COBORDISM OF ALGEBRAIC KNOTS DEFINED BY BRIESKORN POLYNOMIALS, II

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Abstract. In our previous paper, we obtained several results concerning cobordisms of algebraic knots associated with Brieskorn polynomials: for example, under certain conditions, we showed that the exponents are cobordism invariants. In this paper, we further obtain new results concerning the Fox–Milnor type relations, decomposition of the algebraic cobordism class of an algebraic knot associated with a Brieskorn polynomial that has a null-cobordant factor over the field of rational numbers, and cyclic suspensions of knots. As a corollary, we show that a spherical algebraic knot associated with a Brieskorn polynomial has infinite order in the knot cobordism group.

1. Introduction

Let $f : (\mathbf{C}^{n+1}, \mathbf{0}) \to (\mathbf{C}, 0)$, $n \geqslant 1$, be a holomorphic function germ with an isolated critical point at the origin. For a sufficiently small positive real number $\varepsilon > 0$, set $K_f = S^{2n+1}_\varepsilon \cap V_f$, where $V_f = f^{-1}(0)$ is the complex hypersurface in $\mathbf{C}^{n+1}$ with an isolated singularity at the origin and S^{2n+1}_ε is the sphere of radius ε centered at the origin in $\mathbf{C}^{n+1}$ (see Fig. 1). It is known that K_f is an $(n-2)$–connected, oriented $(2n-1)$–dimensional submanifold of $S^{2n+1}_\varepsilon = S^{2n+1}$, that its complement fibers over the circle S^1, and that the isotopy class of K_f in S^{2n+1} is independent of the choice of ε as long as it is sufficiently small (see [16]). Note also that the embedded topology of $V_f \subset \mathbf{C}^{n+1}$ around the origin determines and is determined by the (oriented) isotopy class of $K_f \subset S^{2n+1}$ (see [18]). We call K_f the *algebraic knot* associated with f. In this paper, a *knot* refers to (the isotopy class of) an $(n-2)$–connected, oriented $(2n-1)$–dimensional submanifold in S^{2n+1}.

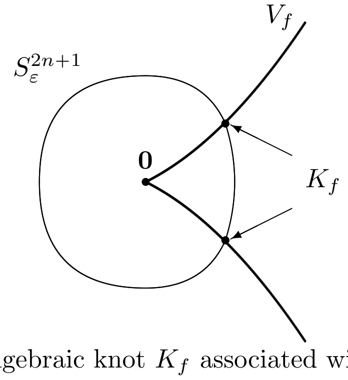

Figure 1. The algebraic knot K_f associated with the singularity at $\mathbf{0}$ of a germ f

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In this paper, we consider Brieskorn polynomials

$$f(z_1, z_2, \dots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}} \tag{1.1}$$

with exponents $a_i \geqslant 2$, $1 \leqslant i \leqslant n+1$, and their associated algebraic knots K_f [4]. We especially focus on the study of their properties concerning cobordisms. Two knots K_0 and K_1 in S^{2n+1} are said to be *cobordant* if there exists a properly embedded oriented submanifold X, diffeomorphic to $K_0 \times [0,1]$, of $S^{2n+1} \times [0,1]$ such that $X \cap (S^{2n+1} \times \{0\}) = K_0$, and $X \cap (S^{2n+1} \times \{1\}) = -K_1^!$, where $-K_1^!$ is the mirror image of K_1 with reversed orientation (see [1, 2]).

In our previous paper [3], we obtained several results concerning cobordisms of algebraic knots associated with Brieskorn polynomials: for example, under certain conditions, we showed that the exponents are cobordism invariants. In this paper, we further obtain new results concerning the Fox–Milnor type relations, decomposition of the algebraic cobordism class of an algebraic knot associated with a Brieskorn polynomial that has a null-cobordant factor over the field of rational numbers, and cyclic suspensions of knots.

The present paper is organized as follows. In §2, we recall several basic definitions and properties concerning invariants and cobordisms of algebraic knots such as Alexander polynomials and Seifert forms.

In §3, we focus on the Fox–Milnor type relations for Alexander polynomials [8, 9] and give a characterization of Brieskorn polynomials that give algebraic knots whose Alexander polynomials satisfy the Fox–Milnor type relation in terms of their exponents. As a consequence, we show that an algebraic knot associated with a Brieskorn polynomial is never null-cobordant: in fact, it turns out that a spherical algebraic knot associated with a Brieskorn polynomial always has infinite order in the knot cobordism group. As far as the authors know, this is the first explicit example of a family of spherical algebraic knots which are not null-cobordant and which have infinite order in the knot cobordism group.

In §4, we consider the linear independence of spherical algebraic knots associated with certain Brieskorn polynomials in the knot cobordism group. In fact, Litherland [14] has shown that the spherical algebraic knots in S^3 associated with Brieskorn polynomials of two variables are linearly independent in the 1–dimensional knot cobordism group by using a certain signature invariant. We will use the same idea to prove a similar linear independence result for higher dimensions.

In §5, we consider the group of algebraic cobordism classes of spherical knots which has been introduced and studied by Levine [12, 13]. We give an explicit example of a spherical algebraic knot associated with a Brieskorn polynomial such that its algebraic cobordism class has a decomposition into those corresponding to the irreducible factors of its Alexander polynomial over the field of rational numbers and that one of them is algebraically null-cobordant. This shows that cobordant spherical algebraic knots associated with Brieskorn polynomials may not share the same irreducible factors of their Alexander polynomials, and therefore the study of cobordism classes of algebraic knots associated with Brieskorn polynomials might be more complicated than expected.

Finally in §6, we consider cyclic suspensions of knots [17] and study its relationship to the cobordisms. Note that the algebraic knot associated with a polynomial of the form $f(z_1, z_2, \dots, z_{n+1}) + z_{n+2}^d$ is the d–fold cyclic suspension of the algebraic knot associated with f. We will see that the cyclic suspension of knots often behaves very badly with respect to cobordisms. For example we show that certain cyclic suspensions of the algebraic knots constructed by Du Bois–Michel in [6], which are cobordant to each other, are not diffeomorphic and are not cobordant.

Throughout the paper, all manifolds and maps between them are smooth of class C^∞.

2. Preliminaries

Let K be a $(2n-1)$–dimensional knot in S^{2n+1}. Suppose that there exists a locally trivial fibration $\varphi : S^{2n+1} \smallsetminus K \to S^1$. We also assume that there is a trivialization $\tau : N(K) \to K \times D^2$ of the normal disk bundle neighborhood $N(K)$ of K in S^{2n+1} such that the composition

$$N(K) \smallsetminus K \xrightarrow{\tau|_{N(K)\smallsetminus K}} K \times (D^2 \smallsetminus \{0\}) \xrightarrow{pr_2} D^2 \smallsetminus \{0\} \xrightarrow{r} S^1$$

coincides with $\varphi|_{N(K)\smallsetminus K}$, where pr_2 is the projection to the second factor and r is the radial projection. Then, we say that K is a *fibered knot*. We call the closure F of a fiber of φ a *fiber*. Note that it is a $2n$–dimensional compact oriented submanifold of S^{2n+1} whose boundary coincides with K. A $(2n-1)$–dimensional fibered knot K is *simple* if it is $(n-2)$–connected and F is $(n-1)$–connected. (Here, for $n = 1$, a manifold is (-1)–connected if it is nonempty.) In this case, F is homotopy equivalent to a bouquet of n–dimensional spheres. Note that an algebraic knot associated with a holomorphic function germ $f : (\mathbf{C}^{n+1}, \mathbf{0}) \to (\mathbf{C}, 0)$ with an isolated critical point at the origin is a simple fibered knot [16]. In this case, a fiber of such an algebraic knot is called the *Milnor fiber* for f.

Let $\psi : F \to F$ be a *geometric monodromy* of the fibration φ; i.e., it is a diffeomorphism which is constructed by integrating an appropriate horizontal vector field on $S^{2n+1} \smallsetminus K$ with respect to φ and which is the identity on the boundary. In other words, $S^{2n+1} \smallsetminus \operatorname{Int} N(K)$ is diffeomorphic to the manifold

$$F \times [0,1]/(x,1) \sim (\psi(x), 0), \quad x \in F.$$

The geometric monodromy is well-defined up to isotopy. The isomorphism

$$\psi_* : H_n(F;\mathbf{Z}) \to H_n(F;\mathbf{Z})$$

is called the *algebraic monodromy*. Its characteristic polynomial $\Delta_K(t)$ is often called the *Alexander polynomial* of K. When K is an algebraic knot associated with a holomorphic function germ f, we often denote $\Delta_{K_f}(t)$ by $\Delta_f(t)$.

Let us consider the multiplicative group $\mathbf{C}^*$ and its group ring $\mathbf{QC}^*$ over the field of rational numbers. For a monic polynomial $\Delta(t)$ with nonzero constant term, we denote by $\operatorname{div}\Delta$ the element

$$\sum m_\xi \langle \xi \rangle \in \mathbf{QC}^*,$$

where ξ runs over all roots of $\Delta(t)$ and m_ξ is its multiplicity. We also denote

$$\Lambda_a = \operatorname{div}(t^a - 1)$$

for a positive integer a. Now, let us consider a Brieskorn polynomial as in (1.1). Then, by Brieskorn [4], it is known that

$$\operatorname{div}\Delta_f = (\Lambda_{a_1} - 1)(\Lambda_{a_2} - 1)\cdots\left(\Lambda_{a_{n+1}} - 1\right).$$

This implies that the roots of $\Delta_f(t)$ are all roots of unity and that $\Delta_f(t)$ is a product of cyclotomic polynomials.

Let K be a $(2n-1)$–dimensional knot. We say that K is *spherical* if K is homeomorphic to the $(2n-1)$–dimensional sphere. When K is a simple fibered $(2n-1)$–knot with $n \neq 2$, it is known that K is spherical if and only if $\Delta_K(1) = \pm 1$. For algebraic knots associated with a Brieskorn polynomial, there is a characterization of spherical knots due to Brieskorn [4] in terms of the exponents (for details, see Theorem 3.12 of the present paper).

Let K be a simple fibered $(2n-1)$–knot with fiber F. We define the bilinear form $\theta_K : H_n(F;\mathbf{Z}) \times H_n(F;\mathbf{Z}) \to \mathbf{Z}$ by $\theta_K(\alpha,\beta) = \operatorname{lk}(a_+, b)$, where a and b are n–cycles representing α and β, respectively, a_+ is the n–cycle in S^{2n+1} obtained by pushing a into the positive normal direction of F, and lk denotes the linking

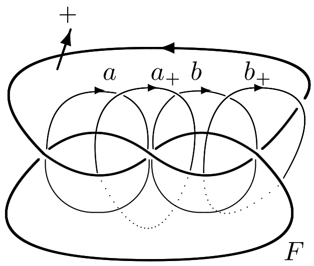

FIGURE 2. Computing a Seifert matrix for the trefoil knot

number of n–cycles in S^{2n+1} (see Fig. 2). The bilinear form θ_K is called the *Seifert form* of K and its representative matrix is called a *Seifert matrix*. It is known that a Seifert form is unimodular, i.e., the determinant of the Seifert matrix is equal to ± 1.

It is known that for $n \geqslant 3$, there is a one-to-one correspondence between the set of isomorphism classes of unimodular bilinear forms over the integers and the set of isotopy classes of simple fibered $(2n-1)$–knots [7, 10].

3. Fox–Milnor type relation

Let $\Delta_f(t)$ and $\Delta_g(t)$ denote the Alexander polynomials for the algebraic knots K_f and K_g associated with f and g, respectively. We say that the Alexander polynomials satisfy the *Fox–Milnor type relation* if there exists a polynomial $\gamma(t)$ with integer coefficients such that $\Delta_f(t)\Delta_g(t) = \pm t^{\deg \gamma}\gamma(t)\gamma(t^{-1})$ ([8, 9]). It is known that if K_f and K_g are cobordant, then their Alexander polynomials satisfy the Fox–Milnor type relation (for details, see [2], for example).

REMARK 3.1. If f and g are Brieskorn polynomials, then the Alexander polynomials $\Delta_f(t)$ and $\Delta_g(t)$ are products of cyclotomic polynomials, which are symmetric. Therefore, their Alexander polynomials satisfy the Fox–Milnor type relation if and only if $\Delta_f(t)\Delta_g(t)$ is a square.

Let
$$f(z) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$$
be a Brieskorn polynomial with $a_j \geqslant 2$ for all j. Set $E_f = \{a_1, a_2, \ldots, a_{n+1}\}$, which may contain the same integer multiple times and is considered to be a multi-set.

DEFINITION 3.2. From E_f, we construct the (non multi-)subset $\overline{E}_f \subset E_f$ by the successive procedure as follows.

(1) Take off all those even integers which appear an even number of times.
(2) Take off the multiple elements except for one in such a way that we get a non multi-set.
(3) Take off a_j if it is an integer multiple of an odd a_k with $k \neq j$.

We call the set $\overline{E}_f$ the *essential exponent set* of f. Note that $\overline{E}_f$ can be empty.

THEOREM 3.3. *Let*
$$f(z) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}} \text{ and } g(z) = z_1^{b_1} + z_2^{b_2} + \cdots + z_{n+1}^{b_{n+1}}$$
be Brieskorn polynomials with $a_j \geqslant 2$ and $b_j \geqslant 2$ for all j. Then, the Alexander polynomials $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor type relation if and only if their essential exponent sets coincide, i.e. $\overline{E}_f = \overline{E}_g$.

EXAMPLE 3.4. For example, consider

$$f(z) = z_1^3 + z_2^4 + z_3^4 + z_4^6 + z_5^9 \text{ and } g(z) = z_1^2 + z_2^2 + z_3^3 + z_4^3 + z_5^{12}.$$

Then, we have

$$E_f = \{3, 4, 4, 6, 9\} \text{ and } E_g = \{2, 2, 3, 3, 12\}.$$

In the process of Definition 3.2, after (1), we get the multi-sets $\{3, 6, 9\}$ and $\{3, 3, 12\}$ for f and g, respectively. After (2), we get the sets $\{3, 6, 9\}$ and $\{3, 12\}$. Finally, after (3), we get the sets $\{3\}$ and $\{3\}$. Hence, we get $\overline{E}_f = \overline{E}_g = \{3\}$ and $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor type relation. In fact, we have

$$\begin{aligned} \operatorname{div}\Delta_f(t) &= 4\Lambda_{12} - \Lambda_3 - 1, \\ \operatorname{div}\Delta_g(t) &= 24\Lambda_{36} + 6\Lambda_{18} - 6\Lambda_{12} - 2\Lambda_9 - 2\Lambda_6 - 2\Lambda_4 + \Lambda_3 - 1, \end{aligned}$$

so we can verify that $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor type relation.

Note that by the signature formula due to Brieskorn [4], we see that the signatures of the 8–dimensional Milnor fibers for f and g are equal to 274 and 30, respectively. Thus, K_f and K_g are not cobordant, since the signature of a fiber of a fibered knot is a cobordism invariant. Nevertheless, their Alexander polynomials satisfy the Fox–Milnor type relation.

On the other hand, for

$$h(z) = z_1^3 + z_2^4 + z_3^4 + z_4^6 + z_5^8,$$

we have $\overline{E}_h = \{3, 8\}$, so $\Delta_f(t)$ (or $\Delta_g(t)$) and $\Delta_h(t)$ do not satisfy the Fox–Milnor type relation. In fact, we have

$$\operatorname{div}\Delta_h(t) = 27\Lambda_{24} - 6\Lambda_{12} + 9\Lambda_8 - 2\Lambda_6 - 2\Lambda_4 + \Lambda_3 - 1.$$

In order to prove Theorem 3.3, let us prepare some preliminary lemmas. Recall that we have

$$\operatorname{div}\Delta_f(t) = \prod_{i=1}^{n+1}(\Lambda_{a_i} - 1) \quad \text{and} \quad \operatorname{div}\Delta_g(t) = \prod_{i=1}^{n+1}(\Lambda_{b_i} - 1)$$

and that $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor type relation if and only if

$$\operatorname{div}\Delta_f(t) \equiv \operatorname{div}\Delta_g(t) \pmod 2.$$

The following lemma is easy to prove by using the basic formula

$$\Lambda_a\Lambda_b = (a, b)\Lambda_{[a,b]},$$

for positive integers a and b, where (a, b) denotes the greatest common divisor of a and b, and $[a, b]$ denotes the least common multiple of a and b.

LEMMA 3.5. *For positive integers a, b and m, we have the following.*

(1) *If a is even, then we have*

$$(\Lambda_a - 1)^m \equiv \begin{cases} 1 \pmod 2, & m\text{: even}, \\ \Lambda_a - 1 \pmod 2, & m\text{: odd}. \end{cases}$$

(2) *If a is odd, then we have*

$$(\Lambda_a - 1)^m \equiv \Lambda_a - 1 \pmod 2$$

for all m.

(3) *If a is odd, then we have*

$$(\Lambda_a - 1)(\Lambda_{ab} - 1) \equiv \Lambda_a - 1 \pmod 2.$$

(4) *If* a_j, $j = 1, 2, \dots, m$, *are even, then we have*

$$\prod_{j=1}^{m}(\Lambda_{a_j} - 1) \equiv \sum_{j=1}^{m} \Lambda_{a_j} - 1 \pmod 2.$$

(5) *If* a *is even and* b *is odd, then we have*

$$\Lambda_a(\Lambda_b - 1) \equiv \Lambda_{[a,b]} - \Lambda_a \pmod 2.$$

Then, we have the following.

LEMMA 3.6. *We have*

$$\prod_{a \in E_f} (\Lambda_a - 1) \equiv \prod_{a \in \overline{E}_f} (\Lambda_a - 1) \pmod 2.$$

REMARK 3.7. When $\overline{E}_f = \varnothing$,

$$\prod_{a \in \overline{E}_f} (\Lambda_a - 1)$$

is understood to be equal to 0 in the group ring $\mathbf{QC}^*$.

Proof of Lemma 3.6. By Lemma 3.5 (1) for m even, even if we perform the procedure Definition 3.2 (1), the modulo 2 class of the product of $\Lambda_a - 1$ over all elements a of the relevant set, corresponding to the divisor of the relevant Alexander polynomial, does not change. Then, by Lemma 3.5 (1) for m odd and (2), the same holds with the procedure of Definition 3.2 (2). Finally, by Lemma 3.5 (3), we see that

$$\operatorname{div} \Delta_f(t) \equiv \prod_{a \in \overline{E}_f} (\Lambda_a - 1) \pmod 2.$$

□

Proof of Theorem 3.3. Suppose that $\overline{E}_f = \overline{E}_g$ holds. Then, by Lemma 3.6, we see that $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor type relation.

Conversely, suppose that $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor type relation. Then, by Lemma 3.6, we have

$$\prod_{a \in \overline{E}_f} (\Lambda_a - 1) \equiv \prod_{b \in \overline{E}_g} (\Lambda_b - 1) \pmod 2.$$

Let $\overline{E}_f^0$ (resp. $\overline{E}_f^1$) be the subset of $\overline{E}_f$ consisting of even (resp. odd) integers. We also define $\overline{E}_g^0$ and $\overline{E}_g^1$ similarly. Then, we have

$$\begin{aligned} &\left(\prod_{a \in \overline{E}_f^0} (\Lambda_a - 1)\right)\left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1)\right) \\ \equiv &\left(\prod_{b \in \overline{E}_g^0} (\Lambda_b - 1)\right)\left(\prod_{b \in \overline{E}_g^1} (\Lambda_b - 1)\right) \pmod 2. \end{aligned}$$

By Lemma 3.5 (4), we have

$$\begin{aligned} &\left(\sum_{a \in \overline{E}_f^0} \Lambda_a - 1\right)\left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1)\right) \\ \equiv &\left(\sum_{b \in \overline{E}_g^0} \Lambda_b - 1\right)\left(\prod_{b \in \overline{E}_g^1} (\Lambda_b - 1)\right) \pmod 2. \end{aligned} \tag{3.1}$$

By comparing the terms of the forms Λ_d with d odd, we have

$$\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1) \equiv \prod_{b \in \overline{E}_g^1} (\Lambda_b - 1) \pmod 2.$$

(For this, see also [3, Lemma 3.3].) As no integer in $\overline{E}_f^1$ (or $\overline{E}_g^1$) is a multiple of another one, by the same argument as in the proof of [3, Theorem 2.7], we see that $\overline{E}_f^1 = \overline{E}_g^1$.

By (3.1), we have

$$\left(\sum_{a \in \overline{E}_f^0} \Lambda_a\right)\left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1)\right) \equiv \left(\sum_{b \in \overline{E}_g^0} \Lambda_b\right)\left(\prod_{b \in \overline{E}_g^1} (\Lambda_b - 1)\right) \pmod 2. \tag{3.2}$$

Then, by considering the terms of the forms Λ_d with d minimal on both sides, we see that

$$\min \overline{E}_f^0 = \min \overline{E}_g^0,$$

which we set as m_0. Consequently, by subtracting

$$\Lambda_{m_0}\left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1)\right) \equiv \Lambda_{m_0}\left(\prod_{b \in \overline{E}_g^1} (\Lambda_b - 1)\right) \pmod 2$$

from both sides of (3.2), we get

$$\left(\sum_{a \in \overline{E}_f^0 \smallsetminus \{m_0\}} \Lambda_a\right)\left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1)\right) \equiv \left(\sum_{b \in \overline{E}_g^0 \smallsetminus \{m_0\}} \Lambda_b\right)\left(\prod_{b \in \overline{E}_g^1} (\Lambda_b - 1)\right) \pmod 2.$$

Repeating this procedure, we finally get $\overline{E}_f^0 = \overline{E}_g^0$. This completes the proof. □

Remark 3.8. By [3, Proposition 2.6], if the Seifert forms of K_f and K_g are Witt equivalent over the real numbers (i.e., if they have the same equivariant signatures), then their Alexander polynomials satisfy the Fox–Milnor type relation. So, by Theorem 3.3, we have $\overline{E}_f = \overline{E}_g$.

Corollary 3.9. *Suppose that the exponents of a Brieskorn polynomial f are all distinct and that no exponent is a multiple of another odd exponent. Let g be an arbitrary Brieskorn polynomial with the same number of variables as f. Then K_f and K_g are cobordant if and only if they have the same set of exponents. In particular, if the exponents of f are all even and all distinct, the same conclusion holds.*

Proof. Under the assumption for f, we easily see that $E_f = \overline{E}_f$. Suppose that K_f and K_g are cobordant. Then, their Alexander polynomials satisfy the Fox–Milnor type relation, and by Theorem 3.3, we have $\overline{E}_f = \overline{E}_g$. As $E_f = \overline{E}_f$ has $n + 1$ distinct elements, so does $\overline{E}_g$. As this is a subset of E_g, we must have $\overline{E}_g = E_g$. Hence we have $E_f = E_g$. This completes the proof. □

We also have the following.

Corollary 3.10. *Let f be a Brieskorn polynomial. Then, the Alexander polynomial $\Delta_f(t)$ of the algebraic knot K_f associated with f is a square if and only if all the exponents are even and each of them appears an even number if times.*

Proof. Note that $\Delta_f(t)$ is a square if and only if $\overline{E}_f = \emptyset$.

If E_f satisfies the condition in the corollary, we see easily that $\overline{E}_f = \emptyset$ and that $\Delta_f(t)$ is a square. On the other hand, if E_f contains an odd integer, it persists in $\overline{E}_f$. Furthermore, if an even integer appears exactly an odd number of times, then one of them persists in $\overline{E}_f$. Hence, in these cases, $\Delta_f(t)$ is not a square. This completes the proof. □

Theorem 3.11. *The algebraic knot K_f associated with a Brieskorn polynomial f is never null-cobordant.*

Proof. Suppose that K_f is null-cobordant. Then, it bounds a $2n$–dimensional disk in D^{2n+2}, so it is diffeomorphic to the standard $(2n-1)$–sphere.

Recall the following result due to Brieskorn [4]. For a Brieskorn polynomial f with the exponent set E_f, we construct a finite graph G_f as follows: the vertices correspond to the elements of E_f, and for $a, b \in E_f$, we connect them by an edge if their greatest common divisor satisfies $(a, b) > 1$. A connected component of G_f is called an *odd 2–component* if its vertex set consists of an odd number of even integers such that each pair of vertices are connected by an edge and their greatest common divisor is always equal to 2. Then we have the following.

Theorem 3.12 (Brieskorn [4]). *Let f be a Brieskorn polynomial of $n+1$ variables. For $n \neq 2$, the algebraic knot K_f is spherical if and only if G_f satisfies one of the following.*

(1) *The graph G_f contains at least two isolated vertices.*
(2) *The graph G_f contains one isolated vertex and an odd 2–component.*

Now. let us go back to the proof of Theorem 3.11. By Theorem 3.12, as K_f is spherical, we see that E_f contains an odd integer. Therefore, $\overline{E}_f$ is never an empty set, and $\Delta_f(t)$ is not a square. Hence, by Remark 3.1 we see that K_f is not null-cobordant. □

In fact, we have a stronger result as follows.

Theorem 3.13. *Let K_f be the algebraic knot associated with a Brieskorn polynomial f. If it is spherical, then it always has infinite order in the knot cobordism group.*

Proof. Suppose K_f is of finite order. Then, its equivariant signatures all vanish. Therefore, by Remark 3.8, its Alexander polynomial must be a square. Then, the rest of the proof is the same as that for Theorem 3.11. □

Proposition 3.14. *Let K_f and K_g be the algebraic knots associated with Brieskorn polynomials f and g, respectively. We assume that they are spherical.*

(1) *If $K_f \sharp (-K_g^!)$ is of finite order in the knot cobordism group, then the order must be equal to 1 or 2.*
(2) *If K_f and K_g have the same equivariant signatures, then $K_f \sharp K_f$ is cobordant to $K_g \sharp K_g$.*

Proof. (1) It is known that $K_f \sharp (-K_g^!)$ is of finite order if and only if its equivariant signatures all vanish. Therefore, by our assumption, the equivariant signatures of K_f and K_g coincide, and by Remark 3.8, the Alexander polynomials of K_f and K_g satisfy the Fox–Milnor type relation. Then, by [5, Theorem 3.4.8], $K_f \sharp (-K_g^!)$

cannot have order 4 in the knot cobordism group. Hence, the order must be equal to 1 or 2.

(2) Since $K_f \sharp(-K_g^!)$ has order 1 or 2, we see that

$$2(K_f \sharp(-K_g^!)) = (K_f \sharp K_f)\sharp(-(K_g \sharp K_g)^!)$$

is null-cobordant, and the result follows. □

We also have the following.

PROPOSITION 3.15. *Let f and g be Brieskorn polynomials. If the algebraic knots K_f and K_g are cobordant, then their Alexander polynomials $\Delta_f(t)$ and $\Delta_g(t)$ share at least one irreducible cyclotomic polynomial factor.*

Proof. By Theorem 3.12, we see that by adding appropriate powers of extra two variables to f, we get a Brieskorn polynomial $\widetilde{f}$ of $n+3$ variables such that $K_{\widetilde{f}}$ is spherical.

Suppose that the equivariant signatures for K_f all vanish. Then, its Seifert form is Witt equivalent to 0 over the real numbers. Since the Seifert form for $K_{\widetilde{f}}$ is the tensor product of that for K_f and a certain matrix, we see that it is also Witt equivalent to 0 over the real numbers. Hence, its equivariant signatures all vanish, which contradicts Theorem 3.13. Hence, an equivariant signature of K_f with respect to a root λ of $\Delta_f(t)$ does not vanish. As an equivariant signature is a cobordism invariant, the equivariant signature of K_g with respect to λ does not vanish, either. This implies that λ is a root of $\Delta_g(t)$. As the Alexander polynomials $\Delta_f(t)$ and $\Delta_g(t)$ are products of cyclotomic polynomials, the result follows. □

4. LINEAR INDEPENDENCE IN THE KNOT COBORDISM GROUP

Litherland [14] has shown that the algebraic knots associated with the Brieskorn polynomials $z_1^p + z_2^q$ with $2 \leqslant p < q$ and $(p,q) = 1$ are linearly independent in the knot cobordism group of dimension 1.

In order to prove a similar result in higher dimensions, let us prepare the following. For a fixed integer $n \geqslant 1$, let $\mathcal{B}$ be a set of exponent sets of $n+1$ elements such that for each exponent set belonging to $\mathcal{B}$, the exponents are relatively prime to each other and that no two of the exponent sets of $\mathcal{B}$ have equal product. In other words, for $\{p_i\}_{i=1}^{n+1} \neq \{q_i\}_{i=1}^{n+1} \in \mathcal{B}$, we have

$$p_1 p_2 \cdots p_{n+1} \neq q_1 q_2 \cdots q_{n+1}.$$

We call such a set $\mathcal{B}$ a *good family of exponent sets*. For example, the set $\mathcal{P}$ of all exponent sets such that the exponents are distinct prime numbers is a good family of exponent sets.

THEOREM 4.1. *Let $\mathcal{B}$ be a good family of exponent sets of $n+1$ elements, and consider the family of Brieskorn polynomials whose exponent sets correspond bijectively to elements of $\mathcal{B}$. Then, for $n \neq 2$, the associated algebraic knots are spherical and are linearly independent in the knot cobordism group of dimension $2n-1$.*

Note that the corresponding algebraic knots are easily seen to be spherical by Theorem 3.12.

For the proof of Theorem 4.1, let us prepare some materials. Let K be a $(2n-1)$-dimensional spherical knot and V its Seifert matrix. For a complex number ζ of modulus 1, let us consider the signature of the Hermitian matrix

$$(1-\zeta)V + (1-\bar{\zeta})V^T.$$

This is independent of the choice of Seifert matrix V. This gives rise to a function $S^1 \to \mathbf{Z}$, where S^1 is the unit circle in $\mathbf{C}$, and it is known to be continuous (and therefore constant) everywhere except at $(-1)^{n+1}$ times the unit roots of the Alexander polynomial $\Delta_K(t)$ (for example, see [5, Chapter 9]). This function is not a cobordism invariant in general: however, the jumps at $(-1)^{n+1}$ times the unit roots of the Alexander polynomial are cobordism invariants (see [12, 15] or [5, Theorem 3.4.7]).

Now let $\{p_i\} = \{p_1, p_2, \ldots, p_{n+1}\}$ be an exponent set in $\mathcal{B}$ and set

$$P = p_1 p_2 \cdots p_{n+1}.$$

Note that the integers $p_1, p_2, \ldots, p_{n+1}$ are relatively prime to each other. For a positive integer r, set

$$L_+\left(\frac{r}{P}\right) = \left\{(k_1, k_2, \ldots, k_{n+1}) \in \mathbf{Z}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} \frac{k_i}{p_i} \equiv \frac{r}{P} \pmod 2, \right.$$
$$\left. 0 < k_i < p_i,\ i = 1, 2, \ldots, n+1 \right\},$$
$$L_-\left(\frac{r}{P}\right) = \left\{(k_1, k_2, \ldots, k_{n+1}) \in \mathbf{Z}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} \frac{k_i}{p_i} \equiv \frac{r}{P} + 1 \pmod 2, \right.$$
$$\left. 0 < k_i < p_i,\ i = 1, 2, \ldots, n+1 \right\}.$$

Then, we see that $L_+(r/P) \cup L_-(r/P)$ contains at most one element, and that $L_+(r/P) = L_-(r/P) = \varnothing$ if and only if r is a multiple of some p_i. Furthermore, the jump at $\exp(2\pi\sqrt{-1}r/P)$ is equal to 1 if $L_+(r/P) \neq \varnothing$, is equal to -1 if $L_-(r/P) \neq \varnothing$, and is equal to 0 if r is a multiple of some p_i (see [5, §9.3]).

Proof of Theorem 4.1. Let us show that the jump functions $j_{\{p_i\}}$ are linearly independent over $\mathbf{Z}$ for $\{p_i\} \in \mathcal{B}$. Suppose there is a nontrivial dependence relation among $j_{\{p_i\}}$. Let M be the maximum of $P = p_1 p_2 \cdots p_{n+1}$ appearing in a nontrivial dependence relation. Note that by the definition of a good family of exponent sets, such maximum is attained only by a unique element $\{q_i\}$ in $\mathcal{B}$. Since $j_{\{p_i\}}(1/M) = 0$ for $p_1 p_2 \cdots p_{n+1} < M$, we see that $j_{\{q_i\}}(1/M) = 0$. This is a contradiction. Therefore, the jump functions corresponding to the elements of $\mathcal{B}$ are linearly independent over $\mathbf{Z}$. Since the jump functions are additive cobordism invariants, the result follows. □

Remark 4.2. The above proof is based on the idea used in [14] for $n = 1$. In Theorem 4.1, we imposed the condition that no two of the exponent sets of $\mathcal{B}$ have equal product. We do not know if this condition is redundant or not.

5. Decomposition of Seifert form

For $a \geqslant 2$, let M_a be the $(a-1) \times (a-1)$ unimodular matrix

$$M_a = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Note that the Seifert form of the algebraic knot associated with the Brieskorn polynomial

$$f = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$$

is given by the tensor product $L = M_{a_1} \otimes M_{a_2} \otimes \cdots \otimes M_{a_{n+1}}$.

Recall that we have

$$S = L+(-1)^n\overline{L^T}, \quad H = (-1)^{n+1}\overline{L^{-1}}L^T, \quad T = (-1)^{n+1}L\overline{(L^{-1})^T}, \quad S = L(I-\overline{H}),$$

where L is the Seifert matrix, S is the sesquilinearized intersection form of the Milnor fiber, H is the homological monodromy matrix, T is the cohomological monodromy matrix, and I is the identity matrix.

Let us consider an explicit example:

$$f(z_1, z_2, z_3) = z_1^3 + z_2^4 + z_3^4.$$

Its characteristic polynomial of the monodromy (or Alexander polynomial) $\Delta_f(t)$ is given by

$$\operatorname{div}\Delta_f = (\Lambda_3 - 1)(\Lambda_4 - 1)(\Lambda_4 - 1) = 2\Lambda_{12} + \Lambda_3 - 2\Lambda_4 - 1,$$

and hence we have

$$\begin{aligned} \Delta_f(t) &= \frac{(t^{12}-1)^2(t^3-1)}{(t^4-1)^2(t-1)} \\ &= \frac{\phi_{12}^2\,\phi_6^2\,\phi_4^2\,\phi_3^2\,\phi_2^2\,\phi_1^2\,\phi_3\,\phi_1}{\phi_4^2\,\phi_2^2\,\phi_1^2\,\phi_1} \\ &= \phi_{12}^2\,\phi_6^2\,\phi_3^3, \end{aligned}$$

where for a positive integer m, $\phi_m(t)$ denotes the m–th cyclotomic polynomial. Note that the degrees of $\phi_{12}, \phi_6, \phi_3$ are equal to $4, 2, 2$, respectively. According to Steenbrink's formula [19], the equivariant signatures corresponding to $\phi_{12}, \phi_6, \phi_3$ are equal to $8, 0, 6$, respectively.

Let us analyze the ϕ_6–primary component. The Seifert form for f is given by the unimodular (18×18)–matrix

$$L = M_3 \otimes M_4 \otimes M_4.$$

The form M_3 is irreducible over $\mathbf{Q}$, since its Alexander polynomial ϕ_3 is irreducible. On the other hand, the Alexander polynomial of M_4 is equal to $\phi_4\,\phi_2$, which is not irreducible. Let us decompose M_4 into the irreducible factors over $\mathbf{Q}$.

By some computations, we see the following:

$$\begin{aligned} M_4 &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \\ M_4^{-1} &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \\ S_4 &= M_4 + M_4^T = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \\ T &= -M_4(M_4^{-1})^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \\ H &= T^T = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} H^T M_4 H &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = M_4. \end{aligned}$$

The eigenvalues of H are $-1, \pm\sqrt{-1}$. Eigenvectors corresponding to the eigenvalues $-1, \sqrt{-1}$ and $-\sqrt{-1}$ are given by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1-\sqrt{-1} \\ -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1 \\ 1+\sqrt{-1} \\ \sqrt{-1} \end{pmatrix},$$

respectively. Therefore, the ϕ_2–primary component is generated by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

and the ϕ_4–primary component is generated by

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(For this, consider the real and the imaginary parts of the corresponding eigenvectors.) Set

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then, we have

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

and

$$P^{-1} H P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

So, we have verified that P gives the correct decomposition of the monodromy into the irreducible components.

Remark 5.1. We can show that we cannot choose an integral unimodular matrix as P as follows. If we choose

$$a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ and } a' \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b' \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

as bases for the ϕ_4–primary component for some integers a, b, a', b', then we can show that the determinant of the corresponding (3×3)–matrix is an even integer.

Then, we have

$$P^T M_4 P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

So, over $\mathbf{Q}$, the bilinear form M_4 is isomorphic to $(1) \oplus R$, where

$$R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then, we have, over $\mathbf{Q}$,

$$\begin{aligned} L &= M_3 \otimes M_4 \otimes M_4 \\ &\cong M_3 \otimes ((1) \oplus R) \otimes ((1) \oplus R) \\ &\cong M_3 \otimes ((1) \oplus R \oplus R \oplus (R \otimes R)) \\ &\cong M_3 \oplus (M_3 \otimes R) \oplus (M_3 \otimes R) \oplus (M_3 \otimes R \otimes R). \end{aligned}$$

The characteristic polynomials corresponding to the 4 irreducible factors are given by

$$\phi_3(t), \phi_{12}(t), \phi_{12}(t) \text{ and } \phi_3(t)^2 \phi_6(t)^2,$$

respectively. So, in order to analyze the ϕ_6–primary component of L, we still need to decompose $M_3 \otimes R \otimes R$, which is an (8×8)–matrix.

Recall that the monodromy matrix H_3 corresponding to M_3 is given by

$$H_3 = -M_3^{-1} M_3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

We have

$$R^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and the monodromy matrix H_R corresponding to R is given by

$$H_R = -R^{-1} R^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Set $\omega = \exp(2\pi\sqrt{-1}/3)$. Eigenvectors of H_3 corresponding to the eigenvalues ω and $\bar{\omega}$ are given by

$$u_1 = \begin{pmatrix} 1 \\ -\omega \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} 1 \\ -\bar{\omega} \end{pmatrix},$$

respectively. Eigenvectors of H_R corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ are given by

$$v_1 = \begin{pmatrix} 1 \\ -\sqrt{-1} \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix},$$

respectively. Therefore, the monodromy matrix $H_{3,R,R}$ associated with $M_3 \otimes R \otimes R$ is diagonalized by the (8×8)–matrix Q consisting of the 8 column vectors

$$u_i \otimes v_j \otimes v_k,$$

$i, j, k = 1, 2$, in such a way that

$$Q^{-1} H_{3,R,R} Q = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Therefore, the ϕ_6–primary component is generated by $u_1 \otimes v_1 \otimes v_1$, $u_1 \otimes v_2 \otimes v_2$, $u_2 \otimes v_1 \otimes v_1$ and $u_2 \otimes v_2 \otimes v_2$ over $\mathbf{C}$. Note that

$$\begin{aligned} u_1 \otimes v_1 \otimes v_1 &= (1, -\sqrt{-1}, -\sqrt{-1}, -1, -\omega, \omega\sqrt{-1}, \omega\sqrt{-1}, \omega)^T, \\ u_1 \otimes v_2 \otimes v_2 &= (1, \sqrt{-1}, \sqrt{-1}, -1, -\omega, -\omega\sqrt{-1}, -\omega\sqrt{-1}, \omega)^T, \\ u_2 \otimes v_1 \otimes v_1 &= (1, -\sqrt{-1}, -\sqrt{-1}, -1, -\bar{\omega}, \bar{\omega}\sqrt{-1}, \bar{\omega}\sqrt{-1}, \bar{\omega})^T, \\ u_2 \otimes v_2 \otimes v_2 &= (1, \sqrt{-1}, \sqrt{-1}, -1, -\bar{\omega}, -\bar{\omega}\sqrt{-1}, -\bar{\omega}\sqrt{-1}, \bar{\omega})^T. \end{aligned}$$

By considering the real and imaginary parts, we see that the ϕ_6–primary component is generated by

$$\begin{aligned} w_1 &= (1,0,0,-1,1/2,-\sqrt{3}/2,-\sqrt{3}/2,-1/2)^T, \\ w_2 &= (0,-1,-1,0,-\sqrt{3}/2,-1/2,-1/2,\sqrt{3}/2)^T, \\ w_3 &= (1,0,0,-1,1/2,\sqrt{3}/2,\sqrt{3}/2,-1/2)^T, \\ w_4 &= (0,1,1,0,-\sqrt{3}/2,1/2,1/2,\sqrt{3}/2)^T \end{aligned}$$

over $\mathbf{R}$. Then, we have

$$\begin{aligned} w_1 + w_3 &= (2,0,0,-2,1,0,0,-1)^T, \\ (w_1 - w_3)/\sqrt{3} &= (0,0,0,0,0,-1,-1,0)^T, \\ (w_2 + w_4)/\sqrt{3} &= (0,0,0,0,-1,0,0,1)^T, \\ -(w_2 - w_4) &= (0,2,2,0,0,1,-,0)^T. \end{aligned}$$

Note that these 4 vectors can be written as

$$\begin{aligned} r_1 &= (2,1)^T \otimes (1,0,0,-1)^T, \\ r_2 &= (0,-1)^T \otimes (0,1,1,0)^T, \\ r_3 &= (0,-1)^T \otimes (1,0,0,-1)^T, \\ r_4 &= (2,1)^T \otimes (0,1,1,0)^T, \end{aligned}$$

respectively. Then, by calculating

$$r_i^T (M_3 \otimes R \otimes R) r_j,$$

$i, j = 1,2,3,4$, we see that the ϕ_6–primary component of the bilinear form $M_3 \otimes R \otimes R$ is isomorphic over $\mathbf{Q}$ to

$$\begin{pmatrix} 0 & -4 & 0 & -12 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \\ 12 & 0 & 4 & 0 \end{pmatrix},$$

which, in turn, is isomorphic to

$$\begin{pmatrix} 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix}$$

over $\mathbf{Q}$.

Consequently, we see that the ϕ_6–primary component of $M_3 \otimes R \otimes R$, hence that of L, is algebraically null-cobordant.

Let p be a positive integer relatively prime to 2 and 3, and consider the Brieskorn polynomial

$$g(z_1, z_2, z_3, z_4) = z_1^3 + z_2^4 + z_3^4 + z_4^p.$$

Then, the algebraic knot K_g associated with g is spherical, i.e. K_f is homeomorphic to the sphere S^7 (see Theorem 3.12). As the Seifert matrix L_g of K_g is given by the tensor product of L and M_p, we see that a certain direct summand of L_g is null cobordant over $\mathbf{Q}$.

Let us recall some known facts about the algebraic knot cobordism group (for details, see [5], for example). Let $\mathcal{G}$ be the algebraic cobordism group which consists of the set of integral square matrices A satisfying $\det(A - A^T) = \pm 1$ up to Witt equivalence. Let us consider the group $\mathcal{G}^{\mathbf{Q}}$: square matrices A with entries in $\mathbf{Q}$ satisfying that $(A - A^T)(A + A^T)$ is non-singular, with the same equivalence relation as in $\mathcal{G}$. It is known that the natural inclusion $\mathcal{G} \to \mathcal{G}^{\mathbf{Q}}$ is injective [13].

It is also known that $\mathcal{G}^{\mathbf{Q}}$ is isomorphic to the group $\mathcal{G}_{\mathbf{Q}}$ of cobordism classes of isometric structures.

For a polynomial $\delta(t) \in \mathbf{Q}[t]$, let $\mathcal{G}_{\mathbf{Q}}^{\delta}$ denote the Witt group of isometric structures over $\mathbf{Q}$ corresponding to a power of δ. Then, it is known that $\mathcal{G}_{\mathbf{Q}} \cong \oplus_{\delta}\mathcal{G}_{\mathbf{Q}}^{\delta}$, where the sum is over all irreducible symmetric polynomials δ.

The above explicit example shows that if we consider the image of the cobordism class of a spherical algebraic knot associated with a Brieskorn polynomial in $\oplus_{\delta}\mathcal{G}_{\mathbf{Q}}^{\delta}$, then there might be a direct summand which vanishes in $\mathcal{G}_{\mathbf{Q}}^{\delta}$ for some δ. This means that even if two algebraic knots are cobordant, the irreducible factors of their Alexander polynomials might be different, although they share at least one irreducible factor according to Proposition 3.15.

6. Cyclic suspension

In this section, we explore cyclic suspensions of simple fibered knots and their properties concerning cobordisms.

Let $K \subset S^{2n+1}$ be a $(2n-1)$–knot. Then, we can move the standard sphere $S^{2n+1} \subset S^{2n+3}$ ambient isotopically to get S' such that S' intersects S^{2n+1} transversely along K. For a positive integer d, we consider the d–fold cyclic branched covering $\widetilde{S}$ of S^{2n+3} branched along S^{2n+1}, which is diffeomorphic to S^{2n+3}. Then the pull-back K_d of S' by the branched covering map in $\widetilde{S}$ is called the *d–fold cyclic suspension* of K. Furthermore, we call the positive integer d the *suspension degree*. Note that K_d itself is diffeomorphic to the d–fold branched covering of S^{2n+1} branched along K, and that it is considered to be a $(2n+1)$–knot in S^{2n+3}. This notion has been introduced by Neumann [17] (see also [11]). Note that if K is a simple fibered knot, then so is K_d.

In this section, we consider the following problem.

Problem 6.1. For a common integer d, let $(K_i)_d$ be the d–fold cyclic suspensions of two knots K_i, $i = 1, 2$. Furthermore, for another common integer e, let $(K_i)_{d,e}$ be the e–fold cyclic suspensions of $(K_i)_d$, $i = 1, 2$. Is it possible to construct examples such that K_i are not cobordant, that $(K_i)_d$ are cobordant and that $(K_i)_{d,e}$ are not cobordant?

If the answer is affirmative, then it would show that the cyclic suspensions do not preserve cobordisms in general.

Recall that the algebraic knot associated with a Brieskorn polynomial $z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$ is the iterated cyclic suspension of the (a_1, a_2)–torus link in S^3. The above problem is closely related to the study of cobordisms of such knots.

Let $n \geqslant 3$ be an integer. For the moment, we will assume that n is odd. Consider the matrices

$$A_1 = \begin{pmatrix} B & C \\ -C^T & \mathbf{0} \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where B is a 2×2 integer matrix with $\det(B + B^T) = \pm 1$, C is a 2×2 integer matrix with $\det C = \pm 1$, and $\mathbf{0}$ denotes the 2×2 zero matrix. So, A_1 is a unimodular (4×4)–matrix and A_2 is a unimodular (2×2)–matrix. Let K_1 and K_2 be the simple fibered $(2n-1)$-knots in S^{2n+1} whose Seifert matrices are given by A_1 and A_2, respectively.

Let $(K_i)_a$ be the a–fold cyclic suspension of the knot K_i, and $(K_i)_{a,b}$ be the b–fold cyclic suspension of $(K_i)_a$, $i = 1, 2$. Then, their Seifert matrices $(A_i)_a$ and

$(A_i)_{a,b}$, respectively, are given by

$$(A_i)_a = A_i \otimes M_a \text{ and } (A_i)_{a,b} = A_i \otimes M_a \otimes M_b.$$

Let us consider the 2–fold cyclic suspensions $(K_i)_2$. As M_2 is the (1×1)–matrix (1), we can identify their Seifert matrices with those of K_i, $i = 1, 2$. As we have

$$S_1 = A_1 + A_1^T = \begin{pmatrix} B + B^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad S_2 = A_2 + A_2^T = \mathbf{0},$$

we see that

$$H_n((K_1)_2; \mathbf{Z}) \cong H_n((K_2)_2; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z} \cong H_{n-1}((K_1)_2; \mathbf{Z}) \cong H_{n-1}((K_2)_2; \mathbf{Z}).$$

Furthermore, as A_1 and A_2 both have metabolizers, so does $A_1 \oplus (-A_2)$.

However, $(K_1)_2$ and $(K_2)_2$ are not cobordant, since the Seifert forms restricted to $H_n((K_i)_2; \mathbf{Z}) = \operatorname{Ker} S_i$, $i = 1, 2$, are not isomorphic (see [1]). Note that these knots are not spherical.

Let us now consider the 3–fold cyclic suspensions $(K_1)_3$ and $(K_2)_3$, respectively. Then, their Seifert matrices are given by

$$(A_1)_3 = A_1 \otimes M_3 = \begin{pmatrix} B \otimes M_3 & C \otimes M_3 \\ -C^T \otimes M_3 & \mathbf{0} \end{pmatrix}$$

and

$$(A_2)_3 = A_2 \otimes M_3 = \begin{pmatrix} \mathbf{0} & M_3 \\ -M_3 & \mathbf{0} \end{pmatrix},$$

respectively. Then, the intersection matrices of their fibers are given by

$$\begin{aligned} (S_1)_3 &= (A_1)_3 + (A_1)_3^T = \begin{pmatrix} B \otimes M_3 + B^T \otimes M_3^T & C \otimes M_3 - C \otimes M_3^T \\ C^T \otimes M_3^T - C^T \otimes M_3 & \mathbf{0} \end{pmatrix} \\ &= \begin{pmatrix} B \otimes M_3 + B^T \otimes M_3^T & C \otimes (M_3 - M_3^T) \\ -C^T \otimes (M_3 - M_3^T) & \mathbf{0} \end{pmatrix} \end{aligned}$$

and

$$(S_2)_3 = (A_2)_3 + (A_2)_3^T = \begin{pmatrix} \mathbf{0} & M_3 - M_3^T \\ -(M_3 - M_3^T) & \mathbf{0} \end{pmatrix},$$

respectively. As we have

$$\det(M_3 - M_3^T) = \det \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 1,$$

we see that both $(S_1)_3$ and $(S_2)_3$ are unimodular. Therefore, the fibered knots $(K_1)_3$ and $(K_2)_3$ are spherical. As their Seifert matrices are obviously algebraically null-cobordant, the knots are, in fact, null-cobordant, and in particular they are cobordant.

We can also show that K_1 and K_2 are not diffeomorphic to each other for an appropriate choice of C. For example, consider

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In this case, the intersection matrices are

$$S_1 = A_1 - A_1^T = \begin{pmatrix} B - B^T & C + C^T \\ -(C + C^T) & \mathbf{0} \end{pmatrix} = \begin{pmatrix} B - B^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

and

$$S_2 = A_2 - A_2^T = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Therefore, the rank of $H_{n-1}(K_1; \mathbf{Z})$ is greater than or equal to 2, while $H_{n-1}(K_2; \mathbf{Z})$ is finite of order 4. So, K_1 and K_2 are not diffeomorphic and hence are not cobordant.

Let us now consider $(K_1)_{2,3} = (K_1)_{3,2}$ and $(K_2)_{2,3} = (K_2)_{3,2}$. Their Seifert forms are given by

$$(A_1)_{3,2} = A_1 \otimes M_3 \otimes M_2 = \begin{pmatrix} B \otimes M_3 & C \otimes M_3 \\ -C^T \otimes M_3 & \mathbf{0} \end{pmatrix}$$

and

$$(A_2)_{3,2} = A_2 \otimes M_3 \otimes M_2 = \begin{pmatrix} \mathbf{0} & M_3 \\ -M_3 & \mathbf{0} \end{pmatrix},$$

respectively. Then, their intersection matrices are

$$(S_1)_{3,2} = (A_1)_{3,2} - (A_1)_{3,2}^T = \begin{pmatrix} B \otimes M_3 - B^T \otimes M_3^T & C \otimes M_3 + C \otimes M_3^T \\ -C^T \otimes M_3 - C^T \otimes M_3^T & \mathbf{0} \end{pmatrix}$$

and

$$(S_2)_{3,2} = (A_2)_{3,2} - (A_2)_{3,2}^T = \begin{pmatrix} \mathbf{0} & M_3 + M_3^T \\ -M_3 - M_3^T & \mathbf{0} \end{pmatrix},$$

respectively. For C as above, we see that

$$|\det(S_1)_{3,2}| = 3^4, \quad |\det(S_2)_{3,2}| = 3^2,$$

and hence $(K_1)_{3,2}$ and $(K_2)_{3,2}$ are not diffeomorphic and are not cobordant.

Summarizing, we have the following.

(1) K_1 and K_2 are not diffeomorphic and are not cobordant.
(2) $(K_1)_2$ and $(K_2)_2$ are diffeomorphic, but are not cobordant.
(3) $(K_1)_3$ and $(K_2)_3$ are spherical and null-cobordant, so they are cobordant to each other.
(4) $(K_1)_{3,2} = (K_1)_{2,3}$ and $(K_2)_{3,2} = (K_2)_{2,3}$ are not diffeomorphic and are not cobordant.

So, this answers Problem 6.1 affirmatively.

Remark 6.2. In general, if two knots K_1 and K_2 are cobordant, and if their cyclic suspensions $(K_1)_d$ and $(K_2)_d$ are spherical, then $(K_1)_d$ and $(K_2)_d$ are cobordant.

Moreover, if K_1 and K_2 are spherical knots which are cobordant, then $(K_1)_{2,2}$ and $(K_2)_{2,2}$ are also cobordant. See [11, §8].

Now, let us consider examples of algebraic knots. In [6], Du Bois and Michel constructed two polynomials

$$f = h_{r,s,p,q}(z_1, z_2, \ldots, z_{n+1}) \text{ and } g = h_{s-8,r+8,p,q}(z_1, z_2, \ldots, z_{n+1})$$

with isolated critical points at the origin such that K_f and K_g are cobordant, although they are not isotopic. Let k be a positive integer called an *exponent* in the sense of [6] for both of f and g: i.e., $(t_f^k - 1)^2$ and $(t_g^k - 1)^2$ both vanish, where t_f and t_g are homological monodromies for the Milnor fibrations for f and g, respectively, and "1" denotes the identity homomorphism. Let us consider the algebraic knots $K_{\tilde{f}}$ and $K_{\tilde{g}}$ associated with

$$\tilde{f}(z_1, z_2, \ldots, z_{n+2}) = f(z_1, z_2, \ldots, z_{n+1}) + z_{n+2}^k$$

and

$$\tilde{g}(z_1, z_2, \ldots, z_{n+2}) = g(z_1, z_2, \ldots, z_{n+1}) + z_{n+2}^k,$$

respectively. Note that they are k–fold cyclic suspensions of K_f and K_g, respectively.

Lemma 6.3. *The homology groups $H_n(K_{\tilde{f}}; \mathbf{Z})$ and $H_n(K_{\tilde{g}}; \mathbf{Z})$ have non-isomorphic torsions.*

Proof. Recall that $K_{\widetilde{f}}$ (resp. $K_{\widetilde{g}}$) is the k–fold cyclic branched cover of S^{2n+1} branched along K_f (resp. K_g). This implies that $K_{\widetilde{f}}$ admits an open book structure with page diffeomorphic to F_f and with algebraic monodromy $t = t_f^k$.

Let $B \subset K_{\widetilde{f}}$ be the branched locus and let E be the complement of an open tubular neighborhood of B in $K_{\widetilde{f}}$. Thus, E is the total space of a fiber bundle over S^1 with fiber F_f and with algebraic monodromy $t = t_f^k$. Then, we have the following Wang exact sequence of homology [20] (see also [16, Lemma 8.4]):

$$H_n(F_f;\mathbf{Z}) \xrightarrow{t-1} H_n(F_f;\mathbf{Z}) \to H_n(E;\mathbf{Z}) \to H_{n-1}(F_f;\mathbf{Z}).$$

Since F_f is $(n-1)$–connected [16], we have $H_{n-1}(F_f;\mathbf{Z}) = 0$ so that we have

$$H_n(E;\mathbf{Z}) \cong H_n(F_f;\mathbf{Z})/\operatorname{Im}(t-1).$$

Then, by the Meyer–Vietoris exact sequence for the pair $(E, N(B))$, where $N(B)$ is the closed tubular neighborhood of B in $K_{\widetilde{f}}$, we have that

$$H_n(\partial N(B);\mathbf{Z}) \to H_n(N(B);\mathbf{Z}) \oplus H_n(E;\mathbf{Z}) \to H_n(K_{\widetilde{f}};\mathbf{Z}) \to H_{n-1}(\partial N(B);\mathbf{Z})$$

is exact. As $N(B) \cong K_f \times D^2$ and K_f is homeomorphic to S^{2n-1} with $n \geqslant 3$, we see that $H_n(\partial N(B);\mathbf{Z})$, $H_n(N(B);\mathbf{Z})$ and $H_{n-1}(\partial N(B);\mathbf{Z})$ all vanish. Therefore, we have $H_n(K_{\widetilde{f}};\mathbf{Z}) \cong H_n(E;\mathbf{Z})$, and hence they are isomorphic to the quotient $H_n(F_f;\mathbf{Z})/(t_f^k - 1)H_n(F_f;\mathbf{Z})$.

On the other hand, $\operatorname{Ker}(t_f^k - 1)$ is a pure submodule of the free abelian group $H_n(F_f;\mathbf{Z})$ of finite rank. Therefore, there exists a free abelian subgroup H_f of $H_n(F_f;\mathbf{Z})$ such that $H_n(F_f;\mathbf{Z}) = H_f \oplus \operatorname{Ker}(t_f^k - 1)$. As $\operatorname{Im}(t_f^k - 1)$ is contained in $\operatorname{Ker}(t_f^k - 1)$, we see that $H_n(K_{\widetilde{f}};\mathbf{Z}) \cong H_n(F_f;\mathbf{Z})/(t_f^k - 1)H_n(F_f;\mathbf{Z})$ is isomorphic to $H_f \oplus \left(\operatorname{Ker}(t_f^k - 1)/\operatorname{Im}(t_f^k - 1)\right)$. Note that a similar isomorphism holds for $H_n(K_{\widetilde{g}};\mathbf{Z})$ as well.

Since the twist groups, which are the torsion subgroups of $\operatorname{Ker}(t_f^k - 1)/\operatorname{Im}(t_f^k - 1)$ and $\operatorname{Ker}(t_g^k - 1)/\operatorname{Im}(t_g^k - 1)$, are not isomorphic according to [6], we see that the torsion subgroups of $H_n(K_{\widetilde{f}};\mathbf{Z})$ and $H_n(K_{\widetilde{g}};\mathbf{Z})$ are not isomorphic. □

The above lemma implies that although K_f and K_g are cobordant, their cyclic suspensions $K_{\widetilde{f}}$ and $K_{\widetilde{g}}$ are not, since they are not diffeomorphic.

If we take further iterated cyclic suspensions appropriately, say $K_{\widehat{f}}$ and $K_{\widehat{g}}$, where

$$\widehat{f}(z_1, z_2, \ldots, z_{n+3}, z_{n+4}) = \widetilde{f}(z_1, z_2, \ldots, z_{n+2}) + z_{n+3}^v + z_{n+4}^w$$

and

$$\widehat{g}(z_1, z_2, \ldots, z_{n+3}, z_{n+4}) = \widetilde{g}(z_1, z_2, \ldots, z_{n+2}) + z_{n+3}^v + z_{n+4}^w$$

for some appropriate prime numbers v and w, then $K_{\widehat{f}}$ and $K_{\widehat{g}}$ are spherical and hence are cobordant.

Summarizing, we have the following.

(1) The algebraic knots K_f and K_g are cobordant, but are not isotopic.
(2) Their k–fold cyclic suspensions $K_{\widetilde{f}}$ and $K_{\widetilde{g}}$ are not diffeomorphic and are not cobordant.
(3) The iterated cyclic suspensions $K_{\widehat{f}}$ and $K_{\widehat{g}}$ of $K_{\widetilde{f}}$ and $K_{\widetilde{g}}$, respectively, are cobordant.

This is yet another example that shows that cyclic suspensions (with a fixed suspension degree) do not behave well with respect to cobordisms. This time, the example shows this phenomenon for algebraic knots.

Remark 6.4. (1) If K_0 and K_1 are cobordant knots, then if their cyclic suspensions $\widetilde{K}_0$ and $\widetilde{K}_1$, respectively, of the same degree are spherical of dimension greater than or equal to 5, then they are cobordant. This is because the Seifert matrices of $\widetilde{K}_i$ are tensor products of those of K_i, which are algebraically cobordant, and the same matrix, and hence they are algebraically cobordant. For spherical higher dimensional knots, this implies that they are cobordant.

(2) Similarly, if K is a spherical knot which has finite order in the knot cobordism group, then if its cyclic suspension $\widetilde{K}$ is spherical, then $\widetilde{K}$ also has finite order in the knot cobordism group. This is because, since the Seifert form of K is Witt equivalent to zero over the real numbers, so is that of $\widetilde{K}$.

(3) Suppose that K is a spherical knot and that its d–fold cyclic suspension $\widetilde{K}$ is also spherical. Let us suppose that $\widetilde{K}$ is null-cobordant. Then, we do not know if K is also null-cobordant or not.

Similarly, suppose that K_0 and K_1 are spherical knots and that their d–fold cyclic suspensions $\widetilde{K}_0$ and $\widetilde{K}_1$, respectively, are also spherical. Let us suppose that $\widetilde{K}_0$ and $\widetilde{K}_1$ are cobordant. Then, we do not know if K_0 and K_1 are also cobordant or not, except for the case $d = 2$.

Since the algebraic knots associated with Brieskorn polynomials are iterated cyclic suspensions of torus knots, the observations in this section may show that by adding extra variables we may encounter a pair of algebraic knots associated with Brieskorn polynomials which are cobordant but which have distinct exponents.

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