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Cell-Centered Asymptotic preserving schemes for linear transport on unstructured meshes

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4. AP schemes for angular approximations of the transport equation

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Introduction

Physical context and objectives

- **IFC simulations** : interaction between the gas modelized by two-temperatures Euler equations and the radiation modelized by a linear transport equation.

Physical context and objectives

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- **IFC simulations** : interaction between the gas modelized by two-temperatures Euler equations and the radiation modelized by a linear transport equation.
- **Grey linear transport equation** : $f(\mathbf{x},\Omega,t) \geq 0$ the distribution function associated to the particles (photons or neutrons) located in $\mathbf{x}$, with a direction Ω. We consider the following equation :

$$\partial_t f(t,\mathbf{x},\Omega) + \Omega.\nabla f(t,\mathbf{x},\Omega) = \sigma \int_{S^2} (f(t,\mathbf{x},\Omega') - f(t,\mathbf{x},\Omega)) d\Omega'.$$

- **Diffusion limit** : for $t >> 1$ and $\sigma >> 1$, the transport equation, tends toward the following diffusion equation

$$\partial_t E(t,\mathbf{x}) - \text{div}\left(\frac{1}{\sigma}\nabla E(t,\mathbf{x})\right) = 0,$$

with $E(t,\mathbf{x}) = \int_{\Omega} f(t,\mathbf{x},\Omega)d\Omega$ and $\mathbf{F}(t,\mathbf{x}) = \int_{\Omega} \Omega f(t,\mathbf{x},\Omega)d\Omega$.

- **Computation cost** : The CPU cost is important, consequently one needs simplified models.

Approximation of transport equation

- Simplified hyperbolic models, depend only on spaces variables.
- **Simplified models** :
  - P_n models : we develop the transport equation on the basis of spherical harmonics.
  - S_n models : we use a quadrature formula to discretize the collision operator.
  - M_n models : non-linear P_n models where the closure is obtained by minimizing the entropy.

P_1 model :

$$\begin{cases} \partial_t E + \dfrac{1}{\varepsilon} \operatorname{div} \mathbf{F} = 0, \\ \partial_t \mathbf{F} + \dfrac{1}{3\varepsilon} \nabla E = -\dfrac{\sigma}{\varepsilon^2} \mathbf{F}. \end{cases}$$

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- **Adapted numerical methods** : asymptotic preserving (AP) finite volume schemes capturing the diffusion limit.

Aims :

Design of cell-centered finite volume schemes for the simplified models capturing the diffusion limit on unstructured meshes.

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AP schemes in 1D and difficulties in 2D

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Staggered and centered schemes

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- The staggered scheme is also asymptotic preserving :

$$\left\{\begin{array}{l} \dfrac{E_j^{n+1}-E_j^n}{\Delta t}+\dfrac{F_{j+\frac{1}{2}}-F_{j-\frac{1}{2}}}{\varepsilon\Delta x}=0, \\ \dfrac{F_{j+\frac{1}{2}}^{n+1}-F_{j+\frac{1}{2}}^n}{\Delta t}+\dfrac{E_{j+1}-E_j}{\varepsilon\Delta x}=-\dfrac{\sigma}{\varepsilon^2}F_{j+\frac{1}{2}}. \end{array}\right.$$

- But the staggered scheme does not preserve the maximum principle $E+F>0$, $E-F>0$ in the transport regime.

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CEA Constraints

- Design of schemes which are equal to the upwind scheme when $\sigma=0$ to preserve the transport properties (maximum principle, entropy etc).
- Design of cell-centered schemes. Indeed the hydrodynamic and diffusion codes are coupled with the transport problem use cell-centered methods.

AP schemes : design and examples

Hyperbolic heat equation :

$$\begin{cases} \partial_t E + \frac{1}{\varepsilon}\partial_x F = 0, \\ \partial_t F + \frac{1}{\varepsilon}\partial_x E = -\frac{\sigma}{\varepsilon^2} F, \end{cases} \qquad \Longrightarrow \partial_t E - \partial_x \frac{1}{\sigma}\partial_x E = 0.$$

- Consistency error of the **upwind** scheme
  - for the first equation : $O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right)$,
  - for the second equation : $O\left(\frac{\Delta x^2}{\varepsilon} + \Delta x + \Delta t\right)$.
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Jin-Levermore scheme

- Principle of design : we introduce the steady state $\partial_x E = -\frac{\sigma}{\varepsilon}F$ in the fluxes.
- We write the relations

$$\begin{cases} E(x_j) = E(x_{j+\frac{1}{2}}) + (x_j - x_{j+\frac{1}{2}})\partial_x E(x_{j+\frac{1}{2}}), \\ E(x_{j+1}) = E(x_{j+\frac{1}{2}}) + (x_{j+1} - x_{j+\frac{1}{2}})\partial_x E(x_{j+\frac{1}{2}}). \end{cases}$$

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Plugging the discrete equivalent of these relations in the fluxes.

$$\begin{cases} F_j + E_j = F_{j+\frac{1}{2}} + E_{j+\frac{1}{2}}, \\ F_{j+1} - E_{j+1} = F_{j+\frac{1}{2}} - E_{j+\frac{1}{2}}. \end{cases}$$

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We obtain

$$\begin{cases} F_j + E_j = F_{j+\frac{1}{2}} + E_{j+\frac{1}{2}} + \frac{\sigma \Delta x}{2\varepsilon} F_{j+\frac{1}{2}}, \\ F_{j+1} - E_{j+1} = F_{j+\frac{1}{2}} - E_{j+\frac{1}{2}} + \frac{\sigma \Delta x}{2\varepsilon} F_{j+\frac{1}{2}}. \end{cases}$$

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We obtain

$$\left\{\begin{array}{l}E_{j+\frac{1}{2}}=\left(\frac{E_j+E_{j+1}}{2}+\frac{F_j-F_{j+1}}{2}\right),\\ F_{j+\frac{1}{2}}=M\left(\frac{F_j+F_{j+1}}{2}+\frac{E_j-E_{j+1}}{2}\right).\end{array}\right.$$

with $M=\frac{2\varepsilon}{2\varepsilon+\sigma\Delta x}$.

AP schemes in 1D

- The Jin-Levermore scheme is

$$\left\{\begin{array}{l} \frac{E_j^{n+1}-E_j^n}{\Delta t} + M\frac{F_{j+1}^n-F_{j-1}^n}{2\varepsilon\Delta x} - M\frac{E_{j+1}^n-2E_j^n+E_{j-1}^n}{2\varepsilon\Delta x} = 0, \\ \frac{F_j^{n+1}-F_j^n}{\Delta t} + \frac{E_{j+1}^n-E_{j-1}^n}{2\varepsilon\Delta x} - \frac{F_{j+1}^n-2F_j^n+F_{j-1}^n}{2\varepsilon\Delta x} + \frac{\sigma}{\varepsilon^2}F_j^n = 0. \end{array}\right. \quad (1)$$

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Gosse-Toscani scheme

- Principle of design : localization of source terms at the interfaces, which induces a stationary wave in the Riemann problem.

$$\begin{cases} \frac{E_j^{n+1}-E_j^n}{\Delta t} + M\frac{F_{j+1}^n - F_{j-1}^n}{2\varepsilon\Delta x} - M\frac{E_{j+1}^n - 2E_j^n + E_{j-1}^n}{2\varepsilon\Delta x} = 0, \\ \frac{F_j^{n+1}-F_j^n}{\Delta t} + M\frac{E_{j+1}^n - E_{j-1}^n}{2\varepsilon\Delta x} - M\frac{F_{j+1}^n - 2F_j^n + F_{j-1}^n}{2\varepsilon\Delta x} + M\frac{\sigma}{\varepsilon^2}F_j^n = 0. \end{cases} \quad (2)$$

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  - for the first equation : $O\left(\varepsilon\Delta x + \Delta x^2 + \Delta t\right)$,
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- **Remark** : The Jin-Levermore scheme (1) with the discretization of the source term $\frac{1}{2}\left(F_{j+1/2} + F_{j-1/2}\right)$ is equal to the Gosse-Toscani scheme.

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- **Remark** : For the two schemes, the numerical viscosity gives the diffusion limit scheme on coarse grids ($\frac{\Delta x}{\varepsilon} >> 1$).

Analysis of AP schemes : modified equations

- To understand the behaviour of the scheme, we use the modified equations method.
- We assume that $||\partial_{t^a,x^b} E|| \leq C_{a,b}$ and $||\partial_{t^a,x^b} F|| \leq \varepsilon C_{a,b}$.
- The modified equation associated to the Upwind scheme is

$$\left\{ \begin{array}{c} \partial_t E + \frac{1}{\varepsilon}\partial_x F - \frac{\Delta x}{2\varepsilon}\partial_{xx} E = 0, \\ \partial_t F + \frac{1}{\varepsilon}\partial_x E - \frac{\Delta x}{2\varepsilon}\partial_{xx} F = -\frac{\sigma}{\varepsilon^2} F. \end{array} \right. \quad (3)$$

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- We plug the relation $\varepsilon\partial_x E + O(\varepsilon^2) = -\sigma F$ in the first equation of (4), we obtain the diffusion limit

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$$\partial_t E - \frac{1}{\sigma}\partial_{xx} E - \frac{\Delta x}{2\varepsilon}\partial_{xx} E = 0.$$

- The scheme does not capture the diffusion limit.
- The modified equation associated to the Gosse-Toscani scheme is

$$\begin{cases} \partial_t E + M\frac{1}{\varepsilon}\partial_x F - M\frac{\Delta x}{2\varepsilon}\partial_{xx} E = 0, \\ \partial_t F + M\frac{1}{\varepsilon}\partial_x E - M\frac{\Delta x}{2\varepsilon}\partial_{xx} F = -M\frac{\sigma}{\varepsilon^2} F. \end{cases} \quad (4)$$

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- We plug the relation $\varepsilon\partial_x E + O(\varepsilon^2) = -\sigma F$ in the first equation of (4), we obtain the diffusion limit

$$\partial_t E - \frac{1}{\sigma}\partial_{xx} E - \frac{\Delta x}{2\varepsilon}\partial_{xx} E = 0.$$

- The scheme does not capture the diffusion limit.
- The modified equation associated to the Gosse-Toscani scheme is

$$\left\{ \begin{array}{c} \partial_t E + M\frac{1}{\varepsilon}\partial_x F - M\frac{\Delta x}{2\varepsilon}\partial_{xx} E = 0, \\ \partial_t F + M\frac{1}{\varepsilon}\partial_x E - M\frac{\Delta x}{2\varepsilon}\partial_{xx} F = -M\frac{\sigma}{\varepsilon^2} F. \end{array} \right. \quad (4)$$

- We plug the relation $M\varepsilon\partial_x E + O(\varepsilon^2) = -M\sigma F$ in the first equation of (4)

$$\partial_t E - \frac{M}{\sigma}\partial_{xx} E - \frac{1-M}{\sigma}\partial_{xx} E = 0$$

Analysis of AP schemes : modified equations

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- To understand the behaviour of the scheme, we use the modified equations method.
- We assume that $||\partial_{t^a,x^b} E|| \leq C_{a,b}$ and $||\partial_{t^a,x^b} F|| \leq \varepsilon C_{a,b}$.
- The modified equation associated to the Upwind scheme is

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- We plug the relation $M\varepsilon\partial_x E + O(\varepsilon^2) = -M\sigma F$ in the first equation of (4)
- On fine grid $\frac{\Delta x}{2\varepsilon} << 1$, $M \to 1$ and the diffusion coefficient is correct.

$$\partial_t E - \frac{1}{\sigma}\partial_{xx} E = 0$$

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- On coarse grid $\frac{\Delta x}{2\varepsilon} >> 1$, $M \rightarrow 0$ and the diffusion coefficient is correct.

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- We plug the relation $M\varepsilon\partial_x E + O(\varepsilon^2) = -M\sigma F$ in the first equation of (4)
- On intermediate grid $\frac{\Delta x}{2\varepsilon} = 0(1)$, the diffusion coefficient is correct.

$$\partial_t E - \frac{1}{\sigma}\partial_{xx} E = 0$$

Analysis of AP schemes : modified equations

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- To understand the behaviour of the scheme, we use the modified equations method.
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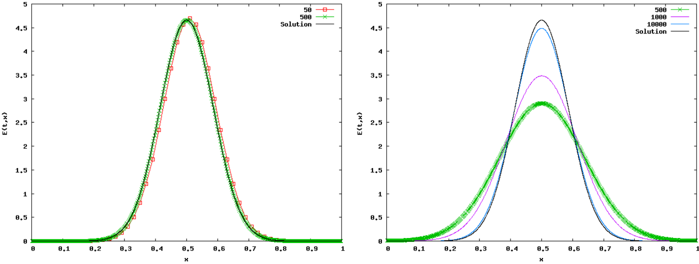
- The scheme captures the diffusion limit (idem for the Jin-Levermore scheme).

Numerical exemple for AP schemes in 1D

- To validate the AP method, we propose the following test case. The parameters are $\sigma = 1$, $\varepsilon = 0.001$. The initial data is given by $E(0,x) = G(x)$ with $G(x)$ a gaussian and $F(0,x) = 0$.



In left results for AP scheme, in right results for upwind scheme

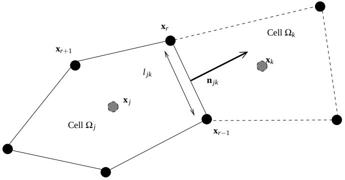
Schemes	Error L^1	Error L^2	Real time	User time
AP scheme, 50 cells	0.0065	0.0110	0m0.054s	0m0.157s
AP scheme, 500 cells	0.0001	0.00018	0m15.22s	1m1.680s
Upwind scheme, 500 cells	0.445	0.647	0m24.317s	1m36.80s
Upwind scheme, 10000 cells	0.0366	0.059	1485m4.26s	5140m56.11s

- The « asymptotic preserving » scheme is significantly more accurate than the upwind scheme.
- In pratice a classical scheme for this type of problem is unusable.

Notations for classical finite volume schemes

- We introduce the notations for the edge formulation of finite volume methods.

Notations



- l_{jk} and $\mathbf{n}_{jk}$ are the lenght and the normal associated to the edge $\partial\Omega_{jk}$.
- $\sum_k l_{jk}\mathbf{n}_{jk} = \mathbf{0}$.
- $(\mathbf{F}_{jk}, \mathbf{n}_{jk})$ and E_{jk} are the fluxes associated to the edge $\partial\Omega_{jk}$.

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2D Extension : difficulties

- **Jin-Levermore method** : modify the upwind schemes, plugging the steady states into the fluxes. We use a Taylor expansion :

$$\begin{cases} E(\mathbf{x}_j) \simeq E(\mathbf{x}_{jk}) + (\mathbf{x}_j - \mathbf{x}_{jk}, \nabla E(\mathbf{x}_{jk})), \\ E(\mathbf{x}_k) \simeq E(\mathbf{x}_{jk}) + (\mathbf{x}_k - \mathbf{x}_{jk}, \nabla E(\mathbf{x}_{jk})). \end{cases}$$

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2D Extension : difficulties

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  Discrete equivalent

$$\left\{ \begin{array}{l} E_j \simeq E_{jk} - \frac{\sigma}{\varepsilon}(\mathbf{F}_{jk}, \mathbf{x}_j - \mathbf{x}_{jk}), \\ E_k \simeq E_{jk} - \frac{\sigma}{\varepsilon}(\mathbf{F}_{jk}, \mathbf{x}_k - \mathbf{x}_{jk}). \end{array} \right.$$

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  Plugging the previous relations in the acoustic solver, we obtain :

$$\begin{cases} (\mathbf{F}_j, \mathbf{n}_{jk}) + E_j = (\mathbf{F}_{jk}, \mathbf{n}_{jk}) + E_{jk}, \\ (\mathbf{F}_k, \mathbf{n}_{jk}) - E_k = (\mathbf{F}_{jk}, \mathbf{n}_{jk}) - E_{jk}. \end{cases}$$

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- To solve this system we need a geometrical assumption.
- **Assumption** : The mesh satisfy the Delaunay condition, therefore :

$$(\mathbf{x}_{jk} - \mathbf{x}_j) = d_{jk}\mathbf{n}_{jk} \text{ et } (\mathbf{x}_{jk} - \mathbf{x}_k) = -d_{kj}\mathbf{n}_{jk}.$$

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Asymptotic limit of Jin-Levermore scheme : Two-Points-Flux scheme

$$|\Omega_j| \frac{E_j^{n+1} - E_j^n}{\triangle t} - \frac{1}{\sigma}\sum_k l_{jk} \frac{E_k^n - E_j^n}{d(\mathbf{x}_j, \mathbf{x}_k)} = 0.$$

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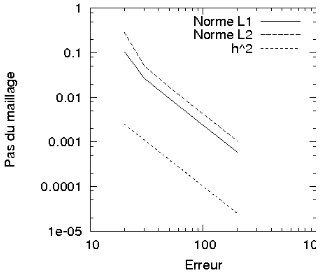
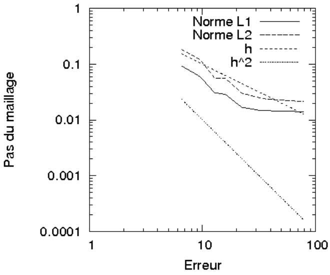
Non convergence of Two-Points-Flux diffusion scheme

- Two-Points-Flux scheme does not converge on distorted meshes.
  Test case : we take as initial condition the fundamental solution of the heat equation at the time $t = 0.001$, final time t_f=0.010.
- Convergence results on Cartesian mesh and Random quadrangular mesh.

Cartesian mesh



Random mesh



- To our knowledge, there were no AP schemes on unstructured meshes before this study.

AP schemes for hyperbolic heat equation in 2D

Notations for nodal finite volume schemes

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Idea :

use nodal formulation of finite volume methods introduced in Lagrangian hydrodynamics to discretize the wave equation and couple this scheme with the Jin-Levermore method.

Notations for nodal finite volume schemes

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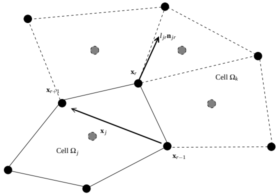
use nodal formulation of finite volume methods introduced in Lagrangian hydrodynamics to discretize the wave equation and couple this scheme with the Jin-Levermore method.

- We introduce the nodal formulation

Notations



- $l_{jr}\mathbf{n}_{jr} = \begin{pmatrix} -y_{r-1} + y_{r+1} \\ x_{r-1} - x_{x+1} \end{pmatrix}$.
- $\sum_j l_{jr}\mathbf{n}_{jr} = \sum_r l_{jr}\mathbf{n}_{jr} = \mathbf{0}$.
- $\mathbf{F}_r$ and $\mathbf{En}_{jr}$ are the fluxes associated to the node $\mathbf{X}_r$.
- V_r is the control volume around the node $\mathbf{X}_r$.

Construction of the nodal scheme

GLACE AP scheme

$$\begin{cases} \mid \Omega_j \mid \partial_t E_j(t) + \dfrac{1}{\varepsilon} \displaystyle\sum_r l_{jr}(\mathbf{F}_r, \mathbf{n}_{jr}) = 0, \\ \mid \Omega_j \mid \partial_t \mathbf{F}_j(t) + \dfrac{1}{\varepsilon} \displaystyle\sum_r l_{jr} \mathbf{E}\mathbf{n}_{jr} = \mathbf{S}_j. \end{cases}$$

- Classical nodal solver :

$$\begin{cases} \mathbf{E}\mathbf{n}_{jr} - E_j \mathbf{n}_{jr} = \widehat{\alpha}_{jr}(\mathbf{F}_j - \mathbf{F}_r), \\ \displaystyle\sum_j l_{jr} \mathbf{E}\mathbf{n}_{jr} = \mathbf{0}, \end{cases}$$

with $\widehat{\alpha}_{jr} = \mathbf{n}_{jr} \otimes \mathbf{n}_{jr}$.

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- Modified nodal solver : plugging $\nabla E = -\frac{\sigma}{\varepsilon} F$ in the fluxes

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GLACE AP scheme

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- Source term discretization : (1) $\mathbf{S}_j = -\frac{\sigma}{\varepsilon^2} \mid \Omega_j \mid \mathbf{F}_j$, (2) $\mathbf{S}_j = -\frac{\sigma}{\varepsilon^2} \sum_r \widehat{\beta}_{jr} \mathbf{F}_r$

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Properties of the nodal schemes

- In 1D, the scheme with the source term (1) is equal to the Jin-Levermore scheme,
- the scheme with the source term (2) is equal to the Gosse-Toscani scheme.

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- The scheme with the discretization (2) of the source term is equal to

$$\begin{cases} \mid \Omega_j \mid \dfrac{E_j^{n+1} - E_j}{\triangle t} + \dfrac{1}{\varepsilon} \sum_r l_{jr} (M_r \mathbf{F}_r, \mathbf{n}_{jr}) = 0, \\ \mid \Omega_j \mid \dfrac{\mathbf{F}_j^{n+1} - \mathbf{F}_j}{\triangle t} + \dfrac{1}{\varepsilon} \sum_r l_{jr} \mathbf{En}_{jr} = -\dfrac{1}{\varepsilon} \left(\sum_r l_{jr} \widehat{\alpha}_{jr} (\widehat{I_d} - M_r) \right) \mathbf{F}_j^{n+1}. \end{cases}$$

with

$$\begin{cases} \mathbf{En}_{jr} - E_j \mathbf{n}_{jr} = \widehat{\alpha}_{jr} M_r (\mathbf{F}_j - \mathbf{F}_r), \\ \left(\sum_j l_{jr} \widehat{\alpha}_{jr} \right) \mathbf{F}_r = \sum_j l_{jr} E_j \mathbf{n}_{jr} + \sum_j l_{jr} \widehat{\alpha}_{jr} \mathbf{F}_j. \end{cases}$$

$$M_r = \left(\sum_j l_{jr} \widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon} \sum_j l_{jr} \widehat{\beta}_{jr} \right)^{-1} \left(\sum_j l_{jr} \widehat{\alpha}_{jr} \right).$$

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- the scheme with the source term (2) is equal to the Gosse-Toscani scheme.
- The scheme with the discretization (2) of the source term is equal to

$$\begin{cases} \mid \Omega_j \mid \dfrac{E_j^{n+1}-E_j}{\triangle t} + \dfrac{1}{\varepsilon}\sum_r l_{jr}(M_r\mathbf{F}_r, \mathbf{n}_{jr}) = 0, \\ \mid \Omega_j \mid \dfrac{\mathbf{F}_j^{n+1}-\mathbf{F}_j}{\triangle t} + \dfrac{1}{\varepsilon}\sum_r l_{jr}\mathbf{E}\mathbf{n}_{jr} = -\dfrac{1}{\varepsilon}\left(\sum_r l_{jr}\widehat{\alpha}_{jr}\left(\widehat{I}_d - M_r\right)\right)\mathbf{F}_j^{n+1}. \end{cases}$$

with

$$\begin{cases} \mathbf{E}\mathbf{n}_{jr} - E_j\mathbf{n}_{jr} = \widehat{\alpha}_{jr}M_r(\mathbf{F}_j - \mathbf{F}_r), \\ \left(\sum_j l_{jr}\widehat{\alpha}_{jr}\right)\mathbf{F}_r = \sum_j l_{jr}E_j\mathbf{n}_{jr} + \sum_j l_{jr}\widehat{\alpha}_{jr}\mathbf{F}_j. \end{cases}$$

$$M_r = \left(\sum_j l_{jr}\widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon}\sum_j l_{jr}\widehat{\beta}_{jr}\right)^{-1}\left(\sum_j l_{jr}\widehat{\alpha}_{jr}\right).$$

- The matrix M_r generalize the coefficient M introduced in the 1D schemes

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Properties of the nodal schemes

- In 1D, the scheme with the source term (1) is equal to the Jin-Levermore scheme,
- the scheme with the source term (2) is equal to the Gosse-Toscani scheme.
- The scheme with the discretization (2) of the source term is equal to

$$\begin{cases} \mid \Omega_j \mid \dfrac{E_j^{n+1}-E_j}{\triangle t} + \dfrac{1}{\varepsilon}\sum_r l_{jr}(M_r\mathbf{F}_r,\mathbf{n}_{jr}) = 0, \\ \mid \Omega_j \mid \dfrac{\mathbf{F}_j^{n+1}-\mathbf{F}_j}{\triangle t} + \dfrac{1}{\varepsilon}\sum_r l_{jr}\mathbf{En}_{jr} = -\dfrac{1}{\varepsilon}\left(\sum_r l_{jr}\widehat{\alpha}_{jr}(\widehat{I}_d - M_r)\right)\mathbf{F}_j^{n+1}. \end{cases}$$

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- The matrix M_r generalize the coefficient M introduced in the 1D schemes
- We implicit the source term to obtain a scheme with a CFL condition independent of ε.

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Diffusion limit

Proposition

When ε tends to zero the limit scheme is :

$$\begin{cases} \mid \Omega_j \mid \partial_t E_j(t) - \sum_r l_{jr}(\mathbf{F}_r, \mathbf{n}_{jr}) = 0, \\ \sigma A_r \mathbf{F}_r = \sum_j l_{jr} E_j \mathbf{n}_{jr}, \quad A_r = -\sum_j l_{jr} \mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j). \end{cases}$$

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$$\| e(t) \|^2_{L^2(\Omega)} = \sum_j \mid \Omega_j \mid (E_j(t) - E(\mathbf{x}_j, t))^2.$$

$$\| \mathbf{f}(t) \|^2_{L^2([0,t]x\Omega)} = \int_0^t \sum_r \mid V_r \mid (\mathbf{F}_r(t) - \nabla E(\mathbf{x}_r, t))^2.$$

Theorem

If the matrix A_r^S satisfies $A_r^S \geq \alpha V_r$ with α a constant then the semi-discrete diffusion scheme is convergent for all time $T > 0$ with the estimation :

$$\| E(t) \|_{L^2(\Omega)} + \| \mathbf{f}(t) \|_{L^2([0,t]x\Omega)} \leq C(T)h.$$

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- The nodal AP scheme with the discretization (2) of the source term is stable in norm L^2.

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AP schemes for angular approximations of the transport equation

Friedrichs systems with stiff source terms

- We introduce the Friedrichs system with stiff source term

$$\partial_t \mathbf{u} + \frac{1}{\varepsilon} A \partial_x \mathbf{u} + \frac{1}{\varepsilon} B \partial_y \mathbf{u} = -\frac{\sigma}{\varepsilon^2} R\mathbf{u},\ \mathbf{u} \in \mathbb{R}^n$$

- A, B, R are symmetric matrices and R is positive.

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Lemma

We note $\mathbf{E}_i$ the eigenvectors of R with $\operatorname{Ker} R = \operatorname{vect}(\mathbf{E}_1 \ldots \mathbf{E}_p)$. There are two particular vectors associated to the eigenvalues λ_{p+1}, λ_{p+2}. We assume that

$$\begin{cases} A\mathbf{E}_i = \gamma_i \mathbf{E}_{p+1},\ \forall i \in \{1..p\}, \\ B\mathbf{E}_i = \delta_i \mathbf{E}_{p+2},\ \forall i \in \{1..p\}, \end{cases}$$

therefore $((\mathbf{u}, \mathbf{E}_1), \ldots, (\mathbf{u}, \mathbf{E}_p))$ tends to $\mathbf{v} \in \mathbb{R}^p$ when ε tends to zero with

$$\partial_t \mathbf{v} - \frac{1}{\lambda_{i_1}\sigma} K_1 \partial_{xx} \mathbf{v} - \frac{1}{\lambda_{i_2}\sigma} K_2 \partial_{yy} \mathbf{v} = \mathbf{0},$$

and K_1, K_2 symmetric positive matrices.

P_n and S_n models

- S_n **models** : (discrete ordinate methods) which approximate the scattering operator with a quadrature formula.
- **Properties of S_n models** : A, B diagonal matrices, $\dim \operatorname{Ker} R = 1$, R symmetric positive for the variable $u_i = \sqrt{w_i} f(\Omega_i)$.
- w_i the quadrature weight, Ω_i the quadrature speed and f the solution of the transport equation.

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- P_n **models** : projection of the transport equation on the spherical harmonics basis.
- **Properties of P_n models** : symmetrizable system, R is defined by $R_{11} = 0$ and $R_{ii} = 1$ $(i \neq 0)$.

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Proposition

The P_n and S_n models satisfy the previous assumption of structure.

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Decomposition of Friedrichs systems

Proposition

Then we write a P_n or S_n model in the eigenvectors basis of R, we obtain

$$\partial_t \mathbf{v} + \frac{1}{\varepsilon} A^{'} \partial_x \mathbf{v} + \frac{1}{\varepsilon} B^{'} \partial_y \mathbf{v} = -\frac{\sigma}{\varepsilon^2} D\mathbf{v} \tag{5}$$

with D a diagonal matrix defined by $D_{11} = 0$ et $D_{ii} = 1$ $(i \neq 0)$. If the assumption of structure is satisfied, then

$$A^{'} = P_{1,x} + A^{''},\ B^{'} = P_{1,y} + B^{''},$$

with $A^{''}_{0,j} = 0$, $A^{''}_{i,0} = 0$, $B^{''}_{0,j} = 0$, $B^{''}_{i,0} = 0$.

- The matrices $P_{1,x}$, $P_{1,y}$ are the matrices associated to the P_1 system.

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- The matrices $P_{1,x}$, $P_{1,y}$ are the matrices associated to the P_1 system.
- **Conclusion** : The P_n and S_n models can be split between a P_1 system and a system which does not play a role in the diffusion regime.
- **Numerical method (micro-macro decomposition ?)** : Split the system, discretize the P_1 system with an AP scheme and the other system with a classical scheme.

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Final algorithm

- Decomposition algorithm for the system

$$\partial_t \mathbf{u} + \frac{1}{\varepsilon} A_1 \partial_x \mathbf{u} + \frac{1}{\varepsilon} A_2 \partial_y \mathbf{u} = -\frac{\sigma}{\varepsilon^2} R\mathbf{u}. \quad (6)$$

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  - **First step** : We write the system (6) in the eigenvectors basis of R to obtain

$$\partial_t \mathbf{v} + \frac{1}{\varepsilon} A_1' \partial_x \mathbf{v} + \frac{1}{\varepsilon} A_2' \partial_y \mathbf{v} = -\frac{\sigma}{\varepsilon^2} D\mathbf{u}, \tag{7}$$

  with $\mathbf{v} = Q^t \mathbf{u}$, $A_1' = Q^t A_1 Q$ et $A_2' = Q^t A_2 Q$.

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    with $\mathbf{v} = Q^t \mathbf{u}$, $A_1^{'} = Q^t A_1 Q$ et $A_2^{'} = Q^t A_2 Q$.
  - **Second step** : We split the diagonalized system (7). We obtain

$$\partial_t \mathbf{v} + \frac{1}{\varepsilon}\left(P_{1,x}\partial_x \mathbf{v} + P_{1,y}\partial_y \mathbf{v}\right) + \frac{1}{\varepsilon}\left(A_1^{''}\partial_x \mathbf{v} + A_2^{''}\partial_y \mathbf{v}\right) = -\frac{\sigma}{\varepsilon^2} D\mathbf{v}. \quad (8)$$

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  - **Third step** : The system (9) is discretized with an AP scheme

$$\partial_t \mathbf{v} + \frac{1}{\varepsilon} (P_{1,x} \partial_x \mathbf{v} + P_{1,y} \partial_y \mathbf{v}) = -\frac{\sigma}{\varepsilon^2} D^{'} \mathbf{v}, \tag{9}$$

    with $D^{'}$ defined by $D^{'}_{22} = D^{'}_{33} = 1$ et $D^{'}_{ii \neq 22, ii \neq 33}$.

Final algorithm

- Decomposition algorithm for the system

$$\partial_t \mathbf{u} + \frac{1}{\varepsilon} A_1 \partial_x \mathbf{u} + \frac{1}{\varepsilon} A_2 \partial_y \mathbf{u} = -\frac{\sigma}{\varepsilon^2} R\mathbf{u}. \quad (6)$$

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    with $D^{'}$ defined by $D_{22}^{'} = D_{33}^{'} = 1$ et $D_{ii\neq 22, ii\neq 33}^{'}$.
  - **Fourth step** : The system (10) is discretized with a classical scheme (Upwind, Rusanov)

$$\partial_t \mathbf{v} + \frac{1}{\varepsilon}\left(A_1^{''}\partial_x \mathbf{v} + A_2^{''}\partial_y \mathbf{v}\right) = -\frac{\sigma}{\varepsilon^2} D^{''} \mathbf{v}, \quad (10)$$

    with $D^{''}$ defined by $D_{11}^{''} = D_{22}^{''} = D_{33}^{''} = 0$ et $D_{ii}^{''} = 1\ i \geq 4$.

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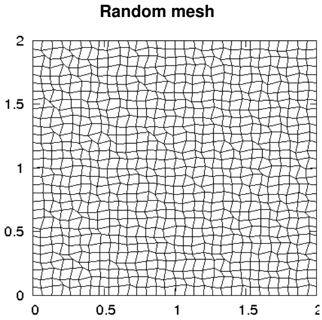
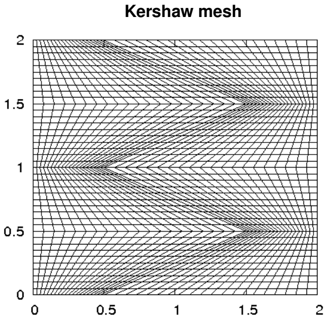
Numerical results

Examples of unstructured meshes

Two classical examples of unstructured meshes.





Numerical results for the P_1 system

- In transport regime ($\varepsilon = O(1)$ and $\sigma = O(1)$), at first order the scheme converges.
- **Diffusion regime** : The initial data is given by the fundamental solution of the heat equation at the time $t = 0.001$. Final time $t_f = 0.010$.

Mesh/ ε	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-7}$
Cartesian 60-120 cells	1.8	2	2.	2.
Cartesian 80-160 cells	1.75	1.97	2	2
Cartesian 120-240 cells	1.7	1.95	2	2
Random quad. 60-120 cells	1.83	2.	2	2
Random quad. 80-160 cells	1.96	2.2	2.2	2.2
Random quad. 120-240 cells	1.73	1.92	2	2
Kershaw 60-120 cells	2	2.1	2.1	2.1
Kershaw 80-160 cells	1.87	1.97	2	2
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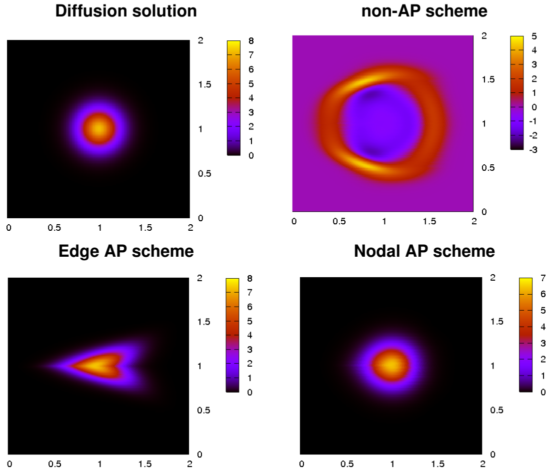
- The scheme converges on triangular meshes with an order between 1 and 2.
- The error between the diffusion solution and the solution of the P_1 model is homogeneous to $O(\varepsilon)$.
- For $\frac{\Delta x}{\varepsilon} = O(1)$ the order decreases. Indeed we compare the numerical solution of P_1 system and the exact diffusion solution.

AP schemes Vs non-AP schemes

- We solve the P_1 model with previous test case and $\varepsilon = 0.001$. The results for the hyperbolic schemes are computed on Kershaw mesh.



Numerical results for Friedrichs systems

- Diffusion regime : previous test case.
- The order of convergence is computed with two meshes (14400 and 57600 cells) :

Mesh/ ε	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
Cartesian	1.8	1.95
Random. quad.	1.85	2
Trig. reg.	1.9	2
Random. trig.	1.35	1.35
Kershaw	1.85	1.95

TAB.: Order for the P_3 numerical solution

Mesh/ ε	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
Cartesian	1.80	1.95
Random. quad.	1.85	2
Trig. reg.	1.9	2
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TAB.: Order for the S_2 numerical solution

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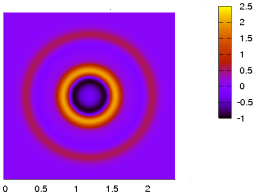
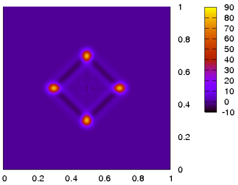
TAB.: Order for the S_2 numerical solution

- Transport test case : fundamental solution

Fundamental solution of P_3



Fundamental solution of S_2



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M_1 model

The non-linear two moments M_1 model, obtained by maximizing the photon entropy, is :

$$\begin{cases} \partial_t E + \frac{1}{\varepsilon}\nabla.\mathbf{F} = 0 \\ \partial_t \mathbf{F} + \frac{1}{\varepsilon}\nabla(\widehat{P}) = -\frac{\sigma}{\varepsilon^2}\mathbf{F}, \end{cases} \quad (11)$$

E is the energy, **F** the radiative flux and

$$\widehat{P} = \frac{1}{2}((1-\chi(\mathbf{f}))Id + (3\chi(\mathbf{f})-1)\frac{\mathbf{f}\otimes\mathbf{f}}{\|\mathbf{f}\|})E \in \mathbb{R}^{2\times 2}$$

the radiative pressure.

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the radiative pressure. We define $\mathbf{f} = |\mathbf{F}|/E$ and $\chi(\mathbf{f}) = \dfrac{3+4\mathbf{f}^2}{5+2\sqrt{4-3\mathbf{f}^2}}$.

The M_1 model satisfies

- the diffusion limit, $\varepsilon \to 0 : \partial_t E - div(\frac{1}{3\sigma}\nabla E) = 0$, First Tools : AP scheme
- the entropy property : $\partial_t S + \frac{1}{\varepsilon} div(\mathbf{Q}) \geq 0$, Second Tools : Reformulation
- the maximum principle : $E > 0$, $|\mathbf{f}| < 1$, like a dynamic gas system

with

$$S = \frac{E^{3/4}(1-|\mathbf{u}|^2)}{(3+|\mathbf{u}|^2)^2},\ \mathbf{u} = \frac{(3\chi-1)\mathbf{f}}{2|\mathbf{f}|^2},\ \mathbf{Q} = \mathbf{u}S.$$

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the radiative pressure. We define $\mathbf{f} = \mid \mathbf{F} \mid / E$ and $\chi(\mathbf{f}) = \dfrac{3+4\mathbf{f}^2}{5+2\sqrt{4-3\mathbf{f}^2}}$.

The M_1 model satisfies

- the diffusion limit, $\varepsilon \to 0 : \partial_t E - div(\frac{1}{3\sigma}\nabla E) = 0$, First Tools : AP scheme
- the entropy property : $\partial_t S + \frac{1}{\varepsilon} div(\mathbf{Q}) \geq 0$, Second Tools : Reformulation
- the maximum principle : $E > 0$, $\mid \mathbf{f} \mid < 1$, like a dynamic gas system

with

$$S = \frac{E^{3/4}(1-\mid \mathbf{u} \mid^2)}{(3+\mid \mathbf{u} \mid^2)^2}, \quad \mathbf{u} = \frac{(3\chi-1)\mathbf{f}}{2\mid \mathbf{f} \mid^2}, \quad \mathbf{Q} = \mathbf{u}S.$$

Idea : we formulate the M_1 model like a dynamic gas system :

- to use Lagrange+remap nodal scheme and obtain a consistent limit diffusion scheme,
- to use the entropy to preserve the maximum principle.

Numerical method

New formulation

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon} div(\rho \mathbf{u}) = 0 & \text{mass conservation} \\ \partial_t \rho \mathbf{v} + \frac{1}{\varepsilon} div(\rho \mathbf{u} \otimes \mathbf{v}) + \frac{1}{\varepsilon} \nabla q = -\frac{\sigma}{\varepsilon^2} \rho \mathbf{v} & \text{momentum conservation} \\ \partial_t \rho e + \frac{1}{\varepsilon} div(\rho \mathbf{u} e + q \mathbf{u}) = 0 & \text{total conservation energy} \\ \partial_t \rho s + \frac{1}{\varepsilon} div(\rho \mathbf{u} s) \geq 0 & \text{Entropy inequality} \end{cases}$$

$\mathbf{F} = \rho \mathbf{v}$ the radiative flux $E = \rho e$ the radiative energy $S = \rho s$.

- $q = \frac{1-\chi}{2} E$, $\mathbf{u} = \frac{3\chi - 1}{2} \frac{\mathbf{f}}{|\,\mathbf{f}\,|^2}$
- The M_1 is independent of the density.
- $\mathbf{F} = uE + qu \qquad \hat{P} = u \otimes \mathbf{F} + qI_d$

Numerical method

New formulation

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon} div(\rho \mathbf{u}) = 0 & \text{mass conservation} \\ \partial_t \rho \mathbf{v} + \frac{1}{\varepsilon} div(\rho \mathbf{u} \otimes \mathbf{v}) + \frac{1}{\varepsilon} \nabla q = -\frac{\sigma}{\varepsilon^2} \rho \mathbf{v} & \text{momentum conservation} \\ \partial_t \rho e + \frac{1}{\varepsilon} div(\rho \mathbf{u} e + q \mathbf{u}) = 0 & \text{total conservation energy} \\ \partial_t \rho s + \frac{1}{\varepsilon} div(\rho \mathbf{u} s) \geq 0 & \text{Entropy inequality} \end{cases}$$

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Numerical discretization

- We use the Lagrange+remap nodal GLACE scheme coupled with the Jin-Levermore method.
- We use a second order advection scheme for the remap part (when ε is small).

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Introduction

AP schemes in 1D and difficulties in 2D

AP schemes for hyperbolic heat equation in 2D

AP schemes for angular approximations of the transport equation

Numerical results

Conclusion

Numerical method

New formulation

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon} div(\rho \mathbf{u}) = 0 & \text{mass conservation} \\ \partial_t \rho \mathbf{v} + \frac{1}{\varepsilon} div(\rho \mathbf{u} \otimes \mathbf{v}) + \frac{1}{\varepsilon} \nabla q = -\frac{\sigma}{\varepsilon^2} \rho \mathbf{v} & \text{momentum conservation} \\ \partial_t \rho e + \frac{1}{\varepsilon} div(\rho \mathbf{u} e + q \mathbf{u}) = 0 & \text{total conservation energy} \\ \partial_t \rho s + \frac{1}{\varepsilon} div(\rho \mathbf{u} s) \geq 0 & \text{Entropy inequality} \end{cases}$$

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- We use the Lagrange+remap nodal GLACE scheme coupled with the Jin-Levermore method.
- We use a second order advection scheme for the remap part (when ε is small).

Properties

- The scheme is entropic and preserve the maximum principle.
- The scheme is AP with a second order positive limit scheme.
- We can define a semi-implicit variant with a CFL condition independant of ε.

Conclusion

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Conclusion

- **Conclusion**
  - We have designed and studied AP schemes for hyperbolic heat equation valid on unstructured meshes ([1] – [2]).
  - Using the previous decomposition we have obtained AP schemes for S_n and P_n models ([3]).
  - Using the proximity between the M_1 model and the Euler equations, we have proposed an AP, entropic and positive scheme for M_1 model (non-linear radiative transfer model) based on a Lagrange+remap method ([4]).

Publications

1 C. Buet, B. Després, E. Franck *Design of asymptotic preserving schemes fore hyperbolic heat equation on unstructured meshes*. Numerish Mathematik, Online.

2 E. Franck, P. Hoch, G. Samba, P. Navarro *An asymptotic preserving scheme for P_1 model using classical diffusion schemes on unstructured polygonal meshes*. ESAIM : Proceedings, October 2011, Vol. 32, p. 56-75.

3 C. Buet, B. Després, E. Franck *AP schemes for Friedrichs systems with stiff relaxation on unstructured meshes. Applications to the angular discretization in transport*. In redaction.

4 C. Buet, B. Després, E. Franck *An asymptotic preserving scheme with the maximum principle for the M_1 model on distorted meshes* . Submitted.

Ongoing works and future works

- **Ongoing works**
  - Design of asymptotic preserving and positive scheme for S_n models based on edge finite volume scheme. We propose to couple a "even-odd" formulation with a nonlinear diffusion scheme (LMP scheme).
  - Extension of the nodal scheme for Euler and Shallow water equations with gravity and friction using a Lagrange+remap approach.
- **Future works**
  - Theoretical study of the AP schemes for the P_1 model and the Euler equations (with C. Buet, B. Després).
  - Extension of the nodal scheme on unstructured conical meshes.
  - Design of asymptotic preserving schemes for multi-groups models.

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Thank you

Thank you for your attention