Cemracs project: Resolution of P1 model on general meshes using asymptotic preserving cell-centered schemes

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CEA-DAM, DIF

26 August 2010

Physical and mathematical backgrounds

- **Physical background** Inertial confinement fusion. Compression of a gaz capsule with a set of laser beams.

- **Radiation hydrodynamics simulation** Interaction between the gas modeled by Euler equations and the photons by transport equation.

- **Radiation** $I(t,x,\mathbf{v}) \geq 0$ The radiative intensity associated to particules located in $\mathbf{x}$ with a velocity $\mathbf{v}$. We consider the following equation of the form

$$\partial_t I(t,x,\mathbf{v}) + \mathbf{v}.\nabla I(t,x,\mathbf{v}) = \sigma_S \int_{S^2} \left(I(t,x,\mathbf{v}^{'}) - I(t,x,\mathbf{v})\right) dv^{'} + \sigma_a(B(T) - I),$$

- **Diffusion limit** The transport equation has, in some regimes, the property to tend towards an equation of diffusion. For example the limit for a long time and $\sigma_S >> \sigma_a$.

$$\partial_t E(t,x) - \frac{1}{\sigma_S + \sigma_a} \triangle E(t,x) = \sigma_a(B(T) - E(t,x)),$$

with $E(t,x) = \int_{S^2} I(t,x,\mathbf{v}) dv$, and $F(t,x) = \int_{S^2} \mathbf{v} I(t,x,\mathbf{v}) dv$.

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Physical and mathematical backgrounds

- **Simplified models** : The solution of transport equation depends on too many variables. We can solve simplified hyperbolic models (P^n, S^n, M^1) with the same diffusion limit.

- **Example P^1 model**

$$\begin{cases} \varepsilon\partial_t E + \nabla.(\mathbf{F}) = 0 \\ \varepsilon\partial_t(\mathbf{F}) + \nabla E + \dfrac{\sigma}{\epsilon}\mathbf{F} = 0 \end{cases}$$

- **Numerical methods** Asymptotic preserving finite volume schemes to capture the diffusion limit.

Aim

Construct asymptotic preserving schemes for the P^1 model on unstructured polygonal meshes given by Lagrangian hydrodynamics.

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Cemracs project

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First step

Improve the diffusion Breil-Maire scheme ([3]) to make it consistent.
Implementation and numerical study of glace nodal scheme (diffusion and P1) in the general unstructured gopp code and comparison with the other diffusion schemes.

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Derivation of a scheme for the P1 equation having the diffusion Breil-Maire scheme in the diffusion limit. Comparison with the P1 glace scheme.

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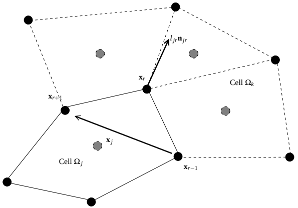
Notations

- We define the notation for the nodal scheme.

nodal notations



Notice that $l_{jr}\mathbf{n}_{jr}$ is equal to the the half of the vector that starts at $\mathbf{x}_{r-1}$ and finish at $\mathbf{x}_{r+1}$. The center of the cell is an arbitrary point inside the cell. $\Rightarrow$ $\mathbf{F}_r$ and E_{jr} are the fluxes associated to the vertex X_r

Definition

The diffusion scheme is :

$$\left\{ \begin{array}{l} \mid V_j \mid \dfrac{E_j^{n+1} - E_j^n}{\triangle t} + \sum_r l_{jr}(\mathbf{F_r}.\mathbf{n_{jr}}) = 0 \\ \sigma(\sum_j l_{jr}\mathbf{n_{jr}} \otimes (\mathbf{x_r} - \mathbf{x_j}))\mathbf{F_r} = \sum_j l_{jr} E_j \mathbf{n_{jr}}. \end{array} \right.$$

We define the following errors :

$$\parallel e(t) \parallel_{L^2(\Omega)} = (\sum_j \mid V_j \mid (E_j(t) - E(x_j, t))^2)^{\frac{1}{2}}$$

$$\parallel f(t) \parallel_{L^2([0,t]x\Omega)} = (\int_0^t \sum_r \mid V_r \mid (\mathbf{F}_r(t) - \nabla E(x_r, t))^2)^{\frac{1}{2}}$$

Theorem

We assume that $E \in W^{3,\infty}(\Omega)$. If there exists a constant α such that $A_r \geq \alpha \mid V_r \mid$, then the semi-discrete diffusion scheme is convergent for all time T > 0,

$$\parallel e(t) \parallel_{L^2(\Omega)} + \parallel f(t) \parallel_{L^2([0,t]x\Omega)} = C(T)h$$

Diffusion glace scheme

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Diffusion maire scheme and its consistant variant

$$\begin{cases} \dfrac{E_j - E_j^n}{\Delta t} V_j = -\sum_r \dfrac{1}{2}(L_{r-1,r}\Phi^j_{r-1/2,r} + L_{r,r+1}\Phi^j_{r,r+1/2}) \\ \qquad \frac{1}{2} L_k(\Phi_k^{k-1} + \Phi_k^k) = 0 \\ \begin{pmatrix} \Phi^j_{r-1/2,r} \\ \Phi^j_{r,r+1/2} \end{pmatrix} = -\dfrac{1}{2\omega_r^j} T_r^j \begin{pmatrix} L_{r-1,r}(\bar{E}_{r-1/2,r} - E_j) \\ L_{r,r+1}(\bar{E}_{r,r+1/2} - E_j) \end{pmatrix} \end{cases}$$

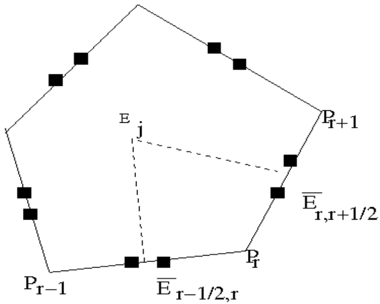

FIG.: Breil-maire scheme stencil

To obtain T_r^j

$$T = \begin{pmatrix} 2\omega_k \dfrac{n_A.OC^\perp}{\beta L_k} & 2\omega_k \dfrac{n_A.OA^\perp}{\beta L_{k+1}} \\ 2\omega_k \dfrac{n_C.OC^\perp}{\beta L_k} & 2\omega_k \dfrac{n_C.OA^\perp}{\beta L_{k+1}} \end{pmatrix}$$

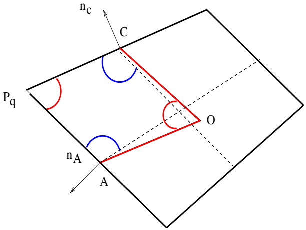

FIG.:

If we assume the quadrilatere is a parallelogram, the classical symmetric scheme is obtained.

P1 AP glace scheme

- Ongoing works done by PhD student Emmanuel Franck at the CEA directed by Christophe Buet and Bruno Després.
- **Idea** Use the nodal scheme "GLACE" constructed for linearized Euler equations analog to P1 model and use this scheme with the Jin-Levermore method to construct a nodal asymptotic preserving scheme.

$$\begin{cases} \mid V_j \mid \dfrac{E_j^{n+1} - E_j^n}{\triangle t} + \dfrac{1}{\varepsilon} \sum_r l_{jr}(\mathbf{F}_r . \mathbf{n}_{jr}) = 0 \\ \mid V_j \mid \dfrac{\mathbf{F}_j^{n+1} - \mathbf{F}_j^n}{\triangle t} + \dfrac{1}{\varepsilon} \sum_r l_{jr} E_{jr} \mathbf{n}_{jr} = -\dfrac{\sigma}{\varepsilon^2} \mathbf{F}_j \end{cases}$$

with the fluxes

$$\begin{cases} E_{jr} = E_j + (\mathbf{F_j} - \mathbf{F_r}, \mathbf{n_{jr}}) - \dfrac{\sigma}{\varepsilon}(\mathbf{F_r}, (\mathbf{x_r} - \mathbf{x_j})) \\ \sum_j l_{jr}(\mathbf{n_{jr}} \otimes \mathbf{n_{jr}} + \dfrac{\sigma}{\varepsilon}(\mathbf{n_{jr}} \otimes (\mathbf{x_r} - \mathbf{x_j})))\mathbf{F_r} = \sum_j (l_{jr} E_j \mathbf{n}_{jr} + l_{jr}(\mathbf{n_{jr}} \otimes \mathbf{n_{jr}})\mathbf{F_j}) \end{cases}$$

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P1 consistant maire scheme

$$
\begin{cases}
\epsilon \dfrac{E_j - E_j^n}{\Delta t} V_j = -\sum_r \dfrac{1}{2}(L_{r-1,r}\Phi^j_{r-1/2,r} + L_{r,r+1}\Phi^j_{r,r+1/2}) \\
\epsilon \dfrac{\mathbf{F_j} - \mathbf{F_j^n}}{\Delta t} V_j + \dfrac{\sigma}{\epsilon}\mathbf{F_j}\mathrm{V_j} = -\sum_{\mathrm{r}} \dfrac{1}{2}(\mathrm{L_{r-1,r}}\mathbf{n^j_{r-1,r}}\bar{\mathrm{E}}_{\mathrm{r-1/2,r}} + \mathrm{L_{r,r+1}}\mathbf{n^j_{r,r+1}}\bar{\mathrm{E}}_{\mathrm{r,r+1/2}}) \\
\dfrac{1}{2} L_k(\Phi_k^{k-1} + \Phi_k^k) = 0 \\
\bar{E}_{r-1/2,r} - E_j + (\Phi^j_{r-1/2,r} - \mathbf{F_j} \cdot \mathbf{n^j_{r-1,r}}) = \mathbf{0} \\
\bar{E}_{r,r+1/2} - E_j + (\Phi^j_{r,r+1/2} - \mathbf{F_j} \cdot \mathbf{n^j_{r,r+1}}) = \mathbf{0}
\end{cases}
$$

Jin Levermore procedure

Replace E_j by $E_j + (\bar{E}_{r-1/2,r} - E_j)$ where $(\bar{E}_{r-1/2,r} - E_j)$ is calculated using the relations of the diffusion scheme

$$\begin{cases} \epsilon \dfrac{E_j - E_j^n}{\Delta t} V_j = -\sum_r \dfrac{1}{2}(L_{r-1,r}\Phi^j_{r-1/2,r} + L_{r,r+1}\Phi^j_{r,r+1/2}) \\ \epsilon \dfrac{\mathbf{F_j} - \mathbf{F_j^n}}{\Delta t} V_j + \dfrac{\sigma}{\epsilon}\mathbf{F_j}\mathrm{V_j} = -\sum_{\mathrm{r}} \dfrac{1}{2}(\mathrm{L_{r-1,r}}\mathbf{n^j_{r-1,r}}\bar{\mathrm{E}}_{\mathrm{r-1/2,r}} + \mathrm{L_{r,r+1}}\mathbf{n^j_{r,r+1}}\bar{\mathrm{E}}_{\mathrm{r,r+1/2}}) \\ \dfrac{1}{2} L_k(\Phi_k^{k-1} + \Phi_k^k) = 0 \end{cases}$$

$$\begin{pmatrix} \Phi^j_{r-1/2,r} \\ \Phi^j_{r,r+1/2} \end{pmatrix} = -\begin{pmatrix} 1 + (S^j_{r,\epsilon})^{xx} & (S^j_{r,\epsilon})^{xy} \\ (S^j_{r,\epsilon})^{yx} & 1 + (S^j_{r,\epsilon})^{yy} \end{pmatrix}^{-1} \begin{pmatrix} \bar{E}_{r-1/2,r} - E_j - \mathbf{F_j} \cdot \mathbf{n^j_{r-1,r}} \\ \bar{E}_{r,r+1/2} - E_j - \mathbf{F_j} \cdot \mathbf{n^j_{r,r+1}} \end{pmatrix}$$

Where $S_{r,\epsilon}$ is a modification to T_r^j dependent of ε and σ

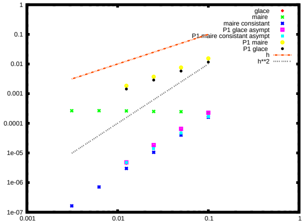
Numericals result for diffusion

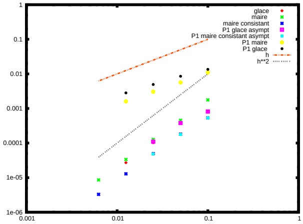
- We solve the heat equation with $E(t=0)=0$, Neumann boundary condition and a source term $Q(x) = (\frac{cos(1)-1}{sin(1)})cos(x) + sin(x)$. The solution is

$$E_{stat}(x) = -x + (\frac{cos(1)-1}{sin(1)})cos(x) + sin(x) + 0.5$$

Result of convergence on Kershaw mesh and Random quadrangular mesh.

Random mesh **Kershaw**

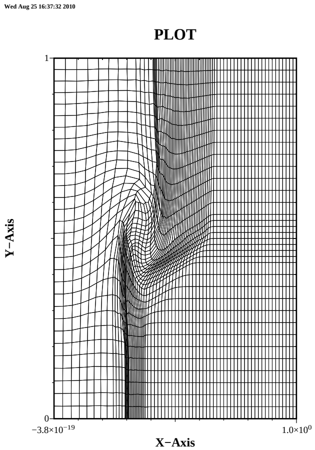


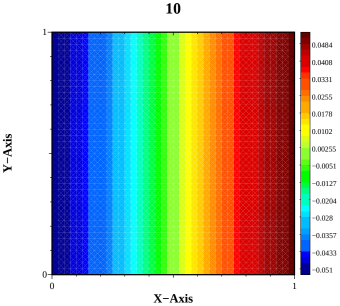


Numericals result for diffusion on polygonal meshe I

Polygonal mesh and solution for a cartesian mesh.

Mesh **solution on cartesian mesh**

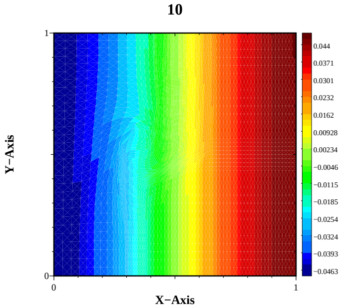


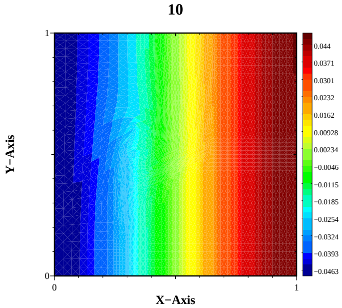


Numericals result for diffusion on polygonal mesh II

Result on polygonal mesh for diffusion glace scheme and modified Maire scheme.

Glace scheme **Maire modified scheme**

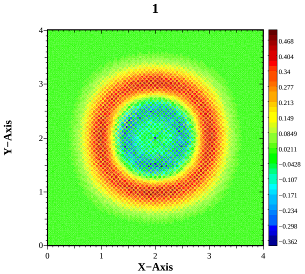


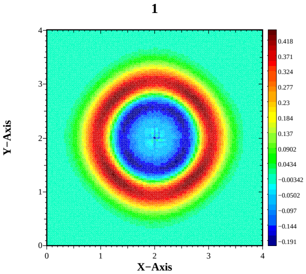


Numericals result P1 I

- We solve the P1 equation with $E(t=0)=\delta$. Result on random quadrangular mesh for the two schemes.

P1 glace scheme **P1 maire scheme**





Numericals result P1 II

. Comparaison with the exact solution in 1D.

1D solution