Asymptotic preserving schemes for moment models on unstructured meshes

Emmanuel Franck

CEA/DAM/DIF/DSSI - UPMC/LJLL

06 Juillet 2010

Supervisors : Christophe Buet (CEA/DAM) and Bruno Després (UPMC/LJLL)

- **Inertial confinement fusion** : Compression of a gaz capsule with a set of laser beams in order to meet the thermonuclear ignition conditions.
- **Radiation hydrodynamics** : Interaction between the gas modeled by Euler equations and the radiance, modeled by a transport equation.
- **Transport equation** : $f(t, \mathbf{x}, \mathbf{v}) \geq 0$ the distribution fonction associated to particules located in $\mathbf{x}$ and with a velocity $\mathbf{v}$. We consider the following equation of the form :

$$\partial_t f(t, \mathbf{x}, \mathbf{v}) + \mathbf{v}.\nabla f(t, \mathbf{x}, \mathbf{v}) = \sigma_S Q(f, f) + \sigma_a S(f),$$

where $Q(f, f)$ is a collision operator (or scattering) and $S(f)$ an absorption/emission term.

Background and Objectives : diffusion limit

- **Diffusion regime** : The transport equation has, in some regimes, the property to tend towards an equation of diffusion on the first moment of f.
- **Example : Non-equilibrium diffusion limit** We study the limit when t is high and $\sigma_S >> \sigma_a$. We begin by rescaling the equation :

$$\tilde{t} = \varepsilon t,\ \tilde{\sigma_S} = \varepsilon \sigma_S,\ \tilde{\sigma_a} = (1/\varepsilon)\sigma_a$$

We obtain

$$\partial_t f(t,\mathbf{x},\mathbf{v}) + \frac{1}{\varepsilon}\mathbf{v}.\nabla f(t,\mathbf{x},\mathbf{v}) = \frac{1}{\varepsilon^2}\sigma_S Q(f,f) + \sigma_a S(f).$$

Proposition : diffusion limit

When ε tends to 0 the previous equation, in the case $\sigma_a = 0$, tends to

$$\partial_t \rho(t,\mathbf{x}) - \frac{1}{\sigma_S}\triangle\rho(t,\mathbf{x}) = 0,$$

with $\rho(t,\mathbf{x}) = \int_\Omega f(t,\mathbf{x},\mathbf{v})dv$.

Moment models

- Simplified hyperbolic models, nammed moment models, depend on spacesvariables.
- **Simplified models** :
  - Pn models : we develop the transport equation on a basis of spherical harmonics.
  - Sn models : We use a quadrature formula for discretized the scattering operator, we obtain one equation for each quadrature point.
  - M1 model : This is the P1 nonlinear model, where the closing is obtained by minnizing the entropy.

Example of P1 model :

$$\left\{\begin{array}{l} \partial_t p + \frac{1}{\varepsilon}\nabla.(\mathbf{u}) = 0 \\ \partial_t(\mathbf{u}) + \frac{1}{3\varepsilon}\nabla p = -\frac{\sigma_S}{\varepsilon^2}\mathbf{u} \end{array}\right. .$$

Thesis objective

The objective is to construct finite volume schemes for the simplified models capturing the diffusion limit on unstructured meshes.

Results for the 1D case

- Consistency error of the standard upwing scheme : $O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right)$
- Consistency error of the Jin-Levermore scheme :
  - for the first equation : $O\left(\Delta x^2 + \varepsilon \Delta x + \Delta t\right)$
  - for the second equation : $O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right)$
- Consistency error to the Gosse-Toscani scheme : $O\left(\Delta x + \Delta t\right)$

  $\Rightarrow$ **Conclusion :** The schemes of Jin-Levermore and Gosse-Toscani are AP since the consitency errors are not dependent of ε
- The Jin-Levermore scheme with the following discretization of the source term : $\frac{1}{2}(u_{j+1/2} + u_{j-1/2})$ is equivalent to the Gosse-Toscani scheme.

  $\Rightarrow$ **Aim :** Construct the equivalent in dimension two of these schemes.

[1] L. Gosse, G. Toscani *An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations* C. R. Acad. Sci Paris,Ser. I 334 (2002) 337-342

S. Jin, D. Levermore *Numerical schemes for hyperbolic conservation laws with stiff relaxation terms.* JCP 126,449-467 ,1996, n°0149

Classical " asymptotic preserving " scheme

First case : P1 system

$$\left\{\begin{array}{l} \partial_t p + \dfrac{1}{\varepsilon}\nabla.(\mathbf{u}) = 0 \\ \partial_t(\mathbf{u}) + \dfrac{1}{3\varepsilon}\nabla p = -\dfrac{\sigma_S}{\varepsilon_2}\mathbf{u} \end{array}\right. .$$

This system is obtained from the transport equation with the scattering term $Q(f,f) = \dfrac{1}{2\pi}\displaystyle\int_{S^1}(f(t,x,\mathbf{v}^{'}) - f(t,x,\mathbf{v}))dv^{'}$.

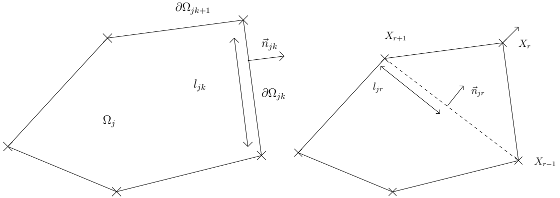
- **Asymptotic limit :** Using a Hilbert expansion, we obtain the following limit : $\partial_t p - \dfrac{1}{3\sigma_S}\triangle p = 0$
- Afterwards, we will study the telegraph equation that corresponds to P1 without coefficient $\frac{1}{3}$.
- This system is equivalent to the telegraph scalar equation :

$$\partial_{tt} p + \frac{1}{\varepsilon}\partial_t p = \frac{1}{\varepsilon}\triangle p.$$

Notations

- We define the notation for the classical scheme and the nodal scheme.

Edge notations **nodal notations**



$\Rightarrow$ $\mathbf{u}_{jk}.\mathbf{n}_{jk}$ and p_{jk} are the fluxes associated to the edge $\partial\Omega_{jk}$.
$\Rightarrow$ $\mathbf{u}_r$ and p_{jr} are the fluxes associated to the vertex X_r

Jin-Levermore method in 2D

- **Principle** : Modify the upwind scheme, incorporing the stationary states associated to the system with an implicit construction of the fluxes. When ε tends to 0, $\nabla p = -(1/\varepsilon)\mathbf{u}$ is not negligible.The classical approximation, constant in each cell, ignores this variation. So we want then to insert this variation using the Taylor formulas.

$$\left\{ \begin{array}{l} p_j \simeq p_{jk} - \dfrac{\sigma}{\varepsilon}(\mathbf{u_{jk}}, \mathbf{x_j} - \mathbf{x_{jk}}) \\ u_k \simeq p_{jk} - \dfrac{\sigma}{\varepsilon}(\mathbf{u_{jk}}, \mathbf{x_k} - \mathbf{x_{jk}}). \end{array} \right.$$

- **Hypothesis** : we suppose that we have a mesh that satisfies the Delaunay condition. Then :

$$(\mathbf{x_{jk}} - \mathbf{x_j}) = d_{jk}\mathbf{n_{jk}} \text{ et } (\mathbf{x_{jk}} - \mathbf{x_k}) = -d_{kj}\mathbf{n_{jk}}.$$

We use this development, with the acoustic solver, to obtain the fluxes :

$$\left\{ \begin{array}{l} \mathbf{u_j}\mathbf{n_{jk}} + p_j = \mathbf{u_{jk}}\mathbf{n_{jk}} + p_{jk} + (\sigma/\varepsilon)d_{jk}\mathbf{u}_{jk}.\mathbf{n}_{jk} \\ \mathbf{u_k}\mathbf{n_{jk}} + p_k = \mathbf{u_{jk}}\mathbf{n_{jk}} - p_{jk} + (\sigma/\varepsilon)d_{kj}\mathbf{u}_{jk}.\mathbf{n}_{jk}. \end{array} \right.$$

Proposition : Jin-Levermore scheme

$$\left\{\begin{array}{l} \mid T_j \mid \dfrac{p_j^{n+1}-p_j^n}{\triangle t} + \dfrac{1}{\varepsilon}\displaystyle\sum_k l_{jk}\mathbf{u}_{jk}.\mathbf{n}_{jk} = 0 \\ \mid T_j \mid \dfrac{u_{1,j}^{n+1}-u_{1,j}^n}{\triangle t} + \dfrac{1}{\varepsilon}\displaystyle\sum_k l_{jk}p_{jk}n_{jk}^x = - \mid T_j \mid \dfrac{\sigma}{\varepsilon^2}u_{1,j}^n \\ \mid T_j \mid \dfrac{u_{2,j}^{n+1}-u_{2,j}^n}{\triangle t} + \dfrac{1}{\varepsilon}\displaystyle\sum_k l_{jk}p_{jk}n_{jk}^y = - \mid T_j \mid \dfrac{\sigma}{\varepsilon^2}u_{2,j}^n, \end{array}\right. \quad (1)$$

with the fluxes

$$\left\{\begin{array}{l} \mathbf{u}_{jk}.\mathbf{n}_{jk} = \dfrac{(\mathbf{u_j}+\mathbf{u}_k)\mathbf{n}_{jk}+(p_j-p_k)}{2+(\sigma/\varepsilon)(d_{jk}+d_{kj})} \\ p_{jk} = \dfrac{(\mathbf{u_j}\mathbf{n}_{jk}+p_j)(1+d_{kj}(\sigma/\varepsilon))-(\mathbf{u}_k\mathbf{n}_{jk}-p_k)(1+d_{jk}(\sigma/\varepsilon))}{2+(\sigma/\varepsilon)(d_{jk}+d_{kj})}. \end{array}\right. \quad (2)$$

Diffusion limit : VF4 scheme

Asymptotic limit of the Jin-Levermore

$$| T_j | \frac{p_j^{n+1} - p_j^n}{\triangle t} - \sum_k l_{jk} \frac{p_k^n - p_j^n}{d(x_j, x_k)} = 0.$$

This scheme is named VF4.

- We obtain an asymptotic preserving scheme, but there are several problems :
  - The VF4 scheme does not converge if we use meshes that do not satisfy the Delaunay condtion.
  - The CFL condition is $\triangle t << \varepsilon h$, where h is the step of the mesh.
- **Conclusion :** A Classical writing of the finite volume scheme, using an edge formulation does not allow to obtain an AP scheme on unstructured meshes.

How to define a good scheme ?

- Using previous results, we want :
  - A limit diffusion scheme that is effective on unstructured meshes and possibly preserving the maximum principle,
  - An implicit scheme to avoid the very restrictive CFL condition.
  - An extensible method in 3D (problem with non-coplanar faces),
  - A scheme able to treat the case of σ variable,

- The principal idea to solve this problem, is to use a nodal formulation. We reconstruct the nodal gradient, and not a gradient in the normal direction, which gives geometric conditions on the mesh.

Nodal schemes

Construction of the nodal schemes

- **Idea :** Use the nodal scheme "GLACE" constructed for linearized Euler equations, by analogy with the P1 model and use this scheme with the Jin-Levermore method.
- **Difficulty :** we do not know the limit diffusion scheme. This may lead at the construction of a new scheme, studed after.

- Solver for linearized Euler equations :

$$\left\{ \begin{array}{l} p_{jr} - p_j = (\mathbf{u_j} - \mathbf{u_r}, \mathbf{n_{jr}}) \\ \sum_j l_{jr} p_{jr} \mathbf{n_{jr}} = 0. \end{array} \right.$$

Nodal solver

Applying to the Jin-Levermore method, we obtain the following solver :

$$\left\{ \begin{array}{l} p_{jr} - p_j = (\mathbf{u_j} - \mathbf{u_r}, \mathbf{n_{jr}}) - \frac{\sigma}{\varepsilon}(\mathbf{u_r}, \mathbf{x_r} - \mathbf{x_j}) \\ \\ \sum_j l_{jr}(\mathbf{n_{jr}} \otimes \mathbf{n_{jr}} + \frac{\sigma}{\varepsilon}(\mathbf{n_{jr}} \otimes (\mathbf{x_r} - \mathbf{x_j})))\mathbf{u_r} = \sum_j l_{jr}(p_j + (\mathbf{u_j}, \mathbf{n_{jr}}))\mathbf{n_{jr}}. \end{array} \right.$$

Invertibility of the matrix associated to the solver

- The fluxes exist if the matrix associated to the nodal solver is invertible. $\Rightarrow$ **Result of C. Mazeran :** The matrix

$$\sum_j l_{jr}(\mathbf{n_{jr}} \otimes \mathbf{n_{jr}})$$

is positive definite if all the cells are nodegenerate.

- the difficulty is to show the invertibility of

$$A_r = \sum_j l_{jr}(\mathbf{n_{jr}} \otimes (\mathbf{x}_r - \mathbf{x}_j))$$

$\Rightarrow$ **Idea** : Try to write A_r as a perturbation of $I_d V$ (V control volume) and prove that if we do not have a mesh with many deformation, this perturbation is small enough, that A_r stays positive definite.

First result

We can write this matrix as

$$A = idV - \frac{1}{4}\sum_j P_j,$$

with $P_j = (\mathbf{x}_{j+1} - \mathbf{x}_{j+1/2})^{\perp} \otimes (\mathbf{x}_{j+1} - \mathbf{x}_{j+1/2}) - (\mathbf{x}_{j+1/2} - \mathbf{x}_j)^{\perp} \otimes (\mathbf{x}_{j+1/2} - \mathbf{x}_j)$.

- We study $(x, A_r x) \geq \parallel x \parallel^2 (\parallel Id_V \parallel - \frac{1}{4} \parallel P \parallel)$
- Cartesian case : $\parallel I_d V \parallel = \triangle x^2$ and $\frac{1}{4} \parallel P \parallel = \frac{\triangle x^2}{2}$. By continuity we can hope the invertibility for deformed cartesian meshes.
- Let $\mathbf{x_j}$ be the cell center, $\mathbf{x}_{j+1/2}$ the edge center. We define V_{T_i} the volume of polygon constructed be with these points and the node r.

Result

If the following condition is verified, the matrix is invertible.

$$V_{T_i} \geq \frac{1}{4} \parallel \mathbf{x}_{j+1/2} - \mathbf{x}_{j-1/2} \parallel \parallel \mathbf{x}_j - \frac{1}{2}(\mathbf{x}_{j+1/2} - \mathbf{x}_{j-1/2}) \parallel$$

- By choosing the gravity center, a quantitative result shows that matrix is invertible if the angles of triangles are superior to approximatively 5 or 6 degrees.

Diffusion limit

- We use a Hilbert expansion on the nodal scheme to obtain the limit scheme.

Proposition

The limit scheme is :

$$\left\{ \begin{array}{l} \mid T_j \mid \dfrac{p_j^{n+1} - p_j^n}{\triangle t} + \sum_r l_{jr}(\mathbf{u_r}.\mathbf{n_{jr}}) = 0 \\ \sigma(\sum_j l_{jr}\mathbf{n_{jr}} \otimes (\mathbf{x_r} - \mathbf{x_j}))\mathbf{u_r} = \sum_j l_{jr} p_j \mathbf{n_{jr}}. \end{array} \right.$$

- Numerically this limit scheme is effective on unstructured mesh.
- The problem of VF4 scheme is associated to the gradient reconstruction in the normal direction. It is possible, under conditions on the mesh. In the nodal formulation, we reconstruct a gradient in any direction that helps to eliminate geometric constraints.

Theorical study of diffusion scheme

We define the following errors :

$$\| e(t) \|_{L^2(\Omega)} = (\sum_j | T_j | (p_j(t) - p(x_j,t))^2)^{\frac{1}{2}}$$

$$\| f(t) \|_{L^2([0,t]x\Omega)} = (\int_0^t \sum_r | V_r | (\mathbf{u}_r(t) - \nabla p(x_r,t))^2)^{\frac{1}{2}}$$

Theorem

We assume that $p \in W^{3,\infty}(\Omega)$. If there exists a constant α such that $A_r^S \geq \alpha V_r$, then the semi-discrete diffusion scheme is convergent for all time $\mathrm{T} > 0$,

$$\| e(t) \|_{L^2(\Omega)} + \| f(t) \|_{L^2([0,t]x\Omega)} = \mathrm{C(T)h}$$

- Ideal of the proof :Classical study of consistancy and Gronwall lemma.
- The semi-discrete scheme is decrasing in L^2 norm.
- The matrix associated to the implicit scheme with Neumann boundary condition, is invertible. The implicit scheme is L^2-stable.

Other approaches with nodal scheme

- In one dimension, the Jin-Levermore scheme with a discretization of the source term using the flux, is equivalent to the Gosse-Toscani scheme.
- Idea : construction of a new scheme equivalent to Gosse-Toscani scheme in dimension two, nammed JL-(b)

$$\left\{\begin{array}{c} \mid T_j \mid \dfrac{p_j^{n+1}-p_j^n}{\triangle t}+\dfrac{1}{\varepsilon}\sum_r l_{jr}(\mathbf{u_r}.\mathbf{n_{jr}})=0 \\ \mid T_j \mid \dfrac{\mathbf{u}_j^{n+1}-\mathbf{u}_j^n}{\triangle t}+\dfrac{1}{\varepsilon}\sum_r l_{jr}p_{jr}\mathbf{n}_{jr}=-\dfrac{\sigma}{\varepsilon^2}\sum_r l_{jr}\mathbf{n}_{jr}\otimes(\mathbf{x}_r-\mathbf{x}_j)\mathbf{u}_r \end{array}\right.$$

- Other approaches studied :
  - The G. Kluth's tensor notation in order to use a CHIC tensor, instead of using GLACE tensor.
  - Change the matrix A_r by $I_d V_r$ to avoid invertibility problem.

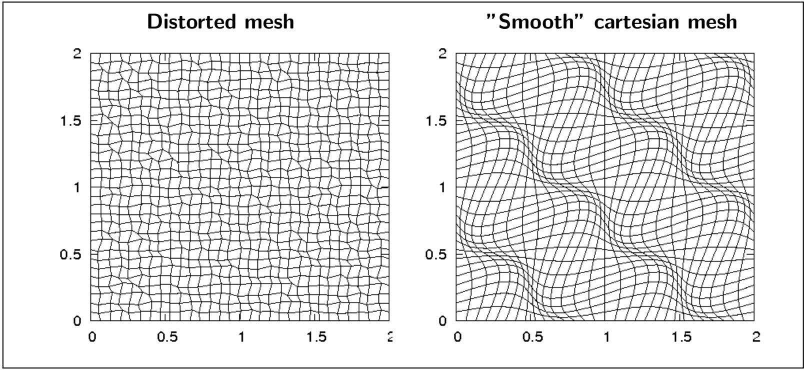
Property

The semi-discrete schemes JL-(a) for $A_r = I_d V_r$ and the JL-(b) scheme for all the variants are decreasing in L^2 norm. The time implicite discretization are L^2 stable.

Numerical results

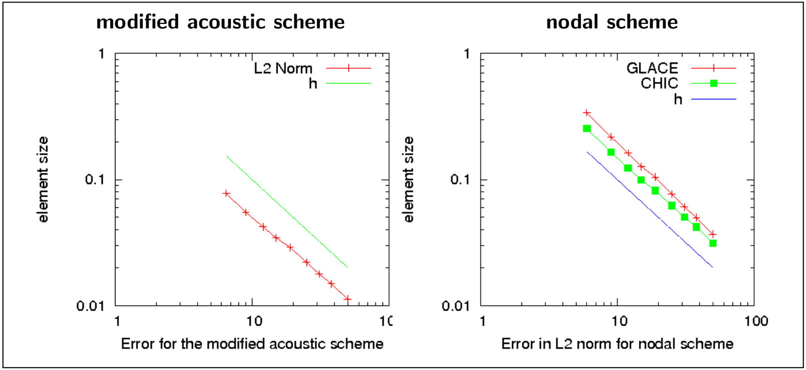
Example of unstructured meshes

Two examples of unsctrutured meshes.



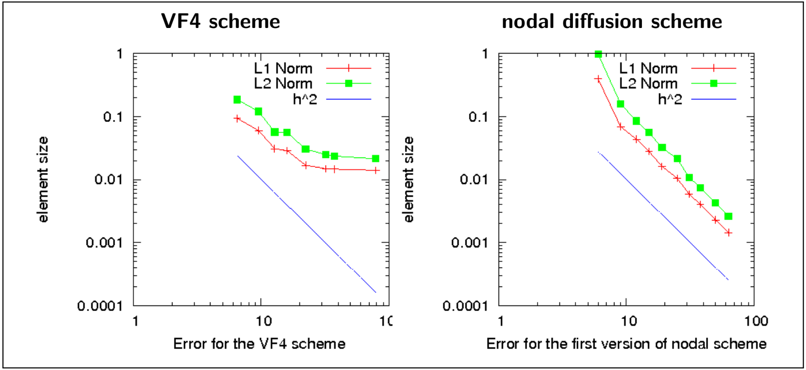
Results for the transport regime

Periodic solution of telegraph equation associated to the initial condition $u_0 = cos(\pi x)cos(\pi y)$ and $v_0 = 0$. Final time = 0.1 sec, $\sigma = 1$, $\varepsilon = 1$.



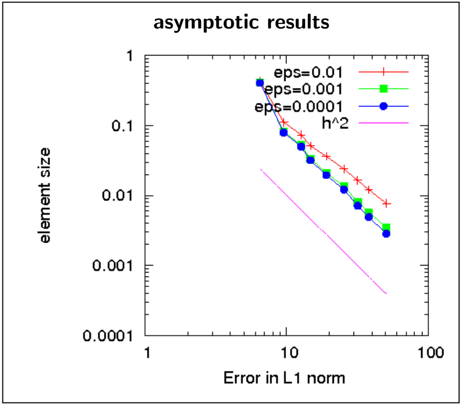
Results for the diffusion scheme

Initial condition : the fundamental solution at time 0.001. Final time 0.011. Type of meshes : cartesian, distorded cartesian, « smooth », mesh of equilateral triangles.



Results for the asymptotic regime

- We take the same hypothesis that for the diffusion, using the AP schemes JL-(a) and JL-(b) for some different values of ε.



The results are the same for the scheme JL-(b) and JL(a).

Conclusion

- We obtain an AP invalid scheme on unstructured meshes, using the Jin-Levermore method and a classical acoustic solver,
- The nodal scheme GLACE, used with the Jin-Levermore method, gives an AP scheme defined on a lot of unstructured meshes.
- Further the limit diffusion mesh is simple, we can give a convergence result allowing to assure the efficiency on unstructured meshes.
- **Problems** : This scheme gives not the VF4 scheme on cartesian mesh and does not respect the maximum principle.
- **Perspectives** :
  - Construct schemes which tend to familiar diffusion schemes (MPFA, Hybrid method....)
  - Extent to the M1 model using the Lagrange+remap method.
  - Construction of AP schemes for the models Pn and Sn.

future preprint

[1] Chritophe Buet, Bruno Després, Emmanuel Franck *Design of asymptotic preserving schemes for the telegraph equation on unsctructured meshes*. proceeding Laboratoire Jacques Louis-Lions. UMPC 2010.

Thank you

Introduction

Classical scheme

Nodal schemes

Numerical results

Conclusion

Thank you for your attention