Design and analysis of cell-centered finite volume schemes in the diffusion limit on distorted meshes

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Outline

Physical and mathematical context

Stiff hyperbolic systems

- **Hyperbolic systems with stiff source terms**:

$$\partial_t \mathbf{U} + \frac{1}{\varepsilon}\partial_x F(\mathbf{U}) + \frac{1}{\varepsilon}\partial_y G(\mathbf{U}) = -\frac{\sigma}{\varepsilon^2} R(\mathbf{U}),\ \mathbf{U} \in \mathbb{R}^n$$

with $\varepsilon \in]0,1]$ et $\sigma > 0$.

- **Diffusion limit** for $\varepsilon \to 0$:

$$\partial_t \mathbf{V} - \mathrm{div}\,(K(\nabla \mathbf{V}, \sigma)) = 0, \quad \mathbf{V} \in \mathrm{Ker}\, R.$$

- Applications: biology, neutronic, fluids dynamic, plasma physic, Radiative Hydrodynamics for IFC (Hydrodynamic + linear transport (talk of C. Hauck)).

Asymptotic preserving schemes

- P_1 Model:

$$\begin{cases} \partial_t E + \frac{1}{\varepsilon}\partial_x F = 0, \\ \partial_t F + \frac{1}{\varepsilon}\partial_x E = -\frac{\sigma}{\varepsilon^2} F, \end{cases} \longrightarrow \partial_t E - \partial_x \left(\frac{1}{\sigma} \partial_x E \right) = 0. \quad (1)$$

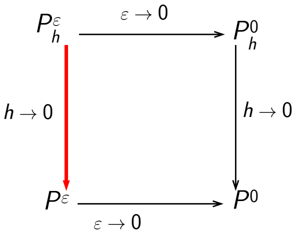


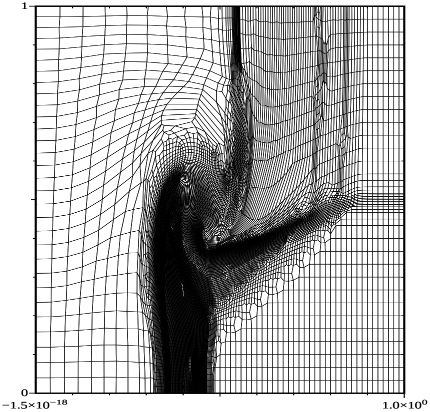
Figure: The AP diagram

- Consistency of **Godunov**-type schemes: $O(\frac{\Delta x}{\varepsilon} + \Delta t)$.
- CFL condition : $\Delta t(\frac{1}{\Delta x \varepsilon} + \frac{\sigma}{\varepsilon^2}) \leq 1$.
- Consistency of AP schemes: $O(\Delta x + \Delta t)$.
- CFL condition: $\Delta t \leq \Delta x^2 + \varepsilon \Delta x$
- AP vs non AP schemes: Very important reduction of CPU cost.

- Godunov-type AP schemes are obtained plugging the source terms in the fluxes (S. Jin-D. Levermore, L.Gosse and al, C. Berthon, R. Turpault and al ...).
- The problem of consistency for the Godunov-type schemes come from to the numerical viscosity homogeneous to $O(\frac{\Delta x}{\varepsilon})$.

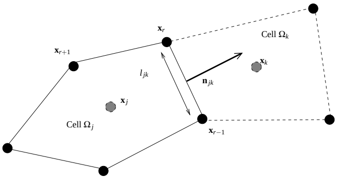
Why unstructured meshes ?

- **Applications** : coupling between radiation and hydrodynamic.
- **Some hydrodynamic codes**: multi-material Langrangian or ALE cell-centered schemes.
- Example of mesh obtained with ALE code (right).
- **Aim**: Design and analyze cell-centered AP schemes for linear transport on general meshes.



2D Asymptotic preserving schemes

- **2D Classical extension Jin-Levermore scheme**: modify the upwind fluxes (1D fluxes write in the normal direction) plugging the steady states into the fluxes (JL method).

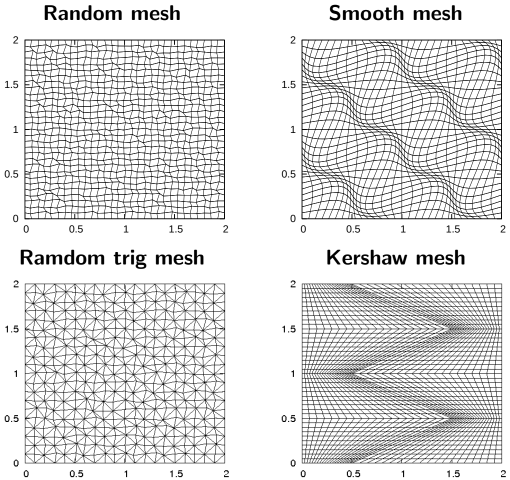


- l_{jk} and $\mathbf{n}_{jk}$ are the lenght and the normal associated to the edge $\partial\Omega_{jk}$.

Asymptotic limit of AP scheme: $$\mid \Omega_j \mid \partial_t E_j(t) - \frac{1}{\sigma}\sum_k l_{jk}\frac{E_k^n - E_j^n}{d(\mathbf{x}_j, \mathbf{x}_k)} = 0.$$

- $||P_h^0 - P_h|| \rightarrow 0$ only under very strong geometrical conditions.
- **Non convergence** of 2D AP schemes on general meshes $\forall\varepsilon$.

Examples of unstructured meshes

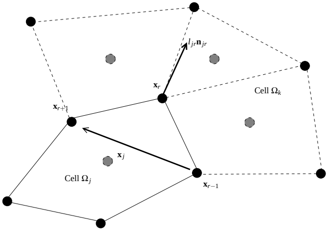


AP schemes on unstructured meshes for the P_1 model

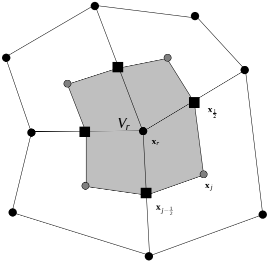
2D AP schemes: Principle and notations

Idea: Nodal formulation of finite volume methods (fluxes localized at the node) for the P_1 model + the Jin-Levermore method.

Geometrical quantities



Control volume



- Geometrical quantities defined by $l_{jr}\mathbf{n}_{jr} = \nabla_{\mathbf{x}_r}|\Omega_j|$ (Left).
- $\sum_j l_{jr}\mathbf{n}_{jr} = \sum_r l_{jr}\mathbf{n}_{jr} = \mathbf{0}$.
- V_r control volume (right).

2D AP schemes

Nodal AP schemes:

$$\begin{cases} \mid \Omega_j \mid \partial_t E_j(t) + \frac{1}{\varepsilon}\sum_r l_{jr}(\mathbf{F}_r, \mathbf{n}_{jr}) = 0, \\ \mid \Omega_j \mid \partial_t \mathbf{F}_j(t) + \frac{1}{\varepsilon}\sum_r l_{jr}\mathbf{En}_{jr} = \mathbf{S}_j. \end{cases}$$

- Classical nodal fluxes:

$$\begin{cases} \mathbf{En}_{jr} - l_{jr}E_j\mathbf{n}_{jr} = \widehat{\alpha}_{jr}(\mathbf{F}_j - \mathbf{F}_r), \\ \sum_j \mathbf{En}_{jr} = \mathbf{0}, \end{cases}$$

  with $\widehat{\alpha}_{jr} = l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}$.

- New fluxes obtained plugging the steady state $\nabla E = -\frac{\sigma}{\varepsilon}F$:

$$\begin{cases} \mathbf{En}_{jr} - l_{jr}E_j\mathbf{n}_{jr} = \widehat{\alpha}_{jr}(\mathbf{F}_j - \mathbf{F}_r) - \frac{\sigma}{\varepsilon}\widehat{\beta}_{jr}\mathbf{F}_r, \\ \left(\sum_j \widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon}\sum_j \widehat{\beta}_{jr}\right)\mathbf{F}_r = \sum_j l_{jr}E_j\mathbf{n}_{jr} + \sum_j \widehat{\alpha}_{jr}\mathbf{F}_j. \end{cases}$$

  with $\widehat{\beta}_{jr} = l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$.

- Source term: (1) $\mathbf{S}_j = -\frac{\sigma}{\varepsilon^2} \mid \Omega_j \mid \mathbf{F}_j$, (2) $\mathbf{S}_j = -\frac{\sigma}{\varepsilon^2}\sum_r \widehat{\beta}_{jr}\mathbf{F}_r$, $\quad \sum_r \widehat{\beta}_{jr} = \hat{I}_d|\Omega_j|$.

Time discretization for AP schemes

- Formulation of the scheme with the source term (2) and local semi-implicit scheme:

$$\left\{\begin{array}{l} \mid \Omega_j \mid \dfrac{E_j^{n+1}-E_j^n}{\triangle t}+\dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}(M_r\mathbf{F}_r,\mathbf{n}_{jr})=0,\\ \mid \Omega_j \mid \dfrac{\mathbf{F}_j^{n+1}-\mathbf{F}_j^n}{\triangle t}+\dfrac{1}{\varepsilon}\displaystyle\sum_r \mathbf{En}_{jr}=-\dfrac{1}{\varepsilon}\left(\sum_r \widehat{\alpha}_{jr}(\widehat{I_d}-M_r)\right)\mathbf{F}_j^{n+1}. \end{array}\right.$$

with

$$\left\{\begin{array}{l} \mathbf{En}_{jr}-l_{jr}E_j\mathbf{n}_{jr}=\widehat{\alpha}_{jr}M_r(\mathbf{F}_j-\mathbf{F}_r),\\ \left(\sum_j \widehat{\alpha}_{jr}\right)\mathbf{F}_r=\sum_j l_{jr}E_j\mathbf{n}_{jr}+\sum_j \widehat{\alpha}_{jr}\mathbf{F}_j. \end{array}\right.$$

$$M_r=\left(\sum_j \widehat{\alpha}_{jr}+\frac{\sigma}{\varepsilon}\sum_j \widehat{\beta}_{jr}\right)^{-1}\left(\sum_j \widehat{\alpha}_{jr}\right)$$

- The semi-implicit scheme is stable on a CFL condition independent to ε (numerically).
- The implicite scheme is stable.

Assumptions for the proof

Geometrical assumptions

- $\left(\mathbf{u}, \left(\sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)\right) \mathbf{u}\right) \geq \alpha V_r(\mathbf{u},\mathbf{u})$,
- $\left(\mathbf{u}, \left(\sum_r l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}\right) \mathbf{u}\right) \geq \beta h(\mathbf{u},\mathbf{u})$.
- $\left(\mathbf{u}, \left(\sum_j l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}\right) \mathbf{u}\right) \geq \gamma h(\mathbf{u},\mathbf{u})$.

Sufficient condition for triangles: all the angles must be bigger than 12 degrees.

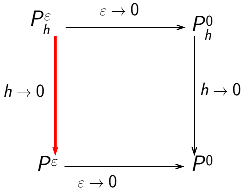
Regularity assumptions and initial data

- $\mathbf{F}(t=0,\mathbf{x}) = -\frac{\varepsilon}{\sigma}\nabla E(t=0,\mathbf{x})$
- Regularity for exact solutions: $\mathbf{V}(t,\mathbf{x}) \in W^{3,\infty}(\Omega)$ and $\mathbf{V}(t=0,\mathbf{x}) \in H^3(\Omega)$
- Regularity for numerical initial solutions: $\mathbf{V}_h(t=0,\mathbf{x}) \in L^2(\Omega)$

Uniform convergence: principle

- Naive convergence estimate : $||P_h^\varepsilon - P^\varepsilon||_{naive} \leq C\varepsilon^{-b}h^c$
- **Idea**: Use intermediate estimates and triangular inequality (Jin-Levermore-Golse).

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq \min(||P_h^\varepsilon - P^\varepsilon||_{naive}, ||P_h^\varepsilon - P_h^0|| + ||P_h^0 - P^0|| + ||P^\varepsilon - P^0||)$$



- intermediate estimates :
  - $||P^\varepsilon - P^0|| \leq C\varepsilon^a$,
  - $||P_h^0 - P^0|| \leq Ch^d$,
  - $||P_h^\varepsilon - P_h^0|| \leq C\varepsilon^e$,
  - $d > c$, $e = a$.

- We obtain:

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq \min(\varepsilon^{-b}h^c, \varepsilon^a + Ch^d + C\varepsilon^e))$$

.

- Comparing ε and $\varepsilon_{threshold} = h^{\frac{ac}{a+b}}$ we obtain the final estimate:

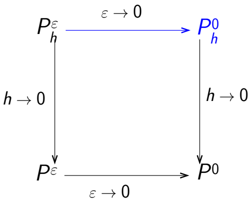
$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq h^{\frac{ac}{a+b}}$$

.

Limit diffusion scheme

Limit diffusion scheme (P_h^0):

$$
\begin{cases}
\mid \Omega_j \mid \partial_t E_j(t) - \sum_r l_{jr}(\mathbf{F}_r, \mathbf{n}_{jr}) = 0, \\
\sum_r l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}\mathbf{F}_j = \sum_r l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}\mathbf{F}_r, \\
\sigma A_r \mathbf{F}_r = \sum_j l_{jr} E_j \mathbf{n}_{jr}, \quad A_r = -\sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j).
\end{cases}
$$

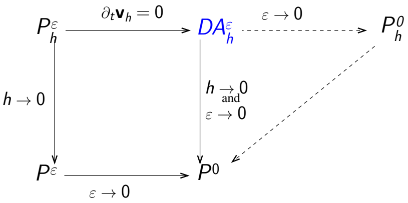


- **Problem** for the estimation of $||P_h^\varepsilon - P_h^0||$.
- We obtain $||P_h^\varepsilon - P_h^0|| \leq C\frac{\varepsilon}{h}$.

Limit diffusion scheme

Limit diffusion scheme (P_h^0):

$$\begin{cases} \mid \Omega_j \mid \partial_t E_j(t) - \sum_r l_{jr}(\mathbf{F}_r, \mathbf{n}_{jr}) = 0, \\ \sum_r l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}\mathbf{F}_j = \sum_r l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}\mathbf{F}_r, \\ \sigma A_r \mathbf{F}_r = \sum_j l_{jr} E_j \mathbf{n}_{jr}, \quad A_r = -\sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j). \end{cases}$$



- **Problem** for the estimation of $||P_h^\varepsilon - P_h^0||$.
- We obtain $||P_h^\varepsilon - P_h^0|| \leq C\frac{\varepsilon}{h}$.
- Introduction of a intermediate diffusion scheme called DA_h^ε.
- DA_h^ε scheme: P_h^ε with $\partial_t \mathbf{F}_j = \mathbf{0}$.
- In the estimates introduced for the proof we replace P_h^0 by DA_h^ε.

Condition H and final result

Condition H: The discrete Hessian matrix of the solution of P_h^0 can be bounded, or the error estimate $\|P_h^\varepsilon - P_h^0\|$ can be made independent of this discrete Hessian.

- Condition H respected : we use P_h^0 in the estimates.
- Condition H non respected : we use DA_h^ε in the estimates
- Condition H respected in Cartesian grids or on non uniform grids in 1D.

Final result: Assuming the geometrical and regularity assumptions are verified, there exist $C(T) > 0$ independent of ε, such that the following estimate holds:

$$\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2([0,T]\times\Omega)} \leq C \min\left(\sqrt{\frac{h}{\varepsilon}}, \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) + h + (h+\varepsilon) + \varepsilon\right) \leq C h^{\frac{1}{4}}. \quad (2)$$

- case $\varepsilon \leq h$: $\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\| \leq C_1 \min(\sqrt{\frac{\varepsilon}{h}}, 1) \leq C_1 h$
- case $\varepsilon \geq h$: $\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\| \leq C_1 \min(\sqrt{\frac{h}{\varepsilon}}, \sqrt{\frac{\varepsilon^3}{h}})$
- Introducing $\varepsilon_{thresh} = h^{\frac{1}{2}}$ we obtain that the worst case is $\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\| \leq C_2 h^{\frac{1}{4}}$.

Intermediary results I

- $\mathbf{V}^{\varepsilon}$ exact solution of P^{ε}, $\mathbf{V}_h^{\varepsilon}$ numerical solution of P_h^{ε}.

Estimate $||\mathbf{V}^{\varepsilon} - \mathbf{V}_h^{\varepsilon}||$:

We assume that the geometrical and regularity assumptions are verified. There exist a constant $C > 0$ such that the following estimate holds:

$$\|\mathbf{V}_h^{\varepsilon} - \mathbf{V}^{\varepsilon}\|_{L^{\infty}((0,T):L^2(\Omega))} \leq C\sqrt{\frac{h}{\varepsilon}}.$$

- Technical proof. Ideas:
  - Control the stability of the discrete quantities $\mathbf{u}_r$ and $\mathbf{u}_j$ by ε
  - We define $E(t) = ||\mathbf{V}^{\varepsilon} - \mathbf{V}_h^{\varepsilon}||_{L^2}$ and estimate $E^{'}(t)$ using Young and Cauchy-Schwartz inequalities, stability estimates, geometrical properties and a lot of calculus.
  - Integration in time of the estimate on $E^{'}(t)$.

Intermediary result II

- $\mathbf{V}_h^0$ solution of DA_h^ε, $\mathbf{V}^0$ exact solution of P^0.

Estimate $||DA_h^\varepsilon - P^0||$:

We assume that the geometrical and regularity assumptions are verified. There exist non negative constant for all time $T > 0$ $C(T)$ such that

$$||\mathbf{V}_h^0 - \mathbf{V}^0||_{L^2(\Omega)} \leq C_1(T)(h + \varepsilon), \qquad 0 < t \leq T. \tag{3}$$

- Ideas of proof:
  - Stability estimates on the discrete quantities E_j and the discrete gradiants at the node.
  - Consistency of the different discrete operators: divergence and gradient.
  - L^2 estimates using the consistency errors + Gronwall Lemma.

Intermediary result III

- $\mathbf{V}_h^\varepsilon$ solution of P_h^ε, $\mathbf{V}_h$ numerical solution of DA_h^ε.

Estimate $||P_h^\varepsilon - DA_h^\varepsilon||$:

We assume that the geometrical and regularity assumptions are verified. There exist a constant for all time $T > 0$, $C(T)$ such that

$$||\mathbf{V}_h^\varepsilon - \mathbf{V}_h||_{L^2(\Omega)} \leq C(T)\varepsilon \max\left(1, \sqrt{\varepsilon h^{-1}}\right) + Ch, \qquad 0 < t \leq T. \tag{4}$$

- $\mathbf{V}^\varepsilon$ solution of P^ε, $\mathbf{V}^0$ numerical solution of P^0.

Estimate $||P^\varepsilon - P^0||$:

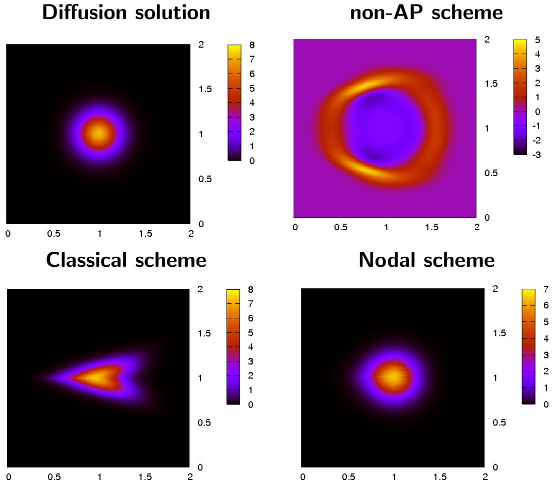
We assume that the geometrical and regularity assumptions are verified. There exist a constant for all time $T > 0$, $C(T)$ such that

$$||\mathbf{V}^\varepsilon - \mathbf{V}^0||_{L^2(\Omega)} \leq C(T)\varepsilon, \qquad 0 < t \leq T. \tag{5}$$

- Idea of Proof
  - Write $P^0 = P^\varepsilon + R$ (resp $DA_h^\varepsilon = P_h^\varepsilon + R$) with R a residue.
  - Obtain a upper bound of the residue by ε.
  - L^2 estimate of the difference between the two models or two schemes.

AP scheme vs non AP scheme

- Test case: Heat fundamental solution, $\varepsilon = 0.001$. Results given by hyperbolic P_1 schemes on Kershaw mesh.



Convergence for P_1 system

- Periodic solution of the P_1 system dependant of ε.
- $E(t,x) = (\alpha(t) + \frac{\varepsilon^2}{\sigma}\alpha'(t))\cos(\pi x)\cos(\pi y)$
- $\mathbf{F}(t,x) = \left(-\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi x)\cos(\pi y), \quad -\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi y)\cos(\pi x)\right)$
- Mesh: random quadrangular mesh.

h/ε	1	0.01	0.001	0.00001
40-80	1.0	1.3	1.9	2.0
80-160	1.05	1.05	1.85	2.0
100-200	1.05	0.8	1.75	2.0
150-300	1.05	0.5	1.65	2.0
240-480	1.05	0.45	1.5	2.0

- Hyperbolic regime $h << \varepsilon$: order 1 (theoretically $\frac{1}{2}$).
- Diffusion regime $h >> \varepsilon$: order 2 (theoretically 1).
- Intermidiate regime $h = O(\varepsilon)$: between $\frac{1}{2}$ and $\frac{1}{4}$.
- Future convergence analysis: $\varepsilon = h$ and $\varepsilon = h^{\frac{1}{2}}$.

Nonlinear extension: M_1 and Euler models

Euler equations with friction and gravity

- Work in progress.
- Euler equations with gravity and friction:

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t \rho \mathbf{u} + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon}\nabla p = \frac{1}{\varepsilon}(\rho \mathbf{g} - \frac{\sigma}{\varepsilon}\rho \mathbf{u}), \\ \partial_t \rho e + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u} e) + \operatorname{div}(p \mathbf{u}) = \frac{1}{\varepsilon}(\rho(\mathbf{g}, \mathbf{u}) - \frac{\sigma}{\varepsilon}\rho(\mathbf{u}, \mathbf{u})). \end{cases}$$

Properties :

- Entropy inequality: $\partial_t \rho S + \frac{1}{\varepsilon}\operatorname{div}(\mathbf{u} S) \geq 0$.
- Steady states :

$$(E_1) \begin{cases} \mathbf{u} = 0, \\ \nabla p = \rho \mathbf{g}. \end{cases} \qquad (E_2) \begin{cases} \mathbf{u} = 0, \quad \rho = \rho_c, \\ \nabla p = \rho_c \mathbf{g} \end{cases} \qquad \text{(simple case)}.$$

- Diffusion limit:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t \rho e + \operatorname{div}(\rho \mathbf{u} e) + p \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u} = \frac{1}{\sigma}\left(\mathbf{g} - \frac{1}{\rho}\nabla p\right). \end{cases}$$

AP scheme for Euler equations

Idea :
Lagrangian + remap nodal scheme coupled with the Jin-Levermore method.

- **Discretization** :
  - Lagrangian part: GLACE scheme (B. Després) or EUCCLYD scheme (P H. Maire).
  - Modification of the fluxes plugging the relation $\nabla p = \rho \mathbf{g} - \frac{\sigma}{\varepsilon} \rho \mathbf{u}$.
  - Discretization of the source term using the nodal fluxes.
  - Nodal advection scheme for remap step.
- **Properties** :
  - AP scheme with a first order limit diffusion scheme (order verified in the isothermal case with linear pressure).
  - Well Balanced for the simple steady state.
  - Positivity of the density on a CFL condition independent to ε.
  - Entropy inequality preserved ???
  - Well Balanced for the general steady states modifying the scheme ???

Well balanced property

Result :
If the initial data satisfy the discrete steady state $\nabla_r p = \rho_r \mathbf{g}$ the steady state is preserved exactly

- ρ_r is a mean at the node of the density
- Continuous steady state: $\rho(\mathbf{x})$, $\mathbf{u}(\mathbf{x})$ and $e(\mathbf{x})$.
- Question : if $\rho_j^0 = \rho(\mathbf{x}_j)$, $\mathbf{u}_j^0 = \mathbf{u}(\mathbf{x}_j)$ and $e_j^0 = e(\mathbf{x}_j)$ the discrete steady state is satisfied ?
  - for ρ constant : yes.
  - for ρ variable : probably not.

- Text case with steady state as initial data: $\mathbf{u}_j = \mathbf{0}$, $\rho_j = 1$ and $e_j = \frac{1}{\gamma-1}(\mathbf{x}_j, \mathbf{g})$.

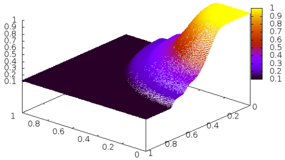
Meshes	Cartesian			Random		
Schemes/cells	40	80	120	40	80	120
LP-AP Ex	0	0	0	0	0	0
LP-AP SI	0	0	0	0	0	0
LP	1×10^{-8}	$\times 10^{-9}$	3×10^{-9}	0.50	0.26	0.17

- L^1 error associated to the differents schemes.

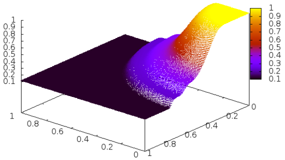
Results for Euler equations

- Test case: Sod problem with $\sigma > 0$, $\varepsilon = 1$ and $g = 0$ (non longer time limit).
- $\sigma = 1$

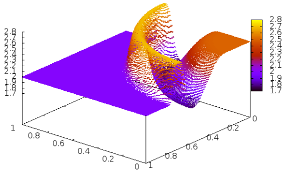
AP scheme, ρ



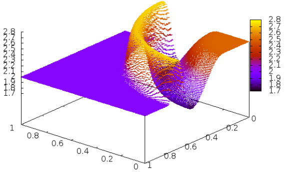
non-AP scheme, ρ



AP scheme, ϵ



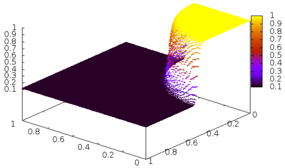
non-AP scheme, ϵ



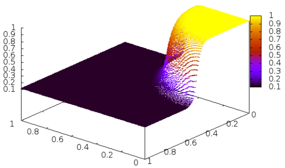
Results for Euler equations

- Test case: Sod problem with $\sigma > 0$, $\varepsilon = 1$ and $g = 0$ (non longer time limit).
- $\sigma = 10^3$

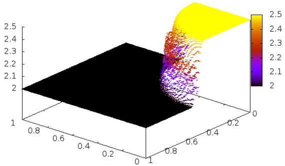
AP scheme, ρ



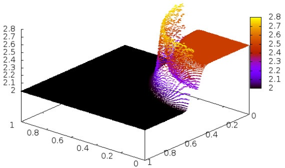
non-AP scheme, ρ



AP scheme, ϵ



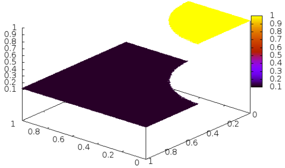
non-AP scheme, ϵ



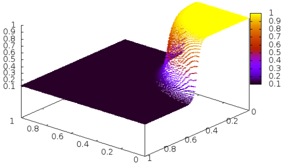
Results for Euler equations

- Test case: Sod problem with $\sigma > 0$, $\varepsilon = 1$ and $g = 0$ (non longer time limit).
- $\sigma = 10^6$

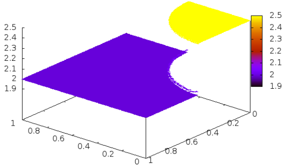
AP scheme, ρ



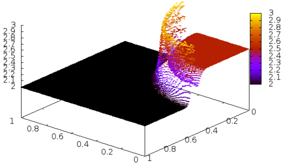
non-AP scheme, ρ



AP scheme, ϵ



non-AP scheme, ϵ



M_1 model and link with Euler equations

- Moment model in radiative transfer : M_1 model.

$$\begin{cases} \partial_t E + \frac{1}{\varepsilon} \operatorname{div} \mathbf{F} = 0, \\ \partial_t \mathbf{F} + \frac{1}{\varepsilon} \nabla(\widehat{P}) = -\frac{\sigma}{\varepsilon^2} \mathbf{F}, \end{cases} \tag{6}$$

- E energy, **F** the flux and $\widehat{P} = \frac{1}{2}((1-\chi(\mathbf{f}))Id + (3\chi(\mathbf{f})-1)\frac{\mathbf{f}\otimes\mathbf{f}}{||\mathbf{f}||})E$ the pressure.
- $\mathbf{f} = ||\mathbf{F}||/E$ and $\chi(\mathbf{f}) = \frac{3+4\mathbf{f}^2}{5+2\sqrt{4-3\mathbf{f}^2}}$.

Properties :

- Diffusion limit, $\varepsilon \to 0$: $\partial_t E - \operatorname{div}(\frac{1}{3\sigma}\nabla E) = 0$,
- Entropy inequality: $\partial_t S + \frac{1}{\varepsilon}\operatorname{div}(\mathbf{Q}) \geq 0$,
- Maximum principle: $E > 0$, $||\mathbf{f}|| < 1$.

M_1 model and link with Euler equations

- Moment model in radiative transfer : M_1 model.

$$\left\{\begin{array}{l}\partial_t E + \frac{1}{\varepsilon}\,\mathrm{div}\,\mathbf{F} = 0,\\[2ex] \partial_t \mathbf{F} + \frac{1}{\varepsilon}\nabla(\widehat{P}) = -\frac{\sigma}{\varepsilon^2}\mathbf{F},\end{array}\right. \iff \left\{\begin{array}{l}\partial_t E + \frac{1}{\varepsilon}\,\mathrm{div}(E\mathbf{u} + q\mathbf{u}) = 0,\\[2ex] \partial_t \mathbf{F} + \frac{1}{\varepsilon}\,\mathrm{div}(\mathbf{u}\otimes\mathbf{F}) + \frac{1}{\varepsilon}\nabla q = -\frac{\sigma}{\varepsilon^2}\mathbf{F}.\end{array}\right. \quad (6)$$

- E energy, $\mathbf{F}$ the flux and $\widehat{P} = \frac{1}{2}((1-\chi(\mathbf{f}))Id + (3\chi(\mathbf{f})-1)\frac{\mathbf{f}\otimes\mathbf{f}}{||\mathbf{f}||})E$ the pressure.
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Properties :

- Diffusion limit, $\varepsilon \to 0$: $\partial_t E - \mathrm{div}(\frac{1}{3\sigma}\nabla E) = 0$,
- Entropy inequality: $\partial_t S + \frac{1}{\varepsilon}\,\mathrm{div}(\mathbf{Q}) \geq 0$,
- Maximum principle: $E > 0$, $||\mathbf{f}|| < 1$.

- Formulation like dynamic gas system:
- $\mathbf{F} = \mathbf{u}E + q\mathbf{u}$, $\quad \hat{P} = \mathbf{u}\otimes\mathbf{F} + q\hat{I}_d$, $\quad q = \frac{1-\chi}{2}E$, $\mathbf{u} = \frac{3\chi-1}{2}\frac{\mathbf{f}}{||\mathbf{f}||^2}$.

Scheme and properties

Idea :
Use the proximity between the new formulation (previous slide) and the Euler equations to use a Lagrange+remap AP scheme.

- Link with the Euler equations:
  - Introduction of artificial density ρ
  - Introduction of Thermodynamics quantities: $E = \rho e$, $\mathbf{F} = \rho \mathbf{v}$ and $\mathbf{S} = \rho s$.
  - We apply the AP scheme for hydrodynamics equations obtained and after we obtain a scheme for the quantities E and $\mathbf{F}$.

- **Properties** :
  - AP scheme with a first order nonlinear limit diffusion scheme.
  - Numerical stability independent to ε (verified numerically).
  - Second order limit diffusion scheme using a MUSCL procedure in the remap step.
  - Entropy inequality preserved by the semi-discrete scheme.
  - Maximum principle preserved by the scheme without MUSCL in the transport regime ($\varepsilon = O(1)$).

Results for M_1 model

- **Diffusion test case**: The data are $E(0,\mathbf{x}) = G(\mathbf{x})$ with $G(\mathbf{x})$ a Gaussian and $\sigma = 1$. Final time $T_f = 0.011$.

Schemes	NL		VF5		Linear		M_1	
Meshes	order	$E_j > 0$	order	$E_j > 0$	order	$E_j > 0$	order	$E_j > 0$
Cartesian	1.9	yes	2	yes	2	yes	2.0	yes
Rand. quad	1.9	yes	0.3	yes	1.98	no	2.	yes
Regular. tri.	2.2	yes	2	yes	2.	yes	2.0	yes
Rand. tri.	2.15	yes	1.	yes	1.32	no	1.9	yes
Kershaw	1.9	yes	0	yes	2	no	1.9	yes

- NL : Limit diffusion scheme of M_1 scheme. M_1 : AP scheme for M_1 model with $\varepsilon = 10^{-3}$.
- **Discrete Maximum principle**: the data are $\sigma = 0$, $E(0,\mathbf{x}) = F_x(0,\mathbf{x}) = \mathbf{1}_{[0.4:0.6]^2}$ et $F_y(0,\mathbf{x}) = 0$. The solution is $E(t,\mathbf{x}) = F_x(t,\mathbf{x}) = \mathbf{1}_{[0.4+t:0.6+t]^2}$ and $F_y(t,\mathbf{x}) = 0$.

Meshes	order	$E_j > 0$	$\parallel \mathbf{f}_j \parallel < 1$
Cartesian	0.5	yes	yes
Rand. quad	0.5	yes	yes
Kershaw	0.49	yes	yes

Conclusion and future works

- Conclusion
  - P_1 **model**: AP nodal scheme on distorted meshes with a stability independant of ε.
  - P_1 **model**: Uniform convergence for the semi discrete scheme on unstructured meshes.
  - **Non linear model** : AP scheme with maximum principle for the M_1 and extension for the Euler equations.
  - **All models** : Spurious mods in few cases (example: Cartesian mesh + initial Dirac data).
- Future works
  - Numerical convergence analysis for test cases with nonlinear diffusion limit (full Euler and isothermal Euler equations).
  - Analysis of the P_1 AP discretization: Time convergence, CFL condition.
  - Analysis of the Euler AP discretization: entropy stability.
  - Extension to Euler scheme for non constant gravity and more complicated steady states.
  - Generic stabilization procedure the for nodal schemes.
- Others works
  - AP schemes for generic linear systems with source terms (P_N, S_N models) using "micro-macro" decomposition.
  - AP scheme for P_1 model based on the MPFA diffusion scheme.

Thank you

Thank you for your attention