square(U)_{\mathsf{m}}$ is again flat over $\mathcal{O}$ and reduced.

Proof. We check that the map π' factors through a map $\bar{R}_S^\psi \to \mathbb{T}_\psi(U)_{\mathsf{m}}$ that takes the universal representation $\bar{\rho}_S^{\mathrm{univ}}$ to ρ_{m} and is surjective.

By the compatibility of the local-global Langlands correspondence proved in [10] and [57] away from p (see also Lemma 7.2), the properties at p in Lemma 7.7 that are available as the residual representation $\bar{\rho}_F$ is irreducible, whenever we have a map $x : \mathbb{T}_\psi(U)_{\mathsf{m}} \to \mathcal{O}'$ with $\mathcal{O}'$ the ring of integers of a finite extension of E (and which gives rise to a representation ρ_x of G_F)

the corresponding map $R_S^\psi \to \mathcal{O}'$, with kernel $\wp_x$, factors through $\bar{R}_S^\psi$. As $\mathbb{T}_\psi(U)_\mathsf{m}$ is flat and reduced, we deduce that $\cap_x \ker(x) = 0$, and thus $\cap_x \wp_x$ is the kernel of the map $\pi' : R_S^\psi \to \mathbb{T}_\psi(U)_\mathsf{m}$. From this we deduce that the map π' factors through $R_S^\psi \to \bar{R}_S^\psi$, and thus we get the desired map $\bar{R}_S^\psi \to \mathbb{T}_\psi(U)_\mathsf{m}$ which takes the universal representation $\bar{\rho}_S^{\text{univ}}$ to ρ_m. From this and the Eichler-Shimura relation it follows that $\bar{R}_S^\psi \to \mathbb{T}_\psi(U)_\mathsf{m}$ is a surjective map.

The other part is proved by similar arguments. We only note that the compatibility of the local-global Langlands correspondence implies that the map $\bar{R}_{S\cup Q_n}^\psi \to \mathbb{T}_{\psi,Q_n}(U_{Q_n})_\mathsf{m}$ takes $\gamma_{\alpha_v}(\pi_v) \to U_v$ for $v \in Q_n$ (using the notation of Proposition 5.11: note that U_v depends on the choice of the uniformiser π_v of F_v). □

Remark: The proof above also yields the surjectivity of $\bar{R}_{S\cup Q_n}^\psi \to \mathbb{T}_{\psi,Q_n}(U_{Q_n})_\mathsf{m}$. Further as the traces of the representation $G_F \to \mathrm{GL}_2(\mathbb{T}_{\psi,Q_n}(U_{Q_n})_\mathsf{m})$ are contained in the image of $\mathbb{T}_\psi(U_{Q_n}) \to \mathbb{T}_{\psi,Q_n}(U_{Q_n})_\mathsf{m}$, and thus it is defined over the image by Théorème 2 of [11], we may deduce from the above proof that the natural map $\mathbb{T}_\psi(U_{Q_n})_\mathsf{m} \to \mathbb{T}_{\psi,Q_n}(U_{Q_n})_\mathsf{m}$ is bijective.

9.1.2. *Patching for p odd.* We assume $p \neq 2$. The following proposition is Kisin's modified version of the Taylor-Wiles systems argument. It is derived directly from the proof of Proposition 3.3.1 of [33], and Proposition 1.3 and Corollary 1.4 of [35].

Proposition 9.2. *Assume the conditions stated at the beginning of this section.*

Let $d = 3|S|$, $h = \dim(H^1_{\{L_v^\perp\}}(S, (\mathrm{Ad}^0)^*(1)))$ (see 4.1.4) and $j = 4|S| - 1$.

(I) We have maps of $\bar{R}_S^{\square,\mathrm{loc},\psi}$-algebras

$$\bar{R}_S^{\square,\mathrm{loc},\psi}[[x_1, \cdots, x_{h+j-d}]] \to R_\infty \to \bar{R}_S^{\square,\psi}$$

with R_∞ a $\mathcal{O}[[y_1, \cdots, y_{h+j}]]$-algebra and a R_∞-module M_∞ such that

(1) Each of the maps is surjective and the map on the right induces an isomorphism $R_\infty/(y_1, \cdots, y_h)R_\infty \simeq \bar{R}_S^{\square,\psi}$ of $\bar{R}_S^{\square,\mathrm{loc},\psi}$-algebras.

(2) M_∞ is a finite free $\mathcal{O}[[y_1, \cdots, y_{h+j}]]$-module, and the action of R_∞ on the quotient $M_\infty/(y_1, \cdots, y_h)M_\infty$ factors through $\mathbb{T}_\psi^\square(U)_\mathsf{m}$ and makes it into a faithful $\mathbb{T}_\psi^\square(U)_\mathsf{m}$ -module.

(II) The ring $\bar{R}_S^{\square,\psi}$ is a finite $\mathcal{O}[[y_{h+1}, \cdots, y_{h+j}]]$-module, and $\bar{R}_S^\psi$ is a finite $\mathcal{O}$-module.

(III) The map $\pi : \bar{R}_S^{\square,\psi} \to \mathbb{T}_\psi^\square(U)_\mathsf{m}$ of Proposition 9.1 is surjective with p-power torsion kernel.

Proof. For each positive integer n choose a set of primes Q_n as in Lemma 5.3. We define $\mathbb{T}^\square_{\psi,Q_n}(U_{Q_n})_\mathsf{m} = \mathbb{T}_{\psi,Q_n}(U_{Q_n}) \otimes_{\bar{R}^\psi_{S\cup Q_n}} \bar{R}^{\square,\psi}_{S\cup Q_n}$.

(I) This is a consequence of Proposition 5.11 and Corollary 7.5 using the patching argument of [60] and [33].

Consider the $\bar{R}_S^{\square,\mathrm{loc},\psi}$-algebra $\bar{R}_{S\cup Q_n}^{\square,\psi}$. The the number of its generators as a $\bar{R}_S^{\square,\mathrm{loc},\psi}$-algebra is controlled by using Proposition 5.5 and is equal to $\dim(H^1_{\{L_v^\perp\}}(S,(\mathrm{Ad}^0)^*(1)))+|S|-1=h+j-d$ and thus we have surjective maps

$$(**)\bar{R}_S^{\square,\mathrm{loc},\psi}[[x_1,\cdots,x_{h+j-d}]]\to\bar{R}_{S\cup Q_n}^{\square,\psi}\to\bar{R}_S^{\square,\psi}.$$

Using Proposition 5.11 we see that $\bar{R}_{S\cup Q_n}^{\square,\psi}$ is a $\mathcal{O}[\Delta'_{Q_n}][[y_{h+1},\cdots,y_{h+j}]]$-module, such that the $\mathcal{O}[\Delta'_{Q_n}]$-covariants is isomorphic to $\bar{R}_S^{\square,\psi}$. The $\mathcal{O}[[y_{h+1},\cdots,y_{h+j}]]$ structure on $\bar{R}_{S\cup Q_n}^{\square,\psi}$ comes from the the framing, i.e., the fact that $\bar{R}_{S\cup Q_n}^{\square,\psi}$ is a power series ring over $\bar{R}_{S\cup Q_n}^{\psi}$ in $j=4|S|-1$ variables (cf. Proposition 4.1).

Consider $S_{k,\psi}(U_{Q_n},\mathcal{O})_{\mathsf{m}}\otimes_{\bar{R}_{S\cup Q_n}^{\psi}}\bar{R}_{S\cup Q_n}^{\square,\psi}$. Here the tensor product is via the map $R_{S\cup Q_n}^{\psi}\to\mathbb{T}_{\psi,Q_n}(U_{Q_n})_{\mathsf{m}}$ composed with $\mathbb{T}_{\psi,Q_n}(U_{Q_n})_{\mathsf{m}}\to\mathrm{End}(S_{k,\psi}(U_{Q_n},\mathcal{O})_{\mathsf{m}})$.

The $S_{k,\psi}(U_{Q_n},\mathcal{O})_{\mathsf{m}}\otimes_{\bar{R}_{S\cup Q_n}^{\psi}}\bar{R}_{S\cup Q_n}^{\square,\psi}$ are $\mathcal{O}[\Delta_{Q_n}][[y_{h+1},\cdots,y_{h+j}]]$-modules where the $\mathcal{O}[[y_{h+1},\cdots,y_{h+j}]]$ structure comes again from the framing, and the $\mathcal{O}[\Delta_{Q_n}]$-action as in Corollary 7.5. Note that from Corollary 7.5 it follows that $S_{k,\psi}(U_{Q_n},\mathcal{O})_{\mathsf{m}}$ is a free $\mathcal{O}[\Delta_{Q_n}]$-module of rank independent of n.

The objects R_∞ and M_∞ are constructed by a patching argument of the type that first occurs in [60], and in the work of Diamond ([16]) and Fujiwara, and that occurs in the form we need it in the proof of Proposition 3.3.1 of [33].

Thus R_∞ gets defined as an inverse limit of suitable finite length $\bar{R}_S^{\square,\mathrm{loc},\psi}$-algebra quotients of $\bar{R}_{S\cup Q_n}^{\square,\psi}$. The module M_∞ is defined by taking an inverse limit over certain finite length quotients of $S_{k,\psi}(U_{Q_n},\mathcal{O})_{\mathsf{m}}\otimes_{\bar{R}_{S\cup Q_n}^{\psi}}\bar{R}_{S\cup Q_n}^{\square,\psi}$.

By virtue of (**) and the construction we get surjective maps

$$\bar{R}_S^{\square,\mathrm{loc},\psi}[[x_1,\cdots,x_{h+j-d}]]\to R_\infty\to\bar{R}_S^{\square,\psi}.$$

As $\bar{R}_{S\cup Q_n}^{\square,\psi}$ is a $\mathcal{O}[\Delta'_{Q_n}][[y_{h+1},\cdots,y_{h+j}]]$-module, such that the $\mathcal{O}[\Delta'_{Q_n}]$-covariants are isomorphic to $\bar{R}_S^{\square,\psi}$, we get from the construction in loc. cit. that R_∞ is a $\mathcal{O}[[y_1,\cdots,y_{h+j}]]$-algebra such that the last map above induces $R_\infty/(y_1,\cdots,y_h)R_\infty\simeq\bar{R}_S^{\square,\psi}$.

By Corollary 7.5, $S_{k,\psi}(U_{Q_n},\mathcal{O})_{\mathsf{m}}$ is a free $\mathcal{O}[\Delta_{Q_n}]$-module of rank independent of n, such that its $\mathcal{O}[\Delta_{Q_n}]$-covariants are isomorphic to $S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}$. Thus by construction we get that M_∞ is a finite flat $\mathcal{O}[[y_1,\cdots,y_{h+j}]]$-module. Further, the action of R_∞ on the quotient $M_\infty/(y_1,\cdots,y_h)M_\infty\simeq S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}}\otimes_{\mathbb{T}_\psi(U)_{\mathsf{m}}}\mathbb{T}_\psi^\square(U)_{\mathsf{m}}$ factors through $\mathbb{T}_\psi^\square(U)_{\mathsf{m}}$ and makes it into a faithful $\mathbb{T}_\psi^\square(U)_{\mathsf{m}}$ -module.

The patching is done in such a way that the natural maps $\bar{R}^{\square,\psi}_{S\cup Q_n} \to \mathbb{T}^{\square}_{\psi,Q_n}(U_{Q_n})_{\mathfrak{m}}$ arising from Proposition 9.1 induce a map $R_\infty \to \mathrm{End}(M_\infty)$ that is compatible with the $\mathcal{O}[[y_1,\cdots,y_{h+j}]]$ action.

(II) The image of $\bar{R}^{\square,\mathrm{loc},\psi}_S[[x_1,...,x_{h+j-d}]]$ in the endomorphisms of M_∞ is a finite, faithful $\mathcal{O}[[y_1,\cdots,y_{h+j}]]$-module and hence of dimension at least $h+j+1$.

We also know that $\bar{R}^{\square,\mathrm{loc},\psi}_S$ is a domain. Since the dimension of $\bar{R}^{\square,\mathrm{loc},\psi}_S[[x_1,...,x_{h+j-d}]]$ is $h+j+1$, we deduce that the composite

$$\bar{R}^{\square,\mathrm{loc},\psi}_S[[x_1,...,x_{h+j-d}]] \to R_\infty \to \mathrm{End}(M_\infty)$$

is injective.

We deduce that $R_\infty \simeq \bar{R}^{\square,\mathrm{loc},\psi}_S[[x_1,...,x_{h+j-d}]]$. Thus R_∞ is a finite $\mathcal{O}[[y_1,\cdots,y_{h+j}]]$-module from which part (II) follows.

We also note the consequence that $R_\infty[\frac{1}{p}]$ is a regular Noetherian domain as $\bar{R}^{\square,\mathrm{loc},\psi}_S[\frac{1}{p}]$ is one.

(III) Consider the map of regular Noetherian domains

$$\mathcal{O}[[y_1,\cdots,y_{h+j}]][\frac{1}{p}] \to R_\infty[\frac{1}{p}],$$

and the module $M_\infty \otimes_{\mathcal{O}} E$ (E is the fraction field of $\mathcal{O}$) that is finite free over $\mathcal{O}[[y_1,\cdots,y_{h+j}]][\frac{1}{p}]$. From Lemma 3.3.4 of [33] (that uses the Auslander-Buchsbaum theorem) it follows that $M_\infty \otimes_{\mathcal{O}} E$ is a finite projective, faithful $R_\infty[\frac{1}{p}]$-module. In particular $M_\infty \otimes_{\mathcal{O}} E/(y_1,\cdots,y_h)M_\infty \otimes_{\mathcal{O}} E$ is a faithful module over

$$R_\infty[\frac{1}{p}]/(y_1,\cdots,y_h) \simeq \bar{R}^{\square,\psi}_S[\frac{1}{p}].$$

Since the action of $\bar{R}^{\square,\psi}_S$ on $M_\infty/(y_1,\cdots,y_h)M_\infty$ factors through $\mathbb{T}^{\square}_{\psi}(U)_{\mathfrak{m}}$ the last part follows. □

9.1.3. *Patching for $p=2$.* We assume $p=2$.

Recall (5.10) that we have an integer n_0 and for each $n \geq n_0+1$, a finite set of auxiliary primes Q_n of fixed cardinality $h^1(S,Ad)-2$ such that in particular the Galois group of the maximal abelian extension of degree a power of 2 which is unramified outside Q_n and is split at S has a fixed number of generators which we note $t(=h^1_{S-split}(Q_n,\mathbb{F}))$. Let $\mathfrak{T} = \mathrm{Hom}(\mathfrak{t},\mathbb{G}_{\mathrm{m}})$ be the torus in the CNL$_{\mathcal{O}}$-category with character the constant free abelian group $\mathfrak{t}$ of rank t.

Proposition 9.3. *Let $d=3|S|$, $h=|Q_n|$ and $j=4|S|-1$.*

(I) We have maps of $\bar{R}^{\square,\mathrm{loc},\psi}_S$-algebras

$$\bar{R}^{\square,\mathrm{loc},\psi}_S[[x_1,\cdots,x_{h+j+t-d}]] \to R'_\infty \to R_\infty \to \bar{R}^{\square,\psi}_S$$

with R_∞ a $\mathcal{O}[[y_1,\cdots,y_{h+j}]]$-algebra and a R_∞-module M_∞ such that

(1) Each of the maps is surjective and the map on the right induces an isomorphism $R_\infty/(y_1,\cdots,y_h)R_\infty \simeq \bar{R}^{\square,\psi}_S$ of $\bar{R}^{\square,\mathrm{loc},\psi}_S$-algebras.

(2) M_∞ is a finite free $\mathcal{O}[[y_1, \cdots, y_{h+j}]]$-module, and the action of R_∞ on the quotient $M_\infty/(y_1, \cdots, y_h)M_\infty$ factors through $\mathbb{T}^\square_\psi(U)_{\mathsf{m}}$ and makes it into a faithful $\mathbb{T}^\square_\psi(U)_{\mathsf{m}}$ -module.

(3) Let X'_∞ and X_∞ be the functors from CNL$_\mathcal{O}$*-algebras to sets defined respectively by R'_∞ and R_∞. We have a free action of the torus $\mathfrak{T}$ on X'_∞ and a map $d : X'_\infty \to \mathfrak{T}$ such that :*

- the closed immersion $X_\infty \hookrightarrow X'_\infty$ identifies X_∞ with the closed subscheme defined by $d(x') = 1$;

- we have for $x' \in X'_\infty$ and $t \in \mathcal{T}$: $d(tx') = t^2 d(x')$.

(II) The ring $\bar{R}^{\square,\psi}_S$ is a finite $\mathcal{O}[[y_{h+1}, \cdots, y_{h+j}]]$-module, and $\bar{R}^\psi_S$ is a finite $\mathcal{O}$-module.

(III) The map $\pi : \bar{R}^{\square,\psi}_S \to \mathbb{T}^\square_\psi(U)_{\mathsf{m}}$ of Proposition 9.1 is surjective with 2*-power torsion kernel.*

Proof. Call $B = \bar{R}^{\square,\text{loc},\psi}_S$.

Let m be an integer > 0. We first give the definition of a patching data of level m. Our definition is a slight modification of the datas appearing in the proposition 3.3.1. of [33] and in the preceeding proposition.

As in prop. 3.3.1. we note $r_m = smp^m(h + j)$ and

$$\mathfrak{c}_m = (\pi_E^m, (1+y_1)^{p^m}-1, \ldots, (1+y_h)^{p^m}-1, y_{h+1}^{p^m}, \ldots, y_{h+j}^{p^m}) \subset \mathcal{O}[[y_1, \ldots, y_{h+j}]].$$

For r an integer and A a CNL$_\mathcal{O}$-algebra, we note $\mathsf{m}_A^{(r)}$ the ideal of A generated by r powers of elements of m_A. Let b be the number of generators of the maximal ideal of $B[[x_1, \ldots, x_{h+j+t-d}]]$ and let $r' = br$. For A a CNL$_\mathcal{O}$-algebra which is a quotient of $B[[x_1, \ldots, x_{h+j+t-d}]]$, we therefore have that $\mathsf{m}_A^{r'} \subset \mathsf{m}_A^{(r)}$. We note $r'_m = br_m$. Recall that G_n is the Galois group of the maximal abelian extension of F which is of degree a power of 2 and is split at S and unramified outside $S \cup Q_n$. Let us call G'_n the maximal quotient of G_n which is killed by 2^{n-2} ; by Lemma 5.10, we know that for $n > n_0$ G'_n is isomorphic to $(\mathbb{Z}/2^{(n-2)}\mathbb{Z})^t$. We choose an isomorphism from G'_n to $\mathfrak{t}/2^{n-2}\mathfrak{t}$. We get an isomorphism of diagonalizable groups between $(G'_n)^*$ and the 2^{n-2}-torsion $\mathfrak{T}_{2^{n-2}}$ of $\mathfrak{T}$. We allow us to enlarge n_0.

We let R and M to be $\bar{R}^{\square,\psi}_S$ and $S_{k,\psi}(U, \mathcal{O})_{\mathsf{m}} \otimes_{\bar{R}^\psi_S} \bar{R}^{\square,\psi}_S$ respectively.

Let m be an integer ≥ 3. Here is the definition of a patching data of level m denoted (D_m, L_m, D'_m).

- (1) A sequence of CNL$_\mathcal{O}$-algebras

  $$\mathcal{O}[[y_1, \ldots y_{h+j}]]/\mathfrak{c}_m \to D_m \to R/(\mathfrak{c}_m R + \mathsf{m}_R^{(r_m)})$$

  where the second map is a surjective map of B-algebras and $\mathsf{m}_D^{(r_m)} = (0)$. We ask that the $\mathcal{O}[[y_1, \ldots y_{h+j}]]$ action which it defines on $R/(\mathfrak{c}_m R + \mathsf{m}_R^{(r_m)})$ coincides with the action coming from the action on R (the action of $y_1, \ldots, y_h$ on R being trivial, and the action of $y_{h+1}, \ldots, y_{h+j}$ coming from the frame).

- (2) A surjection of B-algebras $B[[x_1,\dots,x_{h+j+t-d}]] \to D'_m$, with D'_m a $\mathrm{CNL}_{\mathcal{O}}^{[r'_m]}$-algebra (§2.6), a map d_m in the category of functors represented by $\mathrm{CNL}_{\mathcal{O}}$-algebras $\mathrm{Sp}_{D'_m} \to \mathfrak{T}$ and a closed immersion $\mathrm{Sp}_{D_m} \hookrightarrow \mathrm{Sp}_{D'_m}$, a group action chunk in $\mathrm{CNL}_{\mathcal{O}}^{[r'_m]}$ of $\mathfrak{T}_{2^m}$ on $\mathrm{Sp}_{D'_m}$ which is free, and such that $d_m(lx) = l^2 d_m(x)$ for $l \in \mathfrak{T}_{2^m}(A)$ and $x \in \mathrm{Sp}_{D'_m}(A)$ for A in $\mathrm{CNL}_{\mathcal{O}}^{[r'_m]}$. We ask that the closed immersion $\mathrm{Sp}_{D_m} \hookrightarrow \mathrm{Sp}_{D'_m}$ factors by the closed immersion $\mathrm{Sp}_{D''_m} \hookrightarrow \mathrm{Sp}_{D'_m}$ defined by $d_m(x') = 1$ and that the closed immersion $\mathrm{Sp}_{D_m} \hookrightarrow \mathrm{Sp}_{D''_m}$ has its ideal included in $\mathfrak{m}_{D''_m}^m$.
- (3) A D_m-module L_m which is finite free over $\mathcal{O}[[y_1,\dots,y_{h+j}]]/\mathfrak{c}_m$ of rank which we call s and a surjection of $B[[x_1,\dots,x_{h+j+t-d}]]$-modules $L_m \to M/\mathfrak{c}_m M$.

We suppose $n_0 + 1 \geq 3$. Let n be an integer $\geq n_0 + 1$. For m with $n \geq m \geq 3$, we now define patching datas $(D_{m,n}, L_{m,n}, D'_{m,n})$ of level m.

Let X_n and X'_n be the functors on $\mathrm{CNL}_{\mathcal{O}}$-algebras that are represented by $R_n := \bar{R}^{\square,\psi}_{S\cup Q_n}$ and $R'_n := \bar{R}^{\square}_{S\cup Q_n}$ respectively (§4.1.1) (with the notations of §2.1, $X_n = \mathrm{Sp}_{R_n}$ and $X'_n = \mathrm{Sp}_{R'_n}$).

As for p odd, the framing and the action of inertia groups at places in Q_n induce a structure of $\mathcal{O}[[y_1,\dots,y_{h+j}]]$-module on R_n. More precisely, let v_i for $1 \leq i \leq h$ the elements of Q_n. Recall that Δ'_i is the maximal 2-quotient of the multiplicative group of the residue field k_{v_i} and let δ_i a generator of Δ'_i. The action of y_i on R_n corresponds to the action of $\delta_i - 1$. Let us call M_n the R_n-module $S_{k,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}} \otimes_{\bar{R}^{\psi}_{S\cup Q_n}} \bar{R}^{\square,\psi}_{S\cup Q_n}$ where the module structure arises from the map $\bar{R}^{\psi}_{S\cup Q_n} \to \mathbb{T}_{\psi,Q_n}(U_{Q_n})_{\mathfrak{m}}$ of Proposition 9.1. Let $\mathfrak{b}_n \subset \mathcal{O}[[y_1,\dots,y_{h+j}]]$ the annihilator of M_n. We know by Cor. 7.5 that if a is a fixed integer such that $2^a > N$ where N is the integer appearing in §7.4, for $n > a$, M_n is finite free over $\mathcal{O}[[y_1,\dots,y_{h+j}]]/\mathfrak{b}_n$ of rank s independent of n and that :

$$\mathfrak{b}_n \subset ((1+y_1)^{2^{n-a}} - 1, \dots, (1+y_h)^{2^{n-a}} - 1).$$

We impose $a \geq 2$. Furthermore, M identifies to the quotient of M_n by the ideal $(y_1,\dots,y_h)$.

We take $D_{m,n} = R_{n+a}/\mathfrak{c}_m R_{n+a} + \mathfrak{m}_{R_{n+a}^{(r_m)}}$. and $L_{m,n} = M_{n+a}/\mathfrak{c}_m M_{n+a}$. As in [33], we see that we realize condition (1) and (3) (the only difference is the a).

Let us realize (2). We define $D'_{m,n} = R'^{[r'_m]}_{n+a}$.

By Lemma 5.10, we know that the number of generators of R'_n over B is $2h+1$. We also know by Lemma 5.10 that $t = 2 + h - |S|$, and we see that we have $2h+1 = h+j+t-d$. This gives a surjection of B-algebras $B[[x_1,\dots,x_{h+j+t-d}]] \to D'_{m,n}$.

As the number of generators of G_n is t, we can choose a surjective morphism from the free abelian group $\mathfrak{t}$ of rank t to G_n that is compatible with the chosen morphism $\mathfrak{t} \to G'_n$. It defines an immersion of diagonalizable groups $(G_n)^* \hookrightarrow \mathfrak{T}$ that is compatible with the already chosen immersion $(G'_n)^* \hookrightarrow \mathfrak{T}$. If to a lift ρ_A of $\bar\rho$ with values in a $\mathrm{CNL}_{\mathcal{O}}$-algebra A, one associates the Galois character $\det_{\rho_A} \times (\chi_p\psi)^{-1}$, one defines a map of functors from X'_n to the diagonalizable group $(G_n)^*$. Call it $d_{R'_n}$. With the above immersion $(G_n)^* \hookrightarrow \mathfrak{T}$, we can also see $d_{R'_n}$ as a map from X'_n to $\mathfrak{T}$. We define the map $d_m : \mathrm{Sp}_{D'_m} \to \mathfrak{T}$ as the map deduced by reduction from $d_{R'_{n+a}}$.

The surjection $R'_{n+a} \to R_{n+a}$ defines a surjection $D'_{m,n} \to D_{m,n}$. The immersion $X_n \hookrightarrow X'_n$ identifies X_n as the closed locus $d_{R'_n}(x') = 1$. As furthermore $D_{m,n} = R_{n+a}/(\mathfrak{c}_m R_{n+a} + \mathsf{m}_{R_{n+a}^{(r_m)}})$ and $r_m \geq m$ and $\mathfrak{c}_m \subset \mathsf{m}^m_{\mathcal{O}[[y_1,\dots y_{h+j}]]}$, we see that the closed immersion $\mathrm{Sp}_{D_{m,n}} \hookrightarrow \mathrm{Sp}_{D''_{m,n}}$ has its ideal included in $\mathsf{m}^m_{D''_{m,n}}$.

The twist defines an action of ${G'}^*_{n+a}$ on X'_{n+a} (§5.1), hence, as we have chosen $a \geq 2$ an action of $\mathfrak{T}_{2^n}$ on X'_{n+a}. By Lemma 5.1, one knows that this action is free. By truncation (§2.6), this action defines the group action chunk in $\mathrm{CNL}_{\mathcal{O}}^{[r'_m]}$ of $\mathfrak{T}_{2^n}$, hence of $\mathfrak{T}_{2^m}$, on $\mathrm{Sp}_{D'_{m,n}}$ which is free, and satisfies the claimed identity $d_{m,n}(l.x') = l^2 d_{m,n}(x')$ for $l \in \mathfrak{T}_{2^m}(A)$, $x' \in \mathrm{Sp}_{D'_{m,n}}(A)$ for any $\mathrm{CNL}_{\mathcal{O}}^{[r'_m]}$-algebra A.

We have realized (2).

We define an isomorphism of patching datas in the obvious way ([33]). As $B[[x_1,\dots,x_{h+j+t-d}]]^{[r'_m]}$ is finite, we see that there are only finitely many isomorphic classes of patching data. After extracting a subsequence of (n), we get a sequence (D_m, L_m, D'_m) of patching datas for $m >> 0$ such that for each m, we have for $n \geq m$, $(D_m, L_m, D'_m) \simeq (D_{m,n}, L_{m,n}, D'_{m,n})$. We note $R'_\infty = \lim_m D'_m$, $R_\infty = \lim_m D_m$ and $M_\infty = \lim_m L_m$. The d_m define the map d. The group chunk free actions of $\mathfrak{T}_{2^m}$ on $\mathrm{Sp}_{D'_m}$ define the free action of $\mathfrak{T}$ on X'_∞ by propositions 2.7 and 2.8.

We get (I). To check that X_∞ is defined in X'_∞ by $d(x') = 1$, we use that the maps of $B[[x_1,\dots,x_{h+j+t-d}]] \to D''_m \to D_m$ are surjective and that the second map has its kernel included in $\mathsf{m}^m_{D''_m}$, so that we have $\lim_m D''_m = \lim_m D_m$. We furthermore have isomorphisms of B and $\mathcal{O}[[y_1,\dots,y_{h+j}]]$-algebras and modules $R_\infty/(y_1,\dots,y_h)R_\infty \simeq R$ and of $M_\infty/(y_1,\dots,y_h)M_\infty \simeq M$.

Let us finish the proof of the proposition.

Let R_∞^{inv} be the algebra that represents the orbits for the action of $\mathfrak{T}$ on X'_∞ (prop. 2.5).

Lemma 9.4. *The natural map $R_\infty^{\mathrm{inv}} \to R_\infty$ makes R_∞ a torsor on R_∞^{inv} with group the 2-torsion $\mathfrak{T}_2$ of the torus $\mathfrak{T}$.*

Proof. Let $X'_\infty \times_{\mathfrak{T}} \mathfrak{T}$ be the fiber product of (ρ, λ) such that $d(\rho) = \lambda^2$. It is a torsor on $\mathrm{Sp}_{R_\infty^{\mathrm{inv}}}$ of group $\mathfrak{T} \times \mathfrak{T}_2$, $\mathfrak{T}$ acting diagonally on $X'_\infty \times \mathfrak{T}$ and $\mathfrak{T}_2$ acting trivially on X'_∞ and by translations on $\mathfrak{T}$. The map $(\rho, \lambda) \mapsto (\lambda^{-1}\rho, \lambda)$

identifies the fiber product to $X_\infty \times \mathfrak{T}$. In this identification, $\mathfrak{T}$ acts trivially on X_∞ and by translations on $\mathfrak{T}$. Taking the quotient by the action of $\mathfrak{T}$, we get by proposition 2.6 a free action of $\mathfrak{T}_2$ on X_∞ whose quotient identifies to $\mathrm{Sp}_{R_\infty^{\mathrm{inv}}}$. This proves the lemma.

□

Lemma 9.5. *The surjective map $B[[x_1,\dots,x_{h+j+t-d}]] \to R'_\infty$ is an isomorphism.*

Proof. We know that B is a domain of (relatively to $\mathcal{O}$) dimension d. If the map were not an isomorphism, the dimension of R'_∞ would be $< h+j+t$. It would follow from proposition 2.5 that the dimension of R_∞^{inv} would be $< h+j$. By the preceeding lemma, we would have that the dimension of R_∞ would be also $< h+j$. This is not possible as the action of $\mathcal{O}[[y_1,\dots,y_{h+j}]]$ on M_∞ is faithful and factorizes through R_∞. □

It follows that R'_∞ and R_∞^{inv} are domains flat over $\mathcal{O}$.

The action of R_∞^{inv} on M_∞ is faithful. Otherwise, as R_∞^{inv} is a domain and R_∞ is finite on R_∞^{inv}, the action of R_∞ on M_∞ would factor through a quotient of dimension $< h+j$, which is not the case as $\mathcal{O}[[y_1,\dots,y_{h+j}]]$ acts faithfully on M_∞ and its action factors through R_∞.

Lemma 9.6. *a) The rings $R_\infty[1/2]$ and $R_\infty^{\mathrm{inv}}[1/2]$ are regular.*

b) M_∞ is a faithful R_∞-module.

Proof. For a), as $B[1/2]$ is regular, $B[1/2][[x_1,\dots,x_{h+j+t-d}]]$ is regular. Furthermore, $R_\infty^{\mathrm{inv}} \to R'_\infty \simeq B[[x_1,\dots,x_{h+j+t-d}]]$ is formally smooth. It follows that $R_\infty^{\mathrm{inv}}[1/2]$ is regular (lemma (33.B) of [38]), and as $R_\infty^{\mathrm{inv}}[1/2] \to R_\infty[1/2]$ is etale, $R_\infty[1/2]$ is regular.

Let us prove b). As R_∞ and M_∞ are flat over $\mathcal{O}$, it suffices to prove that $M_\infty[1/2]$ is a faithful $R_\infty[1/2]$-module. As the action of R_∞^{inv} on M_∞ is faithful, the support Supp of the $R_\infty[1/2]$-module $M_\infty[1/2]$ contains a irreductible component of $\mathrm{Spec}(R_\infty[1/2])$. Since $R_\infty^{\mathrm{inv}}[1/2]$ is a regular domain, it follows from Lemma 9.4 that $\mathfrak{T}_2(\mathcal{O}) \simeq (\pm 1)^t$ acts transitively on these irreducible components. So we are reduced to prove that the Supp is stable by the action of $\mathfrak{T}_2(\mathcal{O})$.

For that, we recall that for $n > a$ we have an action of $\mathfrak{T}_2(\mathcal{O})$ by twists on the space of modular forms $S_{k,\psi}(U_{Q_n},\mathcal{O})_{\mathsf{m}}$ (cf paragraph after Proposition 7.6), hence on $M_n = S_{k,\psi}(U_{Q_n},\mathcal{O})_{\mathsf{m}} \otimes_{\bar{R}^{\psi}_{S\cup Q_n}} \bar{R}^{\square,\psi}_{S\cup Q_n}$. It is compatible with the action of twists on the Hecke algebra $\mathbb{T}_{\psi,Q_n}(U_{Q_n})_{\mathsf{m}}$, hence also with the action of twists on R_n. By proposition 5.12, the action of $\mathfrak{T}_2(\mathcal{O})$ on R_n is compatible with the action of $\mathfrak{T}_2(\mathcal{O})$ on $\mathcal{O}[[y_1,\dots,y_{h+j}]]$ given by $\chi.y_i = \chi(\delta_i)(1+y_i)-1$ for $1 \le i \le h$ and δ_i being a generator of Δ'_i, $\chi.y_i = y_i$ for $h+1 \le i \le h+j$ (the definition of Δ'_i was given in 5.6 and was recalled above in this proof). It follows that $\mathfrak{c}_m$ is stable by the action of $\mathfrak{T}_2(\mathcal{O})$ for $m \ge 2$, and we have compatible actions of $\mathfrak{T}_2(\mathcal{O})$ on $D_{m,n} = R_{n+a}/\mathfrak{c}_m R_{n+a} + \mathsf{m}_{R_{n+a}^{(r_m)}}$. and $L_{m,n} = M_{n+a}/\mathfrak{c}_m M_{n+a}$. The annihilator of the $D_{m,n}$ module $L_{m,n}$ is stable by $\mathfrak{T}_2(\mathcal{O})$, and, passing to the projective limit, we see that the support of the R_∞-module M_∞-module is stable by $\mathfrak{T}_2(\mathcal{O})$, hence also Supp. The lemma is proved. □

It follows that R_∞ injects into $\mathrm{End}_{\mathcal{O}[[y_1,\dots,y_{h+j}]]}(M_\infty)$, which is finite on $\mathcal{O}[[y_1,\dots,y_{h+j}]]$, and R_∞ is finite over $\mathcal{O}[[y_1,\dots,y_{h+j}]]$. It follows that $R_\infty/(y_1,\cdots,y_h)R_\infty \simeq \bar{R}_S^{\square,\psi}$ is finite over $\mathcal{O}[[y_{h+1},\dots,y_{h+j}]]$, and $\bar{R}_S^{\psi}$ is finite over $\mathcal{O}$. This proves (II).

Let us prove (III). We apply the lemma (3.3.4) of [33] (using Auslander-Buschsbaum theorem) to get that for each connected component D_i of $R_\infty[1/2]$, the corresponding submodule $M_\infty[1/2]_i$ of $M_\infty[1/2]$ is finite projective over D_i. The modules $M_\infty[1/2]_i$ are non zero as R_∞ acts faithfully on M_∞. It follows that $M_\infty[1/2]$ is a faithful finite projective module over $R_\infty[1/2]$. We deduce that $M_\infty[1/2]/(y_1,\dots,y_h)$ is a faithful $R_\infty[1/2]/(y_1,\dots,y_h) \simeq \bar{R}_S^{\square,\psi}[1/2]$-module. The quotient $M_\infty/(y_1,\dots,y_h)$ is isomorphic to $S_{k,\psi}(U,\mathcal{O})_{\mathsf{m}} \otimes_{\bar{R}_S^{\psi}} \bar{R}_S^{\square,\psi}$ and the image of $R_S^{\square,\psi}$ in the endomorphism ring of $M_\infty/(y_1,\dots,y_h)$ is the Hecke algebra $\mathbb{T}_\psi^{\square}(U)_{\mathsf{m}}$. We see that the natural surjective map $\bar{R}_S^{\square,\psi} \to \mathbb{T}_\psi^{\square}(U)_{\mathsf{m}}$ has 2-power torsion kernel. (III) and the proposition are proved.

□

9.2. Applications to modularity of Galois representations.

Theorem 9.7. *Let F be a totally real field unramified at p, split at p if $\bar{\rho}|_{D_p}$ is locally irreducible or $k(\bar{\rho}) = p+1$, such that $\bar{\rho}_F$ has non-solvable image when $p = 2$ and $\bar{\rho}|_{F(\mu_p)}$ absolutely irreducible for $p > 2$, and assume that $\bar{\rho}_F$ satisfies the assumptions $(\alpha), (\beta)$ if $p \neq 2$, (α) if $p = 2, k(\bar{\rho}) = 2$ and (β) if $p = 2$ (see §8.2). Consider a lift ρ_F of $\bar{\rho}_F$ ramified only at finitely many places, totally odd and such that at all places above p it satisfies one of the conditions (A), (B), (C) (see §8.3.3) when $p > 2$, and when $p = 2$ is either crystalline of weight 2 or semistable non-crystalline of weight 2 with the latter considered only when the residual representation is not finite at places above 2. Then ρ_F is modular.*

Proof. After an allowable base change F'/F, using Lemma 2.2 of [56] we may assume that $\rho_{F'} : G_{F'} \to \mathrm{GL}_2(\mathcal{O})$ is:

- totally odd
- unramified almost everywhere
- $(p = 2)$ is crystalline of weight 2 at all primes above 2, or semistable of weight 2 if the residual representation is not finite
- $(p > 2)$ at all primes above p (at which it has uniform behaviour), it is either simultaneously
  (A) crystalline of weight k, such that $2 \leq k \leq p+1$ and when $k = p+1$ we may assume $k(\bar{\rho}) = p+1$ and F' is split at p
  (B) of weight 2 and crystalline over $\mathbb{Q}_p^{\mathrm{nr}}(\mu_p)$, or

(C) semistable, non-crystalline of weight 2, and for $v|p$, $\rho_F|_{D_v}$ is of the form

$$\begin{pmatrix} \gamma_v\chi_p & * \\ 0 & \gamma_v \end{pmatrix},$$

where χ_p is the p-adic cyclotomic character, and γ_v an unramified character.

- at all (finite) places v not above p at which $\rho_{F'}$ is ramified, $\rho_{F'}|_{D_v}$ is of the form

$$\begin{pmatrix} \gamma_v\chi_p & * \\ 0 & \gamma_v \end{pmatrix},$$

with γ_v an unramified character.
- $\bar{\rho}_{F'}$ is trivial at places above p if $\bar{\rho}$ is unramified at p

Then $\rho_{F'}$ and $\det(\rho_{F'})$ automatically prescribe lifting data and character ψ. Thus $\rho_{F'}$ arises from a morphism $\bar{R}_S^{\square,\psi} \to \mathcal{O}$ with $\bar{R}_S^{\square,\psi}$ a ring of the type considered in Section 9.1. Then after another allowable base change as in Theorem 8.4, we may assume that there is a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_{F'})$ that is discrete series at infinity of parallel weight such that $\rho_{\pi'}$ and $\det(\rho_{\pi'})$ prescribe the same lifting data and character ψ as $\rho_{F'}$ and $\det(\rho_{F'})$. At this point we are done by Propositions 9.2 and 9.3, and solvable base change results of Langlands.

□

10. Proof of Theorems 4.1 and 5.1 of [32]

10.1. **Finiteness of deformation rings.** Consider $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ that we have fixed. (Recall that hence $2 \leq k(\bar{\rho}) \leq p+1$ when $p > 2$, and $\bar{\rho}$ has non-solvable image when $p = 2$ and $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ absolutely irerducible when $p > 2$.) Consider a finite set of places S of $\mathbb{Q}$ that contains p and ∞, and all the places at which $\bar{\rho}$ is ramified. We fix an arithmetic character $\psi : G_{\mathbb{Q}} \to \mathcal{O}^*$ such that $\psi\chi_p$ lifts the determinant of $\bar{\rho}$. For each $v \in S$ we consider deformation rings $\bar{R}_v^{\square,\psi}$ and assume them to be of one of the types considered in Theorem 3.1. Thus for instance at p the representations ρ_x arising from morphisms $x : \bar{R}_v^{\square,\psi} \to \mathcal{O}'$ are either of the type (A) (including the case $k(\bar{\rho}) = p = 2$), (B), or (C) (including the case $p = 2, k(\bar{\rho}) = 4$), and at all infinite places the deformations are odd. (See Section 8.3 for the conditions (A),(B),(C).) When we consider the case (C), we assume that when $p > 2$, we have that $k(\bar{\rho}) = p + 1$.

Consider the corresponding deformation ring $\bar{R}_S^{\square,\psi} = R_S^{\square,\psi} \hat{\otimes}_{R_S^{\square,\mathrm{loc},\psi}} \bar{R}_S^{\square,\mathrm{loc},\psi}$, and $\bar{R}_S^{\psi}$ the image of the universal deformation ring R_S in $\bar{R}_S^{\square,\psi}$.

We have the following corollary of Theorem 8.2, Theorem 6.1 and Propositions 9.2 and 9.3.

Theorem 10.1. *The ring $\bar{R}_S^{\psi}$ is finite as a $\mathbb{Z}_p$-module.*

In the theorem we are allowing $\bar{R}_v^{\square,\psi}$ for v not above p and finite to be any of the rings we have considered in Theorem 3.1, and thus they need not be domains.

Proof. We index for this proof the global deformation rings with the number fields whose absolute Galois group is being represented and thus denote $\bar{R}_{\mathbb{Q},S}^{\square,\psi} = \bar{R}_S^{\square,\psi}$ and $\bar{R}_{\mathbb{Q},S}^{\psi} = \bar{R}_S^{\psi}$.

To prove the finiteness of $\bar{R}_{\mathbb{Q},S}^{\psi}$ as a $\mathcal{O}$-module, we consider a number field F with the following properties. This exists because of the combined effect of Theorem 6.1 (which allows the assumptions (α) and (β) to be verified for $p > 2$, and (α) when $k(\bar{\rho}) = p = 2$ and (β) to be verified for $p = 2$), Theorem 8.2, and Lemma 4.2 of [30]:

- $F/\mathbb{Q}$ is a totally real extension, $\mathrm{im}(\bar{\rho}|_F)$ is non-solvable for $p = 2$ and $\bar{\rho}|_{F(\mu_p)}$ absolutely irreducible when $p > 2$, F is split at p if $\bar{\rho}|_{D_p}$ is irreducible, and unramified otherwise, and ψ_F is unramified at all finite places not above p. The last condition then gives that ψ_F is a character of G_F of the type fixed in Section 8.1.
- if $\bar{\rho}|_{D_p}$ is unramified then for all places $\wp$ of F above p, $\bar{\rho}|_{G_\wp}$ is trivial.
- Strengthened versions of (α) and (β) for $p > 2$, and (α) when $k(\bar{\rho}) = p = 2$ and (β) for $p = 2$, are satisfied:

  – Assume $k(\bar{\rho}) = 2$ when $p = 2$. Then $\bar{\rho}|_{G_F}$ arises from a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_F)$, which is unramified at all places, and is discrete series of weight $k(\bar{\rho})$ at the infinite places. The central character of π' is ψ_F. This π' is used below in the cases corresponding to (A).

  – $\bar{\rho}|_{G_F}$ also arises from a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_F)$, such that π'_v is unramified for all finite places v not above p, such that π'_v, at all places v above p, is of conductor dividing v, and is of weight 2 at the infinite places. The central character of π' is ψ_F. Further when $p = 2, k(\bar{\rho}) = 4$ we may ensure that the representation $\rho_{\pi'}$ at places v of F above 2 arises from $\bar{R}_v^{\square,\psi}$. This type of π' is used below in the cases corresponding to (B) and (C).

  When we need to consider weight $p+1$ liftings, we may assume by Theorem 6.1 and 8.2 that F is split at p.
- The reduction mod p of the universal representation $\bar{\rho}_{\mathbb{Q},S}^{\mathrm{univ}}$ associated to $\bar{R}_{\mathbb{Q},S}^{\psi}$, denoted by τ, when restricted to G_F is unramified outside the places above p and the infinite places. This condition is ensured by using Lemma 4.2 of [30] to see that for each of the finitely many primes $\ell_i \neq p$ at which τ is ramified, there is a finite extension F_{ℓ_i} of $\mathbb{Q}_{\ell_i}$ such that $\tau_{G_{F_{\ell_i}}}$ is unramified and choosing F as in part (c) of Theorem 6.1, such that a completion of F at ℓ_i contains F_{ℓ_i}.

Consider the deformation ring $\bar{R}_F^{\psi_F}$ over F that parametrises (minimal, odd) deformations of $\bar{\rho}_F$ unramified outside places above p and of determinant $\psi_F\chi_p$, and such that at all places above p suitable conditions, *i.e.* one of (A) (including the case $k(\bar{\rho}) = p = 2$), (B) or (C) (including the case $p = 2, k(\bar{\rho}) = 4$), are uniformly satisfied. (Thus the implied set of places S of ramification consists of places above p and the infinite places.)

The representation $\rho_{\pi'}$ prescribes lifting data (where the choice of π' depends on if we are in cases (A),(B) or (C)), and we are in a position to apply Propositions 9.2 and 9.3 to $\bar{R}_F^{\psi_F}$, and conclude that $\bar{R}_F^{\psi_F}$ is finite as a $\mathbb{Z}_p$-module.

As $\bar{\rho}$ and $\bar{\rho}_{|G_F}$ are absolutely irreducible, we have by functoriality CNL$_{\mathcal{O}}$-algebra morphisms $\pi_1 : R_F^{\psi_F} \to \bar{R}_F^{\square,\psi_F}$, $\pi_2 : R_{\mathbb{Q},S}^{\psi} \to \bar{R}_{\mathbb{Q},S}^{\square,\psi}$, and also $\beta : R_F^{\psi_F} \to R_{\mathbb{Q},S}^{\psi}$ and $\alpha : \bar{R}_F^{\square,\psi_F} \to \bar{R}_{\mathbb{Q},S}^{\square,\psi}$, with $\alpha\pi_1 = \pi_2\beta$. As $\pi_1(R_F^{\psi_F}) = \bar{R}_F^{\psi_F}$ and $\pi_2(R_{\mathbb{Q},S}^{\psi}) = \bar{R}_{\mathbb{Q},S}^{\psi}$, β induces a CNL$_{\mathcal{O}}$-algebra morphism $\gamma : \bar{R}_F^{\psi_F} \to \bar{R}_{\mathbb{Q},S}^{\psi}$. The morphism γ takes the universal mod p representation $G_F \to \mathrm{GL}_2(\bar{R}_F^{\psi_F}/(p))$ to the restriction to G_F of the universal mod p representation $\bar{\rho}_{\mathbb{Q},S}^{\mathrm{univ}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{R}_{\mathbb{Q},S}^{\psi}/(p))$. As the representation $G_F \to \mathrm{GL}_2(\bar{R}_F^{\psi_F}/(p))$ has finite image, we deduce that the universal mod p representation $\bar{\rho}_{\mathbb{Q},S}^{\mathrm{univ}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{R}_{\mathbb{Q},S}^{\psi}/(p))$ has finite image. From this we deduce, using 3.14 of [28] or Lemma 3.6 of [31], that $\bar{R}_S^{\psi} = \bar{R}_{\mathbb{Q},S}^{\psi}$ is finite as a $\mathbb{Z}_p$-module. □

10.2. **Proof of Theorem 4.1 of** [32]**.** We remark that some results towards Theorem 4.1 (1) are proved by Dickinson in [20].

Theorem 4.1 (2)(i) for weights $k \leq p-1$ is proven in [19]. The weight $k = p+1$ case is proved in [35] when the lift is non-ordinary at p: note that then residually the representation is irreducible at p of Serre weight 2 by results of Berger-Li-Zhu [3]. Also $\bar{\rho}$ does arise from a newform of level prime to p and weight $p+1$ by the weight part of Serre's conjecture together with multiplication by the Hasse invariant (see 12.4 of [29]), or Corollary 1 of Section 2 of [23].

The case of $k = p+1$ when the lifting is ordinary is treated in [53].

Theorem 4.1 (2) (ii) is proved by Kisin in [33] in the potentially Barsotti-Tate case. The semistable weight 2 case goes back to [62] and [60], and [15].

Thus we need only prove 4.1(1) and 4.1 (2)(i), and the latter only when $k = p$.

Consider a lift ρ of $\bar{\rho}$ as given in Theorem 4.1 of [32]. We are assuming that $\bar{\rho}$ is modular, and thus the assumptions (α) and (β) are fulfilled for $p > 2$, and (α) when $k(\bar{\rho}) = p = 2$ and (β) for $p = 2$, by the weight part of Serre's conjecture proven in [22] and Propositions 8.13 and 8.18 of [25]. (Note that we may assume that the hypothesis $N > 4$ from [25] is fulfilled as we do not care to show that $\bar{\rho}$ arises from optimal prime-to-p level. Also we do not need [25] when $k = p$ but see the remark below.)

At this point we are done by invoking Theorem 9.7.

Remark: The arguments here in fact treat in a self-contained manner all cases of Theorem 4.1(2)(i) except $k = p + 1$ with $k(\bar{\rho}) = 2$, and treats 4.1(2)(ii) when $\rho|_{\mathbb{Q}_p(\mu_p)}$ is semistable weight 2.

10.3. **Proof of Theorem 5.1 of** [32].

10.3.1. *Existence of p-adic lifts of the required type.* We have to first prove the existence of the p-adic deformation $\rho := \rho_p$ of $\bar{\rho}$ asserted in Theorem 5.1 of [32]. We call this a lifting of the *required type*: it has a certain determinant $\psi\chi_p$. The ring $\bar{R}_S^{\psi}$ is defined as in Section 10.1, and has the property that the $\mathcal{O}'$-valued points, for rings of integers $\mathcal{O}'$ of finite extensions of $\mathbb{Q}_p$, of its spectrum correspond exactly to the p-adic deformations of required type.

The existence of such points follows if we know that $\bar{R}_S^{\psi}$ is finite as a $\mathcal{O}$-module as then we may use Corollary 4.7. The finiteness of $\bar{R}_S^{\psi}$ as a $\mathcal{O}$-module follows from Theorem 10.1.

10.3.2. *Existence of compatible systems.* Now we explain how to propagate the lifts we have produced to an almost strictly compatible system as in [21] and [63]. For the definition of an "almost strictly compatible system" see 5.1 of part (I). In fact, in [63] we state that we can propagate to a strictly compatible system, which would follow from the statement of the corollary of the introduction of [36] without the hypothesis of irreducibility of the residual representation, which by a misunderstanding we thought to be unnecessary. When this hypothesis will be removed, we will get a strictly compatible system.

Consider the number field F and the cuspidal automorphic representation π' of the proof of Theorem 10.1. We may assume that $F/\mathbb{Q}$ is Galois which we do. Then Theorem 9.7 yields that $\rho_F := \rho|_{G_F}$ arises from a holomorphic, cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ with respect to the embedding ι_p.

The cuspidal automorphic representation π gives rise to a compatible system, see [57] and [6], such that each member is irreducible, see [58].

Let $G = \mathrm{Gal}(F/\mathbb{Q})$. Using Brauer's theorem we get subextensions F_i of F such that $G_i = \mathrm{Gal}(F/F_i)$ is solvable, characters χ_i of G_i (that we may also regard as characters of G_{F_i}) with values in $\overline{\mathbb{Q}}$ (that we embed in $\overline{\mathbb{Q}}_p$ using ι_p), and $n_i \in \mathbb{Z}$ such that $1_G = \sum_{G_i} n_i \mathrm{Ind}_{G_i}^{G} \chi_i$. Using the base change results of Langlands (as in the last paragraph of the proof of Theorem 2.4 of [56]), we get holomorphic cuspidal automorphic representations π_i of $\mathrm{GL}_2(\mathbb{A}_{F_i})$ such that if ρ_{π_i,ι_p} is the representation of G_{F_i} corresponding to π_i w.r.t. ι_p, then $\rho_{\pi_i,\iota_p} = \rho|_{G_{F_i}}$. Thus $\rho = \sum_{G_i} n_i \mathrm{Ind}_{G_{F_i}}^{G_{\mathbb{Q}}} \chi_i \otimes \rho_{\pi_i,\iota_p}$.

For any prime ℓ and any embedding $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$, we define the virtual representation $\rho_\iota = \sum_{G_i} n_i \mathrm{Ind}_{G_{F_i}}^{G_\mathbb{Q}} \chi_i \otimes \rho_{\pi_i,\iota}$ of $G_\mathbb{Q}$ with the χ_i's now regarded as ℓ-adic characters via the embedding ι. We check that ρ_ι is a true representation just as in the proof of Theorem 5.1 of [30]. For any prime number outside a finite set, the traces of Frob_q in ρ and ρ_ι coincide. It follows that, if F' is a subfield of F such that F/F' is solvable, and if $\pi_{F'}$ is the automorphic form associated to the restriction of ρ to $G_{F'}$, the restriction of ρ_ι to $G_{F'}$ is associated $\pi_{F'}$.

We prove the almost strictly compatibility of (ρ_ι). Let q be a prime number. Let $F^{(q)}$ be the subfield of F fixed by the decomposition subgroup of $\mathrm{Gal}(F/\mathbb{Q})$ for a chosen prime Q of F above q. Let π_q be the local component at Q of the automorphic form corresponding to the restriction of ρ to $G_{F^{(q)}}$. We define the representation r_q of the Weil-Deligne group WD_q as the Frobenius-semisimple Weil-Deligne parameter associated by the local Langlands correspondance to π_q.

Let ι be an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$. and call $r_{q,\iota}$ the Frobenius-semisimple Weil-Deligne parameter associated to the restriction of ρ_ι to D_q.

As the restriction of ρ_ι to $G_{F^{(q)}}$ is associated to $\pi_{F^{(q)}}$, it follows from Carayol and Taylor ([10] and [57]) that, if $q \neq \ell$, $r_{q,\iota}$ and r_q coincide.

If $q = \ell \neq 2$ and r_q is unramified, it follows from Breuil and Berger ([6],[2]) that $r_{q,\iota}$ and r_q coincide.

Let $q = \ell$ and suppose that $\bar{\rho}_\iota$ is irreducible. Let F' be totally real Galois finite extension of $\mathbb{Q}$ such that the restriction of ρ to $G_{F'}$ is associated to an automorphic form π' and F' is linearly disjoint from the field fixed by $\mathrm{Ker}(\bar{\rho}_\iota)$ ((iii) d) of Theorem 6.1). Let us define $F'^{(q)}$ and π'_q as above. As π_q and π'_q correspond to the restriction to D_q of $\rho_{\iota'}$ for ι' an embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}}_{\ell'}$ for $\ell' \neq \ell$, we see that π_q and π'_q are isomorphic. As the restriction of $\bar{\rho}_\iota$ to the Galois group of $F'^{(q)}$ is irreducible, it follows from Kisin that $r_{q,\iota}$ corresponds to π'_q, hence to π_q ([36]). This finishes the proof of the almost strict compatibility of (ρ_ι).

(3), computation of $k(\bar{\rho}_q)$: This follows from the almost strict compatibility and Corollary 6.15 (1) of Savitt's paper [49].

(4), computation of $k(\bar{\rho}_q)$: By the almost strict compatibility of (ρ_ι), and noting that the main theorem of Saito [48] applies to $\rho_q|_{G_{F^{(q)}}}$ (which does not need the irreducibility of $\bar{\rho}_q|_{G_{F^{(q)}}}$), using Corollary 6.15 (2) of Savitt's paper [49], and Serre's definition of weights (see Section 2 of [51]), we conclude the claimed information about $k(\bar{\rho}_q)$.

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