CONFORMAL ACTIONS OF NILPOTENT GROUPS ON PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. We study conformal actions of connected nilpotent Lie groups on compact pseudo-Riemannian manifolds. We prove that if a type-(p, q) compact manifold M supports a conformal action of a connected nilpotent group H, then the degree of nilpotence of H is at most $2p + 1$, assuming $p \leq q$; further, if this maximal degree is attained, then M is conformally equivalent to the universal type-(p, q), compact, conformally flat space, up to finite covers. The proofs make use of the canonical Cartan geometry associated to a pseudo-Riemannian conformal structure.

1. Introduction

Let (M, σ) be a compact *pseudo-Riemannian* manifold—that is, the tangent bundle of M is endowed with a type-(p, q) inner product, where $p+q = n = \dim M$. We will always assume $p \leq q$. The *conformal class* of σ is

$$[\sigma] = \{e^h \sigma \ : \ h : M \to \mathbf{R} \text{ smooth}\}$$

Denote by Conf M the group of *conformal automorphisms* of M—the group of diffeomorphisms f of M such that $f^*\sigma \in [\sigma]$. If $n \geq 3$, then Conf M endowed with the compact-open topology is a Lie group (see [Ko, IV.6.1] for the Riemannian case; the proof is similar for $p > 0$).

A basic question, first addressed by A. Lichnerowicz, is to characterize the pseudo-Riemannian manifolds (M, σ) for which Conf M does not preserve any metric in $[\sigma]$; in this case, Conf M is *essential*. The *Lichnerowicz conjecture*, proved by J. Lelong-Ferrand [LF1], says, *for (M, σ) a Riemannian manifold of dimension ≥ 2, if Conf M is essential, then M is conformally equivalent to the round sphere or Euclidean space.*

Denote by $g_{\mathbf{S}^n}$ the Riemannian metric with curvature $+1$ on $\mathbf{S}^n$. For any (p, q), the manifold $(\mathbf{S}^p \times \mathbf{S}^q)/\mathbf{Z}_2$, where $\mathbf{Z}_2$ acts by the antipodal map, endowed with the conformal structure coming from the metric $-g_{\mathbf{S}^p} \oplus g_{\mathbf{S}^q}$,

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is called the Einstein universe, $\mathrm{Ein}^{p,q}$. From the conformal point of view, $\mathrm{Ein}^{p,q}$ is the most symmetric structure of type (p,q): Conf $\mathrm{Ein}^{p,q}$ is isomorphic to $\mathrm{PO}(p+1,q+1)$, and it is essential. The Einstein spaces are *conformally flat*, locally conformally equivalent to $\mathbf{R}^{p,q}$—that is, $\mathbf{R}^{p+q}$ with the translation-invariant, type-(p,q) metric. More discussion of the Einstein universes appears in section 2.1 below.

Unlike for the Riemannian case, as soon as $p \geq 1$, $\mathrm{Ein}^{p,q}$ is not the only compact type-(p,q) manifold admitting an essential conformal group. For example, it is possible to construct compact Lorentzian manifolds of infinitely-many distinct topological types that admit an essential conformal group. Such examples appear in [Fr2] for any dimension $n \geq 3$, and are all conformally flat. The revised Lichnerowicz conjecture for M compact and pseudo-Riemannian is that if Conf M is essential, then M is conformally flat.

One difficulty for general type (p,q) is that no characterization of essential conformal groups exists. In the Riemannian case, on the other hand, for M compact, Conf M is essential if and only if it is noncompact. For $p \geq 1$, noncompactness is only a necessary condition to be essential. Now, a first approach to the conjecture is to exhibit sufficient conditions on a group of conformal transformations which ensure it is essential, and to test the conjecture on groups satisfying the given condition.

For example, thanks to [Zi1], we know that a simple noncompact real Lie group acting isometrically on a compact pseudo-Riemannian manifold (M,σ) of type (p,q) satisfies rk $H \leq p$, where rk H denotes the real rank. For $H <$ Conf M noncompact and simple, the rank

$$\mathrm{rk}\ H \leq p+1 = \mathrm{rk}\ \mathrm{PO}(p+1,q+1)$$

This was first proved in [Zi1], also in [BN], and for H not necessarily simple in [BFM, 1.3 (1)]. Thus, conformal actions of simple groups H, with rk $H = p+1$, on type-(p,q) compact pseudo-Riemannian manifolds cannot preserve any metric in the conformal class. The results of [BN], together with [FZ], give that when $H <$ Conf M attains this maximal rank, then M is globally conformally equivalent to $\mathrm{Ein}^{p,q}$, up to finite covers when $p \geq 2$; for $p = 1$, M is conformally equivalent to the universal cover $\widetilde{\mathrm{Ein}}^{1,n-1}$, up to cyclic and finite covers. In particular, M is conformally flat, so this result supports the pseudo-Riemannian Lichnerowicz conjecture. The interested reader can find a wide generalization of this theorem in [BFM, 1.5].

Actions of semisimple Lie groups often exhibit rigid behavior partly because the algebraic structure of such groups is itself rigid. The structure of nilpotent Lie groups, on the other hand, is not that well understood; in fact, a classification of nilpotent Lie algebras is available only for small dimensions. From this point of view, it seems challenging to obtain global results similar to those above for actions of nilpotent Lie groups. Observe also that a pseudo-Riemannian conformal structure does not naturally define a volume form, so that the nice tools coming from ergodic theory are not available here.

For a Lie algebra $\mathfrak{h}$, we adopt the notation $\mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}]$, and $\mathfrak{h}_k$ is defined inductively as $[\mathfrak{h}, \mathfrak{h}_{k-1}]$. The *degree of nilpotence* $d(\mathfrak{h})$ is the minimal k such that $\mathfrak{h}_k = 0$. For a connected, nilpotent Lie group H, define the nilpotence degree $d(H)$ to be $d(\mathfrak{h})$. If a connected Lie group H is nilpotent and acts isometrically on a type-(p,q) compact pseudo-Riemannian manifold M, where $p \geq 1$, then the nilpotence degree $d(H) \leq 2p$ (when $p = 0$, then $d(H) = 1$). This was proved in the Lorentzian case in [Zi2], and in broad generality in [BFM, 1.3 (2)]. Theorem 1.3 (2) of [BFM] also implies $d(H) \leq 2p+2$ for $H < \operatorname{Conf} M$. This bound is actually not tight, and the first result of the paper is to provide the tight bound, which turns out to be $2p+1$, the maximal nilpotence degree of a connected nilpotent subgroup in $\operatorname{PO}(p+1, q+1)$.

Theorem 1.1. *Let H be a connected nilpotent Lie group acting conformally on a compact pseudo-Riemannian manifold M of type (p,q), where $p \geq 1$, $p+q \geq 3$. Then $d(H) \leq 2p+1$.*

By theorem 1.3 (2) of [BFM], a connected nilpotent group H such that $d(H) = 2p+1$ cannot act isometrically on a compact pseudo-Riemannian manifold of type (p,q). The following theorem says that if this maximal nilpotence degree is attained in $\operatorname{Conf} M$, then M is a complete conformally flat manifold, providing further support for the pseudo-Riemannian Lichnerowicz conjecture.

Theorem 1.2. *Let H be a connected nilpotent Lie group acting conformally on a compact pseudo-Riemannian manifold M of type (p,q), with $p \geq 1$, $p+q \geq 3$. If $d(H) = 2p+1$, then M is conformally equivalent to $\widetilde{Ein}^{p,q}/\Gamma$, where $\Gamma < \widetilde{O}(p+1, q+1)$ has a finite-index subgroup contained in the center of $\widetilde{O}(p+1, q+1)$.*

Here, $\widetilde{\mathrm{Ein}}^{p,q}$ and $\widetilde{\mathrm{O}}(p+1,q+1)$ denote the universal covers of $\mathrm{Ein}^{p,q}$ and $\mathrm{O}(p+1,q+1)$. Observe that when $p \geq 2$, the center of $\widetilde{\mathrm{O}}(p+1,q+1)$ is finite, so that Γ is itself finite. For $\widetilde{\mathrm{O}}(2,q+1)$, $q \geq 2$, the center has one infinite cyclic factor. We will not treat the Riemannian case in this paper, since the results in this case are a trivial consequence of Ferrand's theorem.

2. Ein^{p,q} as a homogeneous space for PO$(p+1,q+1)$

In this section, we introduce the basic notations used throughout the paper, and provide background on the geometry of the Einstein universe, as well as an algebraic study of nilpotent subalgebras of $\mathfrak{o}(p+1,q+1)$.

2.1. Geometry of $\mathbf{Ein}^{p,q}$.

Let $\mathbf{R}^{p+1,q+1}$ be the space $\mathbf{R}^{p+q+2}$ endowed with the quadratic form

$$Q^{p+1,q+1}(x_0,\ldots,x_{n+1}) = 2(x_0x_{p+q+1} + \cdots + x_px_{q+1}) + \Sigma_{p+1}^{q} x_i^2$$

We consider the null cone

$$\mathcal{N}^{p+1,q+1} = \{x \in \mathbf{R}^{p+1,q+1} \mid Q^{p+1,q+1}(x) = 0\}$$

and denote by $\widehat{\mathcal{N}}^{p+1,q+1}$ the cone $\mathcal{N}^{p+1,q+1}$ with the origin removed. The projectivization $\mathbf{P}(\widehat{\mathcal{N}}^{p+1,q+1})$ is a smooth submanifold of $\mathbf{RP}^{p+q+1}$, and inherits from the pseudo-Riemannian structure of $\mathbf{R}^{p+1,q+1}$ a type-(p,q) conformal class (more details can be found in [Fr1], [BCDGM]). We call the *Einstein universe* of type (p,q), denoted $\mathrm{Ein}^{p,q}$, this compact manifold $\mathbf{P}(\widehat{\mathcal{N}}^{p+1,q+1})$ with this conformal structure. Note that $\mathrm{Ein}^{0,q}$ is conformally equivalent to the round sphere $(\mathbf{S}^q, g_{\mathbf{S}^q})$. When $p \geq 1$, the product $(\mathbf{S}^p \times \mathbf{S}^q, -g_{\mathbf{S}^p} \oplus g_{\mathbf{S}^q})$ is a conformal double cover of $\mathrm{Ein}^{p,q}$.

The orthogonal group of $Q^{p+1,q+1}$, isomorphic to $\mathrm{O}(p+1,q+1)$, acts projectively on $\mathrm{Ein}^{p,q}$ and turns out to be the full conformal group of $\mathrm{Ein}^{p,q}$.

2.1.1. *Lightcones, stereographic projection, and Minkowski charts.*

A *lightlike*, *timelike*, or *spacelike* curve of a pseudo-Riemannian manifold (M,σ) is a C^1 $\gamma : I \to M$ such that $\sigma_{\gamma(t)}(\gamma'(t),\gamma'(t))$ is 0, negative, or positive, respectively, for all $t \in I$. It is clear that the notion of lightlike, timelike and spacelike curves is a conformal one. Lightlike curves are sometimes also called *null*.

It is a remarkable fact that all metrics in $[\sigma]$ have the same null geodesics, as unparametrized curves (see for example [Fr6] for a proof of this fact). Thus it makes sense to speak of null—or lightlike—geodesics for non-Riemannian pseudo-Riemannian conformal structures. Given a point $x \in M$, the *lightcone* of x, denoted $C(x)$, is the set of all lightlike geodesics passing through x.

The lightlike geodesics of $\mathrm{Ein}^{p,q}$ are the projections on $\mathrm{Ein}^{p,q}$ of totally isotropic 2-planes in $\mathbf{R}^{p+1,q+1}$. Hence every null geodesic is closed If $x \in \mathrm{Ein}^{p,q}$ is the projection of $y \in \mathcal{N}^{p+1,q+1}$, the lightcone $C(x)$ is just $\mathbf{P}(y^{\perp} \cap \mathcal{N}^{p+1,q+1})$. Such a lightcone is not smooth, but $C(x) \setminus \{x\}$ is smooth and diffeomorphic to $\mathbf{R} \times \mathbf{S}^{p-1} \times \mathbf{S}^{q-1}$ (see figure 1).

FIGURE 1. the lightcone of a point in $\mathrm{Ein}^{1,2}$

For any (p, q), there is a generalized notion of stereographic projection. Consider $\varphi : \mathbf{R}^{p,q} \to \mathrm{Ein}^{p,q}$ given in projective coordinates of $\mathbf{P}(\mathbf{R}^{p+1,q+1})$ by

$$\varphi : x \mapsto [-\frac{1}{2}Q^{p,q}(x,x) : x_1 : \cdots : x_n : 1]$$

Then φ is a conformal embedding of $\mathbf{R}^{p,q}$ into $\mathrm{Ein}^{p,q}$, called the inverse *stereographic projection* with respect to $[e_0]$. It yields a conformal compactification of $\mathbf{R}^{p,q}$. The image $\varphi(\mathbf{R}^{p,q})$ is a dense open set of $\mathrm{Ein}^{p,q}$ with boundary the lightcone $C([e_0])$. Since the action of $\mathrm{PO}(p+1, q+1)$ is transitive on $\mathrm{Ein}^{p,q}$, it is clear that the complement of any lightcone $C(x)$ in $\mathrm{Ein}^{p,q}$ is conformally equivalent to $\mathbf{R}^{p,q}$. Such an open subset of $\mathrm{Ein}^{p,q}$ will be called a *Minkowski component*, and denoted $\mathbf{M}(x)$. Its identification with $\mathbf{R}^{p,q}$ via stereographic projection with respect to x is a *Minkowski chart*.

For $p \geq 1$, the boundary of a Minkowski component is not merely one point, but a lightcone. The reader will find a detailed description in [Fr1, ch 4] of how a sequence of points going to infinity in $\mathbf{R}^{1,q}$ reaches the boundary in $\mathrm{Ein}^{1,q}$. Here we explain how images of lightlike lines of $\mathbf{R}^{p,q}$ under φ reach the boundary. Lightlike lines of $\mathbf{R}^{p,q}$ are identified via φ with traces on $\mathbf{R}^{p,q}$ of lightlike geodesics in $\mathrm{Ein}^{p,q}$. If $\gamma : \mathbf{R} \to \mathbf{R}^{p,q}$ is a lightlike line, then $\lim_{t\to\infty} \varphi(\gamma(t)) = \lim_{t\to-\infty} \varphi(\gamma(t)) = x_\gamma$, where $x_\gamma \in C([e_0])$ is different from $[e_0]$. For lightlike lines $\gamma(t) = c + tu$ and $\beta(t) = b + tv$, the limits $x_\gamma = x_\beta$ if and only if $u = v$ and $\langle b - c, u\rangle = 0$. In other words, the trace on $\mathbf{M}([e_0])$ of a lightcone $C(x)$, $x \in C([e_0]) \setminus \{[e_0]\}$, is a degenerate affine hyperplane.

2.1.2. *A brief description of* $\mathfrak{o}(p+1,q+1)$. The Lie algebra $\mathfrak{o}(p+1,q+1)$ consists of all $(n+2)\times(n+2)$ matrices X, $n=p+q$, such that

$$X^t J_{p+1,q+1} + J_{p+1,q+1} X = 0$$

where $J_{p+1,q+1}$ is the matrix of the quadratic form $Q^{p+1,q+1}$. It can be written as a sum $\mathfrak{u}^- \oplus \mathfrak{r} \oplus \mathfrak{u}^+$ (see [Ko, IV.4.2] for $p=0$; the case $p>0$ is a straightforward generalization), where

$$\mathfrak{r} = \left\{ \begin{pmatrix} a & & 0 \\ & M & \\ & & -a \end{pmatrix} : \quad \begin{array}{c} a \in \mathbf{R} \\ M \in \mathfrak{o}(p,q) \end{array} \right\}$$

$$\mathfrak{u}^+ = \left\{ \begin{pmatrix} 0 & -x^t.J_{p,q} & 0 \\ & 0 & x \\ & & 0 \end{pmatrix} : \quad x \in \mathbf{R}^{p,q} \right\}$$

and

$$\mathfrak{u}^- = \left\{ \begin{pmatrix} 0 & & \\ x & 0 & \\ 0 & -x^t.J_{p,q} & 0 \end{pmatrix} : \quad x \in \mathbf{R}^{p,q} \right\}$$

Thus $\mathfrak{r} \cong \mathfrak{co}(p,q)$, and there are two obvious isomorphisms i^+ and i^- from $\mathfrak{u}^+$ (resp. $\mathfrak{u}^-$) to $\mathbf{R}^{p,q}$, given by the matrix expressions above.

The standard basis of $\mathbf{R}^{p,q}$ corresponds under i^- to the basis of $\mathfrak{u}^-$

$$U_i = \begin{cases} E_i^0 - E_{n+1}^{n+1-i} & i \in \{1,\ldots,p\} \cup \{q+1,\ldots,n\} \\ E_i^0 - E_{n+1}^i & i \in \{p+1,\ldots,q\} \end{cases}$$

where E_i^j is the $(n+2)$-dimensional square matrix with all entries 0 except for a 1 in the (i,j) place.

The parabolic Lie algebra $\mathfrak{p} \cong \mathfrak{r} \ltimes \mathfrak{u}^+$ is the Lie algebra of the stabilizer P of $[e_0]$ in $\mathrm{PO}(p+1,q+1)$, and similarly for $\mathfrak{p}^- \cong \mathfrak{r} \ltimes \mathfrak{u}^-$, the Lie algebra of the stabilizer of $[e_{n+1}]$. The groups P and P^- are isomorphic to the semidirect product $CO(p,q) \ltimes \mathbf{R}^{p,q}$, and i^+ (respectively i^-) intertwines the adjoint action of P on $\mathfrak{u}^+$ (respectively $\mathfrak{u}^-$) with the conformal action of $CO(p,q)$ on $\mathbf{R}^{p,q}$.

2.1.3. *Translations in* $PO(p+1,q+1)$. Let U^+ be the closed subgroup of $\mathrm{PO}(p+1,q+1)$ with Lie algebra $\mathfrak{u}^+$.

Definition 2.1. *A translation of* $PO(p+1,q+1)$ *is an element which is conjugate in* $PO(p+1,q+1)$ *to an element of* U^+. *A translation of* $\mathfrak{o}(p+1,q+1)$ *is an element generating a* 1*-parameter group of translations of* $PO(p+1,q+1)$.

This terminology is justified because a translation is a conformal transformation of $\mathrm{Ein}^{p,q}$ fixing a point, say x, and reading as a translation in the usual sense under stereographic projection with respect to x. Notice that there are three conjugacy classes of translations in $\mathrm{O}(p+1,q+1)$: lightlike (we will also say null), spacelike, and timelike. An example of a null translation is the element $T = (i^+)^{-1}(1,0,\ldots,0)$ of $\mathfrak{u}^+$.

Since any null translation of $\mathfrak{p}$ is conjugate under P to T, the reader will easily check the following fact, that will be used several times below.

Fact 2.2. *Let* $T \in \mathfrak{p}$ *be a nontrivial null translation and* $\mathfrak{c}(T)$ *the centralizer of* T *in* $\mathfrak{o}(p+1,q+1)$. *Then* $\mathfrak{c}(T) \cap \mathfrak{p}$ *is of codimension one in* $\mathfrak{c}(T)$.

2.2. **Bounds in $\mathbf{PO}(p+1,q+1)$.** The first step for proving theorem 1.1 is to show that any nilpotent subalgebra of $\mathfrak{o}(p+1,q+1)$ has degree $\leq 2p+1$. We will actually prove more:

Proposition 2.3. *For a nilpotent subalgebra* $\mathfrak{h} \subset \mathfrak{o}(p+1,q+1)$, *the degree* $d(\mathfrak{h}) \leq 2p+1$. *Assuming* $p \geq 1$, *if* $d(\mathfrak{h}) = d \geq 2p$, *then* $\mathfrak{h}$ *contains a translation in its center; in fact,* $\mathfrak{h}_{d-1}$ *consists of null translations.*

2.2.1. *Preliminary results.* The following definitions will be relevant below. Let $\mathfrak{l} \subset \mathfrak{gl}(n)$ be a subalgebra. The set of all compositions $\Pi_1^k X_i$, where $X_1,\ldots,X_k \in \mathfrak{l}$, will be denoted $\mathfrak{l}^k$. We say that $\mathfrak{l}$ is a *subalgebra of nilpotents* if there exists $k \geq 1$ such that $\mathfrak{l}^k = 0$. The minimal such k will be called the *order of nilpotence* of $\mathfrak{l}$, denoted $o(\mathfrak{l})$. By Lie's theorem, subalgebras of nilpotents coincide with those subalgebras of $\mathfrak{gl}(n)$, the elements of which are nilpotent matrices. If $\mathfrak{h}$ is a nilpotent Lie algebra, then $\mathrm{ad}\ \mathfrak{h} \subset \mathfrak{gl}(\mathfrak{h})$ is a subalgebra of nilpotents and $d(\mathfrak{h}) = o(\mathrm{ad}\ \mathfrak{h})$.

For V a vector space with form B, a Lie subalgebra $\mathfrak{h} \subset \mathfrak{co}(V)$ is *infinitesimally conformal* if for all $u,v \in V$ and $X \in \mathfrak{h}$,

$$B(Xu,v) + B(u,Xv) = \lambda(X)B(u,v)$$

for some infinitesimal character $\lambda : \mathfrak{h} \to \mathbf{R}$. Of course, the Lie algebra of a subgroup of $\mathrm{CO}(V) \subset \mathrm{GL}(V)$ acts by infinitesimally conformal endomorphisms of V.

Lemma 2.4. *Let $\mathfrak{l}$ be a Lie algebra and V a finite dimensional $\mathfrak{l}$-module. Let $Y \in \mathfrak{h}_{k-1}$ and $v \in V$. Then $Y(v) \in \mathfrak{l}^k(V)$.*

This lemma is easily proved by induction, using the Jacobi identity.

Lemma 2.5. *Let V be a vector space with a symmetric bilinear form B of type (p, q). If $\overline{\mathfrak{u}} \subset \mathfrak{co}(V)$ is a subalgebra of nilpotents, then $o(\overline{\mathfrak{u}}) \leq 2p + 1$.*

Proof: Note that $\overline{\mathfrak{u}}$ is infinitesimally isometric because there are no non-trivial infinitesimal characters $\overline{\mathfrak{u}} \to \mathbf{R}$. If $p = 0$, then $\overline{\mathfrak{u}} \subset \mathfrak{o}(n)$, in which case $\overline{\mathfrak{u}}$ must be trivial.

Now assume $p \geq 1$. Let U be the connected group of unipotent matrices in $CO(p, q)$ with Lie algebra $\overline{\mathfrak{u}}$. Because U consists of unipotent matrices, it lies in a minimal parabolic subgroup of $\mathrm{CO}(p, q)$, hence leaves invariant some isotropic p-plane $N \subset V$. The order of $\overline{\mathfrak{u}}$ on both N and $V/N^{\perp}$ is at most p, because each is dimension p. Because $N^{\perp}/N$ inherits a positive-definite inner product that is infinitesimally conformally invariant by $\overline{\mathfrak{u}}$, the order of $\overline{\mathfrak{u}}$ on it is 1. Then $o(\overline{\mathfrak{u}}) \leq 2p + 1$, as desired. $\diamondsuit$

2.2.2. *Proof of proposition 2.3.* Let H be the connected subgroup of $G = \mathrm{PO}(p + 1, q + 1)$ with Lie algebra $\mathfrak{h}$. We recall some facts from the theory of algebraic groups (see for instance [Bo]). First, the nilpotence degrees of a connected group and its Zariski closure are the same, so that there is no loss of generality assuming H Zariski closed. Then there is an algebraic Levi decomposition of the Lie algebra

$$\mathfrak{h} \cong \mathfrak{r} \times \mathfrak{u}$$

where $\mathfrak{r}$ is abelian and comprises the semisimple elements of $\mathfrak{h}$, and $\mathfrak{u}$ consists of nilpotent. If $\mathfrak{u}$ is trivial, $\mathfrak{h}$ is abelian and proposition 2.3 is proved. If not, $d(\mathfrak{h}) = d(\mathfrak{u})$, so we will assume in what follows that $\mathfrak{h} = \mathfrak{u}$. Moreover, $\mathfrak{u}$ is contained in a minimal parabolic subalgebra of $\mathfrak{o}(p + 1, q + 1)$, and so conjugating $\mathfrak{u}$ if necessary, we have $\mathfrak{u} \subset \mathfrak{p}$. Thanks to i^+ (see 2.1.2), we identify $\mathfrak{u}$ to a subalgebra of $\mathfrak{co}(p, q) \ltimes \mathbf{R}^{p,q}$.

Denote by $\overline{\mathfrak{u}}$ the projection to $\mathfrak{co}(p,q)$, which is actually in $\mathfrak{o}(p,q)$ since $\overline{\mathfrak{u}}$ is a subalgebra of nilpotents. For any natural number k,

$$\mathfrak{u}_k \subseteq \overline{\mathfrak{u}}_k + \overline{\mathfrak{u}}^k(\mathbf{R}^{p,q}) \tag{1}$$

The proof by induction of this relation is straightforward using lemma 2.4 for $\mathfrak{l} = \mathfrak{u}$ and $V = \mathbf{R}^{p,q}$, and is left to the reader.

When $p = 0$, then any nilpotent subalgebra $\overline{\mathfrak{u}} \subset \mathfrak{o}(1, q+1)$ is abelian, by the remarks above, and because the nilpotent radical of the parabolic in $\mathfrak{o}(1,q)$ is itself abelian. We thus have $d(\mathfrak{u}) \leq 2p + 1$ when $p = 0$. Now proceed inductively on p, using lemma 2.5 and relation (1) to obtain $d(\mathfrak{u}) \leq o(\overline{u}) \leq 2p+1$ for any integer p, as desired.

Next suppose that $d(\mathfrak{u}) = d \geq 2p \geq 2$. Since $\overline{\mathfrak{u}}$ is a nilpotent subalgebra of $\mathfrak{o}(p,q)$, its nilpotence index is at most $2p-1$ by the first part of the proof. Since $d \geq 2p$, $\overline{\mathfrak{u}}_{d-1} = 0$ and

$$0 \neq \mathfrak{u}_{d-1} \subseteq \overline{\mathfrak{u}}^{d-1}(\mathbf{R}^{p,q})$$

so an element of $\mathfrak{u}_{d-1}$ can be written

$$w = Y_1 \cdots Y_{d-1}(v) \qquad \text{for } Y_1, \ldots, Y_{d-1} \in \overline{\mathfrak{u}},\ v \in \mathbf{R}^{p,q}$$

Further, any $Y \in \overline{\mathfrak{u}}$ annihilates w. Because $Y_1 \in \overline{\mathfrak{u}}$ is infinitesimally conformal and nilpotent, it is infinitesimally isometric. Then

$$Q^{p,q}(w,w) = \langle Y_1 \cdots Y_{d-1}(v), Y_1 \cdots Y_{d-1}(v)\rangle = -\langle Y_2 \cdots Y_{d-1}(v), Y_1(w)\rangle = 0$$

and so w is a null translation. ♢

2.3. **Conformal structures as Cartan geometries.** In the sequel, it will be fruitful to study pseudo-Riemannian structures in the setting of Cartan geometries. A Cartan geometry modeled on some homogeneous space $\mathbf{X} = G/P$ is a curved analogue of $\mathbf{X}$.

Definition 2.6. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and P a closed subgroup of G such that $Ad\ P$ is faithful on $\mathfrak{g}$. A* Cartan geometry (M, B, ω) *modeled on $(\mathfrak{g}, P)$ is*

1. *a principal P-bundle $\pi : B \to M$*
2. *a $\mathfrak{g}$-valued 1-form ω on B satisfying*
   - *for all $b \in B$, the restriction $\omega_b : T_bB \to \mathfrak{g}$ is an isomorphism*
   - *for all $b \in B$ and $Y \in \mathfrak{h}$, the evaluation $\omega_b(\left.\frac{d}{dt}\right|_0 be^{tY}) = Y$*
   - *for all $b \in B$ and $h \in P$, the pullback $R_h^*\omega = Ad\ h^{-1} \circ \omega$*

For the model $\mathbf{X}$, the canonical Cartan geometry is the triple $(\mathbf{X}, G, \omega_G)$, where ω_G denotes the left-invariant $\mathfrak{g}$-valued 1-form on G, called the *Maurer-Cartan form.*

It is known since E. Cartan that a a conformal structure $(M, [\sigma]))$ of type (p, q) with $p+q \geq 3$, defines, up to isomorphism, a unique *canonical* Cartan geometry (M, B, ω) modeled on $\mathrm{Ein}^{p,q} = \mathrm{PO}(p+1, q+1)/P$ (or equivalently on the pair $(\mathfrak{o}(p+1, q+1), P)$). The interested reader will find the details of this solution of the so-called *equivalence problem* in [Sh, ch 7].

By Aut M, we mean the group of bundle automorphisms of B preserving ω. Any conformal diffeomorphism lifts to an element of Aut M, maybe not unique, but the fibers of the projection from Aut M to the conformal group of M are discrete.

3. General degree bound: proof of theorem 1.1

In this section, we use the interpretation of conformal structures as Cartan geometries to prove theorem 1.1. Let (M, B, ω) a Cartan geometry modeled on $(\mathfrak{g}, P)$, and $H <$ Aut M a connected Lie group. Since H acts on B, each vector $X \in \mathfrak{h}$ defines a Killing field on B, and for every $b \in B$, we will call $X(b)$ the value of this Killing field at b. Thus, each point $b \in B$ determines a linear embedding

$$\begin{aligned} s_b \quad : \quad & \mathfrak{h} \to \mathfrak{g} \\ & X \mapsto \omega_b(X(b)) \end{aligned}$$

The injectivity of s_b comes from the fact that H preserves a framing on B, hence acts freely (see [Ko, I.3.2]). The image $s_b(\mathfrak{h})$ will be denoted $\mathfrak{h}^b$, and, for $X \in \mathfrak{h}$, the image $s_b(X)$ will be denoted X^b. In general, s_b is not a Lie algebra homomorphism, except with respect to stabilizers (see [Sh]): for any $X, Y \in \mathfrak{h}$ and $b \in B$ such that $Y^b \in \mathfrak{p}$,

$$[X, Y]^b = [X^b, Y^b]$$

Observing that Y belongs to the stabilizer $\mathfrak{h}(\pi(b))$ if and only if $Y^b \in \mathfrak{p}$, we deduce the following fact.

Fact 3.1. *If $\mathfrak{h}^b \cap \mathfrak{p}$ is codimension at most 1 in $\mathfrak{h}^b$, then $\mathfrak{h}^b$ is a Lie subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{h}$.*

The following result implies theorem 1.1. It is more precise and will be useful for the proof of theorem 1.2:

Theorem 3.2. *Let $(M, [\sigma])$ be a compact manifold with a type-(p,q) conformal structure, and let (M, B, ω) be the associated Cartan geometry. Let $H < Aut\ M$ be a connected nilpotent Lie group. Then $d(H) \leq 2p+1$. If $d(H) = 2p+1$, then every H-invariant closed subset $F \subset M$ contains a point x such that*

1. *The dimension of the orbit $H.x$ is at most 1.*
2. *For every $b \in \pi^{-1}(x)$, $\mathfrak{h}^b$ is a subalgebra of $\mathfrak{o}(p+1, q+1)$.*
3. *There exists $X \in \mathfrak{h}$ such that X^b is a lightlike translation in $\mathfrak{p}$ for every $b \in \pi^{-1}(x)$, and X^b is in the center of $\mathfrak{h}^b$.*

A consequence of this theorem is that when $d(H) = 2p+1$, there are points with nontrivial stabilizers, because X as in (3) generates a 1-parameter subgroup of the stabilizer $H(x)$. We will study the dynamics near x of this flow in the proof of theorem 1.2.

The starting point for the proof of theorem 3.2 will be the following theorem 4.1 of [BFM].

Theorem 3.3. *Let (M, B, ω) be a Cartan geometry modeled on $(\mathfrak{g}, P)$ with M compact. Assume that $Ad_{\mathfrak{g}}P < Aut\ G$ is Zariski closed. Let $H < Aut\ M$ be a connected amenable subgroup with no compact quotients. Then for each H-invariant closed subset $F \subset M$, there is $x \in F$ and an algebraic subgroup $\check{S} < Ad_{\mathfrak{g}}P$ such that, for all $b \in \pi^{-1}(x)$,*

1. *$\mathfrak{h}^b$ is $\check{S}$-invariant*
2. *s_b intertwines $\overline{Ad\ H}$, the Zariski closure of $Ad\ H$ in $Aut\ \mathfrak{h}$, and $\check{S}\big|_{\mathfrak{h}^b}$*

Actually, in theorem 4.1 of [BFM], we just assume that the group H preserves a finite Borel measure μ on M, and we get the conclusions (1) and (2) for b in the preimage of a subset $\Lambda \subset M$ such that $\mu(\Lambda) = 1$. Here, H is amenable, which implies that on any closed H-invariant subset of the compact manifold M, a finite Borel measure will be preserved. Hence, the conclusions of theorem 4.1 of [BFM] will hold above at least one point of each closed H-invariant subset, thus yielding the statement 3.3.

Proof: (of theorem 3.2)

Let $F \subset M$ be closed and H-invariant, and let $x \in M$ and $\check{S} < \mathrm{Ad}_{\mathfrak{g}}P$ be given by theorem 3.3. Since the adjoint representation of $\mathrm{PO}(p+1, q+1)$ is algebraic and faithful, $\check{S}$ is the image of an algebraic subgroup of P, which we will also denote $\check{S}$. On the Lie algebra level, theorem 3.3 says that for any $X \in \mathfrak{h}$, there exists $\check{X} \in \check{\mathfrak{s}}$ such that for all $Y \in \mathfrak{h}$,

$$[X, Y]^b = [\check{X}, Y^b]$$

Suppose that $d = d(\mathfrak{h}) \geq 2p + 1$. Because $\check{\mathfrak{s}}$ is algebraic, there is a decomposition

$$\check{\mathfrak{s}} \cong \mathfrak{r} \ltimes \mathfrak{u}$$

with $\mathfrak{r}$ reductive and $\mathfrak{u}$ consisting of nilpotent elements (see [Bo]). Because ad $\mathfrak{h}$ consists of nilpotents, the subalgebra $\mathfrak{r}$ is in the kernel of restriction to $\mathfrak{h}^b$, and $\mathfrak{u}$ maps onto ad $\mathfrak{h}$. Therefore, for $l = d(\mathfrak{u})$,

$$2p \leq d - 1 = d(\text{ad } \mathfrak{h}) \leq l \leq 2p + 1$$

where the upper bound comes from proposition 2.3. Also by this proposition, $\mathfrak{u}_{l-1}$ consists of null translations. Whether $l = d - 1$ or d, we will show that $\mathfrak{h}^b$ centralizes a null translation in $\mathfrak{p}$, from which fact we will obtain the bound and points (1) and (3).

First suppose $l = d - 1$. Then $\mathfrak{u}_{d-2}$ consists of null translations and acts on $\mathfrak{h}^b$ as ad $\mathfrak{h}_{d-2}$, which means it centralizes $(\mathfrak{h}_1)^b$. Then by facts 2.2 and 3.1, $(\mathfrak{h}_1)^b$ embeds homomorphically in $\mathfrak{o}(p+1, q+1)$. The order of $\mathfrak{u}$ on $(\mathfrak{h}_1)^b$ is $d - 1$; further, $\mathfrak{u}$ and $(\mathfrak{h}_1)^b$ generate a nilpotent subalgebra $\mathfrak{n}$ of order $d - 1$, in which $(\mathfrak{h}_1)^b$ is an ideal. Since $d - 1 \geq 2p$, proposition 2.3 implies that the commutators $\mathfrak{n}_{d-2}$ are all null translations. But $\mathfrak{n}_{d-2}$ contains

$$\mathfrak{u}^{d-2}(\mathfrak{h}_1)^b = (\mathfrak{h}_{d-1})^b$$

Because $\mathfrak{u}$ preserves $(\mathfrak{h}_1)^b \cap \mathfrak{p}$ and acts by nilpotent transformations on $(\mathfrak{h}_1)^b/((\mathfrak{h}_1)^b \cap \mathfrak{p})$, which is 1-dimensional,

$$\mathfrak{u}^1(\mathfrak{h}_1)^b = (\mathfrak{h}_2)^b \subset \mathfrak{p}$$

Thus $(\mathfrak{h}_k)^b \subset \mathfrak{p}$ as soon as $k \geq 2$, so for any $X \in \mathfrak{h}$, $Y \in \mathfrak{h}_k$, we have $[X, Y]^b = [X^b, Y^b]$. In particular, $(\mathfrak{h}_{d-1})^b$, the image under s_b of the center of $\mathfrak{h}$, commutes with $\mathfrak{h}^b$, so that $\mathfrak{h}^b$ is in the centralizer of a nonzero null translation.

Next suppose $l = d$. Then $\mathfrak{u}_{d-1}$ centralizes $\mathfrak{h}^b$ because it acts as ad $\mathfrak{h}_{d-1}$. By proposition 2.3, $\mathfrak{u}_{d-1}$ consists of null translations, so $\mathfrak{h}^b$ commutes with a nonzero null translation in $\mathfrak{p}$.

Given that $\mathfrak{h}^b$ centralizes a nonzero null translation in $\mathfrak{p}$, 2.2 implies point (1); moreover, $(\mathfrak{h}^b)_1 \subset \mathfrak{p}$. By fact 3.1, $s_b : \mathfrak{h} \to \mathfrak{o}(p+1, q+1)$ is a homomorphic embedding. The assumption $d \geq 2p+1$ and proposition 2.3 forces $d = 2p+1$, proving the bound. Also by proposition 2.3, $(\mathfrak{h}^b)_{2p}$, which is central in $\mathfrak{h}^b$, consists of null translations; finally, $(\mathfrak{h}^b)_{2p} \subset (\mathfrak{h}^b)_1 \subset \mathfrak{p}$.

Finally, point (2) of the theorem is a mere consequence of fact 3.1. $\diamondsuit$

4. Conformal dynamics

This section contains the first steps of the proof of theorem 1.2. We assume that H is a connected nilpotent group acting faithfully and conformally on a compact type-(p, q) pseudo-Riemannian manifold (M, σ), with $d(H) = 2p + 1$.

For $x \in M$, denote by $H(x)$ the stabilizer of x in H. For each $b \in \pi^{-1}(x)$, the action of H by automorphisms of the principal bundle B gives rise to an injective homomorphism $\rho_b : H(x) \to P$. Theorem 3.2 says that each H-invariant closed set F contains a point x_0, such that for some 1-parameter group h^s in $H(x_0)$ and $b_0 \in \pi^{-1}(x_0)$, the image $\rho_{b_0}(h^s)$ is a 1-parameter group of null translations in P. The aim of this section is to understand the dynamics of h^s around x_0. Using the Cartan connection, we will show that these dynamics are essentially the same as those of $\rho_{b_0}(h^s)$ around $[e_0]$ in $\mathrm{Ein}^{p,q}$.

Section 5.1 describes the dynamics of a flow by null translations $\tau^s = \rho_{b_0}(h^s)$ on $\mathrm{Ein}^{p,q}$. In section 5.2, we make the crucial link between the dynamics of τ^s on $\mathrm{Ein}^{p,q}$ and those of h^s on M, via the respective actions on special curves in the two Cartan bundles. The actions on these curves are conjugate locally by the *exponential maps* of the two Cartan geometries. In section 5.3, we deduce from this relationship several precise properties of the h^s-action on M. These dynamical properties will then be used in section 5 to show that the manifold M is actually conformally flat.

4.1. Dynamics of null translations on $\mathrm{Ein}^{p,q}$.

The first task is to describe the dynamics of 1-parameter groups of null translations in the model space $\mathrm{Ein}^{p,q}$. Let T be a null translation in $\mathfrak{p}$, generating a 1-parameter subgroup $\tau^s = e^{sT}$ of P. Up to conjugation in P, we may assume

$$\tau^s = \begin{pmatrix} 1 & & s & 0 \\ & \cdot & & -s \\ & & \cdot & \\ & & & 1 \end{pmatrix}$$

The action of τ^s on $\mathrm{Ein}^{p,q}$ is given in projective coordinates by

$$\tau^s : \ [y_0 : \ \cdots \ : y_{n+1}] \mapsto [y_0 + sy_n : y_1 - sy_{n+1} : y_2 : \ \cdots \ : y_{n+1}]$$

The fixed set is

$$F = \mathbf{P}(e_0^\perp \cap e_1^\perp \cap \mathcal{N}^{p+1,q+1})$$

When $p \geq 2$, then F is homeomorphic to the quotient of an $\mathbf{RP}^2$-bundle over $\mathrm{Ein}^{p-2,q-2} \cong (S^{p-2} \times S^{q-2})/\mathbf{Z}_2$ in which all equatorial circles are identified to a single $\mathbf{RP}^1$; in particular, it has codimension 2. The singular circle is

$$\Lambda = \mathbf{P}(\mathrm{span}\{e_0, e_1\}) \subset F$$

When $p = 1$, then $F = \Lambda$.

If $y \notin F$, then

$$\tau^s.y \to [y_n : -y_{n+1} : 0 : \cdots : 0] \in \Lambda \qquad \text{as } s \to \infty$$

Every point $x \in \mathrm{Ein}^{p,q}$ lies in some $C(y)$ for $y \in \Lambda$, and y is unique when $x \notin F$. We summarize the dynamics of τ^s near Λ; see also figure 2:

Fact 4.1. *The complement of the closed, codimension-2 fixed set F of τ^s in $\mathrm{Ein}^{p,q}$ is foliated by subsets of lightcones $\check{C}(y) = C(y) \setminus (C(y) \cap F)$, for $y \in \Lambda$. Points $x \in \check{C}(y)$ tend under τ^s to y along the lightlike geodesic containing x and y; in particular, τ^s preserves setwise all null geodesics emanating from points of Λ.*

FIGURE 2. local picture of flow by null translation τ^s

4.2. **Geodesics and holonomy.** In this section (M, B, ω) will be a Cartan geometry modeled on G/P. The form ω on B determines special curves, the *geodesics*. Here they will be defined as projections of curves with constant velocity according to ω—that is, $\gamma : (-\epsilon, \epsilon) \to M$ is a geodesic if $\gamma(t) = \pi(\hat{\gamma}(t))$ where

$$\omega(\hat{\gamma}'(t)) = \omega(\hat{\gamma}'(0)) \qquad \text{for all } t \in (-\epsilon, \epsilon)$$

Geodesics on the flat model space $(G/P, G, \omega_G)$ are orbits of 1-parameter subgroups. Note that this class of curves is larger than the usual set of geodesics in case the Cartan geometry corresponds to a pseudo-Riemannian metric or a conformal pseudo-Riemannian structure (see [Fi], [Fri], [Fri-S] for a definition of conformal geodesics).

The *exponential map* is defined on $B \times \mathfrak{g}$ in a neighborhood of $B \times \{0\}$ by

$$\exp(b, X) = \exp_b(X) = \hat{\gamma}_{X,b}(1)$$

where $\hat{\gamma}_{X,b}(0) = b$ and $\omega(\hat{\gamma}'_{X,b}(t)) = X$ for all t; in words, the exponential map at b sends X to the value at time 1 of the ω-constant curve with initial velocity X.

Let $h \in \mathrm{Aut}\ M$, and denote by $\hat{h}$ the corresponding automorphism of B. Because $\hat{h}$ preserves ω,

$$\hat{h} \circ \hat{\gamma}_{X,b} = \hat{\gamma}_{X,\hat{h}(b)}$$

and h carries geodesics in M to geodesics.

Suppose that h^s is a 1-parameter group of automorphisms with lift $\hat{h}^s$ to B. Then for any $b_0 \in B$, the curve parametrized by the flow $\hat{\gamma}(s) = \hat{h}^s.b_0$ projects to a geodesic $\gamma(s)$ in M. The reason is that

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_s \hat{\gamma}(t) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_s h^t.b_0 \\ &= h^s_* \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 h^t.b_0 \right) \end{aligned}$$

and

$$\omega \circ h^s_* = \omega$$

so the derivative of $\hat{\gamma}(s)$ is ω-constant.

Definition 4.2. *If $h \in Aut\ M$ fixes x, and $b \in \pi^{-1}(x)$, then the element $g \in P$ such that $\hat{h}.b = bg$ is the* holonomy of h with respect to b. *More generally, given a local section $\sigma : U \to B$, the* holonomy of $h \in \mathrm{Aut}\ M$ with respect to σ *at some point* $x \in U \cap h^{-1}.U$ *is* g *such that* $\hat{h}.\sigma(x) = \sigma(h.x)g$ *(see figure 3).*

FIGURE 3. the holonomy of h at x with respect to σ is g

If $H < \text{Aut } M$ and $b \in \pi^{-1}(x)$, then the holonomy with respect to b is the monomorphism $\rho_b : H(x) \to P$ mentioned at the beginning of section 4. Replacing b with bp has the effect of post-composing with conjugation by p^{-1}.

For automorphisms fixing a point x_0, the holonomy with respect to some $b_0 \in \pi^{-1}(x_0)$ determines the action in a neighborhood of x_0 via the exponential map composed with projection to M. If, moreover, an automorphism h fixes x_0 and preserves the image of a geodesic γ emanating from x_0, then the holonomy at x_0 determines the holonomy along γ, as follows. For $X \in \mathfrak{g}$, denote by e^X the exponential of X in G.

Proposition 4.3. *Suppose that* $h \in$ *Aut* M *fixes a point* x_0 *and has holonomy* g_0 *with respect to* $b_0 \in \pi^{-1}(x_0)$. *Let* $\gamma(t) = \pi(\exp(b_0, tX))$ *for* $X \in \mathfrak{g}$, *defined on an interval* $(-\epsilon, \epsilon)$. *Suppose there exist*

- $(\alpha, \beta) \subseteq (-\epsilon, \epsilon)$ *containing* 0
- *a path* $g : (\alpha, \beta) \to P$ *with* $g(0) = g_0$
- *a diffeomorphism* $c : (\alpha, \beta) \to (\alpha', \beta')$

such that, for all $t \in (\alpha, \beta)$,

$$g_0 e^{tX} = e^{c(t)X} g(t)$$

Then

(1) *The curves* $\exp(b_0, c(t)X)$ *and* $\gamma(c(t))$ *are defined for all* $t \in (\alpha, \beta)$, *and* $h.\gamma(t) = \gamma(c(t))$; *in particular, if* $\alpha' = -\infty$ *or* $\beta' = \infty$, *then* $\exp(b_0, tX)$ *and* $\gamma(t)$ *are defined on* $(-\infty, 0]$ *or* $[0, \infty)$, *respectively.*
(2) *Viewing* $\exp(b_0, tX)$ *as a section of* B *over* $\gamma(t)$, *the holonomy of* h *at* $\gamma(t)$ *with respect to this section is* $g(t)$.

Proof: In G, reading the derivative of $g_0 e^{tX}$ with ω_G gives (see [Sh, 3.4.12])

$$X = (\text{Ad } g(t)^{-1})(c'(t)X) + \omega_G(g'(t))$$

Because $\hat{h}$ preserves ω, the derivative of

$$\hat{h}.\exp(b_0, tX) = \exp(b_0 g_0, tX)$$

according to ω is X for all $t \in (-\epsilon, \epsilon)$. On the other hand, it is also true in B that whenever $t \in (\alpha, \beta)$ and $\exp(b_0, c(t)X)$ is defined,

$$\omega((\exp(b_0, c(t)X)g(t))') = (\mathrm{Ad}\ g(t)^{-1})(c'(t)X) + \omega_G(g'(t))$$

This formula follows from the properties of ω in the definition 2.6 of a Cartan geometry; see [Sh, 5.4.12]. Therefore, because the two curves have the same initial value and the same derivatives, as measured by ω, both are defined for $t \in (\alpha, \beta)$, and

$$\hat{h}.\exp(b_0, tX) = \exp(b_0, c(t)X)g(t)$$

This formula proves (2); item (1) follows by projecting both curves to M. $\diamondsuit$

We record one more completeness result that will be useful below, for flows that preserve a geodesic, but do not necessarily fix a point on it.

Proposition 4.4. *Let $X \in \mathfrak{g}$ and $b_0 \in B$ be such that $\exp(b_0, tX)$ is defined for all $t \in \mathbf{R}$. Suppose that for some $Y \in \mathfrak{g}$, there exists $g : (\alpha, \beta) \to P$, for $\alpha < 0 < \beta$, such that for all $t \in (\alpha, \beta)$,*

$$e^{tX} = e^{c(t)Y} g(t)$$

in G, where c is a diffeomorphism $(\alpha, \beta) \to \mathbf{R}$ fixing 0. Then $\exp(b_0, tY)$ is defined for all $t \in \mathbf{R}$.

Proof: In G, we have for all $t \in (\alpha, \beta)$

$$Y = \frac{1}{c'(t)}[\mathrm{Ad}\ g(t)][X - \omega_G(g'(t))]$$

Let $c^{-1}(s)$ be the inverse diffeomorphism $(-\infty, \infty) \to (\alpha, \beta)$. Define, for $s \in \mathbf{R}$,

$$\hat{\gamma}(s) = \exp(b_0, c^{-1}(s)X)g(c^{-1}(s))^{-1}$$

The derivative of the right-hand side is

$$\frac{1}{c'(c^{-1}(s))}[\mathrm{Ad}\ g(c^{-1}(s))][X - \omega_G(g'(c^{-1}(s))]$$

For $t = c^{-1}(s)$, this is

$$\frac{1}{c'(t)}[\mathrm{Ad}\ g(t)][X - \omega_G(g'(t))] = Y$$

Since $\hat{\gamma}'(s) = Y$ for all s and $\hat{\gamma}(0) = b_0$, we conclude $\exp(b_0, sY)$ is defined for all $s \in \mathbf{R}$ and equals $\hat{\gamma}(s)$. ♢

4.3. **Dynamics of h^s on M.** We return to the situation described at the beginning of section 4. The point x_0 is given by theorem 3.2 and $h^s.x_0 = x_0$, with the property that if $b_0 \in \pi^{-1}(x_0)$, then the holonomy of h^s with respect to b_0 is a 1-parameter group of lightlike translations. We can choose b_0 in the fiber over x_0 such that this holonomy is τ^s as in the section 4.1.

Let the subalgebra $\mathfrak{u}^-$ complementary to $\mathfrak{p}$ and the basis $U_1, \ldots, U_n$ be as in section 2.1.2. Let $\mathcal{N}(\mathfrak{u}^-)$ be the null cone with respect to $Q^- := (i^-)^*(Q^{p,q})$ (see 2.1.2 for the definition of i^-) in $\mathfrak{u}^-$.

Let $\hat{\Delta}(v) = \exp(b_0, vU_1)$ with domain $I_\Delta \subseteq \mathbf{R}$; let $\Delta = \pi \circ \hat{\Delta}$.

Proposition 4.5. *The flow h^s fixes pointwise the geodesic Δ, and for $v \in I_\Delta$, its holonomy at $\Delta(v)$ with respect to $\hat{\Delta}$ is τ^s.*

Proof: Because Ad τ^s fixes U_1, the corresponding 1-parameter subgroups commute in G:

$$\tau^s e^{vU_1} = e^{vU_1} \tau^s$$

Then by proposition 4.3, the flow h^s fixes Δ pointwise, and the holonomy of h^s with respect to $\hat{\Delta}$ at any $\Delta(v)$, $v \in I_\Delta$, equals τ^s. ♢

Proposition 4.6. *There is an open subset $\mathcal{S} \subset \mathcal{N}(\mathfrak{u}^-)$ such that $\mathcal{S} \cup -\mathcal{S}$ is dense in $\mathcal{N}(\mathfrak{u}^-)$ and for all $v \in I_\Delta$ and $U \in \mathcal{S}$,*

(1) *If the geodesic $\beta(t) = \pi \circ \exp(\hat{\Delta}(v), tU)$ is defined on $(-\epsilon, \epsilon)$, then the flow h^s preserves β and reparametrizes by*

$$c(t) = \frac{t}{1 + st}$$

for $t \in (-\epsilon, \epsilon)$. In particular, for $t > 0$ $(t < 0)$,

$$h^s(\beta(t)) \to \Delta(v) \text{ as } s \to \infty \ (s \to -\infty)$$

Moreover, $\beta(t)$ is complete.

(2) *There is a framing $f_1(t), \ldots, f_n(t)$ of M along $\beta(t)$ for which the derivative*

$$h^s_*(f_i(t)) = \left(\frac{1}{1 + st} \right)^{\sigma(i)} f_i(c(t))$$

where

$$\sigma(i) = \begin{cases} 0 & i = 1 \\ 1 & i \in \{2, \ldots, n-1\} \\ 2 & i = n \end{cases}$$

Before proving this proposition, we establish some algebraic facts in the group G. Let $\mathfrak{r}$ be as in section 2.1.2, a maximal reductive subalgebra of $\mathfrak{p}$.

Lemma 4.7. *Let $R \cong CO(p,q)$ be the connected subgroup of P with Lie algebra $\mathfrak{r}$, and let S be the unipotent radical of the stabilizer in R of U_1.*

(1) *$Fix(Ad\ \tau^s) \cap \mathfrak{u}^- = \mathbf{R}U_1$*
(2) *Let $\mathcal{S} = \mathbf{R}^*_{>0} \cdot S.U_n$. Then $\mathcal{S} \cup -\mathcal{S}$ is open and dense in $\mathcal{N}(\mathfrak{u}^-)$.*
(3) *The subgroups S and τ^s commute.*

Proof:

(1) Recall that T is the infinitesimal generator for τ^s. It suffices to show

$$\mathbf{R}U_1 = \ker(\text{ad } T) \cap \mathfrak{u}^-$$

For $i \in \{1, \ldots, p\} \cup \{q+1, \ldots, n\}$, compute

$$(\text{ad } T)(U_i) = E_1^{n+1-i} - E_i^n + \delta_{in}(E_0^0 - E_{n+1}^{n+1})$$

and for $i \in \{p+1, \ldots, q\}$

$$(\text{ad } T)(U_i) = E_1^i - E_i^n$$

so $(\text{ad } T)(U_i) = 0$ if and only if $i = 1$, while the $(\text{ad } T)(U_i)$ for $2 \leq i \leq n$ are linearly independent.

(2) We will show that $\mathcal{S}$ consists of all $U \in \mathcal{N}(\mathfrak{u}^-)$ with $\langle U, U_1 \rangle > 0$; these elements and their negatives form an open dense subset of $\mathcal{N}(\mathfrak{u}^-)$.

First, if $g \in S$, then $g.U_n \in \mathcal{N}(\mathfrak{u}^-)$ and

$$\begin{aligned} \langle g.U_n, U_1 \rangle &= \langle g.U_n, gU_1 \rangle \\ &= \langle U_n, U_1 \rangle = 1 \end{aligned}$$

Both $\mathcal{N}(\mathfrak{u}^-)$ and the property $\langle U, U_1 \rangle > 0$ are invariant by multiplication by positive real numbers, so $\mathcal{S}$ is contained in the claimed subset.

Next let $U \in \mathcal{N}(\mathfrak{u}^-)$ be such that $\langle U, U_1 \rangle > 0$. Replace U with a positive scalar multiple so that $\langle U, U_1 \rangle = 1$. Define $g \in S$ by

$$\begin{aligned} g \quad : \quad & U_1 \mapsto U_1 \\ & U_n \mapsto U \\ & V \mapsto V - \langle V, U \rangle \cdot U_1 \qquad \text{for } V \in \{U_1, U_n\}^\perp \end{aligned}$$

It is easy to see that g is unipotent and belongs to $O(Q^-)$, and in particular to R. Therefore $U \in \mathcal{S}$.

(3) Both S and τ^s lie in the unipotent radical of P, which, in the chosen basis, is contained in the group of upper-triangular matrices. The commutator of any unipotent element with τ^s is $I_{n+2} + cE_0^{n+1}$ for some $c \in \mathbf{R}$. There is no such element of $\mathrm{O}(p+1, q+1)$ for any nonzero c, so the commutator is the identity. Thus the one-parameter group containing τ is central in the unipotent radical of P, and in particular commutes with S.

$\diamondsuit$

Proof: (of proposition 4.6)

First consider the null geodesic $\alpha(t) = \pi(e^{tU_n})$ in G/P. In projective coordinates on $\mathrm{Ein}^{p,q}$,

$$\begin{aligned} \tau^s.\alpha(t) &= \tau^s.[1 : 0 : \ \cdots \ : t : 0] \\ &= [1 + st : 0 : \ \cdots \ : t : 0] \\ &= [1 : 0 : \ \cdots \ : \frac{t}{1+st} : 0] \\ &= \alpha(c(t)) \end{aligned}$$

Let $\hat{\alpha}(t) = e^{tU_n}$. Now it is possible to compute the holonomy of τ^s along α with respect to $\hat{\alpha}$:

$$\begin{aligned} \tau^s \cdot \hat{\alpha}(t) &= \tau^s \cdot e^{tU_n} \\ &= \hat{\alpha}(c(t)) \cdot e^{-c(t)U_n} \cdot \tau^s \cdot e^{tU_n} \end{aligned}$$

Direct computation gives

$$e^{-c(t)U_n} \cdot \tau^s \cdot e^{tU_n} = \begin{pmatrix} 1+st & & & & & s & 0 \\ & 1+st & & & & & -s \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \frac{1}{1+st} & \\ & & & & & & \frac{1}{1+st} \end{pmatrix}$$

Denote this holonomy matrix by $h(s,t)$.

Now let $S < G$ be as in lemma 4.7, and let $U = (\mathrm{Ad}\ g)(U_n)$ with $g \in S$. Let $\hat{\alpha}(t) = e^{tU}$. Because τ^s commutes with g by lemma 4.7 (3), we can compute the holonomy of τ^s with respect to $\hat{\alpha}$ along α:

$$\begin{aligned} \tau^s \cdot \hat{\alpha}(t) &= \tau^s \cdot e^{tU} \\ &= \tau^s \cdot e^{(\mathrm{Ad}\ g)(tU_n)} \\ &= \tau^s \cdot g \cdot e^{tU_n} \cdot g^{-1} \\ &= g \cdot \tau^s \cdot e^{tU_n} \cdot g^{-1} \\ &= g \cdot e^{c(t)U_n} \cdot h(s,t) \cdot g^{-1} \\ &= e^{(\mathrm{Ad}\ g)(c(t)U_n)} \cdot g \cdot h(s,t) \cdot g^{-1} \\ &= \hat{\alpha}(c(t)) \cdot g \cdot h(s,t) \cdot g^{-1} \end{aligned}$$

Let $\mathcal{S}$ be as in lemma 4.7 (2). Let $U \in \mathcal{S}$. Let $\hat{\beta}(t) = \exp(\hat{\Delta}(v), tU)$ and $\beta = \pi \circ \hat{\beta}$, and assume $\hat{\beta}$ is defined on $(-\epsilon, \epsilon)$. Recall that the holonomy of h^s at $\Delta(v)$ with respect to $\hat{\Delta}$ is τ^s. The above calculation, together with proposition 4.3 (1), implies

$$h^s.\beta(t) = \beta(c(t))$$

for all $t \in (-\epsilon, \epsilon)$. Taking $s = \pm 1/\epsilon$ and again applying proposition 4.3 (1) proves completeness of $\beta(t)$.

By proposition 4.3 (2), the holonomy of h^s at $\beta(t)$ with respect to $\hat{\beta}$ is $g \cdot h(s,t) \cdot g^{-1}$. The adjoint of $h(s,t)$ on $\mathfrak{g}/\mathfrak{p}$ in the basis comprising the images of $U_1, \ldots, U_n$ is

$$\begin{pmatrix} 1 & & & & \\ & \frac{1}{1+st} & & & \\ & & \ddots & & \\ & & & \frac{1}{1+st} & \\ & & & & \frac{1}{(1+st)^2} \end{pmatrix}$$

Since S is contained in P, for $g \in S$, the span of $(\mathrm{Ad}\ g)(U_1), \ldots, (\mathrm{Ad}\ g)(U_n)$ is transverse to $\mathfrak{p}$. The adjoint of $g \cdot h(s,t) \cdot g^{-1}$ in the corresponding basis of $\mathfrak{g}/\mathfrak{p}$ is of course the same diagonal matrix as for $g = 1$. For $\hat{\beta}$ and β as above, define a framing $f_1, \ldots, f_n$ along β by

$$f_i(\beta(t)) = (\pi_* \circ \omega^{-1}_{\hat{\beta}(t)} \circ \mathrm{Ad}\ g)(U_i)$$

Now we can compute the derivative of h^s along β in the framing $(f_1, \ldots, f_n)$. Recall the identity for a Cartan connection

$$\omega_p^{-1} \circ (\mathrm{Ad}\ g) = R_{g^{-1}*} \circ \omega^{-1}_{pg}$$

We will write $f_i(t)$ in place of $f_i(\beta(t))$ below.

$$\begin{aligned} h^s_*(f_i(t)) &= \left(\pi_* \circ \hat{h}^s_* \circ \omega^{-1}_{\hat{\beta}(t)} \circ \mathrm{Ad}\ g\right)(U_i) \\ &= \left(\pi_* \circ \omega^{-1}_{\hat{h}^s \cdot \hat{\beta}(t)} \circ \mathrm{Ad}\ g\right)(U_i) \\ &= \left(\pi_* \circ R_{g^{-1}*} \circ \omega^{-1}_{\hat{h}^s \cdot \hat{\beta}(t) \cdot g}\right)(U_i) \\ &= \left(\pi_* \circ \omega^{-1}_{\hat{\beta}(c(t)) \cdot g \cdot h(s,t)}\right)(U_i) \\ &= \left(\pi_* \circ R_{g \cdot h(s,t)*} \circ \omega^{-1}_{\hat{\beta}(c(t))} \circ \mathrm{Ad}\ (g \cdot h(s,t))\right)(U_i) \\ &= \left(\pi_* \circ \omega^{-1}_{\hat{\beta}(c(t))} \circ \mathrm{Ad}\ (g \cdot h(s,t) \cdot g^{-1}) \circ \mathrm{Ad}\ g\right)(U_i) \\ &= \left(\frac{1}{1+st}\right)^{\sigma(i)} f_i(c(t)) \end{aligned}$$

$\diamondsuit$

5. Maximal degree of nilpotence implies conformal flatness

The aim of this section is to make a step further toward theorem 1.2, and prove the:

Proposition 5.1. *If the group H and the pseudo-Riemannian manifold M statisfy the assumptions of theorem 1.2, then M is conformally flat.*

Recall that a type (p, q) pseudo-Riemannian manifold is *conformally flat* whenever it is locally conformally equivalent to $\mathrm{Ein}^{p,q}$. In dimension ≥ 4 (resp. in dimension 3), conformal flatness is equivalent to the vanishing the *Weyl curvature* W (resp. the *Cotton tensor* C), which is a $(3, 1)$ (resp. a $(3, 0)$) tensor on M (see [AG, p 131]).

If we consider the canonical Cartan geometry (M, B, ω) associated to the conformal structure $(M, [\sigma]))$, the Cartan curvature is defined as follows: the 2-form

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$$

on B vanishes on $u \wedge v$ at b whenever u or v is tangent to the fiber of b. We will define the *Cartan curvature* K to be the resulting function $B \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ (see [Sh, 5.3.22]). Vanishing of K on B is equivalent to M being conformally flat (See [Sh, ch 7] and [Ko, ch IV]).

Now, the proof of proposition 5.1 will work as follows. We introduce the open subset $V \subset M$ on which the Cartan curvature K is nonzero (observe that by the P-equivariance properties of K, its vanishing is an invariant property on the fibers of B, so that it makes sense to say that K vanishes on a subset of M). Proving proposition 5.1 amounts to show that V is empty. Let us assume that it is not the case. Then ∂V is a nonempty H-invariant closed subset. Under the assumptions of theorem 1.2, we have seen that there exist a flow h^s of H, a point $x_0 \in \partial V$, and $b_0 \in B$ above x_0 such that the holonomy of h^s at x_0 with respect to b_0 is the null translation τ^s studied in 4.1. Using the work done in subsection 4.3 about the dynamics of h^s showed , we will show that the Cartan curvature K should vanish on a nonempty neighborhood of x_0 (proposition 5.8): contradiction.

The proof of proposition 5.8 below will require several preliminary results exposed in subsections 5.1 and 5.2.

5.0.1. *Notations.* In the two forthcoming subsections, the points x_0, b_0 and the 1-parameter groups h^s and τ^s are those we just introduced. The infinitesimal generator of τ^s is T given in subsection 4.1. The notations Δ and I_Δ are those of subsection 4.3. The subset $\mathcal{S} \subset \mathcal{N}(\mathfrak{u}^-)$ is as in proposition 4.6. Recall that each curve $\beta(t) = \exp(\hat{\Delta}(v), tU)$ with $U \in \mathcal{S}, v \in I_\Delta$, is defined for all $t \in \mathbf{R}$.

5.1. **Vanishing on lightcones emanating from Δ.** The aim of this subsection is the proof of:

Proposition 5.2. *For every $U \in \mathcal{S}$ and $v \in I_\Delta$, the Cartan curvature of (M, B, ω) vanishes on $\pi^{-1}(\beta(t))$ for all $t \in \mathbf{R}$, where $\beta(t) = \pi \circ \exp(\hat{\Delta}(v), tU)$. Consequently, the Cartan curvature vanishes on the lightcone of each point of Δ, in a sufficiently small neighborhood.*

Proof: Choose $v \in I_\Delta$. We will show that when $p + q \geq 4$, the Weyl curvature vanishes on β, and the Cotton tensor vanishes when $p + q = 3$. These tensors are zero on a closed set, and $\mathcal{S} \cup -\mathcal{S}$ is dense in $\mathcal{N}(\mathfrak{u}^-)$. The neighborhood V can be chosen to be $\pi \circ \exp_{\hat{\Delta}(v)}$, restricted to a neighborhood of the origin in $\mathfrak{u}^-$. Then vanishing on the entire lightcone $C(\Delta(v)) \cap V$ will follow. By the facts cited just above, vanishing of the Weyl and Cotton tensors implies vanishing of the Cartan curvature on the same subset.

Let $f_i(t)$ be the framing along β given by proposition 4.6 (2). We first assume $n \geq 4$ and consider the Weyl tensor. The conformal action of the flow h^s obeys

$$W(h^s_* f_i(t), h^s_* f_j(t), h^s_* f_k(t)) = h^s_* W(f_i(t), f_j(t), f_k(t))$$

The left hand side is

$$\left(\frac{1}{1+st}\right)^{\sigma(i)+\sigma(j)+\sigma(k)} W(f_i(c_s(t)), f_j(c_s(t)), f_k(c_s(t)))$$

We assume $t > 0$, so that $h^s.\beta(t) \to \beta(0) = \Delta(v)$ as $s \to \infty$. (If $t < 0$, then make $s \to -\infty$.) Now

$$W(f_i(0), f_j(0), f_k(0)) = \lim_{s \to \infty} (1+st)^{\sigma(i)+\sigma(j)+\sigma(k)} h^s_* W(f_i(t), f_j(t), f_k(t))$$

If $i = j = k = 1$, then the left side vanishes because W is skew-symmetric in the first two entries. Therefore, we may assume the sum $\sigma(i)+\sigma(j)+\sigma(k) \geq 1$. Boundedness of the right hand side implies

$$h^s_* W(f_i(t), f_j(t), f_k(t)) \to 0 \qquad \text{as } s \to \infty$$

Because $h^s_*(f_1(t)) = f_1(c(t))$, the above limit means $W(f_i(t), f_j(t), f_k(t))$ cannot have a nontrivial component on $f_1(t)$. Then

$$W(f_i(0), f_j(0), f_k(0)) \in \operatorname{span}\{f_2(0), \ldots, f_n(0)\}$$

Note that all the lifted curves $\hat{\beta}$, as U varies in $\mathcal{S}$, have the same initial point $\hat{\Delta}(v)$, so the framings along each β coincide at $\Delta(v)$. Varying U over $\mathcal{S}$, one sees that the Weyl curvature at $\Delta(v)$ has image in

$$\bigcap_{g \in S} \pi_* \omega^{-1}_{\hat{\Delta}(v)}(\text{span}\{(\text{Ad } g)(U_2), \ldots, (\text{Ad } g)(U_n)\})$$

But

$$\text{span}\{(\text{Ad } g)(U_2), \ldots, (\text{Ad } g)(U_n)\} = (\text{Ad } g)(U_n^{\perp}) = (\text{Ad } g)(U_n)^{\perp}$$

By (2) of lemma 4.7, the set of all $(\text{Ad } g)(U_n)$ is a dense set of directions in the null cone $\mathcal{N}(\mathfrak{u}^-)$. Then the intersection

$$\bigcap_{g \in S} (\text{Ad } g)(U_n)^{\perp} = 0$$

and so W vanishes at $\Delta(v)$.

Now

$$0 = \lim_{s \to \infty} (1 + st)^{\sigma(i)+\sigma(j)+\sigma(k)} h^s_* W(f_i(t), f_j(t), f_k(t))$$

If $\sigma(i) + \sigma(j) + \sigma(k) \geq 2$, then

$$W(f_i(t), f_j(t), f_k(t)) = 0$$

because h^s_* cannot contract any tangent vector at $\beta(t)$ strictly faster than $(1 + st)^2$. If $\sigma(i) + \sigma(j) + \sigma(k) = 1$, then we may assume $i = k = 1$, and h^s_* must contract the Weyl curvature strictly faster than $(1 + st)$, which is possible only if

$$W(f_1(t), f_i(t), f_1(t)) \in \mathbf{R} f_n(t)$$

But, in this case, for any inner product $\langle , \rangle$ in the conformal class,

$$\langle W(f_1(t), f_i(t), f_1(t)), f_1(t) \rangle = -\langle W(f_1(t), f_i(t), f_1(t)), f_1(t) \rangle = 0$$

which implies $W(f_1(t), f_i(t), f_1(t)) = 0$, and again W vanishes at $\beta(t)$, as desired.

When $\dim M = 3$, the argument is easier. We will assume again that $t > 0$ and let $s \to \infty$ (If $t < 0$, then consider instead $s \to -\infty$). Then

$$\begin{aligned} C(f_i(t), f_j(t), f_k(t)) &= C(h^s_* f_i(t), h^s_* f_j(t), h^s_* f_k(t)) \\ &= \left(\frac{1}{1 + st} \right)^{\sigma(i)+\sigma(j)+\sigma(k)} C(f_i(c_s(t)), f_j(c_s(t)), f_k(c_s(t))) \end{aligned}$$

As $s \to \infty$,

$$C(f_i(c_s(t)), f_j(c_s(t)), f_k(c_s(t))) \to C(f_i(0), f_j(0), f_k(0))$$

Again we may assume $\sigma(i) + \sigma(j) + \sigma(k) \geq 1$. Taking the limit as $s \to \infty$ gives

$$C(f_i(t), f_j(t), f_k(t)) = 0$$

$\diamondsuit$

5.2. **Vanishing on a neighborhood of x_0.** The previous subsection established vanishing of the Cartan curvature tensor on the union of lightcones emanating from the null geodesic segment Δ containing x_0. This union does not, however, contain a neighborhood of x_0 in general. In this subsection we will show that Δ, or a particular reparametrization of it, is complete, and that lightcones of points on Δ intersect a neighborhood of x_0 in a dense subset. Then vanishing of the Cartan curvature in a neighborhood of x_0 will follow. We keep the notations of the previous section: there is a flow h^s of H, which fixes x_0 with holonomy the lightlike translation τ^s. Recall that T denotes the infinitesimal generator of the one-parameter group τ^s.

Proposition 5.3. *There exists g_θ in the centralizer of T such that $(Ad\ g_\theta)(\mathfrak{u}^-)$ is transverse to $\mathfrak{p}$ and such that the curve $\hat{\Delta}(t) = \exp(b_0, t(Ad\ g_\theta)(U_1))$ in B is defined for all time t.*

Proof: Recall that x_0 and τ^s where obtained by theorem 3.2, which ensured that $\mathfrak{h}^{b_0}$ centralizes the null translation T (see the begining of section 3 for the notation $\mathfrak{h}^{b_0}$). Recall the dynamics on $\mathrm{Ein}^{p,q}$ of the flow τ^s generated by T (fact 4.1): for each y in the null geodesic Λ, an open dense subset of the cone $C(y)$ tends under τ^s to y. Then any flow coming from the centralizer of T must leave Λ setwise invariant; in particular, $\mathfrak{h}^{b_0}$ preserves Λ.

Lemma 5.4. *Let $\mathfrak{n}$ be a nilpotent subalgebra of $o(p+1, q+1)$ fixing two points on Λ. Then the nilpotence degree of $\mathfrak{n}$ is at most $2p$.*

Proof: Because $\mathrm{PO}(p+1, q+1)$ acts transitively on the pairs of distinct points on lightlike geodesics, we may assume $\mathfrak{n}$ fixes $[e_0]$ and $[e_1]$. Fixing $[e_0]$ means $\mathfrak{n}$ is a subalgebra of $\mathfrak{p} \cong \mathfrak{co}(p,q) \ltimes \mathbf{R}^{p,q}$. Recall the embedding of $\mathbf{R}^{p,q}$ in $\mathrm{Ein}^{p,q}$:

$$\varphi : (x_1, \ldots, x_n) \mapsto [-\frac{1}{2}Q^{p,q}(x) : x_1 : \cdots : x_n : 1]$$

Let $u_1, \ldots, u_n$ be the standard basis of $\mathbf{R}^{p,q}$. Then

$$\lim_{t \to \infty} \varphi(tu_1) = [e_1]$$

As in section 2.1.1, the set of lines in $\mathbf{R}^{p,q}$ which tend to $[e_1]$ all have the form

$$\{x + tu_1\} \qquad x \in u_1^{\perp}$$

This set of lines must be invariant by the $\mathfrak{n}$-action on $\mathbf{R}^{p,q}$, which means that the translational components of $\mathfrak{n} \subset \mathfrak{co}(p,q) \ltimes \mathbf{R}^{p,q}$ are all in $u_1^{\perp}$, and the linear components preserve $\mathbf{R}u_1$, and therefore also $u_1^{\perp}$. Now by calculations similar to those in the proof of 2.3, we see that, if $\overline{\mathfrak{n}}$ is the projection of $\mathfrak{n}$ on $\mathfrak{co}(p,q)$, then

$$\mathfrak{n}_k \subseteq \overline{\mathfrak{n}}_k + \overline{\mathfrak{n}}^k(u_1^{\perp})$$

for each positive integer k. But the nilpotence degree of a nilpotent subalgebra $\overline{\mathfrak{n}}$ of $\mathfrak{co}(p,q)$ is at most $2p-1$, while the order of $\overline{\mathfrak{n}}$ on $u_1^{\perp}$ is easily seen to be at most $2p$ (compare with lemma 2.5). $\diamondsuit$

Now consider the restriction of $\mathfrak{h}^{b_0}$ to Λ. Under the identification of $\Lambda = \mathbf{P}(\mathrm{span}\{e_0, e_1\})$ with $\mathbf{RP}^1$, any conformal transformation of $\mathrm{Ein}^{p,q}$ setwise preserving Λ acts as a projective transformation, so the restriction $\mathfrak{a}$ of $\mathfrak{h}^{b_0}$ is a subalgebra of $\mathfrak{sl}(2,\mathbf{R})$. Because $\mathfrak{a}$ is nilpotent, $\dim \mathfrak{a} \leq 1$. By lemma 5.4 above, $\mathfrak{a}$ is nontrivial. Let $L \in \mathfrak{h}^{b_0}$ have nontrivial image in $\mathfrak{a}$; denote this image by $\bar{L}$. Again by lemma 5.4, $\bar{L}$ must generate a 1-parameter subgroup of either parabolic or elliptic type.

First consider the case $\bar{L}$ is elliptic type. Because $\bar{L}$ is conjugate to a rotation of $\mathbf{RP}^1$, the orbit of $[e_0]$ in $\mathrm{Ein}^{p,q}$ under the flow e^{tL} is Λ. Then there exist $(\alpha, \beta) \subset \mathbf{R}$, a diffeomorphism $c : (\alpha, \beta) \to \mathbf{R}$, and a path $g(t) \in P$ such that

$$e^{tL} = e^{c(t)U_1} \cdot g(t)$$

for all $t \in (\alpha, \beta)$. The curve $\exp(b_0, tL)$ is the orbit of b_0 under the lift of a conformal flow, so it is complete, and proposition 4.4 applies to give that $\exp(b_0, tU_1)$ is defined for all $t \in \mathbf{R}$.

Next suppose $\bar{L}$ is parabolic type and that it fixes $[1:0] \in \mathbf{RP}^1$. Then L fixes $[e_0]$ in $\mathrm{Ein}^{p,q}$. The 1-parameter group e^{sL} preserves $\Lambda(t) = \pi(e^{tU_1})$ and reparametrizes it by $t \mapsto \frac{t}{1+st}$. Suppose that $\hat{\Delta}(t) = \exp(b_0, tU_1)$ is defined on $(-\epsilon, \epsilon)$. Take $s_\infty = -1/\epsilon$ and $s_{-\infty} = 1/\epsilon$ and apply proposition 4.3 (1) to see that $\exp(b_0, tU_1)$ is defined for all $t \in \mathbf{R}$.

Next suppose that $\bar{L}$ fixes $[0:1] \in \mathbf{RP}^1$. Then $e^{t\bar{L}}.[1:0] = [1:t]$, and, in $\mathrm{Ein}^{p,q}$, the orbit is $e^{tL}.[e_0] = \Lambda(t)$. Then there exist $g(t) \in P$ such that

$$e^{tL} = e^{tU_1} \cdot g(t) \tag{2}$$

As in the elliptic case, proposition 4.4 gives completeness of $\exp(b_0, tU_1)$. Note that, because the two subgroups e^{tL} and e^{tU_1} have the same restriction to Λ, the path $g(t)$ is in the subgroup $P_\Lambda < P$ pointwise fixing Λ.

Last, consider arbitrary $\bar{L}$ of parabolic type. There exists $\bar{g}_\theta \in \mathrm{PSL}(2, \mathbf{R})$ a rotation such that $(\mathrm{Ad}\ \bar{g}_\theta)(\bar{L})$ fixes $[0:1]$. Let g_θ be the image of $\bar{g}_\theta$ under the standard embedding $\mathrm{SL}(2, \mathbf{R}) \to \mathrm{PO}(p+1, q+1)$ given by

$$\left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & A^{-1} \end{pmatrix} \quad : \quad A \in \mathrm{SL}(2, R) \right\}$$

with respect to which the identification $\mathbf{RP}^1 \to \Lambda$ is equivariant (section 2.1.3). Then g_θ centralizes T. In $\mathrm{PO}(p+1, q+1)$,

$$g_\theta e^{tL} g_\theta^{-1} = e^{tU_1} \cdot g(t)$$

where $g(t) \in P_\Lambda$ is as in (2). So

$$e^{tL} = e^{(\mathrm{Ad}\ g_\theta)(tU_1)} \cdot h(t)$$

where $h(t) = g_\theta g(t) g_\theta^{-1}$. The subgroup P_Λ is normalized by g_θ, so $h(t) \in P_\Lambda$. Proposition 4.4 applies to show $\exp(b_0, (\mathrm{Ad}\ g_\theta)(tU_1))$ is complete, because $\exp(b_0, tL)$ is defined for all t.

To prove the transversality claim, we show $(\mathrm{Ad}\ g_\theta)(\mathfrak{u}^-)$ is still transverse to $\mathfrak{p}$, provided g_θ does not exchange $[e_0]$ and $[e_1]$ in $\mathrm{Ein}^{p,q}$. Then we will take $g_\theta = 1$ when $\bar{L}$ is elliptic or fixes $[1:0]$, and to be the above rotation when $\bar{L}$ is parabolic but does not fix $[1:0]$.

The subalgebra $(\mathrm{Ad}\ g_\theta)(\mathfrak{u}^-)$ is transverse to $\mathfrak{p}$ if the orbit of $[e_0]$ in $\mathrm{Ein}^{p,q}$ under it is n-dimensional. In the Minkowski chart $\mathbf{M}([e_{n+1}])$, the point $[e_0]$ is the origin, and Λ is a null line through the origin, meeting the lightcone at infinity in one point, $[e_1]$. The subalgebra $\mathfrak{u}^-$ acts by translations. If g_θ does not exchange $[e_0]$ and $[e_1]$, then $g_\theta^{-1}[e_0]$ is a point on Λ still contained in $\mathbf{M}([e_{n+1}])$. The orbit

$$(g_\theta \mathfrak{u}^- g_\theta^{-1}).[e_0] = g_\theta(\mathbf{M}([e_{n+1}]))$$

which is n-dimensional. $\diamondsuit$

Proposition 5.5. *Let $g_\theta \in PO(p+1, q+1)$ be given by proposition 5.3, and $\mathcal{S}$ as in proposition 4.6. Let $\hat{\Delta}(v) = \exp(b_0, v(Ad\ g_\theta)(U_1))$ and $\Delta = \pi \circ \hat{\Delta}$. Let $\mathcal{S}' = (Ad\ g_\theta)(\mathcal{S})$. Then*

(1) *$\Delta(v)$ is pointwise fixed by the flow h^s.*

(2) *For each* $U \in \mathcal{S}'$ *and* $v \in \mathbf{R}$, *the curve* $\hat{\beta}(t) = \exp(\hat{\Delta}(v), tU)$ *is complete and projects to a null geodesic.*
(3) *For* Δ *as in (1) and* $\hat{\beta}$ *as in (2), the Cartan curvature vanishes on the fiber of* $\hat{\beta}(t)$ *for all* $t \in \mathbf{R}$.

Proof:

(1) Since g_θ centralizes the null translation T, in $\mathrm{PO}(p+1, q+1)$,

$$\begin{aligned} e^{sT} e^{(\mathrm{Ad}\ g_\theta)(vU_1)} &= e^{sT} g_\theta e^{vU_1} g_\theta^{-1} \\ &= g_\theta e^{vU_1} g_\theta^{-1} e^{sT} \\ &= e^{(\mathrm{Ad}\ g_\theta)(vU_1)} e^{sT} \end{aligned}$$

By proposition 4.3, the geodesic $\Delta(v)$ is pointwise fixed by the flow h^s, and the holonomy of h^s along Δ with respect to $\hat{\Delta}$ is e^{sT} for all v.

(2) Compute that for

$$g_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & & & \\ \sin\theta & \cos\theta & & & \\ & & I_{n-2} & & \\ & & & \cos\theta & \sin\theta \\ & & & -\sin\theta & \cos\theta \end{pmatrix}$$

and $U_n = E_n^0 - E_{n+1}^1$ as above,

$$(\mathrm{Ad}\ g_\theta)(U_n) = U_n$$

Next, note that the subgroup S of lemma 4.7 is contained in P_Λ, the pointwise stabilizer of Λ. Then

$$g_\theta S g_\theta^{-1} < P_\Lambda < P$$

Now any element of $\mathcal{S}'$ is of the form

$$\lambda(\mathrm{Ad}\ g_\theta \circ \mathrm{Ad}\ w)(U_n)$$

for some $\lambda \in \mathbf{R}$ and $w \in S$, and can be written

$$\lambda(\mathrm{Ad}(g_\theta w g_\theta^{-1}))(U_n) = U$$

Since $g_\theta w g_\theta^{-1} \in P$, the element U projects to a null vector in $\mathfrak{g}/\mathfrak{p} \cong \mathbf{R}^{p,q}$. Then $\pi \circ \exp(\hat{\Delta}(v), tU)$ is a null geodesic.

Fix $U = (\mathrm{Ad}\ g_\theta w)(U_n) \in \mathcal{S}'$; it suffices to prove (2) for such U, since the geodesic generated by λU is complete if and only if the geodesic generated by U is. Recall the matrices $h(s,t)$, representing the holonomy of h^s along null geodesics based at $\pi(\exp(b_0, vU_1))$ with initial direction in $\mathcal{S}$ or $-\mathcal{S}$. Straightforward computation shows that g_θ commutes with $h(s,t)$. Recall also that e^{sT} commutes with S. Then in $\mathrm{PO}(p+1,q+1)$,

$$\begin{aligned} e^{sT}e^{tU} &= e^{sT}g_\theta w e^{tU_n} w^{-1} g_\theta^{-1} \\ &= g_\theta w e^{c(t)U_n} h(s,t) w^{-1} g_\theta \\ &= e^{c(t)U}(g_\theta w g_\theta^{-1}) h(s,t) (g_\theta w g_\theta^{-1})^{-1} \end{aligned}$$

where $c(t) = \frac{t}{1+st}$.

The above computation together with proposition 4.3 implies that h^s reparametrizes $\beta(t)$ by $c(t)$ and has holonomy

$$g_\theta w h(s,t) w^{-1} g_\theta^{-1} = (g_\theta w g_\theta^{-1}) h(s,t) (g_\theta w g_\theta^{-1})^{-1} \in P$$

along it with respect to $\hat{\beta}$. Part (1) of proposition 4.3 gives the desired completeness.

(3) As above, we may assume $U = \mathrm{Ad}(g_\theta w)(U_n)$, since the set $\hat{\beta}(t)$, $t \in \mathbf{R}$, is unaffected. Define a framing along $\beta(t)$ as in proposition 4.6 by

$$f_i(\beta(t)) = (\pi_* \circ \omega^{-1}_{\hat{\beta}(t)} \circ (\mathrm{Ad}\ g_\theta w))(U_i)$$

Recall that $(\mathrm{Ad}\ g_\theta)(\mathfrak{u}^-)$ is transverse to $\mathfrak{p}$ by proposition 5.3. Now the derivative of $h(s,t)$ along $\beta(t)$ in this framing is computed as in the proof of 4.6 by the adjoint action of the holonomy $g_\theta w h(s,t) w^{-1} g_\theta^{-1}$ on the $(\mathrm{Ad}\ g_\theta w)(U_i)$, modulo $\mathfrak{p}$. The derivative has the same diagonal form as in proposition 4.6. In fact, all the conclusions of 4.6, and thus also the arguments of proposition 5.2, hold when $\hat{\Delta}(v) = \exp(b_0, (\mathrm{Ad}\ g_\theta)(vU_1))$, and $\mathcal{S}$ is replaced by $\mathcal{S}'$, so we conclude that the Cartan curvature vanishes along the desired geodesics.

$\diamondsuit$

Vanishing of K implies that the developments of homotopic curves in M have the same endpoints in $\mathrm{Ein}^{p,q}$.

Definition 5.6. *Let (M, B, ω) be a Cartan geometry modeled on G/P. Let γ be a piecewise smooth curve in B. The* development *$\mathcal{D}\gamma$ is the piecewise smooth curve in G satisfying $\mathcal{D}\gamma(0) = e$ and $(\mathcal{D}\gamma)'(t) = \omega(\gamma'(t))$ for all but finitely many t.*

Note that for any piecewise smooth curve γ in B, the development $\mathcal{D}\gamma$ is defined on the whole domain of γ, because it is given by a linear first-order ODE on G with bounded coefficients.

Proposition 5.7. *Let (M, B, ω) be a Cartan geometry modeled on G/P. Let $\eta : [0,1] \times [0,1]$ be a fixed-endpoint homotopy between two piecewise smooth curves γ_1 and γ_2 in B—that is,*

$$\eta(0,t) = \gamma_1(t) \qquad \eta(1,t) = \gamma_2(t)$$
$$\eta(s,0) = \gamma_1(0) = \gamma_2(0) \qquad \eta(s,1) = \gamma_1(1) = \gamma_2(1)$$

for all $t, s \in [0,1]$. Suppose that $K = 0$ on the image of η. Then $\mathcal{D}\gamma_1(1) = \mathcal{D}\gamma_2(1)$.

Proof: See [Sh, 3.7.7 and 3.7.8]. $\diamondsuit$

Proposition 5.8. *The Cartan curvature K vanishes on an open set of the form $\pi^{-1}(V)$ for V a neighborhood of x_0 in M.*

Proof: Let $\hat{\Delta}(v) = \exp(b_0, (\mathrm{Ad}\ g_\theta)(vU_1))$, where g_θ is given by proposition 5.3, so $\hat{\Delta}$ is complete. Recall that the curves $\exp(\hat{\Delta}(v), tU)$, where $U \in (\mathrm{Ad}\ g_\theta)(\mathcal{S}) = \mathcal{S}'$ are complete, as well, from proposition 5.5.

To show that the Cartan curvature vanishes on a neighborhood above x_0, it suffices to show that $K = 0$ on $\exp(b_0, V)$, for V a neighborhood of 0 in $(\mathrm{Ad}\ g_\theta)(\mathfrak{u}^-)$, because π_{*b_0} maps $\omega_{b_0}^{-1}(V)$ onto a neighborhood of 0 in $T_{x_0}M$ by proposition 5.3.

First suppose $g_\theta = 1$. Recall from the proof of lemma 4.7 (2) that $\mathcal{S}$ consists of all $U \in \mathcal{N}(\mathfrak{u}^-)$ with $\langle U, U_1 \rangle > 0$. Let

$$Y = aU_1 + X + cU_n \in \mathfrak{u}^-$$

with $X \in \mathrm{span}\{U_2, \ldots, U_{n-1}\}$, and assume that $c \neq 0$. Let $b = \langle X, X \rangle$. Define, for each $0 \leq r \leq 1$, a piecewise smooth curve α_r in B by concatenating

$$\hat{\Delta}(t(b^2/c + 2a)),\ 0 \leq t \leq r/2$$

and

$$\exp(\hat{\Delta}(rb^2/2c + ra), (2t - r)(cU_n + X - b^2/2c\ U_1)),\ r/2 \leq t \leq r$$

Note that $cU_n + X - b^2/2c\ U_1 \in \pm\mathcal{S}$ because $c \neq 0$. Define $\beta(r) = \alpha_r(r)$.

FIGURE 4. components of the homotopy between β and α_1

Because $\mathfrak{u}^-$ is an abelian subalgebra of $\mathfrak{g}$, the development

$$\mathcal{D}\alpha_r(r) = e^{r(\frac{b^2}{2c}+a)U_1} \cdot e^{r(cU_n+X-\frac{b^2}{2c}U_1)} = e^{rY}$$

Denote by $\hat{\beta}^s$ the restriction of β to $[0,s]$ and by $\check{\beta}^s$ the restriction of β to $[s,1]$. The curve β is homotopic to α_1 through the family of concatenations $\alpha_s * \check{\beta}^s$; similarly, $\hat{\beta}^r$ is homotopic to α_r for all $0 \leq r < 1$. Because the curvature K vanishes on the images of these homotopies by proposition 5.2,

$$\mathcal{D}\hat{\beta}^r(r) = \mathcal{D}\alpha_r(r) = e^{rY} \ \forall \ r \in [0,1]$$

But

$$\mathcal{D}\beta(r) = \mathcal{D}\hat{\beta}^r(r) = e^{rY}$$

which means that

$$\beta(r) = \exp(b_0, rY)$$

Then $K(\exp(b_0,Y)) = 0$. Varying Y over all sufficiently small aU_1+X+cU_n with $c \neq 0$ and passing to the closure gives vanishing of K on $\exp(b_0,V)$, for V a neighborhood of 0 in $\mathfrak{u}^-$.

If $g_\theta \neq 1$, then consider

$$Y = (\mathrm{Ad}\ g_\theta)(aU_1 + X + cU_n) \in (\mathrm{Ad}\ g_\theta)(\mathfrak{u}^-)$$

again with $c \neq 0$. Define α_r by concatenating a portion of $\hat{\Delta}$ as above with

$$\exp(\hat{\Delta}(rb^2/2c + ra), (2t-r)(\mathrm{Ad}\ g_\theta)(cU_n + X - b^2/2c\ U_1)), \ r/2 \leq t \leq r$$

note that $(\mathrm{Ad}\ g_\theta)(cU_n + X - b^2/2cU_1) \in \pm\mathcal{S}'$. Therefore, by proposition 5.5 (2) and (3), the curves α_r are defined on $[0,r]$ for all r and K vanishes on them. Then the same argument as above applies to give vanishing of curvature at $\exp(b_0,Y)$. The set of possible Y are dense in a neighborhood V of 0 in $(\mathrm{Ad}\ g_\theta)(\mathfrak{u}^-)$, so we obtain the desired vanishing on $\exp(b_0,V)$. $\diamondsuit$

6. END OF THE PROOF OF THEOREM 1.2: GLOBAL STRUCTURE OF M

In this section, we are again under the assumptions of theorem 1.2: H is a connected nilpotent Lie group acting conformally on a type (p,q) pseudo-Riemanniann manifold (M,σ), and $d(H) = 2p+1$, $p+q \geq 3$. We know by proposition 5.1 that the manifold (M,σ) is locally conformally equivalent to $\mathrm{Ein}^{p,q}$, or in other words is endowed with a $(\widetilde{G}, \widetilde{\mathrm{Ein}}^{p,q})$-structure, where

$\widetilde{G} = \mathrm{Conf}\, \widetilde{\mathrm{Ein}}^{p,q}$, a covering group of $\mathrm{PO}(p+1,q+1)$. We then have a conformal immersion of the universal cover of M

$$\delta : \widetilde{M} \to \widetilde{\mathrm{Ein}}^{p,q}$$

which is called the *developing map* of the structure (see [Th], [Go] for an introduction to (G,X)-structures and the construction of the developing map). This map is unique up to post-composition with an element of $\widetilde{G}$. If $(\widetilde{M}, \widetilde{B}, \tilde{\omega})$ is the canonical Cartan geometry associated to the lifted conformal structure $(\widetilde{M}, \tilde{\sigma})$, then δ lifts to an immersion of bundles $\hat{\delta} : \widetilde{B} \to \widetilde{G}$, where $\widetilde{G}$ is seen as a principal bundle over $\widetilde{\mathrm{Ein}}^{p,q}$. This immersion satisfies $\hat{\delta}^* \omega_{\widetilde{G}} = \tilde{\omega}$, where $\omega_{\widetilde{G}}$ denotes the Maurer-Cartan form on $\widetilde{G}$.

Together with the developing map, we get a *holonomy morphism*

$$\rho : \mathrm{Conf}\, \widetilde{M} \to \widetilde{G}$$

The developing map and the holonomy morphism are related by the equivariance property

$$\rho(\phi) \circ \delta = \delta \circ \phi \quad \forall\ \phi \in \mathrm{Conf}\, \widetilde{M}.$$

The image $\rho(\Gamma)$ is called the *holonomy group* of the structure.

The H-action lifts to a faithful action of a connected covering group of H on $\widetilde{M}$, which we will also denote H. The group $\rho(H) = \check{H}$ is a connected nilpotent subgroup of $\widetilde{G}$, with Lie algebra $\check{\mathfrak{h}}$ isomorphic to $\mathfrak{h}$. In particular $d(\check{\mathfrak{h}}) = 2p+1$. In particular, by proposition 2.3, and conjugating δ if necessary, we may assume in the following that $\check{H}$ contains the 1-parameter group τ^s (see 4.1).

Finally, notice that because Γ centralizes $\mathfrak{h}$, the image $\rho(\Gamma)$ centralizes $\check{\mathfrak{h}}$.

The main step to get theorem 1.2 is now quite clear: we must prove that *the developping map δ is a conformal diffeomorphism between $\tilde{M}$ and $\widetilde{Ein}^{p,q}$*. This will be achieved after some preliminary geometric and algebraic work done in subsections 6.1, 6.2 and 6.3. The final proof of theorem 1.2 will be given in 6.4 for the lorentzian case, and in 6.5 for the other types.

6.1. **More on geometry and dynamics on $\widetilde{\mathbf{Ein}}^{p,q}$.** This section contains necessary facts about $\widetilde{\mathrm{Ein}}^{p,q}$. When $p \geq 2$, $\widetilde{\mathrm{Ein}}^{p,q}$ is just a double cover of $\mathrm{Ein}^{p,q}$. It is $\widehat{\mathcal{N}}^{p+1,q+1}/\mathbf{R}^*_{>0}$. The conformal group $\widetilde{G}$ is $\mathrm{O}(p+1,q+1)$, and the stabilizer of $[e_0]$ is an index-two subgroup of P. A lightcone $C(x)$ in $\widetilde{\mathrm{Ein}}^{p,q}$ has two singular points, and its complement has two connected components, each one conformally equivalent to $\mathbf{R}^{p,q}$.

The Lorentz case $p = 1$ is more subtle since $\widetilde{\mathrm{Ein}}^{1,n-1}$ is no longer compact. It is conformally equivalent to $(\mathbf{R} \times \mathbf{S}^{n-1}, -dt^2 \oplus g_{\mathbf{S}^{n-1}})$. Details about this space are in [Fr1, ch 4.2] and [BCDGM]. The group $\widetilde{G} = \mathrm{Conf}\, \widetilde{\mathrm{Ein}}^{1,n-1}$ is a twofold quotient of $\widetilde{\mathrm{O}}(2,n)$, with center $Z \cong \mathbf{Z}$. The space $\mathrm{Ein}^{1,n-1}$ is the quotient of $\widetilde{\mathrm{Ein}}^{1,n-1}$ by the Z-action.

The lightlike geodesics and lightcones in $\widetilde{\mathrm{Ein}}^{1,n-1}$ are no longer compact. Any lightlike geodesic can be parametrized $\gamma(t) = (t, c(t))$, where $c(t)$ is a unit-speed geodesic of $\mathbf{S}^{n-1}$. Lightlike geodesics are preserved by Z, which acts on them by translations; the quotient is a lightlike geodesic of $\mathrm{Ein}^{1,n-1}$. Any lightcone $C(x) \subset \widetilde{\mathrm{Ein}}^{1,n-1}$ has infinitely-many singular points, which coincide with the Z-orbit of x. The complement of $C(x)$ in $\widetilde{\mathrm{Ein}}^{1,n-1}$ has a countable infinity of connected components, each one conformally diffeomorphic to $\mathbf{R}^{1,n-1}$. The center Z freely and transitively permutes these Minkowski components.

We now take the notations of 4.1. Let τ^s be the flow on $\widetilde{\mathrm{Ein}}^{1,n-1}$ generated by the null translation T. Recall the null geodesic $\Lambda = \mathbf{P}(\mathrm{span}\{e_0, e_1\})$, the fixed set of τ^s on $\mathrm{Ein}^{1,n-1}$. Let $\tilde{\Lambda}$ be the inverse image of Λ in $\widetilde{\mathrm{Ein}}^{1,n-1}$; it is noncompact and connected, and equals the fixed set of τ^s on $\widetilde{\mathrm{Ein}}^{1,n-1}$. Given $\tilde{x} \in \widetilde{\mathrm{Ein}}^{1,n-1} \setminus \tilde{\Lambda}$, there are two distinct points $\tilde{x}^+$ and $\tilde{x}^-$ on $\tilde{\Lambda}$ such that

$$\lim_{s \to \infty} \tau^s.\tilde{x} = \tilde{x}^+ \qquad \text{and} \qquad \lim_{s \to -\infty} \tau^s.\tilde{x} = \tilde{x}^-$$

Details about this material can be found in [Fr1, p 67].

6.2. **About the centralizer of $\check{\mathfrak{h}}$.** For arbitrary (p,q), let $\tilde{\Lambda}$ be the inverse image in $\widetilde{\mathrm{Ein}}^{p,q}$ of Λ; it is connected and fixed by τ^s. Our first task is to find an algebraic restriction on $\rho(\Gamma)$.

Proposition 6.1. *The centralizer $C(\check{\mathfrak{h}})$ of $\check{\mathfrak{h}}$ in $\widetilde{G}$ leaves $\tilde{\Lambda}$ invariant. If $p \geq 2$, the $C(\check{\mathfrak{h}})$-action on $\tilde{\Lambda}$ factors through a finite group. If $p = 1$, it acts on $\tilde{\Lambda}$ by a finite extension of $\mathbf{Z}$, where $\mathbf{Z}$ acts by translations.*

Proof: Let us call $\mathfrak{c}(\check{\mathfrak{h}})$ the Lie algebra of the centralizer of $\check{\mathfrak{h}}$. Observe first that both $\mathfrak{c}(\check{\mathfrak{h}})$ and $\check{\mathfrak{h}}$ centralize τ^s, so they are contained in (see 2.1.3)

$$\left\{ \begin{pmatrix} a & b & -x^t.J_{p-1,q-1} & s & 0 \\ c & -a & -y^t.J_{p-1,q-1} & 0 & -s \\ & & M & y & x \\ & & & a & -b \\ & & & -c & -a \end{pmatrix} : \begin{array}{c} a,b,c,s \in \mathbf{R} \\ x,y \in \mathbf{R}^{p-1,q-1} \\ M \in \mathfrak{o}(p-1,q-1) \end{array} \right\}$$

The projection of $\check{\mathfrak{h}}$ on $\mathfrak{sl}(2,\mathbf{R})$ is a nilpotent subalgebra, so it is 1-dimensional and of parabolic, elliptic, or hyperbolic type. The projection of $\mathfrak{c}(\check{\mathfrak{h}})$ lies in the same subalgebra; we wish to show it is zero.

The hyperbolic case is ruled out by lemma 5.4. We first deal with the parabolic case. We may assume, by conjugating $\check{\mathfrak{h}}$ in the centralizer of τ^s if necessary, that $\mathfrak{c}(\check{\mathfrak{h}})$ and $\check{\mathfrak{h}}$ are subalgebras of

$$\mathfrak{q} = \left\{ \begin{pmatrix} 0 & v & -x^t.J_{p-1,q-1} & s & 0 \\ 0 & 0 & -y^t.J_{p-1,q-1} & 0 & -s \\ & & M & y & x \\ & & & 0 & -v \\ & & & 0 & 0 \end{pmatrix} : \begin{array}{c} v,s \in \mathbf{R} \\ x,y \in \mathbf{R}^{p-1,q-1} \\ M \in \mathfrak{o}(p-1,q-1) \end{array} \right\}$$

This algebra is isomorphic to $(\mathbf{R} \oplus \mathfrak{o}(p-1,q-1)) \ltimes \mathfrak{heis}(2n-3)$. Elements of $\mathfrak{q}$ are denoted $u = (v, M, x, y, s)$, with $v \in \mathbf{R}$, $x, y \in \mathbf{R}^{p-1,q-1}$, and $M \in \mathfrak{o}(p-1,q-1)$. Denote $v = \pi_1(u)$, $M = \pi_2(u)$, and $(x,y) = \pi_3(u)$. Note that if $\pi_i(u) = 0$ for $i = 1,2,3$, then u is in the center of $\mathfrak{q}$. If $u_1 = (v_1, M_1, x_1, y_1, s_1)$ and $u_2 = (v_2, M_2, x_2, y_2, s_2)$ are in $\mathfrak{q}$, then an easy computation yields

- $\pi_1([u_1,u_2]) = 0$
- $\pi_2([u_1,u_2]) = [M_1, M_2]$
- $\pi_3([u_1,u_2]) = (v_1y_2 - v_2y_1 - M_1.x_2 + M_2.x_1, -M_1.y_2 + M_2.y_1)$

Now, if $u_0 = (v_0, M_0, x_0, y_0, s_0)$ is in $\mathfrak{c}(\check{\mathfrak{h}})$, each $u = (v, M, x, y, s) \in \check{\mathfrak{h}}$ must satisfy the relations:

- $[M_0, M] = 0$
- $M_0.y = M.y_0$
- $v_0y - M_0.x = vy_0 - M.x_0$

We claim that $v_0 = 0$. If not, then from the last relation above, whenever u_1 and u_2 are in $\check{\mathfrak{h}}$, then

$$y_1 = \frac{v_1}{v_0} y_0 - \frac{1}{v_0} M_1.x_0 + \frac{1}{v_0} M_0.x_1$$

and

$$y_2 = \frac{v_2}{v_0} y_0 - \frac{1}{v_0} M_2.x_0 + \frac{1}{v_0} M_0.x_2$$

This implies

$$\begin{aligned} \pi_3([u_1, u_2]) &= (\frac{v_1}{v_0}(-M_2.x_0 + M_0.x_2) + \frac{v_2}{v_0}(-M_0.x_1 + M_1.x_0) \\ &- M_1.x_2 + M_2.x_1, -M_1.y_2 + M_2.y_1) \end{aligned}$$

A nilpotent Lie algebra and its Zariski closure have the same nilpotence index, and the same centralizer. Hence we may assume that $\check{\mathfrak{h}}$ is Zariski closed, and write $\check{\mathfrak{h}} \cong \check{\mathfrak{r}} \ltimes \check{\mathfrak{u}}$, where $\check{\mathfrak{r}}$ is reductive and $\check{\mathfrak{u}}$ is an algebra of nilpotents. As already observed, the adjoint action of $\check{\mathfrak{r}}$ on $\check{\mathfrak{h}}$ must be trivial, so that $\check{\mathfrak{r}}$ is central in $\check{\mathfrak{h}}$, and $d(\check{\mathfrak{h}}) = d(\check{\mathfrak{u}})$.

Let $\check{\mathfrak{m}} = \pi_2(\check{\mathfrak{u}})$. It is a nilpotent Lie subalgebra of $\mathfrak{o}(p-1, q-1)$, and also an algebra of nilpotents, since $\check{\mathfrak{u}}$ is so. Using the equation above, we get by induction that $\pi_3(\check{\mathfrak{u}}_k) \subset \check{\mathfrak{m}}^k(\mathbf{R}^{p-1,q-1}) \times \check{\mathfrak{m}}^k(\mathbf{R}^{p-1,q-1})$. Moreover, $\pi_2(\check{\mathfrak{u}}_k) = \check{\mathfrak{m}}_k$, and $\pi_1(\check{\mathfrak{u}}_k) = 0$ as soon as $k \geq 1$. By proposition 2.3, $d(\check{\mathfrak{m}}) \leq 2p-3$, and $o(\check{\mathfrak{m}}) \leq 2p-1$ by lemma 2.5. As a consequence, $\pi_1(\check{\mathfrak{u}}_{2p-1}) = \pi_2(\check{\mathfrak{u}}_{2p-1}) = \pi_3(\check{\mathfrak{u}}_{2p-1}) = 0$, which implies that $\check{\mathfrak{u}}_{2p-1}$ is in the center of $\check{\mathfrak{u}}$, and finally $d(\check{\mathfrak{u}}) \leq 2p$. Since $d(\check{\mathfrak{h}}) = d(\check{\mathfrak{u}})$, we get a contradiction.

Therefore, in the parabolic case, $\mathfrak{c}(\check{\mathfrak{h}})$ is actually a subalgebra of

$$\left\{ \begin{pmatrix} 0 & 0 & -x^t.J_{p-1,q-1} & s & 0 \\ 0 & 0 & -y^t.J_{p-1,q-1} & 0 & -s \\ & & M & y & x \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix} : \begin{array}{c} x, y, s \in \mathbf{R}^{p-1,q-1} \\ M \in \mathfrak{o}(p-1, q-1) \end{array} \right\}$$

Let us now show that the same conclusion holds in the elliptic case. This time, $\mathfrak{c}(\check{\mathfrak{h}})$ and $\check{\mathfrak{h}}$ are subalgebras of

$$\left\{ \begin{pmatrix} 0 & v & -x^t.J_{p-1,q-1} & s & 0 \\ -v & 0 & -y^t.J_{p-1,q-1} & 0 & -s \\ & & M & y & x \\ & & & 0 & -v \\ & & & v & 0 \end{pmatrix} : \begin{array}{c} v, s \in \mathbf{R} \\ x, y \in \mathbf{R}^{p-1,q-1} \\ M \in \mathfrak{o}(p-1, q-1) \end{array} \right\}$$

As above, we denote the elements of this algebra by $u = (v, M, x, y, s)$, and keep the notations π_1, π_2, π_3. In this case, the computations give

$$\pi_3([u_1, u_2]) = (v_1 y_2 - v_2 y_1 - M_1.x_2 + M_2.x_1, -v_1 x_2 + v_2 x_1 - M_1.y_2 + M_2.y_1)$$

If some element $u_0 = (v_0, M_0, x_0, y_0, s_0)$ is in $\mathfrak{c}(\check{\mathfrak{h}})$ and satisfies $v_0 \neq 0$, then as above, for all $u_1, u_2 \in \check{\mathfrak{h}}$,

$$\begin{aligned} \pi_3([u_1, u_2]) \quad = \quad & (\frac{v_1}{v_0}(M_0.x_2 - M_2.x_0) + \frac{v_2}{v_0}(M_1.x_0 - M_0.x_1) - M_1.x_2 + M_2.x_1, \\ & \frac{v_1}{v_0}(M_2.y_0 - M_0.y_2) + \frac{v_2}{v_0}(M_0.y_1 - M_1.y_0) - M_1.y_2 + M_2.y_1) \end{aligned}$$

As before, this would contradict $d(\check{\mathfrak{h}}) = 2p + 1$.

From the above calculation on $\mathfrak{c}(\check{\mathfrak{h}})$, we see that the identity component of $C(\check{\mathfrak{h}})$ acts trivially on $\tilde{\Lambda}$. When $p \geq 2$, the centralizer of $\check{\mathfrak{h}}$ is algebraic, so it has finitely many connected components and its action on $\tilde{\Lambda}$ factors through a finite group. For $p = 1$, the centralizer $C(\check{\mathfrak{h}})$ projects to a finite subgroup $F < \mathrm{PO}(2, n)$ by the same argument, and is an extension of F by Z. But $Z \cong \mathbf{Z}$ and acts by translations on $\tilde{\Lambda}$. This concludes the proof of the proposition. $\diamondsuit$

6.3. **Geometrical properties of the developing map.** There is a 1-parameter group h^s in H such that $\rho(h^s) = \tau^s$. Because M is conformally flat, this exactly means that h^s fixes $x_0 \in M$ and has holonomy τ^s with respect to some b_0 above x_0 in B. We now adopt the notations and results of subsection 4.3. The lightlike geodesic $\Delta(t) = \pi \circ \exp(b_0, tU_1)$ is pointwise fixed by h^s and is locally an attracting set for it (see proposition 4.6 (1)). Choose $\tilde{x}_0 \in \widetilde{M}$ over x_0 and lift h^s to $\widetilde{M}$. Let $\tilde{\Lambda} \subset \widetilde{\mathrm{Ein}}^{p,q}$ be as above: it is a closed subset, pointwise fixed by τ^s, and it is the attracting set for τ^s. Because of these dynamics, $\pi_M^{-1}(\Delta) \subset \delta^{-1}(\tilde{\Lambda})$, which is a closed, Γ-invariant, 1-dimensional, immersed submanifold. Let $\tilde{\Delta}$ be the component of $\delta^{-1}(\tilde{\Lambda})$ containing $\tilde{x_0}$. Denote $\Gamma_0 < \Gamma$ the subgroup leaving $\tilde{\Delta}$ invariant.

The proof of the following fact is easy and left to the reader.

Proposition 6.2. *The image $\pi_M(\tilde{\Delta}) \subset M$ is closed. Therefore, Γ_0 acts cocompactly on $\tilde{\Delta}$.*

As a consequence, we will get:

Proposition 6.3. *The map δ is a covering map from $\tilde{\Delta}$ onto $\tilde{\Lambda}$. When M is Lorentzian, δ is a diffeomorphism between $\tilde{\Delta}$ and $\tilde{\Lambda}$.*

Proof: First note that $\tilde{\Delta}$ is open in $\delta^{-1}(\tilde{\Lambda})$. For if $\delta^{-1}(\tilde{\Lambda})$ were recurrent, then $\tilde{\Lambda}$ would be, as well; but $\tilde{\Lambda}$ is a closed, embedded submanifold of $\widetilde{\mathrm{Ein}}^{p,q}$. Therefore the image $\delta(\tilde{\Delta})$ is a connected open subset of $\tilde{\Lambda}$. By equivarience of δ and the previous proposition, $\rho(\Gamma_0)$ preserves $\delta(\tilde{\Delta})$ and acts cocompactly on it. But $\rho(\Gamma_0)$ centralizes $\check{\mathfrak{h}}$, so its action on $\tilde{\Lambda}$ factors through either a finite group, or the extension of a finite group by $\mathbf{Z}$. In both cases, $\delta(\tilde{\Delta})$ must equal $\tilde{\Lambda}$.

When M has Lorentz type, then $\delta : \tilde{\Delta} \to \tilde{\Lambda}$ must be a diffeomorphism, because all lightlike geodesics in $\widetilde{\mathrm{Ein}}^{1,n-1}$ are embedded copies of $\mathbf{R}$, as described in section 6.1; in particular, they have no self-intersection.

Now assume $p \geq 2$. On one hand, Γ_0 acts cocompactly on $\tilde{\Delta}$; on the other hand, the action of $\rho(\Gamma_0)$ on $\tilde{\Lambda}$ factors through a finite group. Let $\Gamma_0' \lhd \Gamma_0$ be such that $\rho(\Gamma_0')$ is the kernel in $\rho(\Gamma_0)$ of restriction to $\tilde{\Lambda}$. Then the restriction of δ factors

$$\begin{array}{cccc} \delta|_{\tilde{\Delta}} : & \tilde{\Delta} & \to & \tilde{\Lambda} \\ & & \searrow & \uparrow \\ & & & \tilde{\Delta}/\Gamma_0' \end{array}$$

Because Γ_0' acts freely and properly on $\tilde{\Delta}$, the quotient map $\tilde{\Delta} \to \tilde{\Delta}/\Gamma_0'$ is a covering, and because Γ_0' has finite index in Γ_0, its action on $\tilde{\Delta}$ is cocompact. The map $\tilde{\Delta}/\Gamma_0' \to \tilde{\Lambda}$ is surjective because $\delta|_{\tilde{\Delta}}$ is; it is a local diffeomorphism because $\delta|_{\tilde{\Delta}}$ and $\tilde{\Delta} \to \tilde{\Delta}/\Gamma_0'$ are. By compactness of $\tilde{\Delta}/\Gamma_0'$, it follows that $\tilde{\Delta}/\Gamma_0' \to \tilde{\Lambda}$, hence $\tilde{\Delta} \to \tilde{\Lambda}$, is a covering, as desired. $\diamondsuit$

6.4. **Proof of theorem 1.2: the case of Lorentz manifolds.** Suppose now that $p = 1$. Let

$$\Omega = \{\tilde{z} \in \widetilde{M} \setminus \tilde{\Delta} \mid \lim_{s \to \infty} h^s.\tilde{z} \text{ exists and is in } \tilde{\Delta}\}$$

Proposition 6.4. *The set Ω is nonempty and open. It is mapped diffeomorphically by δ onto $\widetilde{Ein}^{1,n-1} \setminus \tilde{\Lambda}$.*

Proof: Let us first check that Ω is nonempty. Recall $\tilde{\Delta}$ is pointwise fixed by h^s. Let $\tilde{z}_\infty \in \tilde{\Delta}$, and choose $\tilde{b}_\infty \in \widetilde{B}$ above $\tilde{z}_\infty$ such that the holonomy of h^s with respect to $\tilde{b}_\infty$ is τ^s. Let $\mathcal{S}$ be as in proposition 4.6 and $U \in \mathcal{S}$. Consider the geodesic $\beta(t) = \pi \circ \exp(\tilde{b}_\infty, tU)$. It is complete by proposition 4.6 (1); further, for $t > 0$,

$$\lim_{s \to \infty} h^s.\beta(t) = \beta(0) = \tilde{z}_\infty$$

Then $\beta(t) \in \Omega$ for $t > 0$.

To prove that Ω is open, choose $\tilde{z}_0 \in \Omega$. There exists $\tilde{z}_\infty \in \tilde{\Delta}$ such that $\lim_{s\to\infty} h^s.\tilde{z}_0 = \tilde{z}_\infty$. Since the orbits of τ^s are lightlike geodesics in $\widetilde{\mathrm{Ein}}^{1,n-1}$, the same is true for the orbits of h^s on $\widetilde{M}$. Then $\tilde{z}_0$ lies on some lightlike geodesic emanating from $\tilde{z}_\infty$. Any such geodesic not fixed by h^s has the form $\pi \circ \exp(\tilde{b}_\infty, tU)$ with $U \in \mathcal{S}$. Then for $s_0 > 0$ there exist $\epsilon > 0$ and a diffeomorphism $c : (s_0, \infty) \to (0, \epsilon)$ such that, for every $s \in (s_0, \infty)$,

$$\pi \circ \exp(\tilde{b}_\infty, c(s)U) = h^s.\tilde{z}_0.$$

There are a neighborhood I of $\tilde{z}_\infty$ in $\tilde{\Delta}$, a segment $\widetilde{I} \subset \widetilde{B}$ lying over I, and an open neighborhood $\mathcal{U}$ of 0 in $\mathfrak{u}^-$ such that the map

$$\begin{aligned} \mu \quad : \quad & I \times (\mathcal{U} \cap \mathcal{S}) \to \widetilde{M} \\ & (\tilde{z}, u) \mapsto \pi \circ \exp(\tilde{b}, u) \end{aligned}$$

where $\tilde{b} \in \widetilde{I}$ lies over $\tilde{z}$, is defined and is a submersion. Choosing s_0 big enough, $c(s_0)U \in \mathcal{U} \cap \mathcal{S}$, so that $V = \mu(I \times (\mathcal{U} \cap \mathcal{S}))$ is an open subset containing $h^{s_0}.\tilde{z}_0$. It follows immediately from proposition 4.6 that $V \subset \Omega$. Since Ω is h^s-invariant, $h^{-s_0}(V) \subset \Omega$. It is an open subset containing $\tilde{z}_0$, which shows that Ω is open.

We now prove that δ is an injection in restriction to Ω. Assume that $\tilde{z}$ and $\tilde{z}'$ are two points of Ω satisfying $\delta(\tilde{z}) = \delta(\tilde{z}')$. Let $\tilde{z}_\infty = \lim_{s\to\infty} h^s.\tilde{z}$ and $\tilde{z}'_\infty = \lim_{s\to\infty} h^s.\tilde{z}'$. Then

$$\delta(\tilde{z}_\infty) = \lim_{s\to\infty} \tau^s.\delta(\tilde{z}) = \lim_{s\to\infty} \tau^s.\delta(\tilde{z}') = \delta(\tilde{z}'_\infty)$$

Because δ is injective on $\tilde{\Delta}$ by proposition 6.3, $\tilde{z}_\infty = \tilde{z}'_\infty$. Choose U an open neighborhood of $\tilde{z}_\infty$ which is mapped diffeomorphically by δ on an open neighborhood V of $z_\infty = \delta(\tilde{z}_\infty)$. There exists $s_0 \geq 0$ such that for all $t \geq s_0$, $h^t.\tilde{z} \in U$ and $h^t.\tilde{z}' \in U$. Moreover, $\delta(h^t.\tilde{z}) = \delta(h^t.\tilde{z}') = \tau^t.\delta(\tilde{z})$. Since δ is an injection in restriction to U, the images $h^t.\tilde{z} = h^t.\tilde{z}'$, so $\tilde{z} = \tilde{z}'$, as desired.

It remains to show that $\delta(\Omega) = \widetilde{\mathrm{Ein}}^{1,n-1} \backslash \tilde{\Lambda}$. The inclusion $\delta(\Omega) \subset \widetilde{\mathrm{Ein}}^{1,n-1} \backslash \tilde{\Lambda}$ follows easily from the definition of Ω. Just note that any $\tilde{z} \in \delta^{-1}(\tilde{\Lambda})$ is fixed by h^s, so Ω cannot meet $\delta^{-1}(\tilde{\Lambda})$. Now, pick $z \in \widetilde{\mathrm{Ein}}^{1,n-1}$. There exists $z_\infty \in \tilde{\Lambda}$ such that $\lim_{s\to\infty} \tau^s.z = z_\infty$. By proposition 6.3, there is a unique $\tilde{z}_\infty \in \tilde{\Delta}$ such that $\delta(\tilde{z}_\infty) = z_\infty$. Also, there is a neighborhood U of $\tilde{z}_\infty$ mapped diffeomorphically by δ on some neighborhood V of z_∞. There exists s_0 such that for $s \geq s_0$, $\tau^s.z \in V$. Let $\tilde{z} \in U$ be such that $\delta(\tilde{z}) = \tau^{s_0}.z$. Then for all $s \geq s_0$, we have $h^s.\tilde{z} \in U$ and $\lim_{s\to\infty} h^s.\tilde{z} = \tilde{z}_\infty$. Thus, $\tilde{z} \in \Omega$.

Moreover, $\delta(h^{-s_0}.\tilde{z}) = z$ and since Ω is h^s-invariant, $z \in \delta(\Omega)$, as desired. $\diamondsuit$

Remark 6.5. *Notice that when we proved that Ω is nonempty, we showed that $\tilde{\Delta}$ is in the closure of Ω.*

The inverse of δ on $\check{\Omega} = \widetilde{\mathrm{Ein}}^{1,n-1} \setminus \tilde{\Lambda}$ is a conformal embedding $\lambda : \check{\Omega} \to \widetilde{M}$. Because $n \geq 3$, $\partial\check{\Omega}$ has codimension at least 2. Then theorem 1.8 of [Fr5] applies in our context. It says,

Theorem 6.6. [Fr5] *Let $\check{\Omega}$ be an open subset of $\widetilde{Ein}^{p,q}$ such that $\partial\check{\Omega}$ is nonempty and has codimension at least 2. Let $\lambda : \check{\Omega} \to (N, \sigma)$ be a conformal embedding, where (N, σ) is a type-(p, q) pseudo-Riemannian manifold. Then there is an open subset $\check{\Omega}' \subset \widetilde{Ein}^{p,q}$ containing $\check{\Omega}$ such that λ extends to a conformal diffeomorphism $\lambda : \check{\Omega}' \to (N, \sigma)$.*

Theorem 6.6 yields an open subset $\check{\Omega}'$ containing $\check{\Omega}$ and a conformal diffeomorphism $\lambda^{-1} : \widetilde{M} \to \check{\Omega}'$, which coincides with δ on Ω. Two conformal maps which are the same on an open set of a connected pseudo-Riemannian manifold of dimension ≥ 3 must coincide, so $\lambda^{-1} = \delta$. Now, $\check{\Omega}'$ contains $\delta(\tilde{\Delta}) = \tilde{\Lambda}$ and $\widetilde{\mathrm{Ein}}^{1,n-1} \setminus \tilde{\Lambda}$, which yields $\check{\Omega}' = \widetilde{\mathrm{Ein}}^{1,n-1}$. Thus M is conformally diffeomorphic to a quotient of $\widetilde{\mathrm{Ein}}^{1,n-1}$ by a discrete group $\Gamma < \widetilde{G}$. Since Γ centralizes $\check{\mathfrak{h}}$, by proposition 6.1, Γ is a finite extension of Z, proving theorem 1.2 in the Lorentz case.

6.5. **Proof of theorem 1.2: the case $p \geq 2$.** The proof in the Lorentz case must be adapted for $p \geq 2$ because in this case, δ is *a priori* just a covering map from $\tilde{\Delta}$ to $\tilde{\Lambda}$ and no longer a diffeomorphism.

Recall that τ^s fixes $\tilde{\Lambda}$ pointwise. Let $p_1 \in \tilde{\Lambda}$. The lightcone $C(p_1)$ has two singular points, p_1, and another point, $p_2 \in \tilde{\Lambda}$, and its complement consists of two Minkowski components, M_1 and M_2. Also, $\tilde{\Lambda} \setminus \{p_1, p_2\}$ has two connected components $\mathcal{I}_1$ and $\mathcal{I}_2$, which can be defined by dynamical properties of τ^s:

$$\forall z \in M_1, \lim_{s \to \infty} \tau^s.z \in \mathcal{I}_1 \quad \text{and} \quad \lim_{s \to -\infty} \tau^s.z \in \mathcal{I}_2$$
$$\forall z \in M_2, \lim_{s \to \infty} \tau^s.z \in \mathcal{I}_2 \quad \text{and} \quad \lim_{s \to -\infty} \tau^s.z \in \mathcal{I}_1$$

If F is the set of fixed points of τ^s, then $C(p_1) \setminus F$ splits into two connected components, C_1 and C_2. Suppose $p_1 = [e_0]$. Then $C(p_1)$ is the quotient

$(e_0^\perp \cap \widehat{\mathcal{N}}^{p+1,q+1})/\mathbf{R}^*_{>0}$ and $p_2 = [-e_0]$. Recall that

$$F = (e_0^\perp \cap e_1^\perp \cap \widehat{\mathcal{N}}^{p+1,q+1})/\mathbf{R}^*_{>0}$$

The components of $C(p_1) \setminus F$ correspond to $\{\langle x, e_1\rangle > 0\}$ and $\{\langle x, e_1\rangle < 0\}$. As in section 4.1, $\tau^s.[x] \to [e_0]$ as $s \to \infty$ if $\langle x, e_1\rangle > 0$ and $\tau^s.[x] \to [-e_0]$ if $\langle x, e_1\rangle < 0$; similarly, $\tau^s.[x] \to [e_0]$ as $s \to -\infty$ if $\langle x, e_1\rangle < 0$ and $\tau^s.[x] \to [-e_0]$ as $s \to -\infty$ if $\langle x, e_1\rangle > 0$.

The dynamics are the same at any $p_1 \in \tilde{\Lambda}$, because there is a conformal automorphism of $\widetilde{\mathrm{Ein}}^{p,q}$ sending $[e_0]$ to p_1 and preserving F:

$$\forall z \in C_1, \ \lim_{s\to\infty} \tau^s.z = p_1 \quad \text{and} \quad \lim_{s\to-\infty} \tau^s.z = p_2$$
$$\forall z \in C_2, \ \lim_{s\to\infty} \tau^s.z = p_2 \quad \text{and} \quad \lim_{s\to-\infty} \tau^s.z = p_1$$

Let $\{\tilde{p}_{2i+1} \ : \ i \in J\} = \delta^{-1}(p_1)$ and $\{\tilde{p}_{2i} \ : \ i \in J\} = \delta^{-1}(p_2)$. Order the points $\tilde{p}_{2i+1}$ and $\tilde{p}_{2i}$ compatibly with an orientation of $\tilde{\Delta}$, and in such a way that $\tilde{p}_{2i}$ is between $\tilde{p}_{2i-1}$ and $\tilde{p}_{2i+1}$. If the covering $\delta : \tilde{\Delta} \to \tilde{\Lambda}$ is finite, then J is finite; in this case, order each set of points cyclically. The segment of $\tilde{\Delta}$ from $\tilde{p}_{2i-1}$ to $\tilde{p}_{2i}$ will be denoted I_{2i-1}, and the segment from $\tilde{p}_{2i}$ to $\tilde{p}_{2i+1}$ will be I_{2i}. Now the set Ω of the previous section will be replaced by the two sets

$$\Omega_1 = \{\tilde{z} \in \widetilde{M} \setminus \tilde{\Delta} \mid \lim_{s\to\infty} h^s.\tilde{z} \text{ exists and is in } I_1\}$$
$$\Omega_2 = \{\tilde{z} \in \widetilde{M} \setminus \tilde{\Delta} \mid \lim_{s\to\infty} h^s.\tilde{z} \text{ exists and is in } I_2\}$$

Using the dynamical characterization of I_1 and I_2 corresponding to that of $\mathcal{I}_1$ and $\mathcal{I}_2$ given above, one can reproduce the proof of proposition 6.4 to obtain

Proposition 6.7. *The sets Ω_1 and Ω_2 are nonempty and open. Each Ω_i is mapped diffeomorphically by δ onto M_i, $i = 1, 2$.*

Lemma 6.8. *$\delta(\partial\Omega_i) \subset C(p_i)$, $i = 1, 2$.*

Proof: Let $\tilde{z} \in \partial\Omega_1$. By proposition 6.7, $\delta(\tilde{z}) \in \overline{M}_1$. If $\delta(\tilde{z}) \in M_1$, the same proposition gives $\tilde{z}' \in \Omega_1$ such that $\delta(\tilde{z}') = \delta(\tilde{z})$. Then if U' is a neighborhood of $\tilde{z}'$ in Ω_1, and if U is a neighborhood of $\tilde{z}$ in $\widetilde{M}$, with $U \cap U' = \emptyset$,

$$\delta(U \cap \Omega_1) \cap \delta(U') \neq \emptyset$$

contradicting the injectivity of δ on Ω_1.

The same proof holds if $\tilde{z} \in \partial\Omega_2$. $\diamondsuit$

If U is an open set of a type-(p,q) pseudo-Riemannian manifold (N,σ), and if $x \in N$, denote by $C_U(x)$ the set of points in U which can be joined to x by a lightlike geodesic contained in U.

Lemma 6.9. *There exists U a neighborhood of $\tilde{p}_1$ in $\widetilde{M}$ such that*

$$U \setminus C_U(\tilde{p}_1) = (U \cap \Omega_2) \bigcup (U \cap \Omega_1)$$

Proof: First choose U a neighborhood of $\tilde{p}_1$ that is geodesically convex for some metric in the conformal class, so $U \setminus C_U(\tilde{p}_1)$ is a union of exactly two connected components U_1 and U_2. (Here we use the assumption $p \geq 2$. In the Lorentz case, there would be three connected components.) We may choose U small enough that δ maps U diffeomorphically on its image V, and $\delta(C_U(\tilde{p}_1)) = V \cap C(p_1)$.

First, $U \cap \Omega_1$ and $U \cap \Omega_2$ are both nonempty: remark 6.5 is easily adapted to the current context to show that $I_1 \subset \overline{\Omega_1}$ and $I_2 \subset \overline{\Omega_2}$. Assume then that $U_1 \cap \Omega_1 \neq \emptyset$. By lemma 6.8, if $U_1 \cap \partial\Omega_1 \neq \emptyset$, then $\delta(U_1 \cap \partial\Omega_1) \subset C(p_1)$. Since δ is injective on U, then $U_1 \cap \partial\Omega_1 \subset C_U(\tilde{p}_1)$, a contradiction. Therefore, $U_1 \cap \partial\Omega_1 = \emptyset$, so $U_1 \subset \Omega_1$. Similarly, $U_2 \subset \Omega_2$. $\diamondsuit$

Let $W_U = C_U(\tilde{p}_1) \setminus F$, and define $W = \bigcup_{s \in \mathbf{R}} h^s.W_U$.

Lemma 6.10. *The set $\Omega = \Omega_1 \cup W \cup \Omega_2 \subset \widetilde{M}$ is open, and is mapped diffeomorphically by δ to $\widetilde{Ein}^{p,q} \setminus F$.*

Proof: We first prove that Ω is open. By lemma 6.9, and the fact that Ω_1 and Ω_2 are open, the set $\Omega_1 \cup W_U \cup \Omega_2$ is open. Now, if $\tilde{z} \in W$, there exists $s_0 \in \mathbf{R}$ such that $h^{s_0}.\tilde{z} \in W_U$. Then there is a neighborhood U' of $h^{s_0}.\tilde{z}$ contained in $\Omega_1 \cup W_U \cup \Omega_2 \subset \Omega$. Then $h^{-s_0}.U'$ is a neighborhood of $\tilde{z}$ contained in Ω.

We now show that δ is injective on Ω. By lemma 6.7, δ is injective on Ω_1 and Ω_2, and because $\delta(\Omega_1) = M_1$ is disjoint from $\delta(\Omega_2) = M_2$, the map δ is actually injective on $\Omega_1 \cup \Omega_2$. Because $\delta(W) \subset C(p_1)$ is disjoint from $\delta(\Omega_1 \cup \Omega_2)$, it suffices to prove that δ is injective on W. Assume $\tilde{z}, \tilde{z}' \in W$ with $\delta(\tilde{z}) = \delta(\tilde{z}')$, and suppose this point is in C_1, so

$$\lim_{s \to \infty} \tau^s.\delta(\tilde{z}) = \lim_{s \to \infty} \tau^s.\delta(\tilde{z}') = p_1$$

Since $\tilde{z} \in W$, either $\lim_{s\to\infty} h^s.\tilde{z} = \tilde{p}_1$ or $\lim_{s\to-\infty} h^s.\tilde{z} = \tilde{p}_1$. But if $\lim_{s\to-\infty} h^s.\tilde{z} = \tilde{p}_1$, then $\lim_{s\to-\infty} \tau^s.\delta(\tilde{z}) = p_1$, contradicting $\delta(\tilde{z}) \in C_1$. Therefore, $\lim_{s\to\infty} h^s.\tilde{z} = \tilde{p_1}$, and for the same reasons, $\lim_{s\to\infty} h^s.\tilde{z}' = \tilde{p}_1$. Then there exists $s_0 > 0$ such that for all $s \geq s_0$, both $h^s.\tilde{z}$ and $h^s.\tilde{z}'$ are in U. Since $\delta(h^s.\tilde{z}) = \tau^s.\delta(\tilde{z}) = \tau^s.\delta(\tilde{z}') = \delta(h^s.\tilde{z}')$, and since δ is injective on U, we get $h^s.\tilde{z} = h^s.\tilde{z}'$ and finally $\tilde{z} = \tilde{z}'$. The proof is similar if $\delta(\tilde{z}) = \delta(\tilde{z}')$ is in C_2.

It remains to understand the set $\delta(\Omega)$. From proposition 6.7, $M_1 \cup M_2 \subset \delta(\Omega)$, and it is also clear that $\delta(\Omega) \subset \widetilde{\mathrm{Ein}}^{p,q} \setminus F$. If $z \in C_1$, then there exists $s > 0$ such that $\tau^s.z \in V$. Hence, there is $\tilde{z} \in U$ such that $\delta(\tilde{z}) = \tau^s.z$, and finally $\delta(h^{-s}.\tilde{z}) = z$. Since $\tilde{z} \in U$, then $h^{-s}.\tilde{z} \in \Omega$, which proves $z \in \delta(\Omega)$. In the same way, we show that if $z \in C_2$, then $z \in \delta(\Omega)$. Finally $\delta(\Omega) = \widetilde{\mathrm{Ein}}^{p,q} \setminus F = M_1 \cup M_2 \cup C_1 \cup C_2$. $\diamondsuit$

The conclusion is essentially the same as in the Lorentz case. Let $\check{\Omega}$ be the complement of F in $\widetilde{\mathrm{Ein}}^{p,q}$. Then inverting δ on $\check{\Omega}$ gives a conformal embedding $\lambda : \check{\Omega} \to \widetilde{M}$. Recall from section 4.1 that $F = \partial\check{\Omega}$ has codimension 2. Then theorem 6.6 gives an open subset $\check{\Omega}'$ containing $\check{\Omega}$ and a conformal diffeomorphism $\lambda^{-1} : \widetilde{M} \to \check{\Omega}'$, which coincides with δ on Ω. As above, $\lambda^{-1} = \delta$. Now, $\tilde{\Lambda} \subset \check{\Omega}'$, and since $\delta : \widetilde{M} \to \check{\Omega}'$ is a diffeomorphism, the action of $\rho(\Gamma)$ on $\check{\Omega}'$ is free and proper. In particular, the map associating to an element of $\rho(\Gamma)$ its restriction to $\tilde{\Lambda}$ is injective. By proposition 6.1, the group $\rho(\Gamma)$ is finite. Because $M = \widetilde{M}/\Gamma$ is compact, $\rho(\Gamma)$ acts cocompactly on $\check{\Omega}'$, so $\check{\Omega}' = \widetilde{\mathrm{Ein}}^{p,q}$. Therefore M is conformally diffeomorphic to a quotient of $\widetilde{\mathrm{Ein}}^{p,q}$ by a finite subgroup of $\widetilde{\mathrm{O}}(p+1, q+1)$, proving theorem 1.2 in the case $p \geq 2$.

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