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Numerical methods for hyperbolic systems

Exercise sheet 1: Advection equation and finite volumes schemes

Exercice 1 We consider the advection equation

$$\begin{cases} \dfrac{\partial u}{\partial t} u + a\dfrac{\partial u}{\partial x} = 0, \quad \forall x \in \mathbb{R}, \quad t > 0, \\ u(t=0,x) = u^0(x), \quad \forall x \in \mathbb{R}, \end{cases} \tag{1}$$

with $u^0(x) \in C^1(\mathbb{R})$.

1. Find the solution using the method of characteristics.

Now we consider the advection equation defined on $\mathbb{R}^+$

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0, \quad \forall x \in \mathbb{R}^+, \quad t > 0, \\ u(t=0,x) = u^0(x), \quad \forall x \in \mathbb{R}^+, \end{cases} \tag{2}$$

with $u^0(x) \in C^1(\mathbb{R})$.

2. Assume that $a < 0$, prove that the equation (2) admits a unique solution.

3. Assume that $a > 0$, explain why the equation have no solution if we do not add a boundary condition $u(t,0) = g(t) \in C^1(\mathbb{R}^+)$. Give the condition on g such as

$$\begin{cases} u^0(x - at), \quad x > at, \\ g(t - \frac{x}{a}), \quad x < at, \end{cases} \tag{3}$$

is solution in $C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ of (2) with $u(t,0) = g(t)$.

3. Assume that u is a function with compact support in $\mathbb{R}$. Prove the following energy estimate

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^+} |u(t,x)|^2 dx\right) = \frac{a}{2}|u(t,x=0)|^2. \tag{4}$$

4. Distinguishing $a < 0$ and $a > 0$, prove the uniqueness of the solution to (2).

Exercice 2 We propose to solve the advection equation on the domain $[0,L]$

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0, \quad \forall x \in [0,L], \quad t > 0, \\ u(t=0,x) = u^0(x), \quad \forall x \in [0,L], \\ u(t,x=0) = u(t,x=L), \end{cases} \tag{5}$$

with $u^0(x) \in C^1([0, L])$.

We consider the upwind scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a + |a|}{2\Delta x}(u_j^n - u_{j-1}^n) + \frac{a - |a|}{2\Delta x}(u_{j+1}^n - u_j^n) = 0, \tag{6}$$

and the centered scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) = 0, \tag{7}$$

with Δt the time step, Δx the step mesh and u_j^n the approximation to $u(n\Delta t, j\Delta x)$ where $n \in \mathbb{N}$, $j \in \mathbb{N}$.

1. The advection equation satisfies the maximum principle

$$\min_{x\in[0,L]} u(t = 0, x) \leq u(t, x) \leq \max_{x\in[0,L]} u(t = 0, x).$$

Prove that the upwind scheme satisfies the discrete maximum principle under a CFL condition

$$\min_{j\in[0,N_x]} u_j^n \leq u_j^{n+1} \leq \max_{j\in[0,N_x]} u_j^n,$$

with $N_x = \frac{L}{\Delta x}$ the number of cells.

2. Prove that the upwind scheme is stable for the L^2 norm using the Neumann analysis.

3. Give the consistency error associated to the upwind scheme.

4. Discuss the discrete maximum principle for the centered scheme.

5. Study the L^2 stability and the consistency error associated to the centered scheme.