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Numerical methods for hyperbolic systems

Numerical Exercise sheet 2: Discontinuous Galerkin explicit solver for Euler equations

Exercice 1
aim: Write a Godunov-type Galerkin Discontinous scheme for the Euler equations in a Matlab program. The high order integrator (TVD Runge Kutta Method) is always present in the initial code.

We consider the Euler equation

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x \rho u^2 + \partial_x p = 0, \\ \partial_t E + \partial_x (u(E+p)). \end{cases} \tag{1}$$

with $p = (\gamma - 1)(E - \rho \frac{1}{2} u^2)$ and $c = \sqrt{\frac{\gamma p}{\rho}}$.

We note $\mathbf{U} = (\rho, \rho u, E)$ and $F(\mathbf{U}) = (\rho u, \rho u^2 + p, u(p + E))$.

1. Write a function which compute the Galerkin Discontinuous scheme for Euler equation with the Rusanov fluxes. The scheme with first order time scheme is given by

$$\sum_{l=0}^{k} \int_{K_i} \phi_l^i \phi_m^i \left(\frac{\mathbf{U}_{l,i}^{n+1} - \mathbf{U}_{l,i}^n}{\Delta t} \right) - \sum_{l=0}^{k} F(\mathbf{U}_{l,i}^n) \int_{K_i} \phi_l^i \partial_x \phi_m^i + \sum_{l=0}^{k} \left[F(\mathbf{U}) \phi_m^i \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \quad \forall 0 \leq m \leq k,$$

with the fluxes

$$\left[F(\mathbf{U}) \phi_m^i \right]_{x_{i+\frac{1}{2}}} = \frac{1}{2} \left(F(\mathbf{U}_{l,i+1}^n) \phi_l^{i+1}(x_{i+\frac{1}{2}}) \phi_m^i(x_{i+\frac{1}{2}}) + F(\mathbf{U}_{l,i}^n) \phi_l^i(x_{i+\frac{1}{2}}) \phi_m^i(x_{i+\frac{1}{2}}) \right)$$

$$+ S_{j+\frac{1}{2}} \frac{1}{2} \left(\mathbf{U}_{l,i}^n \phi_l^i(x_{i+\frac{1}{2}}) \phi_m^i(x_{i+\frac{1}{2}}) - \mathbf{U}_{l,i+1}^n \phi_l^{i+1}(x_{i+\frac{1}{2}}) \phi_m^i(x_{i+\frac{1}{2}}) \right).$$

Contrary to the linear case we must construct the vector $\left\{ F(\mathbf{U}_{l,i}^n) \right\}$. The Rusanov speed is defineb by $S_{i+\frac{1}{2}} = \max\left(\max_l(|u_{l,i}| + c_{l,i}), \max_l(|u_{l,i+1}| + c_{l,i+1})\right)$.

The CFL condition is given by $\Delta t \leq \left(\frac{\Delta x}{\max_i S_{i+\frac{1}{2}}}\right)$.

2. Give the convergence order for the following test case $\rho(t,x) = 1 + 0.99 \sin(\pi(x - t))$, $u(t,x) = 1$ and $p(t,x) = 1$.

3. Write the test case of Sod shock tube on $[-1, 1]$ defined by

$$U^0(x) = \begin{cases} (\rho(t=0) = 1, \quad u(t=0) = 0, \quad p(t=0) = 1), \quad x < 0, \\ (\rho(t=0) = 0.125, \quad u(t=0) = 0, \quad p(t=0) = 0.1), \quad x > 0, \quad \text{else} \end{cases}$$

The exact solution is given in the annex.

4. Explain the problem of stability of the scheme for the Sod test case and P > 0.

5. Add a slope limiter (annex) to stabilize scheme. Test the scheme with slope limiter for the Sod shock test case.