INTERNSHIP REPORT

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Contents

0.0.0.1 Plan (first pass) :

- Linear hyperbolic systems
- Linearization of nonlinear hypeperbolic systems
- The linear Riemann problem
- Nonlinear Riemann problem (shock and the Hugoniot locus)
- Example : Isothermal Euler system

- Jin-Xin relaxation method
- LBM (generalities) [?]
- Vectorial kinetic relaxation in 1D
- Vectorial kinetic relaxation in 2D
- Example : D2Q4 for the barotropic Euler system

- Numerical scheme (transport & relaxation steps,...)
- Numerical stability (CFL, order,...)
- Test : Riemann problem (1D / 2D ?)

- Implementation with C (UML diagrams for classes ?)
- Test : Barotropic Euler system in 2D

- Conclusion (pro - cons)
- Opening : Euler –> MHD –> Plasmas

[Nondimensionalisation, Mach number] ??

1 Mathematical background

The general context of this work is the study of systems of form

$$\begin{aligned} &\partial_t u + \sum_{k=1}^{d} \partial_k F^k(u) = 0 \\ &u(X,0) = u_0(X), \end{aligned}$$

for

$$u(X,t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^m \text{ and } F^k : \mathbb{R}^m \to \mathbb{R}^m,$$

but we will stick to the case $d = 1$ to introduce the theory.

1.1 Linear hyperbolic systems

Let $A \in \mathbb{R}^{m\times m}$ be a constant matrix. The system

$$\begin{aligned} &\partial_t u + A\partial_x u = 0 \\ &u(X,0) = u_0(X), \end{aligned} \tag{1}$$

is said to be *hyperbolic* (resp. strictly hyperbolic) if A is diagonalizable with (resp. distincts) real-valued eigenvalues. From now, we'll assume the strict hyperbolicity of all systems.

Writing $A = R\Lambda R^{-1}$, $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_m)$, and $R = (r_1|\cdots|r_m)$ gives us

$$Ar_p = \lambda_p r_p,\ p = 1, \cdots, m. \tag{2}$$

We start multiplying (1) by R^{-1} to obtain an equation on $v := R^{-1}u$

$$R^{-1}\partial_t u + R^{-1}(R\Lambda R^{-1})\partial_x u = 0 \Leftrightarrow \partial_t v + \Lambda\partial_x v = 0, \tag{3}$$

which is a set of m independents scalar transport equations

$$\begin{aligned} &\partial_t v^p + \lambda_p \partial_x v^p = 0 \\ &p = 1, \cdots, m. \end{aligned} \tag{4}$$

We find the solution using the method of characteristics

$$v^p(x,t) = v^p(x - \lambda_p t, 0) \tag{5}$$

and thus, as $u = Rv = \sum_p v^p r_p$, we get

$$\begin{aligned} u(x,t) &= \sum_{p=1}^{m} v^p(x - \lambda_p t, 0) r_p \\ u_0(x) &= R v_0(x). \end{aligned} \tag{6}$$

We'll call any curve satisfying

$$\begin{aligned} x(t) &= x_0 + \lambda_p t \\ x'(t) &= \lambda_p \end{aligned} \tag{7}$$

a *p-characteristic*, along which $v^p(x,t)$ will remain constant.

A very important property holding here is that any singularity in the initial data can propagate along characteristics only, and that smooth initial data leads to smooth solutions. *This won't necessarily be the case for nonlinear equations.*

1.2 Linearization of Nonlinear system

We now consider a nonlinear system

$$\partial_t u + \partial_x F(u) = 0, \tag{8}$$

and we ask ourselves to what extent could we recover the results of the linear case. Witing $A(u) = F'(u)$, (8) can be written as

$$\partial_t u + A(u)\partial_x u = 0. \tag{9}$$

Unlike the linear case, the hyperobolicity condition depends on the solution u. The pth characteristic now writes

$$\begin{aligned} x'(t) &= \lambda_p(u(x(t),t)) \\ x(0) &= x_0, \end{aligned} \tag{10}$$

and the method we used in the linear case doesn't work anymore. But the expansion $u(x,t) = \bar{u} + \varepsilon u^{(1)}(x,t) + O(\varepsilon^2)$ (with $\varepsilon > 0$ and $\bar{u}$ a constant state) gives us, as $\varepsilon \to 0$, a linear behavior of the first order term of the solution

$$\partial_t u^{(1)}(x,t) + A(\bar{u})\partial_x u^{(1)}(x,t) = 0. \tag{11}$$

We are now able to integrate $x'(t)$ and we obtain a result similar to previously

$$x_p(t) = x_0 + \lambda_p(\bar{u})t \tag{12}$$

We lost the property of the shock to propagate only along characteristics, but it remains (approximately) true for small disturbance of the solution. We could use higher order correction with the same process and so on, but we'll instead use the following :

Theorem 1 (Rankine-Hugoniot jump condition)**.** *The speed s of a discontinuity and the states u_R and u_L are related by the Rankine-Hugoniot jump condition :*

$$F(u_R) - F(u_L) = s(u_R - u_L), \tag{13}$$

which can be written

$$[F] = s[u], \tag{14}$$

Where [.] indicates the jump across the discontinuity.

In the case where $||u_R - u_L|| \equiv \varepsilon << 1$ we are close to the linear theory. The expansion

$$F(u_L) = F(u_R) + F'(u_R)(u_L - u_R) + O(\varepsilon^2), \tag{15}$$

with the RH conditions gives

$$F'(u_R)(u_R - u_L) = s(u_R - u_L) + O(\varepsilon^2), \tag{16}$$

such as

$$\begin{aligned} &A(u_R)z = sz \\ &z := \lim_{\varepsilon \to 0} (u_R - u_L)/\varepsilon, \end{aligned} \tag{17}$$

and the speed of propagation remains an eigenvalue of the jacobian.

1.3 Linear Riemann problem

A Riemann problem is simply a hyperbolic system as (1) with a piece-wise constant initial data

$$\begin{aligned} &\partial_t u + A\partial_x u = 0 \\ &u(x,0) = \begin{cases} u_L\,, x<0 \\ u_R\,, x>0. \end{cases} \end{aligned} \tag{18}$$

We will assume a strict hyperbolicity such that $Sp(A) = \{\lambda_1 < \cdots < \lambda_m\}$.

We decompose u_L and u_R in the diagonalization basis

$$u_L = \sum_{p=1}^{m} \alpha_p r_p\,,\ u_R = \sum_{p=1}^{m} \beta_p r_p, \tag{19}$$

then

$$v^p(x,0) = \begin{cases} \alpha_p\,, x<0 \\ \beta_p\,, x>0, \end{cases} \tag{20}$$

and so

$$v^p(x,t) = \begin{cases} \alpha_p\,, x-\lambda_p t<0 \\ \beta_p\,, x-\lambda_p t>0. \end{cases} \tag{21}$$

Writing $P(x,t)$ the maximal value of p for which $x-\lambda_p t>0$, the solution can be broken down into the following form

$$u(x,t) = \sum_{p=1}^{m} v^p(x,t) = \sum_{p=1}^{P(x,t)} \beta_p r_p + \sum_{p=P(x,t)+1}^{m} \alpha_p r_p. \tag{22}$$

Picture ??

When crossing the pth characteristic the solution jumps with the jump given by

$$[u] = (\beta_p - \alpha_p) r_p. \tag{23}$$

Using the fact that $F(u) = Au$, the Rankine-Hugoniot condition is here written

$$\begin{aligned} [F] &= A[u] \\ &= (\beta_p - \alpha_p) A r_p \\ &= \lambda_p [u]. \end{aligned} \tag{24}$$

NB : We notice that in this case, from the RH condition results the fact that the pth jumps propagates at speed λ_p, which allows us to write the solution in terms of these jumps as

$$u(x,t) = u_L + \sum_{\lambda_p < x/t} (\beta_p - \alpha_p) r_p \tag{25}$$

$$= u_R - \sum_{\lambda_p \geq w/t} (\beta_p - \alpha_p) r_p, \tag{26}$$

from which we have

$$u_R - u_L = \sum_{p=1}^{m} (\beta_p - \alpha_p) r_p. \tag{27}$$

Finding a way to split a jump into a sum of m jumps propagating at constant speed λ_p is what one could call "solving the Riemann problem". The next part is about the generalization to the nonlinear case.

1.4 Nonlinear Riemann problem

Let's get back to the nonlinear system

$$\partial_t u + \partial_x F(u) = 0, \;\; u(x,t) \in \mathbb{R}^m. \tag{28}$$

We place ourselves in a normalized diagonalization basis : $\{r_p(u)\}_{p=1}^m \,/\, ||r_p(u)|| = 1$. Let a discontinuity propagating at the speed s, between the values u_L and u_R. Given the point u_L, we look for the set of all points u_R which can be connected to u_L by a discontinuity satisfying the RH condition (13).

As $u_R \in \mathbb{R}^m$ and $s \in \mathbb{R}$ are $m+1$ unknowns, and RH condition gives m condition, we'll find one parameter families of solutions. The linear case told us that the pth family 's jump was co-linear to r_p, i.e. $[u]_p = \xi r_p$, $\xi \in \mathbb{R}$. Parameterization of these families using this scalar gives us the following solution curves

$$\begin{matrix} u_R(\xi, u_L) = u_L + \xi r_p \\ s_p(\xi, u_L) = \lambda_p \end{matrix} \;, p = 1, \cdots, m. \tag{29}$$

The RH condition now gives

$$F(u_{R,p}(\xi)) - F(u_L) = s_p(\xi)(u_{R,p}(\xi) - u_L), \tag{30}$$

which becomes, after beeing derived with respect to ξ at the origin :

$$f'(u_L)u'_{R,p}(0) = s_p(0)u'_{R,p}(0), \tag{31}$$

meaning that the curve $u_{R,p}(\xi)$ is tangent to $r_p(u_L)$ at the point u_L. If $u_{R,p}$ lies through u_L on the pth curve, called the pth Hugoniot curve, then we say that u_L and $u_{R,p}$ are connected by a p-shock.

2 Vectorial kinetic relaxation method

2.1 Jin-Xin relaxation method

We want to solve the following non-linear transport equation

$$(\mathscr{P}):\ \partial_t\rho + \partial_x F(\rho) = 0,$$

with $\rho = \rho(t,x) \in \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ non-linear in the general case.
We relax the equation with a system of two coupled linear transport equations, which is much easier to solve ($\varepsilon > 0$ is the relaxation parameter and $v \in \mathbb{R}$ the speed) [JIN-XIN]

$$\mathscr{R}_\varepsilon : \begin{cases} \partial_t\rho + \partial_x v = 0 \\ \partial_t v + \alpha^2\partial_x\rho = \frac{1}{\varepsilon}(F(\rho) - v) \end{cases} .$$

Near equilibrium, i.e. $\varepsilon << 1$, we write

$$v^\varepsilon = F(\rho^\varepsilon) + \varepsilon v_1^\varepsilon + O(\varepsilon^2),$$

which implies

$$F(\rho^\varepsilon) = v^\varepsilon - \varepsilon v_1^\varepsilon + O(\varepsilon^2).$$

The system becomes

$$\mathscr{R}_\varepsilon \underset{\varepsilon\to 0}{\to} : \begin{cases} \partial_t\rho^\varepsilon + \partial_x F(\rho^\varepsilon) = O(\varepsilon) \\ \partial_t F(\rho^\varepsilon) + \alpha^2\partial_x\rho^\varepsilon = \frac{1}{\varepsilon}(F(\rho^\varepsilon) - v) \end{cases} .$$

We multiply the first line by $F'(\rho^\varepsilon)$ so it becomes

$$\begin{cases} \partial_t F(\rho^\varepsilon) + |F'(\rho^\varepsilon)|^2 \partial_x \rho^e = O(\varepsilon) \\ \partial_t F(\rho^\varepsilon) + \alpha^2 \partial_x \rho^\varepsilon = -v_1^\varepsilon + O(\varepsilon) \end{cases},$$

and finally, $(L_1)-(L_2)$ gives us

$$v_1^\varepsilon = \left(|F'(\rho^\varepsilon)|^2 - \alpha^2\right)\partial_x \rho^\varepsilon + O(\varepsilon),$$

which leads us to

$$\partial_t \rho^\varepsilon + \partial_x v^\varepsilon = \partial_t \rho^\varepsilon + \partial_x \left[F(\rho^\varepsilon) + \varepsilon v_1^\varepsilon\right] + O(\varepsilon) = 0 \Rightarrow \partial_t \rho^\varepsilon + \partial_x F(\rho^e) = -\varepsilon \partial_x v_1^\varepsilon + O(\varepsilon^2).$$

This gives us the following result

Proposition : Under the hypothesis above, $\mathscr{R}_\varepsilon$ is consistent with the equation

$$\mathscr{P}_\varepsilon : \ \partial_t \rho^\varepsilon + \partial_x F(\rho^e) = \varepsilon \partial_x \left(\left[\alpha^2 - |F'(\rho^\varepsilon)|^2\right]\partial_x \rho^\varepsilon\right) + O(\varepsilon^2).$$

which converge to the initial transport equation under the stability condition

$$\alpha^2 - |F'(\rho^\varepsilon)|^2 > 0 \Leftrightarrow |\alpha| > |F'(\rho^\varepsilon)|.$$

Write Jin-Xin as D1Q2

2.2 Vectorial kinetic relaxation in 1D

We slightly generalize the former method for a set of $M \in \mathbb{N}$ velocities and $N \in \mathbb{N}$ macroscopic variables (such as density, speed, pressure, ...).

We will use $f, f^{eq} \in \mathbb{R}^M$, $w = w(t,x) \in \mathbb{R}^N$, $P \in \mathscr{M}_{N,M}(\mathbb{R})$,
and $\Lambda = \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_M \end{pmatrix} \in \mathscr{M}_M(\mathbb{R}), \quad \lambda_i \in \mathbb{R}$

We introduce the Lattice TOO DOO : definition of a lattice + picture

$$\partial_t f + \Lambda \partial_x f = \frac{1}{\varepsilon}\left(f^{eq}(w) - f\right) \ (*)$$

with two constraints on the moments of f and f^{eq}

$$\begin{cases} Pf = Pf^{eq}(w) = w \\ P\Lambda f^{eq} = F(w) \end{cases}.$$

the constraints gives us the following relations

$$\begin{array}{r} P\times(*): \\ P\Lambda\times(*): \end{array} \quad \begin{cases} \partial_t Pf + \partial_x P\Lambda f = \frac{1}{\varepsilon}\left(Pf^{eq}(w) - Pf\right) = w - w = 0 \\ \partial_t P\Lambda f + \partial_x P\Lambda^2 f = \frac{1}{\varepsilon}\left(P\Lambda f^{eq}(w) - P\Lambda f\right) \end{cases}.$$

Writing $v := P\Lambda f$, the system reads

$$\begin{cases} \partial_t w + \partial_x v = 0 \\ \partial_t v + \partial_x (P\Lambda^2 f) = \frac{1}{\varepsilon}(F(w) - v) \end{cases}.$$

We recognize the relaxation system from the Jin-Xin method.

Ref to Jin-Xin (edit : done)

We continue using the Chapman-Enskog expansion [CHA-ENS], which relies on writing the first order Taylor expansion for each variables

$$w = w_0 + \varepsilon w_1 + O(\varepsilon^2),\ v = v_0 + \varepsilon v_1 + O(\varepsilon^2),\ f = f_0 + \varepsilon f_1 + O(\varepsilon^2),$$

plugging them into both relaxation and Lattice equations

$$\begin{cases} \partial_t w_0 + \varepsilon\partial_t w_1 + \partial_x v_0 + \varepsilon\partial_x v_1 = O(\varepsilon^2) \\ \partial_t v_0 + \varepsilon\partial_t v_1 + \partial_x(P\Lambda^2 f_0) + \varepsilon\partial_x(P\Lambda^2 f_1) = \frac{1}{\varepsilon}\left(F(w_0) + \varepsilon D_{w_0}F w_1 - v_0 - \varepsilon v_1\right) + O(\varepsilon^2) \\ \partial_t f_0 + \varepsilon\partial_t f_1 + \Lambda\partial_x f_0 + \Lambda\partial_x f_1 = \frac{1}{\varepsilon}\left(f^{eq}(w_0) + \varepsilon D_{w_0} f^{eq} w_1 - f_0 - \varepsilon f_1\right) + O(\varepsilon^2) \end{cases},$$

and writing the system at each order of ε.

Keeping only the $O(1/\varepsilon)$ terms leads to

$$\begin{cases} v_0 = F(w_0) \\ f_0 = f^{eq}(w_0) \end{cases},$$

$O(1)$ terms give us

$$\begin{cases} \partial_t w_0 + \partial_x v_0 = \partial_t w_0 + \partial_t F(w_0) = 0 \ (**) \\ \partial_t v_0 + \partial_x (P\Lambda^2 f_0) = \partial_t F(w_0) + \partial_x (P\Lambda^2 f_0) = D_{w_0} F w_1 - v_1 \end{cases},$$

which implies that

$$v_1 = D_{w_0} F w_1 - \partial_t F(w_0) - \partial_x (P\Lambda^2 f^{eq}(w_0)),$$

and, considering the $O(\varepsilon)$ terms, we get

$$\partial_t w_1 + \partial_x (D_{w_0} F w_1) = \partial^2_{xt} F(w_0) + \partial^2 xx (P\Lambda^2 f^{eq}(w_0)) \ (***),$$

We recombine the system using $(**) + \varepsilon(***)$:

$$\partial_t (w_0 + \varepsilon w_1) + \partial_x (F(w_0) + \varepsilon D_{w_0} F w_1) = \varepsilon \left[\partial^2_{xt} F(w_0) + \partial^2_{xx} (P\Lambda^2 f^{eq}(w_0)) \right]$$

i.e.

$$\partial_t w + \partial_x F(w) = \varepsilon \left[\partial^2_{xt} F(w_0) + \partial^2_{xx} (P\Lambda^2 f^{eq}(w_0)) \right] + O(\varepsilon^2) \ : \ (\mathscr{P}_\varepsilon)$$

We want the error term to be written to the form $\varepsilon \partial_x [D_f] \partial_x w_0$, with D_f the so-called diffusion tensor, associated to the Lattice f.

$D_{w_0} F \times (**)$ leads to

$$\begin{aligned} & D_{w_0} F \partial_t w_0 + D_{w_0} F \partial_x F(w_0) = 0 \\ & \Rightarrow \partial_t F(w_0) + \left[D_{w_0} F \right]^2 \partial_x w_0 = 0 \\ & \Rightarrow \partial_t F(w_0) = - \left[D_{w_0} F \right]^2 \partial_x w_0 \end{aligned}$$

Also note that

$$\partial^2_{xx} \left[P\Lambda^2 f^{eq}(w_0) \right] = \partial_x \left[P\Lambda^2 \partial_x f^{eq}(w_0) \right] = \partial_x \left[P\Lambda^2 D_{w_0} f^{eq} \partial_x w_0 \right]$$

We thus obtain

$$D_f = P\Lambda^2 D_{w_0} f^{eq}(w_0) - \left[D_{w_0} F \right]^2,$$

and the approximated system is then written as :

$$(\mathscr{P}_\varepsilon) : \ \partial_t w + \partial_x F(w) = \varepsilon \partial_x \left[\left[P\Lambda^2 D_{w_0} f^{eq} - \left[D_{w_0} F \right]^2 \right] \partial_x w_0 \right] + O(\varepsilon^2)$$

which converge to the initial problem :

$$(\mathscr{P}_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (\mathscr{P}) : \ \partial_t w + \partial_x F(w) = 0$$

with a stability condition relating to the eigenvalues of $A := D_{w_0}F$:

$$|\lambda_{max}(A)| > \max_k |\lambda_k|$$

NB : Note that, the system being hyperbolic, A is diagonalizable with reals eigenvalues.

2.3 Vectorial kinetic method in 2D

This paragraph tells the exact same thing as the previous one. Its aim is to generalize all the notations and concepts in order to write them for any dimension.

With respect to the previous notations, we write

$$f, f^{eq} \in \mathbb{R}^M, \ w = w(t,x,y) \in \mathbb{R}^N, \ \Lambda_\alpha = \begin{pmatrix} \lambda_1^\alpha & & O \\ & \ddots & \\ O & & \lambda_M^\alpha \end{pmatrix} \in \mathscr{M}_M(\mathbb{R}), \ \forall \alpha \in \{x,y\},$$

$$\Lambda = \begin{pmatrix} \Lambda_x & 0 \\ 0 & \Lambda_y \end{pmatrix} \in \mathscr{M}_{2M}, \ P \in \mathscr{M}_{N,M}(\mathbb{R})$$

The lattice is then written as follows :

$$\begin{cases} \partial_t f + \Lambda_x \partial_x f + \Lambda_y \partial_y f = \frac{1}{\varepsilon}\left(f^{eq}(w) - f\right) \\ Pf = Pf^{eq}(w) = w \\ P\Lambda_\alpha f^{eq} = F^\alpha(w) \end{cases}$$

With a very similar method than the one dimensional case (see Appendix), we show that for $\varepsilon \to 0$, this kinetic model is consistent with the following equation

$$\partial_t w + \nabla \cdot F(w) = \varepsilon \nabla \cdot \left[\left[P\bar{\bar{\Lambda}} D_{w_0} f^{eq} - \overline{\overline{D_{w_0} F}} \right] \nabla w_0 \right] + O(\varepsilon^2)$$

We call $D_f = P\bar{\bar{\Lambda}} D_{w_0} f^{eq} - \overline{\overline{D_{w_0} F}}$ the diffusion tensor, which has the same structure than before and must verify a similar stability condition.

2.4 Example: D2Q4 for the barotropic Euler system

We'll now study the particular case of the approximation of the barotropic Euler system with a D2Q4 scheme for each macroscopic variable.[1]

Barotropic Euler system reads :

$$\begin{cases} \partial_t\rho + \partial_x(\rho u^x) + \partial_y(\rho u^y) = 0 \\ \partial_t(\rho u) + \nabla\cdot(\rho u \otimes u + p\,\mathrm{Id}) = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_t\rho + \partial_x(\rho u^x) + \partial_y(\rho u^y) = 0 \\ \partial_t \begin{pmatrix} \rho u^x \\ \rho u^y \end{pmatrix} + \partial_x \begin{pmatrix} \rho u^x u^x + P \\ \rho u^y u^x \end{pmatrix} + \partial_y \begin{pmatrix} \rho u^x u^y \\ \rho u^y u^y + P \end{pmatrix} = 0 \end{cases}$$

Writing

$w = (\rho, \rho u^x, \rho u^y)^T$, $F^x(w) = (\rho u^x u^x + P, \rho u^x u^y)^T$ and $F^y(w) = (\rho u^y u^x, \rho u^y u^y + P)^T$,

we recognize the non-linear tranport equation : $\partial_t w + \nabla\cdot F(w) = 0$

As we calculate the equilibrium for each macroscopic variable separately, we'll use the following notations :

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix},\ \Lambda_x = \begin{pmatrix} \lambda & & & \\ & 0 & & \\ & & -\lambda & \\ & & & 0 \end{pmatrix},\ \Lambda_y = \begin{pmatrix} 0 & & & \\ & -\lambda & & \\ & & 0 & \\ & & & \lambda \end{pmatrix},\ \lambda \in \mathbb{R}$$

Using the relations from the previous section, we already got 3 equations per variable. We add the constraint $P\Lambda_x^2 f_{w_i}^{eq} = w_i\lambda^2/2$, in order to close the system on f^{eq}.

Equation on ρ All constraints writes :

$$\begin{cases} Pf_\rho^{eq} = \rho \\ P\Lambda_x f_\rho^{eq} = \rho u^x \\ P\Lambda_y f_\rho^{eq} = \rho u^y \\ P\Lambda_x^2 f_\rho^{eq} = \rho\lambda^2/2 \end{cases} \Leftrightarrow \begin{cases} \sum_i f_{i,\rho}^{eq} = \rho \\ \lambda(f_1^{eq} - f_3^{eq}) = \rho u^x \\ \lambda(f_4^{eq} - f_2^{eq}) = \rho u^y \\ \lambda^2(f_1^{eq} + f_3^{eq}) = \lambda^2\rho/2 \end{cases} \Leftrightarrow \overbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda & 0 & -\lambda & 0 \\ 0 & -\lambda & 0 & \lambda \\ \lambda^2 & 0 & \lambda^2 & 0 \end{pmatrix}}^{A} f_\rho^{eq} = \overbrace{\begin{pmatrix} \rho \\ \rho u^x \\ \rho u^y \\ \lambda^2\rho/2 \end{pmatrix}}^{b}$$

which gives us :

$$f_\rho^{eq} = A^{-1}b = \frac{1}{2}\begin{pmatrix} 0 & 1/\lambda & 0 & 1/\lambda^2 \\ 1 & 0 & -1/\lambda & -1/\lambda^2 \\ 0 & -1/\lambda & 0 & 1/\lambda \\ 1 & 0 & 1/\lambda & -1/\lambda^2 \end{pmatrix} b \Leftrightarrow \boxed{f_\rho^{eq} = \frac{\rho}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2\lambda}\begin{pmatrix} \rho u^x \\ -\rho u^y \\ -\rho u^x \\ \rho u^y \end{pmatrix}}$$

[1]This is called [D2Q4][3]

We repeat the same operations for ρu^x and ρu^y.

Equation on ρu^x

$$\begin{cases} P f^{eq}_{\rho u^x} = \rho u^x \\ P\Lambda_x f^{eq}_{\rho u^x} = \rho u^x u^x + P \\ P\Lambda_y f^{eq}_{\rho u^x} = \rho u^x u^y \\ P\Lambda_x^2 f^{eq}_{\rho u^x} = \rho u^x \lambda^2/2 \end{cases} \Leftrightarrow \overbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda & 0 & -\lambda & 0 \\ 0 & -\lambda & 0 & \lambda \\ \lambda^2 & 0 & \lambda^2 & 0 \end{pmatrix}}^{A} f^{eq}_{\rho} = \overbrace{\begin{pmatrix} \rho u^x \\ \rho u^x u^x + P \\ \rho u^x u^y \\ \rho u^x \lambda^2/2 \end{pmatrix}}^{b}$$

$$\Leftrightarrow f^{eq}_{\rho u^x} = A^{-1}b = \frac{\rho u^x}{4}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{2\lambda}\begin{pmatrix} \rho u^x u^x + P \\ \lambda\rho u^x - \rho u^x u^y \\ -\rho u^x u^x - P \\ \lambda\rho u^x + \rho u^x u^y \end{pmatrix} \Leftrightarrow \boxed{f^{eq}_{\rho u^x} = \frac{\rho u^x}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2\lambda}\begin{pmatrix} \rho u^x u^x + P \\ -\rho u^x u^y \\ -\rho u^x u^x - P \\ \rho u^x u^y \end{pmatrix}}$$

Equation on ρu^y

$$\begin{cases} P f^{eq}_{\rho u^y} = \rho u^y \\ P\Lambda_x f^{eq}_{\rho u^y} = \rho u^y u^x \\ P\Lambda_y f^{eq}_{\rho u^y} = \rho u^y u^y + P \\ P\Lambda_x^2 f^{eq}_{\rho u^y} = \rho u^y \lambda^2/2 \end{cases} \Leftrightarrow \overbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda & 0 & -\lambda & 0 \\ 0 & -\lambda & 0 & \lambda \\ \lambda^2 & 0 & \lambda^2 & 0 \end{pmatrix}}^{A} f^{eq}_{\rho} = \overbrace{\begin{pmatrix} \rho u^y \\ \rho u^y u^x \\ \rho u^y u^y + P \\ \rho u^y \lambda^2/2 \end{pmatrix}}^{b}$$

$$\Leftrightarrow f^{eq}_{\rho u^y} = A^{-1}b \Leftrightarrow \boxed{f^{eq}_{\rho u^y} = \frac{\rho u^y}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2\lambda}\begin{pmatrix} \rho u^y u^x \\ -\rho u^y u^y - P \\ -\rho u^y u^x \\ \rho u^y u^y + P \end{pmatrix}}$$

We finally gather all those results and obtain $f^{eq} = (f^{eq}_{\rho}, f^{eq}_{\rho u^x}, f^{eq}_{\rho u^y})^T$

Matrix diffusion error Since we know the equilibrium function, we are now able to determine the error of our relaxation by writing the viscosity term D_f of the continuity equation.

As $\Lambda_{xy} = \Lambda_{yx} = 0$, we get $\overline{\overline{\Lambda}} = \begin{pmatrix} \Lambda_x^2 & 0 \\ 0 & \Lambda_y^2 \end{pmatrix}$

From now on we'll write $q^\alpha = \rho u^\alpha$ and $A := D_{u_0}F$.

The viscosity then reads :

$$D_f = P\bar{\bar{\Lambda}} D_{u_0} f^{eq} - \overline{\overline{D_{u_0}F}} = P\bar{\bar{\Lambda}} D_{u_0} f^{eq} - \overline{\overline{A}}.$$

And we now want to calculate the following term :

$$P\bar{\bar{\Lambda}} D_{u_0} f^{eq} = \begin{pmatrix} P\Lambda_x^2 D_{u_0} f^{eq} & 0 \\ 0 & P\Lambda_y^2 D_{u_0} f^{eq} \end{pmatrix},$$

which we are going to write in two steps, thanks to the diagonal structure of the tensor :

$$\begin{cases} P\Lambda_x^2 D_{u_0} f_{u_k}^{eq} = \begin{pmatrix} \lambda^2 & 0 & \lambda^2 & 0 \end{pmatrix} D_{u_0} f_{u_k}^{eq}, \\ P\Lambda_y^2 D_{u_0} f_{u_k}^{eq} = \begin{pmatrix} 0 & \lambda^2 & 0 & \lambda^2 \end{pmatrix} D_{u_0} f_{u_k}^{eq}. \end{cases}$$

We start by calculating the partial derivatives of f_ρ^{eq} :

$$\partial_\rho f_\rho^{eq} = \partial_\rho \left[\frac{\rho}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2\lambda} \begin{pmatrix} q^x \\ -q^y \\ -q^x \\ q^y \end{pmatrix} \right] = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\partial_{q^x} f_\rho^{eq} = \partial_{q^x} \left[\frac{\rho}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2\lambda} \begin{pmatrix} q^x \\ -q^y \\ -q^x \\ q^y \end{pmatrix} \right] = \frac{1}{2\lambda} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \partial_{q^y} f_\rho^{eq} = \partial_{q^y} \left[\frac{\rho}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2\lambda} \begin{pmatrix} q^x \\ -q^y \\ -q^x \\ q^y \end{pmatrix} \right] = \frac{1}{2\lambda} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

We then get the Jacobian of f_ρ^{eq} which gives us the first part of the viscosity :

$$D_{u_0} f_\rho^{eq} = \begin{pmatrix} \partial_\rho f_\rho^{eq} & \partial_{q^x} f_\rho^{eq} & \partial_{q^y} f_\rho^{eq} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2\lambda} & 0 \\ \frac{1}{4} & 0 & \frac{-1}{2\lambda} \\ \frac{1}{4} & \frac{-1}{2\lambda} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2\lambda} \end{pmatrix} \Rightarrow P\Lambda_x^2 D_{u_0} f_\rho^{eq} = \lambda^2 \begin{pmatrix} 1/2 & 0 & 0 \end{pmatrix}$$

Doing the same with q^x and q^y gives us :

$$\begin{cases} P\Lambda_x^2 D_{u_0} f_\rho^{eq} = \lambda^2 \begin{pmatrix} 1/2 & 0 & 0 \end{pmatrix} \\ P\Lambda_x^2 D_{u_0} f_{q^x}^{eq} = \lambda^2 \begin{pmatrix} 0 & 1/2 & 0 \end{pmatrix} \\ P\Lambda_x^2 D_{u_0} f_{q^y}^{eq} = \lambda^2 \begin{pmatrix} 0 & 0 & 1/2 \end{pmatrix} \end{cases}$$

and :

$$\begin{cases} P\Lambda_y^2 D_{u_0} f_\rho^{eq} = \lambda^2 \begin{pmatrix} 1/2 & 0 & 0 \end{pmatrix} \\ P\Lambda_y^2 D_{u_0} f_{q^x}^{eq} = \lambda^2 \begin{pmatrix} 0 & 1/2 & 0 \end{pmatrix} \\ P\Lambda_y^2 D_{u_0} f_{q^y}^{eq} = \lambda^2 \begin{pmatrix} 0 & 0 & 1/2 \end{pmatrix} \end{cases}$$

So we finally have the following result :

$$D_f = \frac{\lambda^2}{2} I_6 - \bar{\bar{A}}$$

which allows us to write the stability condition of the system

$$D_f = \frac{\lambda^2}{2} I_6 - \bar{\bar{A}} \ > \ 0 \Leftrightarrow \lambda^2 > 2\ \lambda_{max}(\bar{\bar{A}}) \Leftrightarrow |\lambda| > \sqrt{2}\ \ |\lambda_{max}(A)|$$

2.5 Nondimensionalization of the barotropic Euler system

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0 \\ \partial_t \rho u + \nabla\cdot(\rho u \otimes u + P) = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_t \rho + u\cdot\nabla\rho + \rho\nabla\cdot u = 0 \\ \partial_t u + (u\cdot\nabla)u + \frac{\nabla P}{\rho} = 0 \end{cases}$$

$t = t_0 \hat{t} \Rightarrow \partial_t = \frac{1}{t_0}\partial_{\hat{t}}$

$\nabla = \frac{\hat{\nabla}}{L}$

$\rho = \rho_0\hat{\rho},\ \ u = u_0\hat{u}, P = P_0\hat{P}$

$$\Rightarrow \begin{cases} \frac{\rho_0}{t_0}\partial_{\hat{t}}\hat{\rho} + \frac{u_0\rho_0}{L}\hat{u}\cdot\hat{\nabla}\hat{\rho} + \frac{\rho_0 u_0}{L}\hat{\rho}\hat{\nabla}\cdot\hat{u} = 0 & (L_1) \\ \frac{u_0}{t_0}\partial_{\hat{t}}\hat{u} + \frac{u_0^2}{L}(\hat{u}\cdot\hat{\nabla})\hat{u} + \frac{P_0}{\rho_0 L}\frac{\hat{\nabla}\hat{P}}{\hat{\rho}} = 0 & (L_2) \end{cases}$$

$$\begin{matrix} \frac{t_0}{\rho_0}\times L_1 : \\ \frac{t_0}{u_0}\times L_2 : \end{matrix} \quad \begin{cases} \partial_t \rho + \frac{t_0 u_0}{L} u\cdot\nabla\rho + \frac{t_0 u_0}{L}\rho\nabla\cdot u = 0 \\ \partial_t u + \frac{t_0 u_0}{L}(u\cdot\nabla)u + \frac{t_0 P_0}{u_0\rho_0 L}\frac{\nabla P}{\rho} = 0 \end{cases} \qquad (\hat{\mathscr{T}} \equiv \mathscr{T})$$

$$\Rightarrow \begin{cases} \partial_t \rho + \frac{t_0 u_0}{L}(u\cdot\nabla)\rho + \frac{t_0 u_0}{L}\rho\nabla\cdot u = 0 \\ \partial_t u + \frac{t_0 u_0}{L}(u\cdot\nabla)u + \frac{t_0 c_0^2}{u_0 L}\frac{\nabla P}{\rho} = 0 \end{cases} \qquad (P = c^2\rho)$$

$$\Rightarrow \quad \begin{cases} \partial_t \rho + \frac{u_0}{v_0}(u\cdot\nabla)\rho + \frac{u_0}{v_0}\rho\nabla\cdot u = 0 \\ \partial_t u + \frac{u_0}{v_0}(u\cdot\nabla)u + \frac{c_0^2}{u_0 v_0}\frac{\nabla P}{\rho} = 0 \end{cases} \qquad (v_0 := \frac{L}{t_0})$$

$M := \frac{u_0}{\rho_0}$ (Mach number)

$u_0 = v_0$ (we are interested in the overall speed of the fluid, i.e. the convection)

$$\Rightarrow \begin{cases} \partial_t \rho + (u\cdot\nabla)\rho + \rho\nabla\cdot u = 0 \\ \partial_t u + (u\cdot\nabla)u + \frac{1}{M^2}\frac{\nabla P}{\rho} = 0 \end{cases}$$

$P = P^{(0)} + M^2\Pi$

$u = u^{(0)} + M^2 u^{(1)}$

$\rho = \rho^{(0)} + M^2\rho^{(1)}$

$O(1/M^2)$: $\frac{1}{M^2}\frac{\nabla P^{(0)}}{\rho^{(0)}} = 0 \Rightarrow P^{(0)} \equiv cte \Rightarrow \rho^{(0)} \equiv cte$ (continuity gives us the constancy over time)

$$O(1): \quad \begin{cases} \partial_t \rho^{(0)} + u^{(0)}\cdot\nabla\rho^{(0)} + \rho^{(0)}\nabla\cdot u^{(0)} = \rho^{(0)}\nabla\cdot u^{(0)} = 0 \\ \partial_t u^{(0)} + (u^{(0)}\cdot\nabla)u^{(0)} + \frac{\nabla\Pi}{\rho^{(0)}} = 0 \end{cases}$$

We obtain Euler equations (incompressible case) :

$$(\mathscr{E}): \quad \begin{cases} \partial_t u^{(0)} + (u^{(0)}\cdot\nabla)u^{(0)} + \nabla\Pi = 0 \\ \nabla\cdot u^{(0)} = 0 \end{cases}$$

3 Appendix

3.1 Equivalent equation for the vectorial kinetic method in 2D

with P the moment matrix which gives us the following relations :

$$\begin{array}{r} P \times (*) : \\ P\Lambda_x \times (*) : \\ P\Lambda_y \times (*) : \end{array} \quad \begin{cases} \partial_t w + \partial_x v^x + \partial_y v^y = 0 \\ \partial_t v^x + \partial_x P \Lambda_x^2 f + \partial_y P \Lambda_x \Lambda_y f = \frac{1}{\varepsilon} \left[F^x(w) - v^x \right] \\ \partial_t v^y + \partial_x P \Lambda_y \Lambda_x f + \partial_y P \Lambda_y^2 f = \frac{1}{\varepsilon} \left[F^y(w) - v^y \right] \end{cases}$$

We use once again the Taylor expansion :

$w = w_0 + \varepsilon w_1 + O(\varepsilon^2)$

$v^\alpha = v_0^\alpha + \varepsilon v_1^\alpha + O(\varepsilon^2)$

$f = f_0 + \varepsilon f_1 + O(\varepsilon^2)$

We also introduce the following notations :

$DF^\alpha(w_\beta) = \mathrm{Jac} F^\alpha(w_\beta)$

$\Lambda_{\alpha\beta} = \Lambda_\alpha \Lambda_\beta \ \forall \alpha, \beta \in \{x, y\}$

Plugging everything in our set of equations :

$$\begin{cases} \partial_t w_0 + \varepsilon \partial_t w_1 + \partial_x v_0^x + \varepsilon \partial_x v_1^x + \partial_x v_0^y + \varepsilon \partial_x v_1^y = O(\varepsilon^2) \\ \partial_t v_0^x + \varepsilon \partial_t v_1^x + \partial_x (P \Lambda_x^2 f_0) + \varepsilon \partial_x (P \Lambda_x^2 f_1) + \partial_y (P \Lambda_{xy} f_0) + \varepsilon \partial_y (P \Lambda_{xy} f_1) \\ \qquad = \frac{1}{\varepsilon} \left[F^x(w_0) + \varepsilon D_{w_0} F^x w_1 - v_0^x - \varepsilon v_1^x \right] + O(\varepsilon^2) \\ \partial_t v_0^y + \varepsilon \partial_t v_1^y + \partial_x (P \Lambda_{yx} f_0) + \varepsilon \partial_x (P \Lambda_{yx} f_1) + \partial_y (P \Lambda_y^2 f_0) + \varepsilon \partial_y (P \Lambda_y^2 f_1) \\ \qquad = \frac{1}{\varepsilon} \left[F^y(w_0) + \varepsilon D_{w_0} F^y w_1 - v_0^y - \varepsilon v_1^y \right] + O(\varepsilon^2) \\ \partial_t f_0 + \varepsilon \partial_t f_1 + \Lambda_x \partial_x f_0 + \Lambda_x \partial_x f_1 + \Lambda_y \partial_y f_0 + \Lambda_y \partial_y f_1 \\ \qquad = \frac{1}{\varepsilon} \left[f^{eq}(w_0) + \varepsilon D_{w_0} f^{eq}(w_0) w_1 - f_0 - \varepsilon f_1 \right] + O(\varepsilon^2) \end{cases}$$

As in the previous section, we write the system at each order of ε :

$$O(1/\varepsilon): \begin{cases} v_0^\alpha = F^\alpha(w_0) \\ f_0 = f^{eq}(w_0) \end{cases}$$

$$O(1): \begin{cases} \partial_t w_0 + \partial_x v_0^x \partial_y v_0^y = \boxed{\partial_t w_0 + \partial_x F^x(w_0) + \partial_y F^y(w_0) = 0} \quad (**) \\ \partial_t v_0^x + \partial_x(P\Lambda_x^2 f_0) + \partial_y(P\Lambda_{xy} f_0) = \partial_t F^x(w_0) + \partial_x(P\Lambda_x^2 f_0) + \partial_y(P\Lambda_{xy} f_0) \\ \qquad = D_{w_0} F^x w_1 - v_1^x \\ \partial_t v_0^y + \partial_x(P\Lambda_{yx} f_0) + \partial_y(P\Lambda_y^2 f_0) = \partial_t F^y(w_0) + \partial_x(P\Lambda_{yx} f_0) + \partial_y(P\Lambda_y^2 f_0) \\ \qquad = D_{w_0} F^y w_1 - v_1^y \end{cases}$$

$$\Rightarrow \boxed{v_1 = D_{w_0} F w_1 - \partial_t F(w_0) - \vec{\nabla} \cdot (P\bar{\bar{\Lambda}} f^{eq}(w_0))} \quad (\mathcal{K})$$

$$O(\varepsilon): \begin{cases} \partial_t w_1 + \partial_x v_1^x + \partial_y v_1^y = \boxed{\partial_t w_1 + \nabla \cdot v_1 = 0} \quad (***) \\ \partial_t v_1^x + \partial_x(P\Lambda_x^2 f_1) + \partial_y(P\Lambda_{xy} f_1) = 0 \\ \partial_t v_1^y + \partial_x(P\Lambda_{yx} f_1) + \partial_y(P\Lambda_y^2 f_1) = 0 \end{cases}$$

In a conciseness concern, we introduce the *tensorization* and the application of the divergence to matrices :

$$\bar{\bar{\Lambda}} := \begin{pmatrix} \Lambda_x^2 & \Lambda_{xy} \\ \Lambda_{yx} & \Lambda_y^2 \end{pmatrix} = \Lambda \otimes \Lambda = \Lambda\Lambda^T \quad , \quad \vec{\nabla} \cdot \begin{pmatrix} \vec{H_1} \\ \vec{H_2} \end{pmatrix} := \begin{pmatrix} \nabla \cdot (H_{11}, H_{12}) \\ \nabla \cdot (H_{21}, H_{22}) \end{pmatrix} = \begin{pmatrix} \partial_x H_{11} + \partial_y H_{12} \\ \partial_x H_{21} + \partial_y H_{22} \end{pmatrix}$$

Using these, $(***)$ &$(\mathcal{K})$ leads us to :

$$\boxed{\partial_t w_1 + \nabla \cdot (DF(w_0) w_1) = \nabla \cdot (\partial_t F(w_0) + \vec{\nabla} \cdot P\bar{\bar{\Lambda}} f_0)} \quad (\widetilde{***})$$

and then, we write $(**) + \varepsilon(\widetilde{***})$:

$$\boxed{\partial_t w + \nabla \cdot F(w) = \varepsilon \nabla \cdot (\partial_t F(w_0) + \vec{\nabla} \cdot P\bar{\bar{\Lambda}} f_0) + O(\varepsilon^2)}$$

We now want to show the diffusion tensor in the error term :

$$\begin{array}{l} D_{w_0} F^x \times (**) : \\ \\ D_{w_0} F^y \times (**) : \end{array} \begin{cases} D_{w_0} F^x \partial_t w_0 + D_{w_0} F^x \partial_x F^x(w_0) + D_{w_0} F^x \partial_y F^x(w_0) = 0 \\ \Rightarrow \boxed{\partial_t F^x(w_0) + \left[D_{w_0} F^x\right]^2 \partial_x w_0 + D_{w_0} F^x D_{w_0} F^y \partial_y w_0 = 0} \\ D_{w_0} F^y \partial_t w_0 + D_{w_0} F^y \partial_x F^x(w_0) + D_{w_0} F^y \partial_y F^x(w_0) = 0 \\ \Rightarrow \boxed{\partial_t F^y(w_0) + D_{w_0} F^y D_{w_0} F^x \partial_x w_0 + \left[D_{w_0} F^y\right]^2 \partial_y w_0 = 0} \end{cases}$$

We then obtain $\boxed{\partial_t F(w_0) = -\overline{\overline{D_{w_0} F}}\ \nabla w_0}$, $\left(\text{recall } \overline{\overline{DF}}(w_0) := \left[D_{w_0} F^\alpha D_{w_0} F^\beta\right]_{\alpha,\beta}\right)$, and the approximated system is now written as :

$$(\mathscr{P}_\varepsilon) : \boxed{\partial_t w + \nabla \cdot F(w) = \varepsilon \nabla \cdot \left[\left[P\bar{\bar{\Lambda}} D_{w_0} f^{eq} - \overline{\overline{D_{w_0} F}}\right] \nabla w_0\right] + O(\varepsilon^2)}$$

with $D_f := P\bar{\bar{\Lambda}} D_{w_0} f^{eq} - \overline{\overline{D_{w_0} F}}$ which must be positive for the stability. We also get the convergence to the transport equation :

$$(\mathscr{P}_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} (\mathscr{P}) : \boxed{\partial_t w + \nabla \cdot F(w) = 0}$$

References