INTERTWINING OF RAMIFIED AND UNRAMIFIED ZEROS OF IWASAWA MODULES

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1. Introduction

Let F be a totally real field, $p > 2$ a prime, $F_\infty \subset F(\mu_{p^\infty})$ the cyclotomic $\mathbb{Z}_p$-extension of F with Galois group $\Gamma = \mathbb{Z}_p = \langle\gamma\rangle$. This short note is a sequel to [5]. The sole aim is to point out the ubiquity of a phenomenon dicussed in a particular case in our previous paper. Namely, the (arithmetic) eigenvalues of γ acting on Galois groups of maximal p-abelian *unramified* extensions of $F(\mu_{p^\infty})$ intertwine with the eigenvalues acting on inertia subgroups of *ramified* p-abelian extensions of $F(\mu_{p^\infty})$. We make this vague philosophy precise in the text below after alluding for *mise-en-scène* to the p-adic L-functions that lurk suggestively in the wings, but do not play an explicit role in the algebraic computations of this note.

Let ψ be an even Dirichlet character of F. Consider the p-adic L function $\zeta_{F,p}(s,\psi) = L_p(s,\psi)$, $s \in \mathbb{Z}_p$, which is characterised by the interpolation property $L_p(1-n,\psi) = L(1-n,\psi\omega^{-n})\Pi_{v|p}(1-\psi\omega^{-n}(v)N(v)^{n-1})$, for $n \geq 1$ a positive integer. When $\psi = 1$, we denote the corresponding L-function by $\zeta_{F,p}(s)$.

It is known that $L_p(s,\psi)$ is holomorphic outside $s = 1$, is holomorphic everywhere when $\psi \neq 1$, and otherwise has at most a simple pole at $s = 1$. This pole is predicted to exist by the conjecture of Leopoldt, which asserts the non-vanishing of the the p-adic regulator of units of F. The residue of $\zeta_{F,p}$ at $s = 1$ has been computed by Pierre Colmez:

$$\mathrm{Res}_{s=1}\zeta_{F,p}(s) = \frac{2^d R_{F,p} h_F}{2\sqrt{D_F}},$$

where $R_{F,p}$ is the p-adic regulator for F. The non-vanishing of $R_{F,p}$ is the Leopoldt conjecture for F and p.

We recall a folklore conjecture that is a very particular case of the general conjectures of Jannsen ([4]) about the non-vanishing of higher regulators.

Conjecture 1.1. *(Non-vanishing of higher p-adic regulators) For an integer $m \neq 0$, $L_{F,p}(m,\psi) \neq 0$ if either $m \neq 1$ or $\psi \neq 1$. Furthermore, $\zeta_{F,p}(s)$ has a pole at $s = 1$.*

CK was partially supported by NSF grants.
JPW is member of the Institut Universitaire de France.

As a supplement in the case $m = 0$, the case of "trivial zeros", the multiplicity of the zero at $s = 0$ of $L_p(s, \psi)$ is conjectured to be given by the number of $v|p$ such that $(1 - \psi\omega^{-1}(v)) = 0$.

We call the zeros of $\zeta_{F,p}(s)$ *unramified zeros* following a similar usage in [8].

The main conjecture of Iwasawa theory realises the zeros of the L-functions as roots of the characteristic polynomial of γ acting on "unramified arithmetic spaces". Nevertheless it gives no direct information about the zeros, belying the Hilbert-Polya philosophy in this case!

Our basic observation, coming from [5] that we reinforce here, is that the unramified zeros (of the p-adic L-function) always intertwine with (the eigenvalues of γ acting on) ramification at p. Further the Leopoldt zeros, i.e. eigenvalues of Γ that correspond on a finite index subgroup to its action on p-power roots of unity, intertwine with ramification at Q, for finite sets of primes away from p, for a *generic* choice of Q.

This shows that the non-vanishing of p-adic regulators is equivalent to splitting of ramification in naturally occuring exact sequences of Iwasawa modules. The reader is referred to Theorems 4.2, 4.6 and 4.7 for precise statements.

The proofs of these theorems rely on numerical coincidences between

– dimensions of certain Galois cohomology groups whose computation result from the work of Soulé and Poitou-Tate duality,

– dimensions of Iwasawa modules that follows from theorems of Iwasawa describing the structure of inertia at p (resp. at a set Q of auxiliary primes) in the Galois group of the maximal odd abelian p-extension of $F(\mu_{p^\infty})$ that is unramified outside p (resp. Q).

2. Galois cohomology

2.1. In our previous paper [5] we paid attention to integral questions, while here we work over $\mathbb{Q}_p$ exclusively. We consider F a totally real field, an odd prime p. We consider a sufficiently large finite extension K of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ (that will contain values of the character under consideration).

Let S_p be the set of places of F above p and ∞ and let S be a finite subset of places of F containing S_p. Let G_S be a Galois group of the maximal algebraic extension of F unramified outside S. We consider a potentially crystalline, or arithmetic, character χ of G_S, of (parallel) weight m and thus of the form $\eta\chi_p^m$ where χ_p is the p-adic cyclotomic character, and η a finite order character. We impose that $\chi(c)$ is independent of the choice of complex conjugation $c \in G_F$, and call it odd or even according as this value is either -1 or 1. We denote by ω the Teichmüller character. We consider the cohomology subgroup $H^1_{(p)f}(G_S, K(\eta\chi_p^m))$ of $H^1(G_S, K(\eta\chi_p^m))$ defined by imposing for $v \in S$, $v \notin S_p$ the condition to be unramified (although χ might be ramified at these v). We denote by $H^1_f(G_S, K(\eta\chi_p^m))$ their Bloch-Kato subgroups, where for v primes over p, we impose the Bloch-Kato finiteness condition ([2]). We denote the dimensions over K of $H^1_{(p)f}(G_S, K(\eta\chi_p^m))$ and $H^1_f(G_S, K(\eta\chi_p^m))$ by $h^1(\chi)$ and $h^1_f(\chi)$. We use $h^1_{split}(\chi)$ to denote dimensions of cohomology groups where we ask that the classes are split locally at all places above p and unramified at other primes. We have the tables.

2.2. Odd χ.

$\chi = \eta\chi_p^m$, χ *odd*	$m > 1$	$m = 1$	$m \leq 0$
$h^1(\chi)$	d	$d + \sum_{v\mid p} h^0(G_v, \eta^{-1}) - \delta_{\chi,\chi_p}$	d
$h^1_f(\chi)$	d	$d - \delta_{\chi,\chi_p}$	0

2.3. Even χ.

$\chi = \eta\chi_p^m$, χ *even*	$m > 1$	$m = 1$	$m \leq 0$
$h^1(\chi)$	0	$\sum_{v\mid p} h^0(G_v, \eta^{-1})$	$\delta_{\chi,id} + h^1_{split}(\chi_p\chi^{-1})$
$h^1_f(\chi)$	0	0	0

Remark. The situation for even χ is thus not satisfactory as we do not have an explicit formula in all cases for $h^1(\chi)$. The conjectures in [4], concerning non-vanishing of higher p-adic regulators, predict the vanishing of $h^1_{split}(\chi_p\chi^{-1})$ for χ an even arithmetic character of G_F (see also [1] 5.2). This follows from the main conjecture of Iwasawa theory if the weight of χ is > 0.

2.4. Ingredients of the computation. We justify the values in the tables. We need the following ingredients:

- (Bloch-Kato) $h^1_f(\chi) = \dim_K K(\chi)^{G_F} + \mathrm{ord}_{s=1-m} L(\eta^{-1}, s)$
- (global duality) χ even:

$$h^1(\chi) = h^1_{split}(\chi_p\chi^{-1}) + \delta_{\chi,id} + \sum_{v\mid p} h^0(G_v, \chi_p\chi^{-1})$$

- (global duality) χ odd:

$$h^1(\chi) = d + h^1_{split}(\chi_p\chi^{-1}) - \delta_{\chi,\chi_p} + \sum_{v\mid p} h^0(G_v, \chi_p\chi^{-1})$$

The (global duality) equalities follow from Theorem 8.7.9. of [6] and:

- (local Euler Poincaré characteristic)

  $$h^1(G_v, V) = h^0(G_v, V) + h^2(G_v, V) + [F_v : \mathbb{Q}_p]\dim_K(V)$$

  where $v\mid p$
- (local duality) $h^0(G_v, V^*(1)) = h^2(G_v, V)$

The Bloch-Kato formula, which directly implies the bottom row of both tables, follows from a theorem of Soulé (Theorem 1 of [7]) and duality as we justify.

– The theorem of Soulé directly implies Bloch-Kato formula for $h^1_f(\chi)$ for $m > 1$.

– The case of $m = 1$ for the bottom rows follows from Kummer theory.

– We have the equality for $m \leq 0$ and χ even:

$$h^1_f(\chi) = h^1_f(\chi_p \chi^{-1}) - d + \delta_{\chi,\mathrm{id}},$$

and for $m \leq 0$ and χ odd:

$$h^1_f(\chi) = h^1_f(\chi_p \chi^{-1}).$$

These equalities follow from Theorem 8.7.9 of [6] and the fact that for $m \leq 0$, $H^1_f(G_v, \chi)$ coincides with the unramified cohomology $H^1_{ur}(G_v, \chi)$. These two equalities allow to deduce the case $m \leq 0$ of the Bloch-Kato formula from the Theorem of Soulé (see also [1], §4.3.1).

This checks the second rows of both the tables.

Let us check the first row. For the first column in the case $m > 1$, we use the fact that $h^1(G_v, \chi) = h^1_f(G_v, \chi)$. The case $m \leq 0$ even χ follows from (global duality) as $h^0(G_v, \chi_p\chi^{-1}) = 0$ for all v above p. For the case $m = 1$, we use global duality and that $h^1_{split}(\eta^{-1}) = 0$ which follows easily from the fact that a $\mathbb{Z}_p$-extension of a number field has to be ramified at a place above p. For χ odd and $m \leq 0$, we use global duality and the fact that $h^1_{split}(\chi_p\chi^{-1}) \leq h^1_f(\chi_p\chi^{-1}) = 0$.

2.5. Galois groups. We consider ε totally odd and ψ totally even finite order characters of G_F, such that $\varepsilon\psi = \omega$. We set $\psi(n) = \psi(\omega^{-1}\chi_p)^n := \psi\kappa^n$, and likewise for $\varepsilon(n)$. We consider characters ψ_ζ of Γ that send a chosen generator γ of Γ to a p-power root of unity ζ and consider $\psi(n)\psi_\zeta$, $\varepsilon(n)\psi_\zeta$.

We assume that ψ is of type S, i.e. we assume that the field F_ψ cut out by ψ is linearly disjoint from the cyclotomic $\mathbb{Z}_p$-extension F_∞ of F. We denote by Γ the Galois group of F_∞/F with choice of generator γ, consider Λ the completed group algebra $\mathbb{Z}_p[[\Gamma]]$ that is isomorphic to $\mathbb{Z}_p[[T]]$ via the homomorphism which sends $\gamma \to 1 + T$.

We let $\Lambda_K = \Lambda \otimes K$. We consider the Galois group $\mathcal{G}_\psi = \mathcal{G}_\varepsilon = \mathrm{Gal}(F_\psi(\mu_p)/F) \times \mathrm{Gal}(F_\infty/F)$ of $F_\psi(\mu_{p^\infty})$ over F. A continuous character χ of $\mathcal{G}_\psi$ with values in $\mathcal{O}^*$ is the product of a finite character χ_ψ of $\mathrm{Gal}(F_\psi(\mu_p)/F)$ and a character χ_Γ of Γ. The character χ_Γ induces a K-algebra map $\Lambda_K \to K$. We denote by P_χ the corresponding prime ideal of Λ_K kernel of this map. It is generated by $\gamma - \chi(\gamma)$. The character χ_ψ induces a morphism $f_\chi : \mathbb{Z}_p[\mathrm{Gal}(F_\psi(\mu_p)/F)] \to \mathcal{O} \subset K$. The morphism $\mathbb{Z}_p[[\mathcal{G}_\psi]] \to K$ induced by χ is the composite of the map $\mathbb{Z}_p[[\mathcal{G}_\psi]] \to \Lambda_K$ induced by f_χ and the morphism $\Lambda_K \to K$.

We consider the maximal, abelian pro-p extension $\mathcal{L}_\infty$ of $F_\psi(\mu_{p^\infty})$ unramified everywhere: we denote its Galois group by X'_∞ and set $X_\infty = X'_\infty \otimes K$. We set $X_{\infty,\varepsilon}$ to be the maximal quotient of X_∞ on which $\mathrm{Gal}(F_\psi(\mu_p)/F)$ acts by ε.

We also consider the analogous extensions $\mathcal{L}_{\infty,\varepsilon}(Q)$ and corresponding Galois group $X_{\infty,\varepsilon,Q}$ when Q is any set of places of F and we allow ramification above Q. There are two natural cases to consider:

- Q is all the places above p, and then we replace Q by p in the notation;
- Q is a finite set of places disjoint from the places above p.

Let $F_{\psi,\infty}$ be the cyclotomic $\mathbb{Z}_p$-extension of the totally real field F_ψ. We denote by Y'_∞ the Galois group of the maximal abelian prop-p extension of $F_{\psi,\infty}$ that is unramified outside p, and $Y_\infty = \mathbb{Q}_p \otimes Y'_\infty$. As above, we denote by $Y_{\infty,\psi}$ the related quotient on which $\mathrm{Gal}(F_\psi(\mu_p)/F)$ acs by ψ, and recall the perfect Γ-equivariant Iwasawa pairing

$$Y_{\infty,\psi} \times X_{\infty,\varepsilon} \to K(1).$$

Lemma 2.1. *Let ψ be an even character of G_F of type S and $n \in \mathbb{Z}$.*

(1) *We have $h^1(\psi_\zeta\psi(n)) = \dim_K(Y_{\infty,\psi}/P_{\psi_\zeta\psi(n)})$ if $\psi_\zeta\psi(n)$ is not trivial, and $h^1(\psi_\zeta\varepsilon(n)) = \dim_K(X_{\infty,\varepsilon,p}/P_{\psi_\zeta\varepsilon(n)})$.*

(2) *We have that*

$$\dim_K((X_{\infty,\varepsilon})_{P_{\psi_\zeta\varepsilon(-n)}}/P_{\psi_\zeta\varepsilon(-n)}) = \dim_K((Y_{\infty,\psi})_{P_{\psi_\zeta^{-1}\psi(n+1)}}/P_{\psi_\zeta^{-1}\psi(n+1)}).$$

Proof. For (i) we use the inflation-restriction sequence relative to $G_S \to \mathcal{G}_\psi$. Recall that the unramified condition for $\eta \in H^1_{(p)f}(G_S, K(\varepsilon\chi_p^m))$ at v not above p and such that ψ is ramified at v (section 2.1). We check that this condition is equivalent to that the restriction of η to the kernel of the map $G_S \to \mathcal{G}_\psi$ is unramified at v.

For (ii), we invoke the pairing of Iwasawa. We use the cyclicity of Γ to identify dimensions of twisted invariants and covariants for Γ.

□

We rederive a standard result about trivial zeros of p-adic L-functions, usually proved using genus theory, which is a corollary to the lemma.

Proposition 2.2. *Suppose ε is an odd character of* $\mathrm{Gal}(F_\psi(\mu_p)/F)$ *as before. Then*

$$\dim_K((X_{\infty,\varepsilon})_{P_{\psi_\zeta\varepsilon}}/P_{\psi_\zeta\varepsilon}) = h^1(\chi_p\psi_\zeta^{-1}\varepsilon^{-1}) = \sum_{v|p} h^0(G_v, \psi_\zeta^{-1}\varepsilon^{-1}).$$

Proof. We deduce the first equality using (2) of earlier lemma for $n = 0$ and the second equality using the above table for $m = 1$ even χ. □

Note that $\sum_{v|p} h^0(G_v, \psi_\zeta^{-1}\varepsilon^{-1})$ is the number of places $v|p$ such that the Euler factor $(1 - \psi_\zeta^{-1}\varepsilon^{-1}(v))$ is 0. It is conjectured by Greenberg that the trivial zeros occur semisimply i.e. $\dim_K(X_{\infty,\varepsilon}/P_{\psi_\zeta\varepsilon}) = \dim_K(X_{\infty,\varepsilon})_{P_{\psi_\zeta\varepsilon}}$.

3. Main conjecture and higher regulators

We consider the Deligne-Ribet p-adic L-function $\zeta_{F,p}(s,\psi)$ for an even character ψ of F of type S. It is defined on $\mathbb{Z}_p$ when ψ is non-trivial, and on $\mathbb{Z}_p\backslash\{1\}$ when ψ is trivial. It is characterised by the interpolation formula that for integers $n \geq 1$,

$$\zeta_{F,p}(1-n,\psi) = L^p(1-n,\psi\omega^{-n}),$$

where the superscript denotes that we have dropped the Euler factors at p. There is a power series $W_\psi(T)$ in $\Lambda_K = \mathbb{Z}_p[[\Gamma]] \otimes K$ (the latter by the isomorphism that sends a chosen generator γ of $\Gamma = \mathrm{Gal}(F_\infty/F)$ to $1+T$ and $u := \chi_p(\gamma)$), with the property that:

$$\frac{W_\psi(u^s-1)}{u^{1-s}-1} = \zeta_{F,p}(s,\psi)$$

when ψ is trivial, and

$$W_\psi(u^s-1) = \zeta_{F,p}(s,\psi)$$

otherwise. Furthermore we have :

$$W_{\psi\psi_\zeta}(T) = W_\psi(\zeta^{-1}(1+T)-1);$$

see the introduction of [8], where the notation is $G_\psi(T)$ for $W_\psi(u(1+T)^{-1}-1)$.

Then the main conjecture asserts for characters ψ of type S that

$$(W_\psi(T)) = \mathrm{char}_{\Lambda_K}(X_{\infty,\psi^{-1}\omega}),$$

i.e. the characteristic polynomial of the action of γ on the finite dimensional K-vector space $X_{\infty,\varepsilon}$ generates the same ideal as $W_\psi(T)$. (We ignore μ-invariants via this formulation.)

When m is an integer $\neq 0$, Conjecture 1.1 is equivalent via the main conjecture to the statement that the generalised ζu^m-eigenspace of $X_{\infty,\psi^{-1}\omega} \otimes \mathbb{Q}_p$ for the action of γ is trivial . When $m=0$, via the main conjecture, we have that the generalised ζ-eigenspace is of dimension given by the number of $v|p$ of F such that $(1-\psi_\zeta^{-1}\varepsilon^{-1}(v))=0$

4. Intertwining of ramified and unramified zeros

4.1. Ramification at p. We consider the exact sequences:

$$0 \to I_Q \to X_{\infty,\varepsilon,Q} \to X_{\infty,\varepsilon} \to 0,$$

of finitely generated Λ_K-modules with $Q=p$ (see 2.5).

4.1.1. *Ramification at p and intertwining with non-trivial unramified zeros.* Next theorem follows from Th. 25 of [3] :

Theorem 4.1. *We have an ismorphism of Λ_K-modules*

$$I_p = \frac{\{\Lambda_K^d \bigoplus \oplus_{j=1}^s \mathrm{Ind}_{G_{\wp_j}}^{\Gamma} K(1)\}}{K(1)},$$

where $G_\wp$ are the decomposition groups of the places $\wp$ above p in Γ.

We deduce from this and the computations in Galois cohomology earlier:

Theorem 4.2. *Let $\chi = \varepsilon\psi_\zeta\kappa^m$ be an odd arithmetic character of $\mathcal{G}_\varepsilon$ of weight m, and let P_χ be the corresponding prime ideal of $\Lambda_{\mathcal{O}}$ for $\mathcal{O}$ such that χ is valued in $\mathcal{O}^*$. Then the exact sequence*

$$0 \to (I_p)_{P_\chi} \to (X_{\infty,\varepsilon,p})_{P_\chi} \to (X_{\infty,\varepsilon})_{P_\chi} \to 0,$$

of finitely generated Λ_K-modules splits if and only if $(X_{\infty,\varepsilon})_{P_\chi}$ vanishes. This is equivalent to that $\zeta_{F,p}(m, \psi_\zeta^{-1}\varepsilon^{-1}\omega) \neq 0$ when $\chi \neq \chi_p$, and when $\chi = \chi_p$, to that $\zeta_{F,p}(s)$ has a pole at $s = 1$.

Proof. One direction is trivial. For the other direction, one notes by the theorem of Iwasawa that $\dim_K((I_p)_{P_\chi}/P_\chi)$ is d if $m \neq 1$ and $d + \sum_{v|p} h^0(G_v, \varepsilon^{-1}) - \delta_{\chi,\chi_p}$ if $m = 1$. We have the numerical coincidence :

$$\dim_K((I_p)_{P_\chi}/P_\chi) = h^1(\chi).$$

Further one notes from Lemma 2.1 that $h^1(\chi) = \dim_K(X_{\infty,\varepsilon,p})_{P_\chi}/P_\chi$. Thus if the exact sequence in the theorem splits, we deduce that $(X_{\infty,\varepsilon})_{P_\chi}/P_\chi = 0$, which is equivalent to the vanishing of $(X_{\infty,\varepsilon})_{P_\chi}$.

By the main conjecture proved by Wiles, the vanishing of $(X_{\infty,\varepsilon})_{P_\chi}/P_\chi$ is equivalent to the vanishing of W_ψ at $\zeta u^m - 1$. It is equivalent to the vanishing of $\zeta_p(m, \psi\psi_\zeta^{-1}) = \zeta_{F,p}(m, \psi_\zeta^{-1}\varepsilon^{-1}\omega)$. □

Remarks.

1. We also deduce that an equivalent formulation of the non-vanishing of the higher regulator conjecture is the conjecture that the exact sequence

$$0 \to (I_p)_{P_\chi} \to (X_{\infty,\varepsilon,p})_{P_\chi} \to (X_{\infty,\varepsilon})_{P_\chi} \to 0,$$

of finitely generated Λ_K-modules splits for *all* odd arithmetic characters χ of $\mathcal{G}$ of non-zero weight.

2. As trivial zeros (in weight 0) do occur we get examples of Iwasawa modules in which ramification is allowed at p such that γ acts non-semisimply.

We have a conditional result for any character $\chi = \varepsilon\kappa^s$ of $\mathcal{G}_\varepsilon$ not necessarily arithmetic, and that follows by similar arguments.

Proposition 4.3. *Let $\chi = \varepsilon\kappa^s$ be any odd character of $\mathcal{G}_\varepsilon$. Assume further that $H^1_{split}(G_{F,p}, \chi_p\chi^{-1})$ is trivial. Then the exact sequence*

$$0 \to (I_p)_{P_\chi} \to (X_{\infty,\varepsilon,p})_{P_\chi} \to (X_{\infty,\varepsilon})_{P_\chi} \to 0,$$

of compact finitely generated Λ_K-modules splits if and only if $(X_{\infty,\varepsilon})_{P_\chi} = 0$.

Remark: Note that the vanishing of $H^1_{split}(G_{F,p}, \chi_p\chi^{-1}) = 0$ is predicted by Greenberg's conjecture that the p-part of the the class group of the cyclotomic $\mathbb{Z}_p$-extension F_∞ of F is finite. The above proposition suggests that there may be a formulation of the main conjecture using $\mathrm{Ext}_{\Lambda_K}(X_{\infty,\varepsilon}, I_p)$.

4.2. Ramification away from p.

4.2.1. *ψ and ψ_ζ trivial.* Consider the maximal abelian p-extension L_∞ of F, and denote its $\mathbb{Z}_p$-rank by $1+\delta$. The Leopoldt conjecture asserts that $\delta = 0$.

Definition 4.4. *(generic sets Q) We say that a finite set of primes Q of cardinality r away from p is generic if the rank r_Q of the subgroup generated by the* Frob_q*'s for $q \in Q$ in* $\mathrm{Gal}(L_\infty/F)$ *is* $\min(r, 1+\delta)$.

The terminology is meant to reflect the fact that when $\delta > 0$, the $\mathrm{Frob}_{q_1}, \mathrm{Frob}_{q_2}$ will be linearly independent in $\mathbb{Z}_p^{1+\delta} = \mathrm{Gal}(L_\infty/F)$ for most choices of q_1, q_2. If $r = 2$ and we choose a prime q_1 freely, then for a density one set of primes q_2, the set $Q = \{q_1, q_2\}$ is generic.

We now show the intertwining of the Leopoldt zero $u = u_\gamma$ with the ramification at Q provided Q is a generic set of primes with $|Q| > 1$.

Proposition 4.5. *Let Q be a finite set of primes of F away from p. Then the subgroup I_Q of $X_{\infty,\omega,Q}$ generated by the conjugacy class of the inertia groups I_q for $q \in Q$ is isomorphic as a* $\mathrm{Gal}(\mathcal{F}_\infty/F) = \mathrm{Gal}(F(\mu_p)/F) \times \Gamma$*-module to*

$$\frac{(\Pi_{q\in Q}\mathrm{Ind}_{G_q}^{\Gamma} K(1))}{K(1)}$$

where G_q is the decomposition subgroup at q in $\Gamma = \mathrm{Gal}(F_\infty/F)$*, and where we declare that* $\mathrm{Gal}(F(\mu_p)/F)$ *acts by ω.*

Proof. This follows from class field theory. □

Note that when the primes q are inert in F_∞/F then $I_Q = K(1)^{r-1}$.

Theorem 4.6. *Let Q be generic set of primes Q of cardinality $r \geq 2$. Then the exact sequence*

$$0 \to (I_Q)_{P_{\chi_p}} \to (X_{\infty,\omega,Q})_{P_{\chi_p}} \to (X_{\infty,\omega})_{P_{\chi_p}} \to 0$$

splits if and only if $(X_{\infty,\omega})_{P_{\chi_p}} = 0$, i.e., if and only if the Leopoldt conjecture is true.

Thus if there is a Leopoldt zero, then it intertwines with the ramified zeros at Q for a generic set of primes (away from p) with $|Q| \geq 2$.

Proof. If Leopoldt conjecture is true, $(X_{\infty,\omega})_{P_{\chi_p}} = 0$ and the exact sequence splits.

Let us prove the converse.

For a finite dimensional vector space V over K endowed with a continuous action of G_F that is unramified outside a finite set of places, and a set of Selmer conditions $\mathcal{L} = \{\mathcal{L}_v\}$ for $\mathcal{L}_v \subset H^1(F_v, V)$ where $\mathcal{L}_v$ is outside a finite set of places the unramified subgroup, we have the formula :

$$h^1_{\mathcal{L}}(F,V) - h^1_{\mathcal{L}^\perp}(F,V^*(1)) = h^0(F,V) - h^0(F,V^*(1)) + \sum_v (\dim_K \mathcal{L}_v - h^0(F_v,V)).$$

We apply this formula for $V = K(1)$, and with the Selmer conditions $\mathcal{L}$ to be unramified everywhere. In particular the Selmer condition is trivial at places v above p as $V^{I_v} = 0$ for v above p. We get :

$$h^1_{\mathcal{L}}(F,V) - h^1_{\mathcal{L}^\perp}(F,V^*(1)) = -1.$$

Furthermore, we have $h^1_{\mathcal{L}^\perp}(F,K) = 1 + \delta$.

Consider the Selmer conditions $\mathcal{L}_Q$ that arise when we allow ramification at Q, i.e., $(\mathcal{L}_Q)_v = \mathcal{L}_v$ for $v \notin Q$ and $(\mathcal{L}_Q)_v = H^1(G_v, K(1))$ for $v \in Q$. We get :

$$h^1_{\mathcal{L}_Q}(F,V) - h^1_{\mathcal{L}_Q^\perp}(F,V^*(1)) = -1 + r.$$

Furthermore, we have $h^1_{\mathcal{L}_Q^\perp}(F,K) = 1 + \delta - r_Q$. We see that :

$$h^1_{\mathcal{L}_Q}(F,K(1)) = h^1_{\mathcal{L}}(F,K(1)) + r - r_Q.$$

If the exact sequence splits, it remains exact after reduction modulo P_{χ_p}, hence we have :

$$h^1_{\mathcal{L}_Q}(F,K(1)) = h^1_{\mathcal{L}}(F,K(1)) + r - 1.$$

Thus we get $r_Q = 1$. As Q is generic and $\mid Q \mid \geq 2$, i.e., $r_Q = \min(r, 1+\delta)$ with $r \geq 2$, we get that $1 + \delta = 1$, thus $\delta = 0$ and Leopoldt conjecture is true.

□

4.2.2. *Weight 1, ψ or ψ_ζ non-trivial.*

Theorem 4.7. *Consider the exact sequence*

$$0 \to (I_q)_{P_{\psi_\zeta \varepsilon \kappa}} \to (X_{\infty,\varepsilon,q})_{P_{\psi_\zeta \varepsilon \kappa}} \to (X_{\infty,\varepsilon})_{P_{\psi_\zeta \varepsilon \kappa}} \to 0.$$

Assume $\varepsilon \neq \omega$ or that $\zeta \neq 1$. Then the sequence splits for all choices of primes q of F away from p, if and only if $(X_{\infty,\varepsilon})_{P_{\psi_\zeta \varepsilon \kappa}} = 0$.

Proof. We only sketch the proof as its very similar to the proof of Theorem 4.6.

By 2) of lemma 2.1, if $(X_{\infty,\varepsilon})_{P_{\psi_\zeta \varepsilon \kappa}} \neq 0$, the maximal abelian p-extension L of $F_{\varepsilon^{-1}\psi_\zeta^{-1}\omega} = F_{\psi\psi_\zeta^{-1}}$ on which $\mathrm{Gal}(F_{\psi\psi_\zeta^{-1}}/F)$ acts by $\psi\psi_\zeta^{-1}$ and which is unramified outside p, has Galois group that is of positive rank as a $\mathbb{Z}_p$-module. Let q be a prime of F away from p, that splits in $F_{\psi\psi_\zeta^{-1}}$, such that the Frobenius at a prime above q of $F_{\psi\psi_\zeta^{-1}}$ is a non-torsion element in $\mathrm{Gal}(L/F_{\psi\psi_\zeta^{-1}})$. With $V = K(\psi_\zeta \varepsilon \kappa)$ and the Selmer conditions as above, it follows that $h^1_{\mathcal{L}^\perp} - h^1_{\mathcal{L}_q^\perp} = 1$.

Using the above formula, we get $h^1_{\mathcal{L}_q} - h^1_{\mathcal{L}_q^\perp} - h^1_{\mathcal{L}} + h^1_{\mathcal{L}^\perp} = 0$. If the exact sequence were to split, we would have $h^1_{\mathcal{L}_q} = h^1_{\mathcal{L}}$, which is a contradiction. □

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