Asymptotic preserving finite volumes scheme for the M_1 model of radiative transfer on unstructured meshes

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Physical context

- **Radiation hydrodynamics** : Interaction between the gas modeled by Euler equations and radiation, modeled by a transport equation.

- Valid method on unstructured meshes is necessary for lagrangian radiative hydrodynamics simulation.

- **Grey transport equation** : $I(t, \mathbf{x}, \mathbf{\Omega}) \geq 0$ the distribution function associated to particles located in $\mathbf{x}$ and with a direction $\mathbf{\Omega}$. We consider the following equation of the form :

$$\frac{1}{c}\partial_t I(t, \mathbf{x}, \mathbf{\Omega}) + \mathbf{\Omega}.\nabla I(t, \mathbf{x}, \mathbf{\Omega}) = \sigma_S(E - I) + \sigma_a S(T, I)$$

  where $E = \int_{S^2} I(t, \mathbf{x}, \mathbf{\Omega}^{'}) d\mathbf{\Omega}^{'}$ the energy, σ_S, σ_a the matter opacity and $S(T, I)$ a coupling term with the matter.

- **Diffusion limit** : Where σ_S or σ_a are high, the transport equation tends to a diffusion equation.

- **Computation cost** : The CPU very important, consequently one needs simplified models.

Model M_1

The non-linear two moments M_1 model, obtained by maximizing the photon entropy, is :

$$\left\{\begin{array}{l} \partial_t E + \frac{1}{\varepsilon}\nabla.\mathbf{F} = 0 \\ \partial_t \mathbf{F} + \frac{1}{\varepsilon}\nabla(\widehat{P}) = -\frac{\sigma}{\varepsilon^2}\mathbf{F}, \end{array}\right. \quad (1)$$

E is the energy, **F** the radiative flux and

$$\widehat{P} = \frac{1}{2}((1-\chi(\mathbf{f}))Id + (3\chi(\mathbf{f})-1)\frac{\mathbf{f}\otimes\mathbf{f}}{\|\mathbf{f}\|})E \in \mathbb{R}^{2\times 2}$$

the radiative pressure. We define $\mathbf{f} = \mid \mathbf{F} \mid / E$ and $\chi(\mathbf{f}) = \dfrac{3+4\mathbf{f}^2}{5+2\sqrt{4-3\mathbf{f}^2}}$.

The M_1 model satisfies

- the diffusion limit, $\varepsilon \to 0$: $\partial_t E - div(\frac{1}{3\sigma}\nabla E) = 0$, First Tools : AP scheme
- the entropy property : $\partial_t S + \frac{1}{\varepsilon} div(\mathbf{Q}) \geq 0$, Second Tools : Reformulation
- the maximum principle : $E > 0$, $\mid \mathbf{f} \mid < 1$, like a dynamic gas system

with

$$S = \frac{E^{3/4}(1-\mid \mathbf{u} \mid^2)}{(3+\mid \mathbf{u} \mid^2)^2}, \ \mathbf{u} = \frac{(3\chi-1)\mathbf{f}}{2\mid \mathbf{f} \mid^2}, \ \mathbf{Q} = \mathbf{u}S$$

Introduction

Definition of the scheme

Properties of the scheme

Numerical results

Other radiation models

Asymptotic Preserving schemes

- The classical Godunov scheme has a consistency error in $O(\frac{\Delta x}{\varepsilon})$.
- It does not converge on coarse grids.

Asymptotic preserving (AP) scheme :

Convergence independently of ε

References in 1D or 2D Cartesian.

- C. Berthon, P. Charrier and B. Dubroca, An HLLC scheme to solve the M_1 model of radiative transfer in two space dimensions. J. Scie. Comput.
- C. Buet, B. Després A gas dynamics scheme for a two moments model of radiative tranfert, SMF.

Study :

Desing of asymptotic preserving schemes to capture the diffusion limit on unstructured meshes

- **Difficulty** : The classical diffusion scheme is not consistent on unstructured meshes.
- **Method to obtain an AP scheme** : We use the Jin-Levermore procedure which consists in incorporating the steady state into the fluxes.
- This method is equivalent to modify the numerical viscosity.

Reformulation like a dynamic gas system

We formulate the M_1 model like a dynamic gas system :

- to use Lagrange+remap nodal scheme and obtain a consistent limit diffusion scheme,
- to use the entropy to preserve the maximum principle.

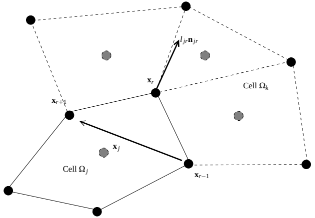
$$\left\{\begin{array}{ll} \partial_t \rho + \frac{1}{\varepsilon} div(\rho \mathbf{u}) = 0 & \text{mass conservation} \\ \partial_t \rho \mathbf{v} + \frac{1}{\varepsilon} div(\rho \mathbf{u} \otimes \mathbf{v}) + \frac{1}{\varepsilon} \nabla q = -\frac{\sigma}{\varepsilon^2} \rho \mathbf{v} & \text{momentum conservation} \\ \partial_t \rho e + \frac{1}{\varepsilon} div(\rho \mathbf{u} e + q \mathbf{u}) = 0 & \text{total conservation energy} \\ \partial_t \rho s + \frac{1}{\varepsilon} div(\rho \mathbf{u} s) \geq 0 & \text{Entropy inequality} \end{array}\right.$$

$\mathbf{F} = \rho \mathbf{v}$ the radiative flux $E = \rho e$ the radiative energy $S = \rho s$.

- $q = \frac{1-\chi}{2} E$
- $\mathbf{u} = \frac{3\chi - 1}{2} \frac{\mathbf{f}}{\mid \mathbf{f} \mid^2}$ with $\mathbf{f} = \frac{\mid \mathbf{v} \mid}{e}$
- The M_1 is independent of the density.
- $\mathbf{F} = uE + qu \qquad \hat{P} = u \otimes \mathbf{F} + qI_d$

Notation and AP schemes

We define the notations for the nodal scheme



l_{jr} and $\mathbf{n}_{jr}$ are the lenght and the normal associated to $\mathbf{X}_r$

- $\mathbf{F}_r$ and $\mathbf{G}_{jr}$ fluxes associated to X_r.
- We define the GLACE viscosity matrix $\widehat{\alpha}_{jr} = l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}$,
- We define the AP viscosity matrix $\widehat{\beta}_{jr} = \frac{V_{jr}}{V_r} \sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$.
- V_r is the control volume associated to the node r. V_{jr} is the fragment of V_r associated to the cell j.

Lagrange+remap scheme

- We use a nodal scheme for the Lagrange step (GLACE scheme) and remap step

$$\begin{cases} \mid \Omega_j \mid \partial_t \rho_j + \frac{1}{\varepsilon}\left(\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})\rho_{jr}\right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j \mathbf{v}_j + \frac{1}{\varepsilon}\left(\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})(\rho \mathbf{v})_{jr}\right) + \frac{1}{\varepsilon}\sum_r \mathbf{G}_{jr} = -\frac{\sigma}{\varepsilon^2}\sum_r k_r \beta_{jr}\mathbf{u}_r \\ \mid \Omega_j \mid \partial_t \rho_j e_j + \frac{1}{\varepsilon}\left(\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})(\rho e)_{jr}\right) + \frac{1}{\varepsilon}\sum_r (\mathbf{u}_r, \mathbf{G}_{jr}) = 0 \end{cases}$$

The lagrangian fluxes

$$\begin{cases} \mathbf{G}_{jr} = l_{jr} q_j \mathbf{n}_{jr} + r_j \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) - \frac{\sigma}{\varepsilon} k_r \hat{\beta}_{jr}\mathbf{u}_r \\ (\sum_j r_j \widehat{\alpha_{jr}} + \frac{\sigma}{\varepsilon} k_r \widehat{\beta}_{jr})\mathbf{u}_r = \sum_j l_{jr} q_j \mathbf{n}_{jr} + r_{jr}\hat{\alpha}_{jr}\mathbf{u}_j \end{cases} \quad (2)$$

The upwind flux is defined by $f_{jr} = 1_{((\mathbf{u}_r, \mathbf{n}_{jr})>0)} f_j + 1_{((\mathbf{u}_r, \mathbf{n}_{jr})<0)} \frac{\sum_j l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) f_j}{\sum_j l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})}$.

$$k_r = \frac{2E_r \mid \mathbf{f_r} \mid^2}{(3\chi - 1)} \quad r_j = \frac{4}{\sqrt{3}}\frac{E_j}{3+|\mathbf{u}_j|^2}$$

Limit regime

- Using a Hilbert expansion we obtain the following non-linear positive limit diffusion scheme

$$\left\{\begin{array}{l} \partial_t E_j(t) + \sum_r \frac{1}{12\sigma}((l_{jr}E_j\mathbf{n}_{jr} - \sigma\widehat{\beta}_{jr}\frac{\tilde{\mathbf{u}}_r}{E_r})\tilde{\mathbf{u}}_r, \mathbf{n}_{jr}) + \frac{1}{4\sigma}\sum_r l_{jr}(\frac{\tilde{\mathbf{u}}_r}{E_r}, \mathbf{n}_{jr})E_{jr} = 0 \\ \sigma\left(\sum_j \hat{\beta}_{jr}\right)\tilde{\mathbf{u}}_r = \sum_j l_{jr}E_j\mathbf{n}_{jr}. \end{array}\right. \quad (3)$$

- E_{jr} is given by the upwind flux, E_r is a mean of E_j around r.
- the vector $\tilde{\mathbf{u}}(\mathbf{x}_r)$ defined by

$$\sigma\left(\sum_j \hat{\beta}_{jr}\right)\tilde{\mathbf{u}}(\mathbf{x}_r) = \sum_j l_{jr}E(\mathbf{x}_j)\mathbf{n}_{jr}$$

  is a first order approximation to $-\frac{1}{\sigma}\nabla E(\mathbf{x}_r)$.
- the second term of (3) is homogeneous to $(E(\mathbf{x}_j) - (\mathbf{x}_r - \mathbf{x}_j, \nabla E(\mathbf{x}_r)))(\frac{\nabla E(\mathbf{x}_r)}{E(\mathbf{x}_r)}, l_{jr}\mathbf{n}_{jr}) \simeq E(\mathbf{x}_r)(\frac{\nabla E(\mathbf{x}_r)}{E(\mathbf{x}_r)}, l_{jr}\mathbf{n}_{jr}) = (\nabla E(\mathbf{x}_r), l_{jr}\mathbf{n}_{jr})$.

Limit regime and spurious modes

- **Remark** : In the reformulation like a dynamic gas system, we obtain a non-linear equation on E.
- Therefore we obtain a non-linear positive diffusion scheme

- The limit diffusion scheme is first order.
- To obtain a second order scheme, we use a MUSCL procedure with flux limiter for keeping the positivity.

- **Remark** : We can use other advection scheme (classical edge upwind scheme, anti-dissipative scheme, etc.).

- This scheme exhibits spurious modes (non convergence for Dirac initial data) on Cartesian mesh.
- With an other definition of the $\mathbf{n}_{jr}$ and l_{jr} we keep the convergence and kill the spurious modes.

Entropy and maximum principle

Lemma

Assuming

$$E_j(t=0) > 0, \ \parallel \mathbf{f}_j(t=0) \parallel < 1 \tag{4}$$

the semi-discrete scheme is entropic

$$\partial_t(\rho_j s_j)(t) + \frac{1}{\varepsilon}\left(\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})(\rho s)_{jr}\right) \geq 0, \text{ for all time} \tag{5}$$

Sketch of proof :

- A classical calculus shows that the scheme is entropic if $\widehat{\beta}_{jr}$ is positive.
- $S = \rho s = \frac{E^{3/4}(1-|\mathbf{u}|^2)}{(3+|\mathbf{u}|^2)^2} > 0$, we show that E and $(1- \mid \mathbf{u} \mid^2)$ are positive and bounded.
- With initial data (4), using classical results for the dynamics system, we prove the lemma for all time.

Remark : $\widehat{\beta}_{jr}$ is positive on a lot of meshes.

The initial condition is the fundamental solution of the heat equation at t=0.001.
The final time is $T_f = 0.011$.
K is a deformation coefficient for the Kershaw mesh.

Scheme	Non-linear		VF5		Linear	
Mesh	order	Nb $E_j < 0$	order	$E_j < 0$	order	$E_j < 0$
Cartesian	1.92	0	2	0	2	0
Rand. quad	1.9	0	0.31	0	1.98	4
Cartesian tri.	2.23	0	2	0	2.	0
Rand tri.	2.16	0	0.96	0	1.32	4453
Kershaw K=1	1.93	0	0	0	2	96664
Kershaw K=1.5	2.02	0	0	0	1.94	224403

TAB.: Order of convergence for the limit diffusion scheme.

We compare the numerical solution of the M_1 scheme with the diffusion solution for different random meshes. $\varepsilon = 0.0001$.

number of cell	40	50	80	100
Error	0.0328	0.02228	0.00901	0.00610

TAB.: Error for different mesh. Order $\simeq 1.85$

- To test the maximum principle, we propose a transport test, $\sigma = 0$ $E(0) = F_x(0) = \mathbf{1}_{[0.4:0.6]^2}$ and $F_y(0) = 0$. The solution is $E(t) = F_x(t) = \mathbf{1}_{[0.4+t:0.6+t]^2}$ and $F_y(t) = 0$.
- The order is computed with two meshes 100*100 and 200*200.

Mesh	order	Nb coef $E < 0$	Nb coef $\parallel \mathbf{f} \parallel > 1$
Cartesian mesh	0.45	0	0
Rand. quad mesh	0.43	0	0
Kershaw K=1	0.4	0	0

TAB.: Order of convergence for the M_1 scheme.

- The theoretical order for discontinous solution is 0.5.
- The numerical viscosity involves the Lagrangian part.

M_1 model with coupling matter

- we introduce the M_1 model coupled with an energy matter equation.

$$\left\{\begin{array}{c} \partial_t E + \frac{1}{\varepsilon}\nabla.\mathbf{F} = \frac{\sigma_a}{\varepsilon^2}(aT^4 - E) \\ \partial_t \mathbf{F} + \frac{1}{\varepsilon}\nabla(\hat{P}) = -\frac{\sigma_a}{\varepsilon^2}\mathbf{F} \\ \rho C_v \partial_t T = \frac{\sigma_a}{\varepsilon^2}(E - aT^4) \end{array}\right. \quad (6)$$

- We define the radiative temperature $E = aT_r^4$ with a the Stefan-Boltzmann constant.
- To treat this model, we use a splitting strategy. The M_1 part is solved with the previous scheme.
- The absorption/emission coupling is solved with an implicit fixed point procedure.
- This strategy preserves $E > 0$ but not $\mid \mathbf{f} \mid < 1$.

Numerical results for M_1 model with coupling

- We consider a test case described by Berthon, Turpault and co workers, We consider a material initially cold and at radiative equilibrium. A heat wave enters the domain and we observe this evolution.

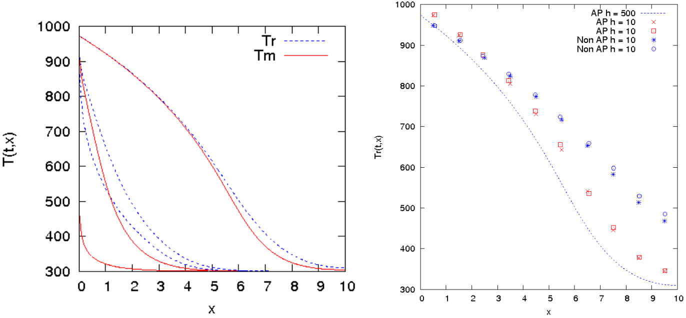


FIG.: At left, the material and radiative temperature for the three times. At right the final solution on Cartesian (cross and point) and random meshes (square and circle) with 10 cells.

- In the first figure we plot the solution on cartesian mesh with 500 cells at the time $t = 1.333 \times 10^{-9}, 1.333 \times 10^{-8}, 1.333 \times 10^{-7}$.

Conclusion

- The reformulation like a dynamics gas system gives a scheme which preserves the maximum principle.
- The scheme is valid for all regimes.
- We obtain a new positive second order non-linear diffusion scheme.

- Nodal asymptotic preserving scheme
  - For the linear P_1 model the classical finite volume scheme is not consistent in the diffusion regime.
  - With a nodal scheme it is easy to obtain AP scheme because the viscosity is consistent.
  - The extension in 3D is natural for the GLACE scheme.
  - **Future works** Find a semi-implicit or implicit time discretization independent to ε.

- Edge asymptotic preserving scheme
  - Modifying the viscosity of the classical upwind we can obtain AP edge schemes for the P_1 model (works with G. Samba and P. Hoch)
  - **Future works** : Construction of edge schemes for the M_1 models for any Eddington tensors.
- Using nodal scheme we haved design asymptotic preserving scheme for other radiation models (P_1, P_n, S_n)

The P_1 model

- This is a two moments linear model for radiation transport

$$\left\{\begin{array}{l}\partial_t E(t) + \dfrac{1}{\varepsilon}\nabla.\mathbf{F} = 0\\ \partial_t \mathbf{F}(t) + \dfrac{1}{\varepsilon}\nabla E = -\dfrac{\sigma_S}{\varepsilon^2}\mathbf{F}\end{array}\right. , \quad \left\{\begin{array}{l} \mid \Omega_j \mid \partial_t E_j(t) + \frac{1}{\varepsilon}\sum_r l_{jr}(\mathbf{F}_r.\mathbf{n}_{jr}) = 0\\ \mid \Omega_j \mid \partial_t \mathbf{F}_j(t) + \frac{1}{\varepsilon}\sum_r G_{jr} = -\frac{\sigma}{\varepsilon^2}\sum_r \widehat{\beta}_{jr} F_r\end{array}\right.$$

with fluxes

$$\left\{\begin{array}{l} G_{jr} = l_{jr}E_j\mathbf{n}_{jr} + \widehat{\alpha}_{jr}(\mathbf{F}_j - \mathbf{F}_r) - \frac{\sigma}{\varepsilon}\widehat{\beta}_{jr}\mathbf{F}_r\\ (\displaystyle\sum_j \widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon}\widehat{\beta}_{jr})\mathbf{F}_r = \sum_j l_{jr}E_j\mathbf{n}_{jr} + \widehat{\alpha}_{jr}\mathbf{F}_j\end{array}\right.$$

- The nodal matrix is positive under sufficiently condition (all the angles inferior to 11 degrees for the triangles)
- We prove that the limit diffusion scheme converges with order one
- The limit diffusion scheme is convergent numerically with order two
- We prove that the implicit scheme is L^2 stable and tends at Δx fixed to the diffusion scheme
- A reformulation gives a semi-implicit local scheme with the CFL condition $\Delta t < \frac{\Delta x^2}{\sigma} + \varepsilon\Delta x$ (The CFL condition to the upwind scheme is $\Delta t < \frac{\varepsilon^2}{\sigma} + \varepsilon\Delta x$)

The P_n model

- The P_n model is obtained by projection of the transport equation on the harmonics spherical base.
- We rewrite these models as a Friedrich's system

$$\partial_t \mathbf{u} + \frac{1}{\varepsilon} A \partial_x \mathbf{u} + \frac{1}{\varepsilon} B \partial_y \mathbf{u} = -\frac{\sigma}{\varepsilon^2} R \mathbf{u}$$

- For all the P_n models, R is diagonal with $R_{11} = 0$ et $R_{ii} = 1$ $(i \neq 0)$
- We can split the matrix A and B as

$$A = P_{1,x} + A^{'},\ B = P_{1,y} + B^{'}$$

with $A^{'}_{0,j} = 0,\ A^{'}_{i,0} = 0, B^{'}_{0,j} = 0,\ B^{'}_{i,0} = 0.$

- To obtain an AP scheme for the P_n equation, we use the asymptotic preserving scheme for the P_1 part and classical scheme for the other part (Rusanov, upwind scheme).
- Theoretically : the first moment is in (1), second moment in $O(\varepsilon)$ and the other moment $O(\varepsilon^2)$.
- Numerically : the first moment is in (1), second moment in $O(\Delta x)$ and the other moment $O(\Delta x \varepsilon)$.

Numerical results for the P_3 model

- Results for diffusion limit. Same test case that for the diffusion scheme

Mesh	0.001	0.0001
Cartesian	1.81	1.97
Random quad.	1.85	1.98
Triang reg.	1.9	1.99
Random trig.	1.37	1.37
Kershaw K=1	1.85	1.97

- Fundamental solution for the P_3 equation.

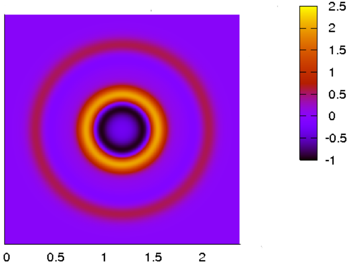


FIG. : Solution fundamental