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Numerical methods for hyperbolic systems

Part 2 of correction exercise sheet 2: Galerkin discontinuous for advection equation

Exercice 1 In this exercise we propose to study the DG approximation for the advection equation with $a > 0$. Ω is the domain. The mesh Ω_h is defined by $n+1$ points x_i and n cells $K_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$. We call a generic cell K. Finally the test function are defined by $v \in V_h = \{v/v|_K \in \mathbb{P}^p(K)\}$ with $\mathbb{P}^p(K)$ a space of p-order polynomials on K.

1. Write the weak formulation of the advection equation for an element K.

We multiply the advection equation $\partial_t u + a\partial_x u = 0$ by a test function $v \in V$ a function space and we integrate on a volume K. We obtain

$$\int_K v\partial_t u dx + a\int_K v\partial_x u dx = 0.$$

Integrating by parts we obtain

$$\int_K v\partial_t u dx - a\int_K u\partial_x v dx + a\,[uv]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0,$$

which is equivalent to

$$\int_K v\partial_t u dx - a\int_K u\partial_x v dx + a\left(u(x_{i+\frac{1}{2}})v(x_{i+\frac{1}{2}}) - u(x_{i-\frac{1}{2}})v(x_{i-\frac{1}{2}})\right) = 0.$$

We define $\left\{\phi_l^i\right\}_{l=0}^p$ a basis of V_h. The numerical solution in the element K_i is noted $u_h^i = u_{h_{|K_i}} = \sum_{l=0}^p u_l^i\phi_l^i$.

The DG-centered scheme is given by

$$\partial_t \int_{K_i} u_h^i\phi_m^i - a\int_{K_i} u_h^i\partial_x\phi_m^i + a\left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \quad 0 \le m \le k, \tag{1}$$

with

$$\left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \frac{1}{2}(u_h^i(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}}) + u_h^{i+1}(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}}))$$

$$-\frac{1}{2}(u_h^{i-1}(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}}) + u_h^i(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}})),$$

and in matrix form by

$$\sum_{l=0}^{p} \partial_t u_l^i \int_{K_i} \phi_l^i \phi_m^i - a \sum_{l=0}^{p} u_l^i \int_{K_i} \phi_l^i \partial_x \phi_m^i + a \sum_{l=0}^{p} \left[u\phi_m^i \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \quad 0 \le m \le k, \tag{2}$$

with

$$\begin{aligned}\left[u\phi_m^i \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} &= \frac{1}{2}(u_l^i \phi_l^i(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}}) + u_l^{i+1}\phi_l^{i+1}(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}})) \\ &-\frac{1}{2}(u_l^{i-1}\phi_l^{i-1}(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}}) + u_l^i \phi_l^i(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}})).\end{aligned}$$

2. Assuming that boundary conditions are periodic, prove that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 dx = \frac{1}{2}\sum_i \frac{d}{dt}\int_\Omega |u_h^i|^2 dx = 0.$$

We start with

$$\partial_t \int_{K_i} u_h^i \phi_m^i - a \int_{K_i} u_h^i \partial_x \phi_m^i + a \left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \quad 0 \le m \le k, \tag{3}$$

where

$$\left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \frac{1}{2}(u_h^i(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}}) + u_h^{i+1}(x_{i+\frac{1}{2}})\phi_m^i(x_{i+\frac{1}{2}}))$$
$$-\frac{1}{2}(u_h^{i-1}(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}}) + u_h^i(x_{i-\frac{1}{2}})\phi_m^i(x_{i-\frac{1}{2}})).$$

Since $u_h^i \in V_h$ we can choose $\phi_m^i = u_h^i$. In this case we obtain

$$\frac{1}{2}\partial_t \int_{K_i} u_h^{i,2} - a\frac{1}{2} \int_{K_i} \partial_x u_h^{i,2} + a \left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \tag{4}$$

with

$$\left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \frac{1}{2}(u_h^{i,2}(x_{i+\frac{1}{2}}) + u_h^{i+1}(x_{i+\frac{1}{2}})u_h^i(x_{i+\frac{1}{2}}))$$
$$-\frac{1}{2}(u_h^{i-1}(x_{i-\frac{1}{2}})u_h^i(x_{i-\frac{1}{2}}) + u_h^{i,2}(x_{i-\frac{1}{2}})).$$

Now we remark that

$$0 = \int_{K_i} \partial_x a u_h^{i,2} = -\int_{K_i} a\partial_x u_h^{i,2} + a[u_h^{i,2}(x_{i+\frac{1}{2}}) - u_h^{i,2}(x_{i-\frac{1}{2}})].$$

Consequently

$$\frac{1}{2}\partial_t \int_{K_i} u_h^{i,2} = a[u_h^{i,2}(x_{i+\frac{1}{2}}) - u_h^{i,2}(x_{i-\frac{1}{2}})] - \left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}.$$

After simplification we obtain

$$\frac{1}{2}\partial_t \int_{K_i} u_h^{i,2} = \frac{a}{2}(u_h^{i+1}(x_{i+\frac{1}{2}})u_h^i(x_{i+\frac{1}{2}}) - u_h^{i-1}(x_{i-\frac{1}{2}})u_h^i(x_{i-\frac{1}{2}})). \tag{5}$$

To finish we sum on all cells the quantities (5). Since the boundary conditions are periodic we obtain

$$\frac{1}{2}\partial_t \sum_i \int_{K_i} u_h^{i,2} = 0. \tag{6}$$

3. How obtain the classical finite volumes centered scheme starting the DG scheme (7).

The finite volumes method is based on an constant approximation by cell thus we propose to take $V_h = \mathbb{P}^0(K)$ with $\phi_0^i = 1$. Using this choice of basis and $\partial_x \phi_0^i = 0$ we obtain

$$\partial_t u_0^i \int_{K_i} dx + a \left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \tag{7}$$

with

$$\left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \frac{1}{2}(u_0^i + u_0^{i+1}) - \frac{1}{2}(u_0^{i-1} + u_0^i).$$

Now we remark that $\int_{K_i} = \Delta x_j = [x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}]$, consequently we obtain

$$\partial_t u_0^i + a \frac{u_0^{i+1} - u_0^{i-1}}{\Delta x_j} = 0.$$

4. Design a flux $\left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$ which is the DG extension of the upwind finite volumes scheme.

A flux can be write on the form

$$\left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = u_l^{i+\frac{1}{2}} \phi_l^{i+\frac{1}{2}}(x_{i+\frac{1}{2}})\phi_l^i(x_{i+\frac{1}{2}}) - u_l^{i-\frac{1}{2}} \phi_l^{i-\frac{1}{2}}(x_{i-\frac{1}{2}})\phi_l^i(x_{i-\frac{1}{2}}).$$

For the finite volume centered approximation the flux $F_{j+\frac{1}{2}} = \frac{1}{2}(u_j + u_{j+1})$, thus the DG extension is

$$F_{j+\frac{1}{2}} = u_l^{i+\frac{1}{2}} \phi_l^{i+\frac{1}{2}} = \frac{1}{2}(u_h^i(x_{i+\frac{1}{2}}) + u_h^{i+1}(x_{i+\frac{1}{2}})) = \sum_l^p \frac{1}{2}(u_l^i \phi_l^i(x_{i+\frac{1}{2}}) + u_l^{i+1} \phi_l^{i+1}(x_{i+\frac{1}{2}})).$$

For the finite volume upwind approximation for $a > 0$ the flux is $F_{j+\frac{1}{2}} = u_j$, thus the DG extension is

$$F_{j+\frac{1}{2}} = u_l^{i+\frac{1}{2}} \phi_l^{i+\frac{1}{2}} = u_h^i(x_{i+\frac{1}{2}}) = \sum_l^p u_l^i \phi_l^i(x_{i+\frac{1}{2}}).$$

The final flux is given by

$$\left[u\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = u_l^i \phi_l^i(x_{i+\frac{1}{2}})\phi_l^i(x_{i+\frac{1}{2}}) - u_l^{i-1} \phi_l^{i-1}(x_{i-\frac{1}{2}})\phi_l^i(x_{i-\frac{1}{2}}).$$