COMPTES RENDUS

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The conformal boundary of Margulis space–times

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Abstract

In this Note, we show how to construct the conformal boundary of Margulis space–times $\mathbf{R}^{1,2}/\Gamma$ when Γ is an affine Schottky group. ***To cite this article: C. Frances, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***
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Résumé

Le bord conforme des espaces–temps de Margulis. Dans cette Note, nous montrons comment construire le bord conforme des espaces–temps de Margulis $\mathbf{R}^{1,2}/\Gamma$ lorsque Γ est un groupe de Schottky affine. ***Pour citer cet article : C. Frances, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***
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En 1977, Milnor posait dans [7] la question suivante : *le groupe fondamental d'une variété affine complète plate est-il toujours virtuellement résoluble* ?

Six années plus tard, Margulis apporta une réponse négative à cette question en exhibant des groupes libres de transformations affines agissant proprement discontinûment sur $\mathbf{R}^3$. Pour cela, il montra que certains groupes de Schottky $H \subset \mathrm{SO}^0(1,2)$ admettent des déformations affines propres Γ dans $\mathrm{SO}^0(1,2) \ltimes \mathbf{R}^3$, c'est-à-dire telles que le groupe déformé obtenu agisse proprement discontinûment sur l'espace de Minkowski $\mathbf{R}^{1,2}$ (voir [6] et [3]). On appela les variétés quotients $\mathbf{R}^{1,2}/\Gamma$ ainsi construites *espace–temps de Margulis* et depuis, ce nom désigne toute variété obtenue en quotientant $\mathbf{R}^{1,2}$ par un groupe libre d'isométries lorentziennes. Les espaces–temps de Margulis sont des variétés non compactes naturellement dotées d'une métrique lorentzienne complète plate et notre but dans cette Note est d'en comprendre l'infini conforme en construisant leur bord conforme. Rappelons que l'espace de Minkowski admet lui-même une compactification conforme : l'univers d'Einstein, noté $\mathbf{Ein}_3$. Nous donnerons plus de détails sur l'espace $\mathbf{Ein}_3$ dans la deuxième section de cette Note, mais pour le moment, précisons juste qu'il est muni d'une structure *conforme lorentzienne* canonique. Pour cette structure, l'espace de Minkowski se plonge conformément comme un ouvert de $\mathbf{Ein}_3$, dont la frontière $\mathbf{C}_\infty$ s'interprète comme le bord conforme de $\mathbf{R}^{1,2}$.

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Le lien avec les espaces–temps de Margulis vient du fait qu'il existe sur **Ein**$_3$ un analogue au théorème de Liouville sur la sphère, qui assure que toute transformation conforme de $\mathbf{R}^{1,2}$ s'étend en une transformation conforme globale de **Ein**$_3$. Ainsi, si l'on considère un espace–temps de Margulis $\mathbf{R}^{1,2}/\Gamma$, le groupe Γ agit sur **Ein**$_3$ comme un groupe discret de transformations conformes. Cette action est proprement discontinue sur un ouvert de **Ein**$_3$ conformément équivalent à l'espace de Minkowski et une question naturelle est de savoir si l'action de Γ reste proprement discontinue sur un ouvert strictement plus grand, rencontrant la frontière $\mathbf{C}_\infty$. Nous allons répondre positivement à cette question dans le cas où Γ est *un groupe de Schottky affine*, comme ceux introduits par Drumm dans [1]. Il s'agit de sous-groupes $\Gamma \subset \mathrm{Isom}(\mathbf{R}^{1,2})$ ayant une dynamique de type *ping–pong* (voir Section 2.1), pour lesquels Drumm décrivit des domaines fondamentaux bordés par ce qu'il appela des « *crooked planes* » [1].

Notre résultat principal peut s'énoncer comme suit :

Théorème 0.1. *Pour tout groupe de Schottky affine $\Gamma \subset \mathrm{Is}(\mathbf{R}^{1,2})$, l'espace–temps de Margulis $M_\Gamma = \mathbf{R}^{1,2}/\Gamma$ admet un bord conforme ∂M_Γ. Plus précisément, il existe un fermé strict de $\mathbf{C}_\infty$ noté Λ_Γ^∞ tel que :*

(i) *L'action de Γ est libre et proprement discontinue sur l'ouvert $\Omega_\Gamma = \mathbf{Ein}_3 \setminus \Lambda_\Gamma^\infty$, et Ω_Γ est le plus grand ouvert de **Ein**$_3$ contenant $\mathbf{R}^{1,2}$ à avoir cette propriété.*

(ii) *L'espace quotient $(\mathbf{R}^{1,2} \cup \Omega_\Gamma^\infty)/\Gamma$ (où $\Omega_\Gamma^\infty = \mathbf{C}_\infty \setminus \Lambda_\Gamma^\infty$) est une variété à bord, lisse, et dont l'intérieur est conformément équivalent à l'espace–temps de Margulis M_Γ. Le bord $\Omega_\Gamma^\infty/\Gamma$ est constitué d'un nombre fini de cylindres $\mathbf{R} \times \mathbf{S}^1$, chacun muni d'une classe conforme de métriques dégénérées.*

(iii) *Il existe une compactification de $\Omega_\Gamma^\infty/\Gamma$ par deux points, notée ∂M_Γ. L'espace $M_\Gamma \cup \partial M_\Gamma$ est singulier en ces deux points. Toute demi-géodésique de M_Γ, qui sort de tout compact dans M_Γ, converge vers un unique point de ∂M_Γ.*

1. Introduction

The interest for Margulis space–times comes from the study of complete flat affine manifolds, especially from questions about their fundamental group. In 1977, Milnor asked in [7] if the fundamental group of such manifolds is always solvable. Together, he gave a way to find possible counter-examples. The idea was to take a free group inside $\mathrm{SO}^0(1,2)$ and to show that it possesses some affine deformation in $\mathrm{SO}^0(1,2) \ltimes \mathbf{R}^3$, which acts properly discontinuously on Minkowski space $\mathbf{R}^{1,2}$. One had then to wait six years, before Margulis managed to let this method work, proving that some Schottky groups $H \subset \mathrm{SO}^0(1,2)$ really admit *proper deformations* in $\mathrm{Isom}(\mathbf{R}^{1,2}) = \mathrm{SO}^0(1,2) \ltimes \mathbf{R}^3$ (see [6,3]). The quotient manifolds were called *Margulis space–times* and since that time, so was called any manifold obtained as a quotient of $\mathbf{R}^{1,2}$ by a free subgroup of Lorentzian isometries. Afterwards, Drumm showed how to obtain wide classes of Margulis space–times, constructing *affine Schottky groups*. These are subgroups $\Gamma \subset \mathrm{Isom}(\mathbf{R}^{1,2})$ having a "ping–pong-like" dynamics (see Section 2.1). Drumm also exhibited some fundamental domains for these groups, bounded by what he called *crooked planes* [1]. Thanks to these techniques, he proved in [2] that every free discrete group of $\mathrm{SO}^0(1,2)$ admits some proper affine deformation in $\mathrm{Isom}(\mathbf{R}^{1,2})$.

Our aim in this Note is, leaving the affine framework aside, to see Margulis space–times as flat, and in particular conformally flat, Lorentz manifolds (i.e., the geometry of which is locally modeled on the conformal structure of Minkowski space). Now, conformally flat Lorentz manifolds admit, among them, a "universal space", called *Einstein universe* and denoted by **Ein**$_3$. It is universal in the following sense: every conformally flat Lorentz manifold develops in it. Thus, Einstein universe is the Lorentzian conformal analogue of the sphere. We will give more details on **Ein**$_3$ in Section 2 of this Note, but for the moment, we only need to know that the space $\mathbf{R}^{1,2}$ is conformally equivalent to an open subset of Einstein universe. The boundary of this open subset, denoted by $\mathbf{C}_\infty$, is called the conformal boundary of Minkowski space. It is a torus pinched in two points and thus, it is singular. The link with Margulis space–times comes from a Lorentzian analogue, on **Ein**$_3$, of Liouville's theorem on the sphere: any conformal transformation of $\mathbf{R}^{1,2}$ extends to a global conformal transformation of **Ein**$_3$. Thus, if we consider a Margulis space–time $\mathbf{R}^{1,2}/\Gamma$, the group Γ acts on **Ein**$_3$ as a discrete group of conformal transformations. This

action is properly discontinuous on a subset conformally equivalent to $\mathbf{R}^{1,2}$, and a natural question is to know if the action remains properly discontinuous on a greater open set, intersecting the boundary $\mathbf{C}_\infty$. In this Note, we answer positively to this question in the case where Γ is an affine Schottky group, like those introduced by Drumm. In particular, it allows us to build a conformal boundary to the Margulis space–times associated to these groups. Let us precise that the notion of conformal boundary is in general quite difficult to deal with, especially for Lorentzian manifolds. For us, constructing a conformal boundary of a Lorentzian manifold M, means finding a manifold with boundary $\widehat{M}$, endowed with a Lorentzian conformal structure, such that the interior of $\widehat{M}$ is conformally equivalent to M. Then, we call $\partial\widehat{M}$ the conformal boundary of M. Notice that this boundary may be singular, as it is the case for Minkowski space.

Our main result can be stated as follows:

Theorem 1.1. *For each affine Schottky group $\Gamma \subset \mathrm{Is}(\mathbf{R}^{1,2})$, the Margulis space–time $M_\Gamma = \mathbf{R}^{1,2}/\Gamma$ admits a conformal boundary ∂M_Γ. More precisely, there is a closed subset Λ_Γ^∞ of $\mathbf{C}_\infty$, such that*:

 (i) *The action of Γ is free and properly discontinuous on $\Omega_\Gamma = \mathbf{Ein}_3 \setminus \Lambda_\Gamma^\infty$, and Ω_Γ is the biggest open subset of $\mathbf{Ein}_3$ containing $\mathbf{R}^{1,2}$, having this property.*
 (ii) *The quotient space $(\Omega_\Gamma^\infty \cup \mathbf{R}^{1,2})/\Gamma$ (where $\Omega_\Gamma^\infty = \mathbf{C}_\infty \setminus \Lambda_\Gamma^\infty$) is a smooth manifold with boundary, whose interior is conformally equivalent to the Margulis space–time M_Γ. The boundary $\Omega_\Gamma^\infty/\Gamma$ is a finite union of cylinders $\mathbf{R} \times \mathbf{S}^1$, each one endowed with a conformal class of degenerate metrics.*
 (iii) *There is a two points compactification of $\Omega_\Gamma^\infty/\Gamma$ denoted by ∂M_Γ. The space $M_\Gamma \cup \partial M_\Gamma$ is singular in these two points. Each semi-geodesic of M_Γ leaving any compact subset of M_Γ, tends to a unique point of ∂M_Γ.*

2. Drumm's affine Schottky groups

2.1. Generalities on Schottky dynamics

A discrete finitely generated group $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ acting by diffeomorphisms on a manifold X is called a *Schottky group* if it admits a *ping-pong dynamics*, namely there exists $2n$ open sets $U_i^\pm$ ($i = 1, \ldots, n$) with disjoint closures $\mathrm{cl}(U_i^\pm)$ in X such that for each i, $\gamma_i(X \setminus \mathrm{cl}(U_i^-)) = U_i^+$. The complement D of the union of the $U_i^\pm$ is called *a fundamental domain* for Γ and one has the following lemma (see, for example, [5]):

Lemma 2.1. *The set $\Omega = \bigcup_{\gamma \in \Gamma} \gamma(D)$ is open and Γ acts freely properly discontinuously on Ω.*

2.2. Drumm's crooked planes

Following [1] and [4], we begin with some notations.

- Minkowski space $\mathbf{R}^{1,2}$ is the space $\mathbf{R}^3$ endowed with the quadratic form $q^{1,2}(x) = -x_1^2 + x_2^2 + x_3^2$.
- The set of future oriented isotropic vectors of $\mathbf{R}^{1,2}$, of Euclidean norm 1, is a circle denoted by $\mathcal{C}$.
- Given a spacelike vector w (i.e., a vector such that $q^{1,2}(w) > 0$), we can define two isotropic vectors x_w^- and x_w^+ of $\mathcal{C}$ as follows: (i) x_w^- and x_w^+ are both $q^{1,2}$-orthogonal to w. (ii) (w, x_w^-, x_w^+) is a right-handed basis.
- Given an isotropic vector $u \in \mathcal{C}$, we call *positive half plane orthogonal to u* the set $P(u) = \{w \in u^\perp \mid q^{1,2}(w, w) = 0 \text{ or } q^{1,2}(w, w) > 0 \text{ and } v_w^+ = u\}$.
  Now the data of two distinct isotropic lines Δ_1 and Δ_2, intersecting at a point $\mathbf{p}$, defines a unique crooked plane $C(\Delta_1, \Delta_2)$ given as the union of: (i) two *wings* defined by $W_1 = \mathbf{p} + P(u_1)$ and $W_2 = \mathbf{p} + P(u_2)$, where u_1 and u_2 are in $\mathcal{C}$ and correspond to the respective directions of Δ_1 and Δ_2. (ii) a *stem* constituted by the points $\mathbf{q}$ included in the affine plane containing Δ_1 and Δ_2 and such that $q^{1,2}(\mathbf{q} - \mathbf{p}, \mathbf{q} - \mathbf{p}) < 0$.

2.3. Drumm's theorem

We now consider a Schottky group $H = \langle h_1, \ldots, h_n \rangle \subset \mathrm{SO}^0(1, 2)$ acting on the circle $\mathcal{C}$. By definition, there are $2n$ disjoint open intervals $\mathcal{I}_1^\pm, \ldots, \mathcal{I}_n^\pm$ of $\mathcal{C}$ such that for each i, the closure in $\mathcal{C}$ of $h_i(\mathcal{I}_i^-)$ is $\mathcal{C} \setminus \mathcal{I}_i^+$. The extremal points of each $\mathcal{I}_i^\pm$ are denoted by $v_{i1}^\pm$ and $v_{i2}^\pm$, in such a way that $h_i(v_{ij}^-) = v_{ij}^+$ ($j \in \{1, 2\}$). In [1], Drumm proved the following theorem:

Theorem 2.2 (Drumm [1]). *Given a Schottky group $H = \langle h_1, \ldots, h_n \rangle$ as above, one can find an affine deformation Γ of H which is an affine Schottky group. More precisely, there are $2n$ mutually disjoint crooked planes $C_i^\pm$ such that*:

(i) *Each $C_i^\pm$ is of the type $C(\Delta_{i1}^\pm, \Delta_{i2}^\pm)$ where for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, 2\}$, $\Delta_{ij}^\pm$ is an isotropic line of direction $v_{ij}^\pm$.*

(ii) *For each i, $\gamma_i(X \setminus \mathrm{cl}(U_i^-)) = U_i^+$, where $U_i^\pm$ is the connected component of $\mathbf{R}^{1,2} \setminus C_i^\pm$ which does not contain $C_i^\mp$.*

(iii) *The action of Γ on $\mathbf{R}^{1,2}$ is free and properly discontinuous. Moreover, if D is the complement in $\mathbf{R}^{1,2}$ of the union of the $U_i^\pm$'s, then $\mathrm{cl}(D)$ is a fundamental domain for the action of Γ on $\mathbf{R}^{1,2}$.*

Free groups Γ obtained as in the theorem above are called *affine Schottky groups*.

3. Geometry of Einstein universe

The properties of Einstein universe have been studied in details (and in all dimensions) in [5]. In all what follows, we restrict ourselves to the 3-dimensional case.

3.1. Definition

We define the 3-dimensional *Einstein universe* to be the product $\mathbf{S}^1 \times \mathbf{S}^2$ endowed with the Lorentzian conformal structure associated to the product metric $-\mathrm{d}t^2 + g_{\mathrm{can}}$, where g_{can} is the canonical metric of $\mathbf{S}^2$. This space admits a projective model that can be described as follows. We call $\mathbf{R}^{2,3}$ the space $\mathbf{R}^5$ endowed with the quadratic form $q^{2,3} = -2x_1 x_5 + 2x_2 x_4 + x_3^2$. The projection of the isotropic punctured cone $C^{2,3}$ of $q^{2,3}$ into the projective space $\mathbf{RP}^4$ is a smooth hypersurface of $\mathbf{RP}^4$ on which $q^{2,3}$ induces canonically a *conformal class of Lorentzian metrics*. The space $\mathbf{Ein}_3$ is just a double covering of this hypersurface. Thus, it is just the space $C^{2,3}/\mathbb{R}^+$ and we call π the projection from $C^{2,3}$ onto $\mathbf{Ein}_3$.

3.2. Stereographic projection

A striking property of Einstein universe is that it is conformally flat. In fact, we are going to show that each of its points has a neighbourhood which is conformally equivalent to the whole $\mathbf{R}^{1,2}$. To do this, we have to introduce two important geometrical objects on $\mathbf{Ein}_3$.

- A *lightlike geodesic* of $\mathbf{Ein}_3$ is the image by π of a 2-plane $P \subset \mathbf{R}^{2,3}$ on which $q^{2,3}$ is zero. Thus, lightlike geodesics of $\mathbf{Ein}_3$ are circles.
- A *lightcone* of $\mathbf{Ein}_3$ is the set of lightlike geodesics passing through a same point, called the vertex of the cone. Lightcones are obtained by projecting on $\mathbf{Ein}_3$ the intersection between $C^{2,3}$ and degenerate hyperplanes of $\mathbf{R}^{2,3}$. Topologically, the lightcone $\mathbf{C}(\mathbf{p})$ of vertex $\mathbf{p}$, is a singular hypersurface of $\mathbf{Ein}_3$, homeomorphic to a torus $\mathbf{T}^2$ with two pinched points. One of these points is $\mathbf{p}$ itself and the other is its *antipodal point* $\mathbf{p}'$, namely the image of $\mathbf{p}$ by the product of antipodal maps of $\mathbf{S}^1$ and $\mathbf{S}^2$. When we remove to $\mathbf{C}(\mathbf{p})$ both $\mathbf{p}$ and $\mathbf{p}'$, we obtain two cylinders $\mathbf{R} \times \mathbf{S}^1$, switched by the antipodal map, and endowed with a conformal class of degenerate metrics, for which the foliation $\mathbf{R} \times \{x\}$ is lightlike.

When we remove $\mathbf{C}(\mathbf{p})$ from $\mathbf{Ein}_3$, we get two connected components, switched by the antipodal map. Each of these components is conformally equivalent to Minkowski space. To see this, we identify Minkowski space with the subspace of $\mathbf{R}^{2,3}$ spanned by (e_2, e_3, e_4), we denote by $\langle x, x \rangle$ the restriction of $q^{2,3}$ to this subspace, and we consider the map:

$$\bar{\rho} : \mathbf{R}^{1,2} \to \mathbf{C}^{2,3}, \quad x \mapsto e_1 + 2x + \langle x, x \rangle e_5.$$

Let us call $\mathbf{C}_\infty$ the lightcone of vertex $\mathbf{p}_\infty = \pi(e_5)$. This notation is consistent with that of the introduction, as we have:

Proposition 3.1. *The map $\rho = \pi \circ \bar{\rho}$ is a conformal diffeomorphism between $\mathbf{R}^{1,2}$ and one of the two connected components of $\mathbf{Ein}_3 \setminus \mathbf{C}_\infty$. It is called stereographic projection of pole $\mathbf{p}_\infty$.*

Thanks to the projection ρ we now see Minkowski space as an open subspace of $\mathbf{Ein}_3$, that we still call $\mathbf{R}^{1,2}$.

Proposition 3.2. *The stereographic projection ρ gives a natural biunivoque correspondence between the isotropic directions of $\mathbf{R}^{1,2}$ and the lightlike geodesics of $\mathbf{C}_\infty$.*
For each isotropic direction u, we call $\hat{u}$ the associated lightlike geodesic of $\mathbf{C}_\infty$.

The proposition follows from simple computations involving the stereographic projection. One can see that every isotropic line $\Delta \subset \mathbf{R}^{1,2}$ is the trace on $\mathbf{R}^{1,2}$ of a unique lightlike geodesic of $\mathbf{Ein}_3$, intersecting $\mathbf{C}_\infty$ in exactly two antipodal points. Through these points passes a unique lightlike geodesic of $\mathbf{C}_\infty$, called $\hat{\Delta}$. One checks moreover that $\hat{\Delta}_1 = \hat{\Delta}_2$ if and only if Δ_1 and Δ_2 have the same direction u and so we define $\hat{u} = \hat{\Delta}_1 = \hat{\Delta}_2$.

Definition 3.1. If $\mathcal{E}$ is a subset of $\mathcal{C}$, the set of isotropic directions of Minkowski space, we define $\widehat{\mathcal{E}} = \bigcup_{u \in \mathcal{E}} \hat{u}$.

Definition 3.2. If E is a subset of $\mathbf{Ein}_3$, we define $E^\star = E \setminus \{\mathbf{p}_\infty \cup \mathbf{p}'_\infty\}$.

3.3. The conformal group $\mathrm{Conf}(\mathbf{Ein}_3)$

Any linear transformation of $\mathbf{R}^{2,3}$ preserving $q^{2,3}$ naturally preserves the conformal structure of the Einstein universe. Actually, the converse is also true, so that the conformal group of $\mathbf{Ein}_3$ is exactly $\mathrm{O}(2, 3)$. The group $\mathrm{Conf}(\mathbf{Ein}_3)$ has a center consisting in the identity and the antipodal map. A fundamental fact about conformal transformations of $\mathbf{Ein}_3$ is the following Liouville's theorem:

Theorem 3.3. *Every conformal transformation between open subsets of $\mathbf{Ein}_3$ is the restriction of a unique element of $\mathrm{Conf}(\mathbf{Ein}_3)$.*

This theorem and the stereographic projection allow us to see every isometry $\gamma \in \mathrm{Isom}(\mathbf{R}^{1,2})$ as an element of $\mathrm{Conf}(\mathbf{Ein}_3)$. Such an isometry acts by its linear part L_γ both on the space $\mathcal{C}$ (that we see as the space of isotropic directions of $\mathbf{R}^{1,2}$) and on the set of lightlike geodesics of $\mathbf{C}_\infty$. In fact, we have:

Lemma 3.4. *Let $u \in \mathcal{C}$, then $\gamma(\hat{u}) = \widehat{L_\gamma(u)}$.*

4. Conformal boundary of Drumm's examples

We show in this section that the affine Schottky groups admit a ping–pong dynamics not only on $\mathbf{R}^{1,2}$ but also on $\mathbf{Ein}_3^\star$.

4.1. Generalized crooked planes

To each crooked plane $C(\Delta_1, \Delta_2) \subset \mathbf{R}^{1,2}$, we associate a *generalised crooked plane* $\overline{C}(\Delta_1, \Delta_2)$ defined as the closure in $\mathbf{Ein}_3$ of $C(\Delta_1, \Delta_2) \cup C'(\Delta_1, \Delta_2)$, where $C'(\Delta_1, \Delta_2)$ is the image of $C(\Delta_1, \Delta_2)$ by the antipodal map. This closure is described by the following lemma:

Lemma 4.1. *If $C(\Delta_1, \Delta_2)$ is a crooked plane, the associated generalised crooked plane $\overline{C}(\Delta_1, \Delta_2)$ is the union* $C(\Delta_1, \Delta_2) \cup C'(\Delta_1, \Delta_2) \cup \hat{\Delta}_1 \cup \hat{\Delta}_2$.

Moreover, one can check that each generalised crooked plane divides $\mathbf{Ein}_3$ into two connected components.

4.2. Affine Schottky groups as Schottky groups of $\mathrm{Conf}(\mathbf{Ein}_3^\star)$

Let H be a Schottky group of $\mathrm{SO}^0(1, 2)$. It is well known that H has a limit set $\mathcal{L}_H$ strictly included in $\mathcal{C}$. The action of H on $\mathcal{O}_H = \mathcal{C} \setminus \mathcal{L}_H$ is proper and cocompact. A fundamental domain for this action is the closure in $\mathcal{C}$ of the complementary of $\bigcup_{i=1}^n \mathcal{I}_i^\pm$ (see Section 2.3 for the notations). We call $\Lambda_\Gamma^\infty = \hat{\mathcal{L}}_H$ and $\Omega_\Gamma^\infty = \widehat{\mathcal{O}}_H^\star$.

Now, let $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ be an affine deformation of H as in Theorem 2.2. Using the notations of Theorem 2.2, we call $U_i^\pm$ the connected component of $\mathbf{Ein}_3 \setminus \overline{C}_i^\pm$ not containing $\overline{C}_i^\mp$. One can check that for all i, $\gamma_i(\mathbf{Ein}_3^\star \setminus \mathrm{cl}(U_i^-)) = U_i^+$, where the closures are taken in $\mathbf{Ein}_3^\star$. Since the closures of the $U_i^\pm$'s in $\mathbf{Ein}_3^\star$ are disjoint, Γ is a Schottky group of $\mathrm{Conf}(\mathbf{Ein}_3^\star)$, in the sense of Section 2.1. In Section 2.1, we also defined a fundamental domain $\overline{D}$ for Γ, as the closure in $\mathbf{Ein}_3^\star$ of the complementary of $\bigcup_{i=1}^n (U_i^+ \cup U_i^-)$. Moreover, we have:

Lemma 4.2. *The set $\overline{D}$ is just the union $D \cup D' \cup \widehat{\mathcal{D}}^\star$ (where D is the fundamental domain of Theorem 2.2).*

Now, applying Theorem 2.1, we obtain that Γ acts properly on $\Omega_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma(\overline{D})$. But since $\overline{D} = D \cup D' \cup \widehat{\mathcal{D}}^\star$, we get $\Omega_\Gamma = \mathbf{R}^{1,2} \cup \mathbf{R}^{1,2'} \cup \Omega_\Gamma^\infty = \mathbf{Ein}_3 \setminus \Lambda_\Gamma^\infty$. It proves the part (i) of Theorem 1.1. We call $\Omega_\Gamma^\infty = \mathbf{C}_\infty \setminus \Lambda_\Gamma^\infty$. Then $\Omega_\Gamma^\infty \cup \mathbf{R}^{1,2}$ is a smooth manifold with boundary, included in Ω_Γ. Thus, the action of Γ on it is proper and since Γ preserves Ω_Γ^∞, the space $(\Omega_\Gamma^\infty \cup \mathbf{R}^{1,2})/\Gamma$ is a smooth manifold with boundary. The interior of this manifold is the Margulis space–time $\mathbf{R}^{1,2}/\Gamma$. Finally, since $\mathcal{O}_H/H$ is a finite union of circles, the boundary $\Omega_\Gamma^\infty/\Gamma$ is a finite union of degenerate cylinders.

The two points compactification of $\Omega_\Gamma^\infty/\Gamma$ is obtained by adding $\mathbf{p}_\infty$ and $\mathbf{p}'_\infty$ in the construction above. Of course, this compactification is not smooth.

References

[1] T.A. Drumm, Fundamental polyhedra for Margulis space–times, Topology 31 (4) (1992) 677–683.
[2] T.A. Drumm, Linear holonomy of Margulis space–times, J. Differential Geom. 38 (3) (1993) 679–690.
[3] T.A. Drumm, W.M. Goldman, Complete flat Lorentz 3-manifolds with free fundamental group, Internat. J. Math. 1 (2) (1990) 149–161.
[4] T.A. Drumm, W.M. Goldman, The geometry of crooked planes, Topology 38 (2) (1999) 323–351.
[5] C. Frances, Géométrie et dynamique lorentziennes conformes, Thèse, available at www.umpa.ens-lyon.fr/cfrances/.
[6] G. Margulis, Free properly discontinuous groups of affine transformations, Dokl. Akad. Nauk SSSR 272 (1983) 937–940.
[7] J. Milnor, On fundamental group of complete affinely flat manifolds, Adv. Math. 25 (1977) 178–187.