MURASUGI NUMBERS OF NON-SPHERICAL KNOTS

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Abstract. In this paper, we study Murasugi numbers $g(K)$ of $(2n-1)$–dimensional (not necessarily spherical) knots K in $S^{2n+1} = \partial D^{2n+2}$ for $n \geqslant 2$, where $g(K)$ is the minimal middle Betti number among those of properly embedded compact oriented $2n$–dimensional submanifolds of D^{2n+2} homotopy equivalent to a bouquet of n–spheres bounded by K. We obtain lower and upper bounds of $g(K)$ by algebraic invariants, i.e. the Witt co-index and the pure Witt co-index of a Seifert form, for a large class of $(2n-1)$–dimensional knots for $n \geqslant 3$. As a consequence, we completely describe the Murasugi number by an algebraic invariant of a Seifert form for spherical $(2n-1)$–knots for $n \geqslant 3$. We also show that if $n \geqslant 4$ is even, then for the algebraic knot K_f associated with an isolated critical point of a holomorphic function germ $f : (\mathbf{C}^{n+1}, \mathbf{0}) \to (\mathbf{C}, 0)$, we have $\mu(K_f) = g(K_f)$ if and only if the singularity of f is simple or almost-simple, provided that $H_{n-1}(K_f; \mathbf{Z})$ is torsion free, generalizing a result due to Michel for the case of vanishing $H_{n-1}(K_f; \mathbf{Z})$, where $\mu(K_f)$, the Milnor number, is the middle Betti number of the Milnor fiber for f.

1. Introduction

Throughout the paper, we work in the smooth category and a knot (or more precisely, a $(2n-1)$–knot) refers to the isotopy class of an $(n-2)$–connected closed oriented $(2n-1)$–dimensional submanifold in S^{2n+1}, $n \geqslant 1$. Note that K may not be homeomorphic to the $(2n-1)$–sphere S^{2n-1}. When K is homeomorphic to S^{2n-1}, we say that K is *spherical*; otherwise, *non-spherical.*

A typical example of a knot is the algebraic knot K_f associated with a holomorphic function germ $f : (\mathbf{C}^{n+1}, \mathbf{0}) \to (\mathbf{C}, 0)$ with an isolated critical point at the origin [19]. Note that the isotopy class of K_f determines and is determined by the embedded topology of the complex hypersurface $f^{-1}(0)$ in a small neighborhood of the origin in $\mathbf{C}^{n+1}$ (see [26]). Recall that such a knot K_f bounds a compact $2n$–dimensional submanifold F_f of S^{2n+1}, called the *Milnor fiber*, which is homotopy equivalent to a bouquet of n–spheres, and the middle Betti number $b_n(F_f)$ is a very important topological invariant of the singularity, called the *Milnor number.*

The main objective of this paper is to study the minimal middle Betti number of a properly embedded compact oriented $2n$–dimensional submanifold of D^{2n+2} bounded by a given knot $K \subset S^{2n+1} = \partial D^{2n+2}$ homotopy equivalent to a bouquet of n–spheres. This minimal number is called the *Murasugi number* of the knot K, and is denoted by $g(K)$ [18] (see also [21]). When, $n = 1$ and K is connected, $g(K)$ is always even and $g(K)/2$ is often called the 4*–ball genus*, *slice genus*, or *Murasugi genus.* Note that the Murasugi number is a cobordism invariant for $(2n-1)$–knots.

Michel [18] studied the Murasugi numbers of algebraic knots K_f as above for $n \geqslant 3$, especially when K_f is spherical. In particular, it is shown that when K_f is

Date: May 28, 2026.

2020 *Mathematics Subject Classification.* Primary 57K45; Secondary 32S55, 57K10.

Key words and phrases. High dimensional knot, Murasugi number, algebraic knot, Milnor number, cobordism, co-index, simple singularity, almost-simple singularity.

spherical, $g(K_f)$ is equal to the Milnor number of f if and only if the singularity of f at the origin is simple in the sense of [6].

In this paper, we first study the Murasugi numbers of $(2n-1)$–knots K for $n \geqslant 3$ using algebraic invariants of their Seifert forms θ_K. These invariants are the Witt co-index $\beta(\theta_K)$ introduced, for example, in [8], and its variant, called the pure Witt co-index $\beta_{\mathrm{p}}(\theta_K)$ (for details, see §3 of the present paper). We will show that for a large class of $(2n-1)$–knots K, called *exact knots* introduced in [4], we have

$$\beta(\theta_K) \leqslant g(K) \leqslant \beta_{\mathrm{p}}(\theta_K)$$

for $n \geqslant 3$, where θ_K is the Seifert form of K with respect to an arbitrary exact Seifert hypersurface for K in the sense of [4] (see Proposition 3.14). As a corollary, we show that for spherical $(2n-1)$–knots for $n \geqslant 3$, the Murasugi number coincides with the Witt co-index of the Seifert form (see Theorem 3.15). Furthermore, we also generalize the above-mentioned result of Michel [18] for a more general class of algebraic knots K_f for $n \geqslant 4$ even, using the notions of simple and almost-simple singularities (see Theorem 4.4).

The paper is organized as follows. In §2, we recall several materials related to $(2n-1)$–knots and their cobordisms. In §3, we introduce the algebraic invariants, the Witt co-index and the pure Witt co-index, and study the Murasugi numbers of $(2n-1)$–knots. We give lower and upper bounds for the Murasugi numbers (see Propositions 3.13 and 3.14) and prove Theorem 3.15 about the Murasugi numbers of spherical knots. In §4, we study the Murasugi numbers of algebraic $(2n-1)$–knots K_f for $n \geqslant 4$ even such that $H_{n-1}(K_f)$ is torsion free, and prove Theorem 4.4.

Throughout the paper, we work in the smooth category. Homology and cohomology groups are with integer coefficients unless otherwise stated. For a $\mathbf{Z}$–module H, we denote by $\operatorname{Tors} H$ the torsion submodule of H.

2. Preliminaries

Let K be a $(2n-1)$–dimensional closed oriented submanifold of S^{2n+1} with $n \geqslant 1$. We will assume that K is $(n-2)$–connected. (Here, when $n = 1$, a submanifold is (-1)–connected if it is nonempty.) In this case, we say that K (or its isotopy class) is a *knot* (or a $(2n-1)$*–knot*). Note that K may not be homeomorphic to S^{2n-1}. If K is homeomorphic to S^{2n-1}, then K is called a *spherical knot*, otherwise a *non-spherical knot*.

For such a $(2n-1)$–knot K, it is known that there exists a compact oriented $2n$–dimensional submanifold F of S^{2n+1} such that $K = \partial F$. Such an F is called a *Seifert hypersurface* of K.

Definition 2.1. Let K be a $(2n-1)$–knot in S^{2n+1} and F be a Seifert hypersurface for K. We define the bilinear form

$$\theta_{K,F} : (H_n(F)/\operatorname{Tors} H_n(F)) \times (H_n(F)/\operatorname{Tors} H_n(F)) \to \mathbf{Z}$$

by $\theta_{K,F}(\alpha, \beta) = \operatorname{lk}(a_+, b)$, where a and b are n–cycles representing α and β, respectively, a_+ is the n–cycle in S^{2n+1} obtained by pushing a into the positive normal direction of F, and lk denotes the linking number of n–cycles in S^{2n+1}. The bilinear form $\theta_{K,F}$ is called the *Seifert form* of K (with respect to F) and its representative matrix is called a *Seifert matrix.* We also use the notation $\theta_K = \theta_{K,F}$ when there is no confusion, or when the Seifert hypersurface F is clear from the context.

Definition 2.2. For a $(2n-1)$–knot K, if there exists a Seifert hypersurface F which is $(n-1)$–connected, then K is called a *simple knot.* We can show that such an F is homotopy equivalent to a bouquet of finitely many n–spheres by using our assumption that K is $(n-2)$–connected.

Definition 2.3. Two $(2n-1)$–knots K_0 and K_1 in S^{2n+1} are said to be *cobordant* if there exists a properly embedded $2n$–dimensional oriented submanifold X, abstractly diffeomorphic to $K_0 \times [0,1]$, of $S^{2n+1} \times [0,1]$ such that $X \cap (S^{2n+1} \times \{0\}) = K_0 \times \{0\}$, and $X \cap (S^{2n+1} \times \{1\}) = -K_1^! \times \{1\}$, where $-K_1^!$ is the mirror image of K_1 with reversed orientation (see [2, 3]). In this case, X is called a *cobordism* between K_0 and K_1.

Note that if K_0 and K_1 are isotopic, then they are cobordant. However, the converse does not hold in general.

It has been proved that every $(2n-1)$–knot is cobordant to a simple knot by Levine [15] (see also [4]). Note that Levine proves the assertion for spherical knots; however, the same argument using embedded surgeries and an engulfing theorem works equally well for an arbitrary $(2n-1)$–knot. This implies the following.

Proposition 2.4. *Let K be a $(2n-1)$–knot in $S^{2n+1} = \partial D^{2n+2}$. Then, there exists a properly embedded compact oriented $2n$–dimensional submanifold G in D^{2n+2} homotopy equivalent to a bouquet of finitely many n–spheres such that $\partial G = K$.*

Proof. Let us identify $S^{2n+1} \times [0,1]$ with a collar neighborhood of ∂D^{2n+2} in D^{2n+2}. Then, as K is cobordant to a simple knot K', there exists a cobordism X in $S^{2n+1} \times [0,1]$ between K and K'. As K' is a simple knot, there is a Seifert hypersurface F' for K' in $S^{2n+1} \times \{1\}$ such that $\partial F' = K'$ and F' is $(n-1)$–connected, i.e., F' is homotopy equivalent to a bouquet of finitely many n–spheres. Then, $G = X \cup F'$ is a desired $2n$–dimensional submanifold of D^{2n+2} with $\partial G = K$. □

Definition 2.5. Let K be a $(2n-1)$–knot in $S^{2n+1} = \partial D^{2n+2}$. Then, the minimum nonnegative integer among $b_n(G)$ for all properly embedded compact oriented $2n$–dimensional submanifolds G in D^{2n+2} homotopy equivalent to a bouquet of finitely many n–spheres with $\partial G = K$, is called the *Murasugi number* of K, and is denoted by $g(K)$ [18] (see also [21]). Here, $b_n(G)$ is the n–th Betti number, i.e., the rank of $H_n(G)$. In other words, G is homotopy equivalent to the bouquet of $b_n(G)$ copies of n–spheres. By Proposition 2.4, $g(K)$ is a well-defined nonnegative integer. (For $n = 2$, see also [25].)

It is easy to see that the Murasugi number is a cobordism invariant:

Lemma 2.6. *If two $(2n-1)$–knots K_0 and K_1 are cobordant, then we have $g(K_0) = g(K_1)$.*

Definition 2.7. Let K be a simple $(2n-1)$–knot in S^{2n+1}. Then, the minimum nonnegative integer among $b_n(F)$ for all $(n-1)$–connected Seifert hypersurfaces for K in S^{2n+1}, is called the *Milnor number* of K, and is denoted by $\mu(K)$.

Lemma 2.8. *If K is a simple $(2n-1)$–knot in S^{2n+1}, then we always have $g(K) \leqslant \mu(K)$.*

Proof. Let F be an $(n-1)$-connected Seifert hypersurface for K in $S^{2n+1} = \partial D^{2n+2}$ such that $b_n(F) = \mu(K)$. Then, by pushing the interior of F into the interior of D^{2n+2}, we get a properly embedded submanifold of D^{2n+2} with boundary K, and with n–th Betti number equal to $\mu(K)$. Hence the result follows by virtue of the definition of the Murasugi number. □

Remark 2.9. Let G be a properly embedded compact oriented $2n$–dimensional submanifold in D^{2n+2} with $\partial G = K$ being a $(2n-1)$–knot in S^{2n+1}. We do not know if $G \setminus \operatorname{Int} C$ can be engulfed into the boundary of a $(2n+2)$–disk embedded in $\operatorname{Int} D^{2n+2}$, as in the proof of [15, Lemma 4] (see also [10, Theorem 2]), where C is a collar neighborhood of ∂G in G. If the answer is positive, then the Murasugi number of K would be equal to the minimum of Milnor numbers among all simple $(2n-1)$–knots cobordant to K.

Let us recall the following definition.

Definition 2.10. Let K be a $(2n-1)$–knot in S^{2n+1} with $n \geqslant 2$. A Seifert hypersurface F of K is said to be *exact* if the sequence

$$0 \to H_n(K) \to H_n(F)/\operatorname{Tors} H_n(F) \to H_n(F,K)/\operatorname{Tors} H_n(F,K) \to H_{n-1}(K) \to 0,$$

derived from the homology exact sequence for the pair (F,K), is well defined and exact [4]. Note that the homomorphism $H_n(F,K)/\operatorname{Tors} H_n(F,K) \to H_{n-1}(K)$ may not be well defined in general. Here, we impose the condition that this map should be well defined. A $(2n-1)$–knot K is said to be *exact* if it admits an exact Seifert hypersurface.

It has been known that simple $(2n-1)$–knots, fibered $(2n-1)$–knots, and spherical $(2n-1)$–knots are always exact [4].

In the following, for a finitely generated $\mathbf{Z}$–module H, we denote by $\gamma(H)$ the minimal number of its generators as $\mathbf{Z}$–module.

Proposition 2.11. *Let K be a $(2n-1)$–knot, $n \geqslant 2$. Then we always have $g(K) \geqslant \gamma(H_{n-1}(K))$.*

Proof. Let G be a compact oriented $2n$–dimensional manifold properly embedded in D^{2n+2} homotopy equivalent to a bouquet of n–spheres such that $\partial G = K$ and $b_n(G) = g(K)$. Then we have the following homology exact sequence for the pair (G,K):

$$H_n(G,F) \to H_{n-1}(K) \to 0,$$

where $H_n(G,F) \cong H^n(G) \cong \mathbf{Z}^{b_n(G)}$ by Poincaré–Lefschetz duality. This implies that $\gamma(H_{n-1}(K)) \leqslant b_n(G) = g(K)$. This completes the proof. □

Example 2.12. For $n \geqslant 2$, we can embed $S^n \times D^n$ in S^{2n+1}. We denote the embedded image by F and set $K = \partial F$, which is a $(2n-1)$–knot. Since the Seifert hypersurface F is $(n-1)$–connected, it is a simple knot. Then, since $H_{n-1}(K) \cong \mathbf{Z}$, we have, by Proposition 2.11,

$$1 \leqslant g(K) \leqslant \mu(K) \leqslant 1,$$

which implies that $g(K) = \mu(K) = 1$.

3. Co-index and pure co-index

Let A be a free $\mathbf{Z}$–module of finite rank and

$$\theta : A \times A \to \mathbf{Z}$$

a bilinear form, which may not be symmetric or non-degenerate. In the following, we fix a positive integer n and set $S = \theta + (-1)^n\theta^T$, i.e., $S(a,b) = \theta(a,b) + (-1)^n\theta(b,a)$ for $a, b \in A$. Later, such a bilinear form θ will be a Seifert form θ_K of a $(2n-1)$–knot K.

Definition 3.1. A submodule $U \subset A$ is said to be *totally isotropic* if $\theta(a,b) = 0$ for all $a, b \in U$. The maximal rank of a totally isotropic submodule for θ is called the *index* (or the *Witt index*) of θ and is denoted by $\operatorname{ind}(\theta)$. The *co-index* (or the *Witt co-index*) of θ, denoted by $\beta(\theta)$, is defined as

$$\beta(\theta) = \operatorname{rank} A - 2\operatorname{ind}(\theta).$$

See [8], for example.

Remark 3.2. Suppose U is a totally isotropic submodule for θ. Then, $\widehat{U} \subset A$, which is the smallest pure submodule of A containing U, is also totally isotropic. Here, a submodule of A is *pure* if it is a direct summand of A.

Definition 3.3. A submodule $U \subset A$ is said to be *purely totally isotropic* if the following two conditions are satisfied.

(1) The submodule U is pure.
(2) There exist a base $\alpha_1, \alpha_2, \dots, \alpha_k$ of U and a set of elements $\alpha_1^*, \alpha_2^*, \dots, \alpha_k^*$ of A such that $\theta(\alpha_i, \alpha_j) = 0$ and $S(\alpha_i, \alpha_j^*) = \pm\delta_{ij}$ for all $1 \leqslant i, j \leqslant k$, where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Note that then $\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_1^*, \alpha_2^*, \dots, \alpha_k^*$ constitute a base of a pure submodule of A.

The maximal rank of a purely totally isotropic submodule for θ is called the *pure index* (or the *pure Witt index*) of θ and is denoted by $\mathrm{ind}_{\mathrm{p}}(\theta)$. The *pure co-index* (or the *pure Witt co-index*) of θ, denoted by $\beta_{\mathrm{p}}(\theta)$, is defined as

$$\beta_{\mathrm{p}}(\theta) = \operatorname{rank} A - 2\,\mathrm{ind}_{\mathrm{p}}(\theta).$$

Remark 3.4. In Definition 3.3, the submodule U is totally isotropic. Hence, we always have

$$\mathrm{ind}(\theta) \geqslant \mathrm{ind}_{\mathrm{p}}(\theta) \quad \text{and} \quad \beta(\theta) \leqslant \beta_{\mathrm{p}}(\theta).$$

Example 3.5. For a positive integer m, let us consider the $2m \times 2m$ integer matrix L of the form

$$L = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix. We identify L with an integral bilinear form on $\mathbf{Z}^{2m}$. Since $\det L \neq 0$, we see that $\mathrm{ind}(L) = m$ (see [8, Definition 2.2] and the subsequent paragraph). However, since $S = L + L^T = 2L$ for n even and $S = L - L^T = 0$ for n odd, we see that there exist no pair of elements $x, y \in \mathbf{Z}^{2m}$ with $S(x, y) = \pm 1$. Hence we have $\mathrm{ind}_{\mathrm{p}}(L) = 0$. Consequently, we have

$$\beta(L) = 0 < 2m = \beta_{\mathrm{p}}(L).$$

Note that there exists a simple $(2n-1)$–knot K in S^{2n+1} with Seifert form isomorphic to L for $n \geqslant 3$. Since L is unimodular, such a knot K is a simple fibered knot (see [7]). For n even, we have $H_{n-1}(K) \cong (\mathbf{Z}/2\mathbf{Z})^{2m}$, and for n odd, we have $H_{n-1}(K) \cong \mathbf{Z}^{2m}$. Hence, K is non-spherical. Furthermore, by using Proposition 2.11, we can show that $g(K) = \mu(K) = 2m$.

Recall that if a spherical $(2n-1)$–knot K, $n \geqslant 3$, satisfies $\beta(\theta_K) = 0$ for a Seifert form θ_K (see Definition 3.10), then we have $g(K) = 0$ [15].

These observations show that, in the case of a general not necessarily spherical $(2n-1)$–knot, the co-index of a Seifert form may not be efficient for determining the Murasugi number.

Lemma 3.6. *Let θ be an integral bilinear form on a free $\mathbf{Z}$–module of finite rank. If $S = \theta + (-1)^n\theta^T$ is unimodular, then we have*

$$\mathrm{ind}(\theta) = \mathrm{ind}_{\mathrm{p}}(\theta) \quad \textit{and} \quad \beta(\theta) = \beta_{\mathrm{p}}(\theta).$$

Proof. Let $U \subset A$ be a totally isotropic submodule A of maximal rank. By Remark 3.2, we may assume that it is pure. Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be a base of U, where $k = \operatorname{rank} U$. Note that $\theta(\alpha_i, \alpha_j) = 0$ for all $1 \leqslant i, j \leqslant k$. Furthermore, since S is unimodular, we can find a set of elements $\alpha_1^*, \alpha_2^*, \dots, \alpha_k^*$ of A such that $S(\alpha_i, \alpha_j^*) = \pm\delta_{ij}$ for all $1 \leqslant i, j \leqslant k$. Therefore, U is purely totally isotropic. Hence, by definition, we get the desired equalities. □

Definition 3.7. Two integral bilinear forms are said to be *S-equivalent* if their representative matrices are transformed to each other by a finite sequence of congruences and the replacements:

$$A \leftrightarrow B = \begin{pmatrix} A & 0 & u \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A \leftrightarrow C = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ v & 0 & 0 \end{pmatrix}, \tag{3.1}$$

where u (or v) is a column (resp. row) vector with integer entries. (See also [16, §2], [8, §2.5].)

The Seifert form $\theta_{K,F}$ depends on the choice of the Seifert hypersurface F for K. However, by [16, Theorem 1], it is known that the S-equivalence class of a Seifert form is well-defined for a given $(2n-1)$–knot K, provided that K is spherical.

Remark 3.8. In general, even up to S-equivalence, the Seifert form is not well-defined. Here is an example.

Let $K = S^{n-1} \times S^n$ be the $(2n-1)$–knot embedded trivially in $S^{2n} \subset S^{2n-1}$, $n \geqslant 2$: i.e., $K = \partial D^n \times S^n \subset D^n \times S^n \subset \partial(D^n \times D^{n+1}) = \partial D^{2n+1} = S^{2n} \subset S^{2n+1}$. Then, K bounds two Seifert hypersurfaces $F_0 = D^n \times S^n$ and $F_1 = S^{n-1} \times D^{n+1}$, both of which are embedded in $S^{2n} \subset S^{2n+1}$. Since F_0 is $(n-1)$–connected, K is a simple $(2n-1)$–knot. Furthermore, since F_0 is exact, K is an exact $(2n-1)$–knot. (Refer also to Example 2.12.) On the other hand, we see easily that F_1 is not exact, since $H_n(S^{n-1} \times S^n) \to H_n(S^{n-1} \times D^{n+1})$ is not a monomorphism. Furthermore, $H_n(F_0) \cong \mathbf{Z}$, while $H_n(F_1) = 0$. Therefore, the $\mathbf{Z}$–module for which the Seifert form is defined with respect to F_0 and that for F_1 have distinct ranks modulo 2. Therefore, the Seifert forms are not S-equivalent to each other.

This means that the Seifert form is not well-defined in general up to S-equivalence even for simple or exact $(2n-1)$–knots.

The following lemma is proved in [8, Proposition 2.14].

Lemma 3.9. *Let θ_1 and θ_2 be integral bilinear forms on free $\mathbf{Z}$–modules of finite ranks. If they are S-equivalent, then we have $\beta(\theta_1) = \beta(\theta_2)$.*

We do not know if $\beta_{\mathrm{p}}(\theta_1) = \beta_{\mathrm{p}}(\theta_2)$ holds for S-equivalent integral bilinear forms θ_1 and θ_2.

Definition 3.10. Let K be a spherical $(2n-1)$–knot in S^{2n+1}. The *co-index* $\beta(K)$ of K is the Witt co-index $\beta(\theta_{K,F})$ for a Seifert form $\theta_{K,F}$ of K with respect to a Seifert hypersurface F. This does not depend on the choice of F and is well-defined by virtue of [16, Theorem 1] and Lemma 3.9.

Remark 3.11. (1) Let L be an integral square matrix. We say that L is *admissible* if $\det(L - tL^T)$ is non-zero as a polynomial in t. Note that L is admissible if and only if $\det(L + tL^T) \neq 0$. Two integral square matrices L and L' are said to be *concordant* if $\beta(L \oplus (-L')) \leqslant 0$. By [8, Proposition 2.1], if two admissible matrices L and L' are concordant, then we have $\beta(L) = \beta(L')$.

(2) Let K be a $(2n-1)$–knot in S^{2n+1}, which may not be spherical. Then, its Alexander polynomial $\Delta_K(t)$ can be defined by using the n–th homology group of the the infinite cyclic covering of $S^{2n+1} \smallsetminus K$. In general, it is well-defined as an element of $\mathbf{Q}[t, t^{-1}]$ up to a multiple of rt^m for some $r \in \mathbf{Q} \smallsetminus \{0\}$ and $m \in \mathbf{Z}$ (for details, see [23], for example). See also [8], where a normalized Alexander polynomial is defined as an element of $\mathbf{Q}[t]$.

(3) By the above remark (1), for $(2n-1)$–knots K and K' whose Alexander polynomials are non-zero, if K and K' are concordant, then we have $\beta(K) = \beta(K')$.

On the other hand, for the example $K = S^{n-1} \times S^n$ given in Remark 3.8, we see that the n–th homology of the infinite cyclic covering of $S^{2n+1} \smallsetminus K$ is isomorphic to

$\mathbf{Q}[t, t^{-1}]$, hence $\Delta_K(t) = 0$. Furthermore, we see easily that $\beta(\theta_{K,F_0}) = -1$, while we have $\beta(\theta_{K,F_1}) = 0$. Therefore, even for a given $(2n-1)$–knot K, if $\Delta_K(t) = 0$, then the co-index of its Seifert form may not be well-defined.

(4) For $n = 1$, it is known that if K and K' are 1–knots and if the Alexander polynomial of one of the two knots is non-zero, then so is that of the other one (see [13]). We do not know if the same is true for $n \geqslant 2$ as well.

Remark 3.12. Let L be an integral square matrix. If $\beta(L) < 0$, then we can show that $\det(L + (-1)^n tL^T) = 0$. In particular, if the Alexander polynomial of a $(2n-1)$–knot K is nonzero, then $\beta(\theta_K) \geqslant 0$.

Proposition 3.13. *Let K be a simple $(2n-1)$–knot in S^{2n+1} with $n \geqslant 3$. Then, we have*

$$\beta(\theta_K) \leqslant g(K) \leqslant \beta_{\mathrm{p}}(\theta_K),$$

where $\theta_K = \theta_{K,F}$ is the Seifert form of K with respect to an arbitrary $(n-1)$-connected Seifert hypersurface F for K.

Proof. The inequality $g(K) \leqslant \beta_{\mathrm{p}}(\theta_K)$ follows from an argument using the embedded surgery technique as in [14, 18], since $n \geqslant 3$.

Let us now show $\beta(K) \leqslant g(K)$. Let G be a properly embedded compact oriented $2n$–dimensional submanifold in D^{2n+2} homotopy equivalent to a bouquet of finitely many n–spheres with $\partial G = K$ such that $b_n(G) = g(K)$. Let F be an $(n-1)$-connected Seifert hypersurface for K with respect to which $\theta_K = \theta_{K,F}$ is defined. Note that $F \cap G = K$. Then, by a standard argument, we see that there exists a compact orientable $(2n+1)$–dimensional submanifold W of D^{2n+2} such that $\partial W = F \cup G$. We set $X = \partial W$. We see easily that X is a closed orientable $(n-1)$-connected $2n$–dimensional manifold with $b_n(X) = b_n(F) + b_n(G)$.

Let us consider the homology exact sequence for the pair (W, X) with integer coefficients:

$$\begin{aligned} & H_{n+1}(X) \xrightarrow{j_*} H_{n+1}(W) \to H_{n+1}(W, X) \\ \to \quad & H_n(X) \xrightarrow{j_*} H_n(W) \to H_n(W, X) \to H_{n-1}(X), \end{aligned}$$

where $j : X \to W$ is the inclusion and we have $H_{n+1}(X) = H_{n-1}(X) = 0$, since $n \geqslant 2$. Hence, we have the exact sequences

$$\begin{aligned} & 0 \to H_{n+1}(W) \to H_{n+1}(W, X) \to \operatorname{Ker} j_* \to 0, \text{ and} \\ & 0 \to \operatorname{Ker} j_* \to H_n(X) \to H_n(W) \to H_n(W, X) \to 0, \end{aligned}$$

where we have $H_{n+1}(W, X) \cong H^n(W)$ and $H_n(W, X) \cong H^{n+1}(W)$ by virtue of the Poincaré–Lefschetz duality. Therefore, we have

$$\begin{aligned} & \operatorname{rank} \operatorname{Ker} j_* = b_n(W) - b_{n+1}(W) \quad \text{and} \\ & b_{n+1}(W) = b_n(W) - b_n(X) + \operatorname{rank} \operatorname{Ker} j_*, \end{aligned}$$

which immediately implies

$$\operatorname{rank} \operatorname{Ker} j_* = b_n(X)/2 = (b_n(F) + b_n(G))/2. \tag{3.2}$$

For the moment, we consider homology with rational coefficients. By the exact sequence for the pair (X, F),

$$H_{n+1}(X, F; \mathbf{Q}) \to H_n(F; \mathbf{Q}) \xrightarrow{i_*} H_n(X; \mathbf{Q}),$$

we see that $i_* : H_n(F; \mathbf{Q}) \to H_n(X; \mathbf{Q})$ is injective, where $i : F \to X$ is the inclusion and $H_{n+1}(X, F; \mathbf{Q}) \cong H_{n+1}(G, \partial G; \mathbf{Q}) \cong H^{n-1}(G; \mathbf{Q}) = 0$ by excision and the Poincaré–Lefschetz duality. Hence, we may regard $H_n(F; \mathbf{Q})$ as a subspace of $H_n(X; \mathbf{Q})$.

Now, let us consider the $\mathbf{Q}$-vector spaces

$$\operatorname{Ker} j_* + H_n(F;\mathbf{Q}) \subset H_n(X;\mathbf{Q}),$$

where $j_* : H_n(X;\mathbf{Q}) \to H_n(W;\mathbf{Q})$. Since they are $\mathbf{Q}$–vector spaces, we have

$$\dim(\operatorname{Ker} j_* + H_n(F;\mathbf{Q})) \leqslant \dim H_n(X;\mathbf{Q}),$$

where dim denotes the dimension over $\mathbf{Q}$. Hence, we have

$$\dim \operatorname{Ker} j_* + b_n(F) - \dim(\operatorname{Ker} j_* \cap H_n(F;\mathbf{Q})) \leqslant b_n(F) + b_n(G).$$

By (3.2) we have

$$\dim(\operatorname{Ker} j_* \cap H_n(F;\mathbf{Q})) \geqslant (b_n(F) - b_n(G))/2.$$

This implies

$$\operatorname{rank}(\operatorname{Ker} j_* \cap H_n(F)) \geqslant (b_n(F) - b_n(G))/2,$$

where $j_* : H_n(X) \to H_n(W)$. Note that $\operatorname{Ker} j_* \cap H_n(F)$ is a totally isotropic submodule for θ_K, which can be seen by the same argument as in the proof of [18, Lemme 1]. Therefore, we have

$$\operatorname{ind}(\theta_K) \geqslant (b_n(F) - b_n(G))/2,$$

which implies

$$\beta(\theta_K) = b_n(F) - 2\operatorname{ind}(\theta_K) \leqslant b_n(G) = g(K).$$

This completes the proof. □

We do not know if a statement as in Proposition 3.13 is true for general $(2n-1)$–knots which may not be simple. However, for exact knots (see Definition 2.10), we have the following.

Proposition 3.14. *Let K be an exact $(2n-1)$–knot in S^{2n+1} with $n \geqslant 3$. Then, we have*

$$\beta(\theta_K) \leqslant g(K) \leqslant \beta_{\mathrm{p}}(\theta_K),$$

where $\theta_K = \theta_{K,F}$ is the Seifert form of K with respect to an arbitrary exact Seifert hypersurface F for K.

Proof. By [4, Proposition 3.8], K is cobordant to a simple $(2n-1)$–knot K' such that the Seifert form $\theta_{K'} = \theta_{K',F'}$ of K' associated with an $(n-1)$–connected Seifert hypersurface F' is isomorphic to θ_K. By Proposition 3.13, we have

$$\beta(\theta_{K'}) \leqslant g(K') \leqslant \beta_{\mathrm{p}}(\theta_{K'}).$$

Now, by by Lemma 2.6, we have $g(K) = g(K')$. Furthermore, we have $\beta(\theta_{K'}) = \beta(\theta_K)$ and $\beta_{\mathrm{p}}(\theta_{K'}) = \beta_{\mathrm{p}}(\theta_K)$, since $\theta_{K'}$ is isomorphic to θ_K. Hence, the required result follows. □

Since, spherical knots are exact (see [4, Lemma 3.3]), combining Proposition 3.14 with Lemma 3.6, we get the following, which is implicitly obtained in [18] in the case of simple spherical knots.

Theorem 3.15. *Let K be a spherical $(2n-1)$–knot in S^{2n+1} with $n \geqslant 3$. Then, we have*

$$g(K) = \beta(\theta_K),$$

where $\theta_K = \theta_{K,F}$ is the Seifert form of K with respect to an arbitrary Seifert hypersurface F for K. In particular, we have

$$g(K) = \beta(K).$$

When K is a $\mathbf{Z}$–homology 3–sphere, see also [25, Theorem 3.3], where an upper bound for the Murasugi number is given in terms of the signature of K and the existence of a certain abstract 4–dimensional manifold bounded by K.

Remark 3.16. Theorem 3.15 does not hold for $n = 2$ in general. For example, let K be a simple spherical 3–knot such that its Seifert form $\theta_K = \theta_{K,F}$ with respect to a 1–connected Seifert hypersurface F is S-equivalent to a bilinear form τ (in the sense of [16, §2]) such that $\tau + \tau^T$ is a positive definite unimodular bilinear form with rank 16. (For example, consider the form corresponding to $L_1 \oplus L_1$ for the matrix L_1 constructed in [24, §3].) The existence of such a 3–knot follows from [16, Theorem 2]. By [8, Proposition 2.14], the co-indices for S-equivalent forms are the same. On the other hand, if the index of τ is positive, then we see easily that $\tau + \tau^T$ is not definite, which is a contradiction. Therefore, we have $\mathrm{ind}(\tau) = 0$, and hence $\beta(\theta_K) = \beta(\tau) = 16$. Suppose that $g(K) = 16$. Let G be a properly embedded compact 1–connected submanifold of D^6 bounded by K with $b_2(G) = g(K) = 16$. Then, we can show that $G \cup D^4$ is a smooth closed 1–connected spin 4–dimensional manifold with positive definite intersection form of rank 16, where we attach G and D^4 along their sphere boundaries. This contradicts the celebrated theorem of Donaldson [5]. Hence, we have $\beta(\theta_K) < g(K)$, although K is a simple spherical 3–knot.

Remark 3.17. For a $(2n-1)$–knot K in S^{2n+1} let $K(2)$ denote its 2–fold cyclic suspension, which is a $(2n+1)$–knot in S^{2n+3} (see [12, 22]). Furthermore, we denote by $K(2,2)$ the 2–fold cyclic suspension of $K(2)$, which is a $(2n+3)$–knot in S^{2n+5}. It is known that an arbitrary Seifert form θ_K of K is isomorphic to a Seifert form $\theta_{K(2,2)}$ of $K(2,2)$. Hence, we have $\beta(\theta_K) = \beta(\theta_{K(2,2)})$. Therefore, if K is spherical, then $K(2,2)$ is also spherical and if $n \geqslant 3$, then we have

$$g(K) = \beta(K) = \beta(K(2,2)) = g(K(2,2))$$

by Theorem 3.15. We do not know if $g(K) = g(K(2,2))$ holds for non-spherical $(2n-1)$–knots K in general.

4. Simple and almost-simple singularities

Let $f : (\mathbf{C}^{n+1}, \mathbf{0}) \to (\mathbf{C}, 0)$, $n \geqslant 1$, be a holomorphic function germ with an isolated critical point at the origin. For a sufficiently small positive real number $\varepsilon > 0$, set $K_f = S_\varepsilon^{2n+1} \cap V_f$, where $V_f = f^{-1}(0)$ is the complex hypersurface in $\mathbf{C}^{n+1}$ with an isolated singularity at the origin and S_ε^{2n+1} is the sphere of radius ε centered at the origin in $\mathbf{C}^{n+1}$. It is known that K_f is an $(n-2)$–connected, closed oriented $(2n-1)$–dimensional submanifold of $S_\varepsilon^{2n+1} = S^{2n+1}$, that its complement fibers over the circle S^1, and that the isotopy class of K_f in S^{2n+1} is independent of the choice of ε as long as it is sufficiently small (see [19]). Note also that the embedded topology of $V_f \subset \mathbf{C}^{n+1}$ around the origin determines and is determined by the (oriented) isotopy class of $K_f \subset S^{2n+1}$ (see [26]). We call K_f the *algebraic knot* associated with f. The closure of a fiber of the fibration $S_\varepsilon^{2n+1} \smallsetminus K_f \to S^1$ mentioned above is called the *Milnor fiber* for f and is denoted by F_f. It is a Seifert hypersurface for K_f and is $(n-1)$–connected. Furthermore, we can show that $\mu(K_f) = b_n(F_f)$, the *Milnor number* of f.

Definition 4.1. The critical point (or the singularity) of f is *simple* (resp., *almost-simple*) if the symmetric bilinear form $Q_f = \theta_f + \theta_f^T$ is definite (resp., not definite, but semi-definite), where $\theta_f = \theta_{K_f,F_f}$ is the Seifert form for K_f with respect to the Seifert hypersurface F_f [6].

Michel [18] showed that when K_f is spherical with $n \geqslant 3$, then $g(K_f) = \mu(K_f)$ if and only if the singularity of f is simple.

Remark 4.2. Note that for $n \geqslant 3$, K_f is spherical if and only of $S = \theta_f + (-1)^n \theta_f$ is unimodular. On the other hand, in the proof provided in [18], the author argues that if $Q_f = \theta_f + \theta_f^T$ is indefinite over the integers, then there exists a non-zero element $x \in H_n(F_f)$ such that $Q_f(x, x) = 0$. As Q_f may not be unimodular when n is odd, this seems to be nontrivial. In fact, if $\mu(K_f) \geqslant 5$, then the existence of such an x follows from Meyer's Theorem [17] (see also [9, Corollary 5.10], for example). For $\mu(K_f) \leqslant 4$, according to the classification of isolated critical points of small Milnor numbers (see [1] or [6, Table 2]), the singularity is simple, so that Q_f is never indefinite. Hence, the argument in [18] works even for n odd.

Remark 4.3. When n is odd, consider the unimodular matrix

$$L = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

There exists a simple $(2n-1)$–knot K with $(n-1)$–connected Seifert hypersurface such that the associated Seifert matrix is given by L (for example, see [7, 11]). Then, as $L - L^T$ is unimodular, we see that K is spherical. On the other hand, we have

$$Q = L + L^T = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix},$$

which is indefinite. However, it is an easy exercise to show that there exists no non-zero $x \in \mathbf{Z}^2$ with $x^T Q x = 0$.

Therefore, even in the case of a spherical knot K_f, for n odd, we do not know if there exists an x with $Q_f(x, x) = 0$ in general, for $\mu(K_f) \leqslant 4$.

Now, we have the following theorem, which generalizes Michel's result [18] mentioned above for n even.

Theorem 4.4. *Suppose $n \geqslant 4$ is even and that $H_{n-1}(K_f)$ is torsion free. Then $g(K_f) = \mu(K_f)$ if and only if the singularity of f is simple or almost-simple.*

Proof. First, suppose that the singularity of f is simple or almost-simple. Then, the bilinear form $Q_f = S = \theta_f + \theta_f^T$ is definite or semi-definite: in other words, $|\sigma(F_f)| = |\sigma(S)| = \operatorname{rank} S$, where σ denotes the signature of a $2n$–dimensional manifold or that of a symmetric integer matrix. Now, suppose $g(K_f) < \mu(K_f)$ and there exists a properly embedded compact oriented $2n$–dimensional submanifold G of D^{2n+2} such that $\partial G = K_f$ and $b_n(G) < b_n(F_f)$. As $F_f \cup (-G)$ bounds a compact oriented $(2n+1)$–dimensional submanifold W of D^{2n+2}, we see that

$$\sigma(F_f \cup (-G)) = \sigma(F_f) - \sigma(G) = 0$$

by the Novikov additivity and hence $\sigma(F_f) = \sigma(G)$. On the other hand, consider the homology exact sequences of the pairs (F_f, K_f) and (G, K_f), respectively:

$$H_n(K_f) \to H_n(F_f) \to H_n(F_f, K_f) \to H_{n-1}(K_f), \text{ and}$$
$$H_n(K_f) \to H_n(G) \to H_n(G, K_f) \to H_{n-1}(K_f).$$

We see that the image of $H_n(K_f)$ in $H_n(F_f)$ (resp., in $H_n(G)$) coincides with the kernel of the intersection form of F_f (resp., of G) by virtue of the Poincaré duality. Hence, we have $|\sigma(F_f)| = b_n(F_f) - b_n(K_f)$ and $|\sigma(G)| \leqslant b_n(G) - b_n(K_f)$. As we have $\sigma(F_f) = \sigma(G)$, this implies that $b_n(F_f) \leqslant b_n(G)$, which is a contradiction. Hence, we must have $g(K_f) = \mu(K_f)$.

Conversely, suppose that $g(K_f) = \mu(K_f)$ holds. Let us assume that $S = Q_f$ is not definite, nor semi-definite. By the above exact sequence, we see that the intersection form S of F_f is isomorphic to a symmetric bilinear form of the form $S' \oplus 0_\ell$, where S' is an integral non-degenerate symmetric bilinear form on a pure free submodule A' of $H_n(F_f)$ of rank $b_n(F_f) - \ell$, and 0_ℓ is the zero form on the pure free submodule of rank $\ell = b_n(K_f)$. By our assumption that $H_{n-1}(K_f)$ is torsion free, we see that $S' : A' \times A' \to \mathbf{Z}$ is unimodular. As S' is indefinite and is unimodular, we see that there exists a non-zero element $x' \in A'$ such that

$S'(x', x') = 0$. This implies that $\theta_f(x', x') = 0$. Since S' is unimodular, with the help of x', we can find a purely totally isotropic submodule of rank 1 for θ_f. This implies that the pure co-index of θ_f is smaller than or equal to $\mu(K_f)-2$. Hence, by Proposition 3.13, we see that $g(K_f) \leqslant \mu(K_f) - 2$. This is a contradiction. Hence, Q_f must be definite or semi-definite, and the singularity of f must be simple or almost-simple [6]. This completes the proof. □

We do not know if Theorem 4.4 holds for $n \geqslant 3$ odd, or when $H_{n-1}(K_f)$ is not free.

Example 4.5. Let $2 \leqslant p < q$ be co-prime integers and consider the Brieskorn polynomial

$$f(z_1, z_2, \ldots, z_{n+1}) = z_1^p + z_2^q + z_3^{pq} + z_4^2 + \cdots + z_{n+1}^2,$$

where $n \geqslant 2$ is even. When $n = 2$, the link K_f is the so-called Brieskorn 3–manifold $\Sigma(p, q, pq)$ (or $M(p, q, pq)$). By [20, Theorem 7.3], $\Sigma(p, q, pq)$ is a circle bundle of Euler number (or Chern number) -1 over the closed connected orientable surface of Euler characteristic

$$pq(p^{-1} + q^{-1} + (pq)^{-1} - 1) = 2 - (p - 1)(q - 1).$$

Hence, by a straightforward Meyer–Vietoris exact sequence argument, for example, we see that $H_1(K_f)$ is free of rank $(p - 1)(q - 1)$. In particular, it has no torsion. Hence, for general $n \geqslant 2$ even, by the periodicity of the link homology of singularities, we see that $H_{n-1}(K_f)$ is free of rank $(p - 1)(q - 1) > 0$.

According to [6], the singularity of f is never simple, and it is almost-simple if and only if $(p, q, pq) = (2, 3, 6)$. Otherwise, the singularity of f is not almost-simple. Hence, by our Theorem 4.4 we see that for $(p, q, pq) \neq (2, 3, 6)$ the Murasugi number $g(K_f)$ is strictly smaller than the Milnor number $\mu(K_f)$ provided $n \geqslant 4$ is even. On the other hand, for $(p, q, pq) = (2, 3, 6)$, we have $g(K_f) = \mu(K_f)$. Note that the singularity of f at the origin for $(p, q, pq) = (2, 3, 6)$ is almost-simple and that $H_{n-1}(K_f)$ is torsion free.

When $(p, q, pq) \neq (2, 3, 6)$, we do not know if there exists a simple $(2n - 1)$–knot K' cobordant to K_f such that $\mu(K') < \mu(K_f)$.

Example 4.6. Recall that in [25], the following example for $n = 2$ has been given. For the Brieskorn polynomial

$$f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^{11},$$

K_f is a $\mathbf{Z}$–homology 3–sphere, and the singularity of f is not simple nor almost-simple. Nevertheless, we have $g(K_f) = \mu(K_f)$. Such a phenomenon happens because of the non-existence of a certain compact 4–dimensional manifold with boundary K_f, which follows from the gauge theory. This example shows that Theorem 4.4 does not hold for $n = 2$.

Acknowledgment

The second author would like to thank the Institut de Recherche Mathématique Avancée (IRMA, UMR 7501) of CNRS and the University of Strasbourg for the hospitality during the preparation of the manuscript. This work has been supported in part by JSPS KAKENHI Grant Numbers JP22K18267, JP23H05437, JP25K24558.

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