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Numerical methods for hyperbolic systems

Correction 1 of exercise sheet: advection equation and finite volumes schemes

Exercice 1 We consider the advection equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0, \quad \forall x \in \mathbb{R}, \quad t > 0, \\ u(t=0,x) = u^0(x), \quad \forall x \in \mathbb{R}, \end{cases} \tag{1}$$

with $u^0(x) \in C^1(\mathbb{R})$.

1. Find the solution using the method of characteristics.

The characteristic method is a method to find the solutions for some scalar hyperbolic equations (advection equation and some nonlinear scalar equations).
The characteristic curve $C = \{(t, X(t)) \in \mathbb{R}^+ \times \mathbb{R}\}$ is a curve where the PDE can be reduced to an ODE. For advection problem the curve is defined by

$$\begin{cases} X^{'}(t) = a, \quad t > 0, \\ X(0) = X_0. \end{cases} \tag{2}$$

It is trivial to prove that $X(t) = X_0 + at$. Now we note that

$$\frac{du(t, X(t))}{dt} = \partial_t u + X^{'}(t)\partial_x u = 0.$$

Consequently the solutions of the PDE (1) are solutions of the ODE $\frac{du(t,X(t))}{dt} = 0$ given by $u(t, X(t)) = u(0, X(0)) = u_0(X_0)$. We note that $X_0 = x - at$ is the unique X_0 such as the solution $u(t, x)$ is defined in $\mathbb{R}$ by

$$u(t, X_0 + at) = u_0(X_0) \Longleftrightarrow u(t, x) = u_0(x - at).$$

The unique solution is given by the foot (beginning) X_0 of the characteristic curve.

Now we consider the advection equation defined on $\mathbb{R}^+$

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0, \quad \forall x \in \mathbb{R}^+, \quad t > 0, \\ u(t=0,x) = u^0(x), \quad \forall x \in \mathbb{R}^+, \end{cases} \tag{3}$$

with $u^0(x) \in C^1(\mathbb{R}^+)$.

2. Assume that $a < 0$, prove that the equation (3) admits a unique solution.

We use the method of characteristics to find the solution. The characteristic curves with a foot $X_0 \in \mathbb{R}^+$ generate (x,t) in the space $\mathbb{R}^+ \times \mathbb{R}^+$ thus we obtain the solution

$$u(t,x) = u_0(x - at), \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$$

3. Assume that $a > 0$, explain why the equation have no solution if we do not add a boundary condition $u(t,0) = g(t)$. Give the condition on g such as

$$\begin{cases} u^0(x - at), & x > at, \\ g(t - \frac{x}{a}), & x < at, \end{cases} \tag{4}$$

is solution in $C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ of (3) with $u(t,0) = g(t)$.

We use the method of characteristics to find the solution. The characteristic curves with a foot $X_0 \in \mathbb{R}^+$ do not generate the space $\mathbb{R}^+ \times \mathbb{R}^+$.
Indeed $x - at \notin \mathbb{R}^+$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$. Since $u^0(x) \in C^1(\mathbb{R}^+)$ the solution $u^0(x - at)$ is not correctly defined. Consequently we add a boundary condition

$$\begin{cases} \frac{\partial u}{\partial t} u + a \frac{\partial u}{\partial x} = 0, & \forall x \in \mathbb{R}^+, \quad t > 0, \\ u(t = 0, x) = u^0(x), & \forall x \in \mathbb{R}^+, \\ u(t, x = 0) = g(t), & \forall t \in \mathbb{R}^+. \end{cases} \tag{5}$$

After we plug the function (4) in the equation. This function (4) is a solution. She is included in $C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ if some conditions are verified. To finish we give these conditions
The solution (4) is $C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ if $g(0) = u^0$ and $g^{'}(0) + au^{'}(0) = 0$.

We obtain these conditions using $g(t - \frac{x}{a}) = u^0(x - at)$, $\partial_x g(t - \frac{x}{a}) = \partial_x u^0(x - at) = 0$ for $x = at$.

4. Assume that $u(x)$ is a function with compact support in $\mathbb{R}$. Prove the following energy estimate

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^+} |u(t,x)|^2 dx\right) = \frac{a}{2}|u(t, x = 0)|^2. \tag{6}$$

We multiply by $u(t,x)$ and integrate the equation (3) to obtain

$$\int_{\mathbb{R}^+} u(t,x)\partial_t u(t,x) dx + a \int_{\mathbb{R}^+} u(t,x)\partial_x u(t,x) dx = 0,$$

equivalent to

$$\partial_t \int_{\mathbb{R}^+} |u(t,x)|^2 dx = -\frac{a}{2}\partial_x \int_{\mathbb{R}^+} |u(t,x)|^2 dx = -\left[|u(t,x)|^2\right]_0^{+\infty}.$$

For obtain this result we use the derivate formula $2f(x)f^{'}(x) = (f(x)^2)^{'}$. Since the solution is defined in a compact space, the solution is equal to zero when x is close to infinity. We obtain the result.

5. Distinguishing $a > 0$ and $a < 0$, prove the uniqueness of the solution to (3).

We define two solutions u_1, u_2 of (3) and study the equation of the difference between u_1 and u_2. We apply the previous computations to obtain

$$\partial_t E(t) = \partial_t \int_{\mathbb{R}+} |u_1(t,x) - u_2(t,x)|^2 dx = \frac{a}{2}|u_1(t,0) - u_2(t,0)|^2.$$

If $a < 0$, $E(t) = \int_{\mathbb{R}+} |u_1(t,x) - u_2(t,x)|^2$ decreases ($\partial_t E(t) \leq 0$). Since the initial data are the same for the two solutions we obtain the uniqueness. Indeed the energy is equal to zero for $t = 0$. For $t > 0$ the energy decreases and is non negative consequently $E(t) = 0$.

If $a > 0$, the energy increases. But if we add a boundary condition (the same boundary condition for each solution) $\frac{a}{2}|u_1(t,0) - u_2(t,0)|^2 = 0$, then $E(t) = C$. Since $E(t = 0) = 0$ we obtain the uniqueness.