Time implicit scheme for JOREK MHD code: Newton procedure, continuation and preconditioning

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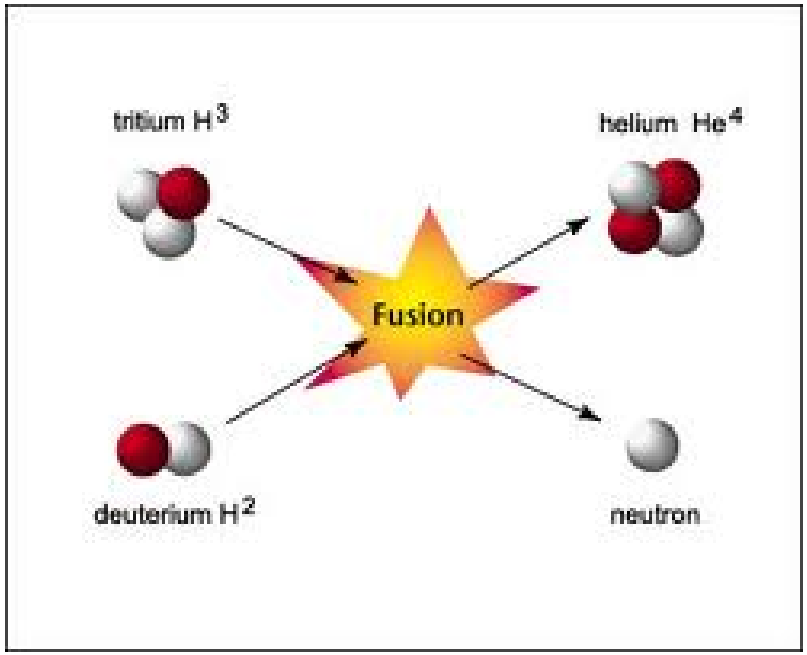
Physical context and models

Iter

- **Fusion DT**: Reaction between Deuterium and tritium which product Helium and energy. The deuterium and tritium form a plasma (ionized gas).



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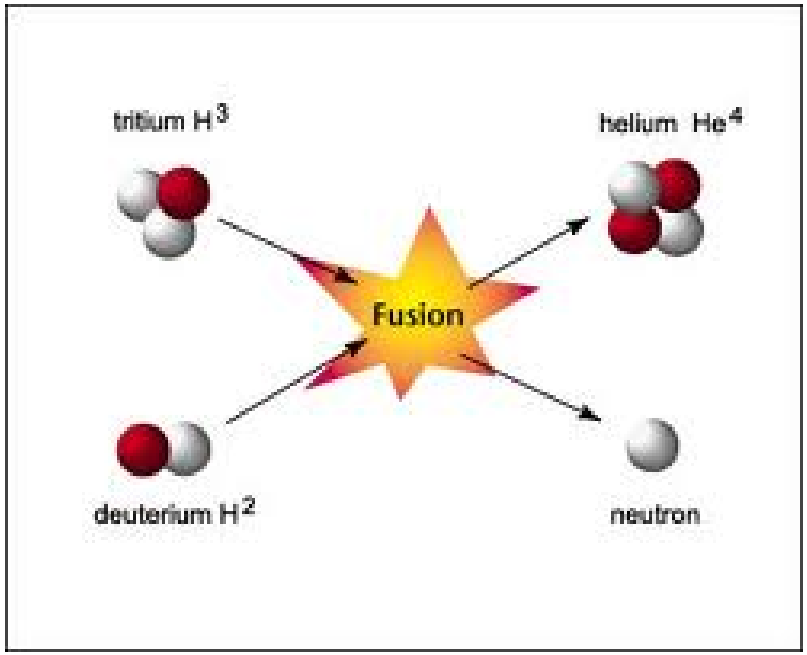

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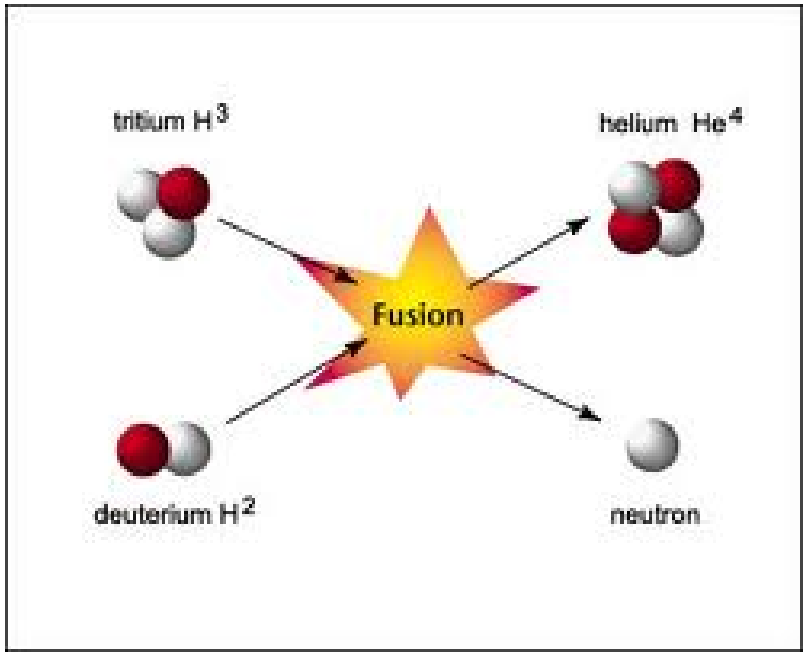
- **Magnetic confinement**: The plasma obtained by the reaction is confined in the center of the reactor (tokamak) using a powerful magnetic field.



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- **Tokamak**: Toroidal room used for the plasma confinement.

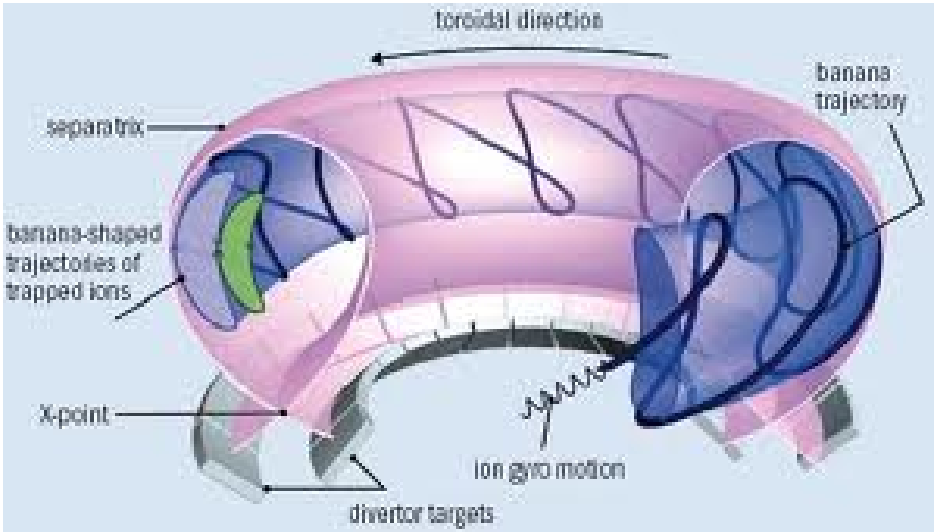


Figure: Tokamak

Models for Iter

- The dynamic of the plasmas in Iter is a very difficult multiscale problem.

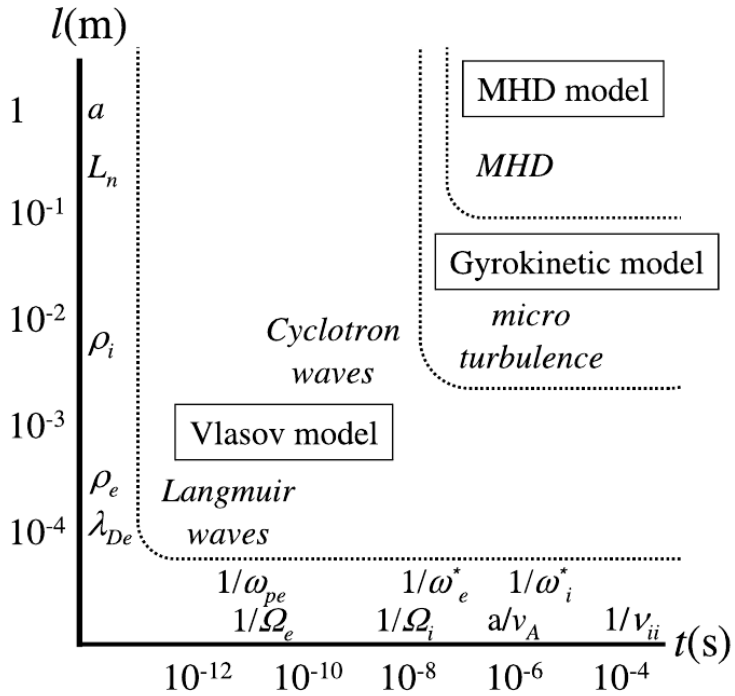


Figure: Spatial and time scales

Models for Iter

- The dynamic of the plasmas in Iter is a very difficult multiscale problem.
- We have different models for the different time and space scales :
  - Kinetic Vlasov-Maxwell equation not use for instance in the codes.
  - Gyrokinetic approximation of the Vlasov-Maxwell equation used for the turbulence in the core tokamak
  - MagnetoHydrodynamics fluids models (resistive MHD, two fluids MHD) used to simulate the edge instabilities.

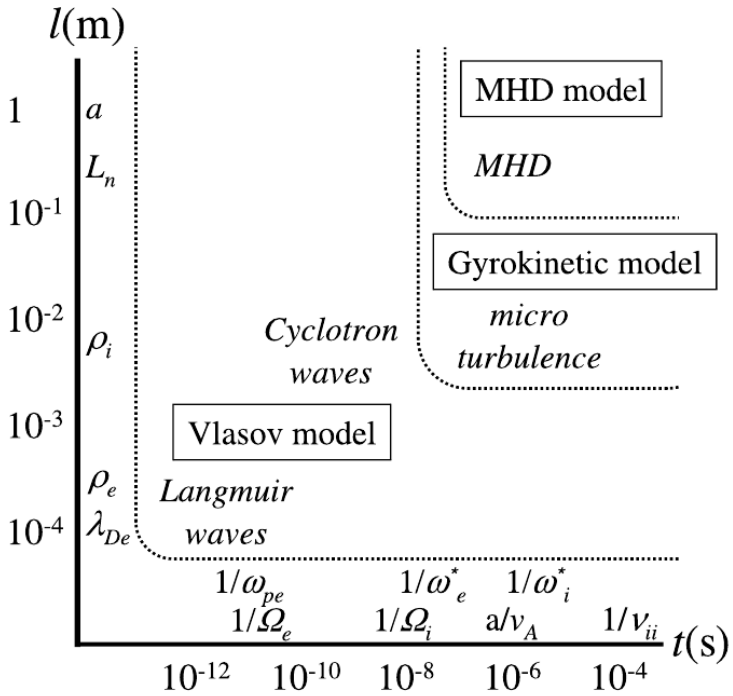


Figure: Spatial and time scales

ELMs and instabilities

- An edge-localized mode ("ELM's") is a disruptive instability occurring in the edge region of a tokamak plasma.
- The development of edge-localized modes poses an important challenge in magnetic fusion research with tokamaks. Instabilities can damage wall components due to their extremely high energy transfer rate.
- **Aim**: simulate the ELM's to estimate the amplitude of these instabilities and understand how control these.

- *MHD stability in X-point Geometry: simulation of ELMs*, G. Huysmans, O. Czarny, Nuclear fusion, 2007.
- *Reduced magnetohydrodynamic simulation of toroidally and poloidally localized edge localized modes*, M. Hölzl and co-workers, Phys. of Plasmas, 2012.

MHD model

- The full- resistive MHD model is given by

$$\begin{cases} \partial_t \rho + \nabla.(\rho \mathbf{v}) = \nabla.(D_{||}\nabla_{||}\rho + D_{\perp}\nabla_{\perp}\rho) + S_p \\ \rho\partial_t \mathbf{v} + \rho\mathbf{v}.\nabla\mathbf{v} + \nabla(\rho T) = \mathbf{J}\times\mathbf{B} + \nu\triangle\mathbf{v} \\ \rho\partial_t T + \rho\mathbf{v}.\nabla T + (\gamma-1)\rho T\nabla\mathbf{v} = \nabla.(K_{||}\nabla_{||}T + K_{\perp}\nabla_{\perp}T) + S_h \\ \partial_t \mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \nabla\times\eta\mathbf{J} + S_c \\ \nabla.\mathbf{B} = \mathbf{0} \end{cases} \quad (1)$$

with ρ the density, $\mathbf{v}$ the velocity, T the temperature, $\mathbf{B}$ the magnetic field and $\mathbf{J} = \nabla\times\mathbf{B}$ the current.

- The terms $D_{||}$, $D_{\perp}$, $K_{||}$, $K_{\perp}$ are anisotropic diffusion tensors.
- Source terms: S_c correspond to the current source, S_h correspond to the heat source, S_p correspond to the particle source.

Reduced MHD: assumption and derivation

- We consider the cylindric coordinates $(R, Z, \phi) \in \Omega \times [0, 2\pi]$.
- (R, Z) correspond to the poloidal plan and ϕ the toroidal direction.

Reduced MHD: assumptions

$$\mathbf{B} = \frac{F_0}{R}\mathbf{e}_\phi + \frac{1}{R}\nabla\Psi \times \mathbf{e}_\phi \quad \mathbf{v} = -R\nabla u \times \mathbf{e}_\phi + v_{||}\mathbf{B}$$

with u the electrical potential and ψ the poloidal magnetic flux.

- For the reduced MHD the quantities are ρ, T, Ψ, u,$v_{||}$ the parallel velocity, w the vorticity and z_j the toroidal current.
- Derivation: Plug $\mathbf{B}$ and $\mathbf{v}$ in the density, magnetic and energy equations. For the equations on u and $v_{||}$ we use

$$R\mathbf{e}_\phi.\nabla \times (\rho\partial_t\mathbf{v} + \rho\mathbf{v}.\nabla\mathbf{v} + \nabla(\rho T) = \mathbf{J} \times \mathbf{B} + \nu\triangle\mathbf{v})$$

and

$$\mathbf{B}.(\rho\partial_t\mathbf{v} + \rho\mathbf{v}.\nabla\mathbf{v} + \nabla(\rho T) = \mathbf{J} \times \mathbf{B} + \nu\triangle\mathbf{v}).$$

Reduced MHD without $v_{||}$: most simple model

- With $v_{||} = 0$ we obtain the model considered in this talk.
- We solve $\partial_t A(\mathbf{U}) = B(\mathbf{U}, t)$ with

$$B(\mathbf{U}) = \begin{pmatrix} [\Psi, u] - \epsilon \frac{F_0}{R} \partial_\phi u + \frac{\eta(T)}{R}(z_j - S_c(\Psi)) - \eta_n \nabla.(\nabla z_j) \\ \frac{1}{2}[R^2 ||\nabla u||^2, \hat{\rho}] + [R^2 \hat{\rho} w, u] + [\Psi, z_j] - \epsilon \frac{F_0}{R} \partial_\phi z_j - [R^2, p] \\ + \nabla.(R\nu(T)\nabla w) - \nu_n \nabla.(\nabla w) \\ \frac{1}{R^2} z_j - \nabla.(\frac{1}{R^2} \nabla \Phi) \\ w - \nabla.(\nabla u) \\ R^2[\rho, u] + 2R\rho \partial_Z u + \nabla.(D_{||} \nabla_{||} \rho + D_\perp \nabla_\perp \rho) + S_p(\Psi) \\ R^2[T, u] + 2(\gamma - 1) R T \partial_Z u + \nabla.(K_{||} \nabla_{||} T + K_\perp \nabla_\perp T) + S_h(\Psi) \end{pmatrix}$$

with $\hat{\rho} = R^2 \rho$ and $\partial_t A(\mathbf{U}) = \left(\frac{1}{R}\partial_t \Psi,\ R\nabla.(\hat{\rho}\nabla(\partial_t u)),\ 0,\ 0,\ R\partial_t \rho,\ R\partial_t T\right)$.

- $\eta(T)$ and η_n are the physical and numerical resistivity. $\nu(T)$ and ν_n are viscosity coefficients.

Jorek Code: description

Description of the jorek code I

- Jorek: Fortran 90 code, parallel (MPI+OpenMP) + algebraic libraries (Pastix, MUMPS ...)
- Initialization
- Determinate the equilibrium
  - Define the boundary of the computational domain
  - Create a first grid which is used to compute the aligned grid
  - Compute $\psi(R, Z)$ in the new grid.
- Compute equilibrium
  - Solve the Grad-Shafranov equation

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \Psi}{\partial R}\right) + \frac{\partial^2 \Psi}{\partial Z^2} = -R^2\frac{\partial p}{\partial \Psi} - F\frac{\partial F}{\partial \Psi}$$

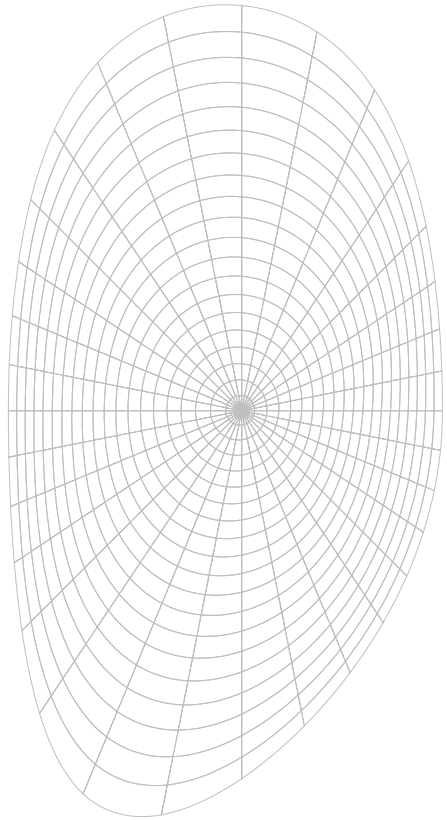


Figure: unaligned grid

Description of the jorek code II

- Computation of aligned grid
  - Identification of the magnetic flux surfaces
  - Create the aligned grid (with x-point)
  - Interpolate $\Psi(R, Z)$ in the new grid.
- Recompute equilibrium of the new grid.
- Time-stepping (restart)
  - Construction of the matrix and some profiles (diffusion tensors, sources terms)
  - Solve linear system
  - Update solutions
  - Plot kinetic magnetic energies and restart files.

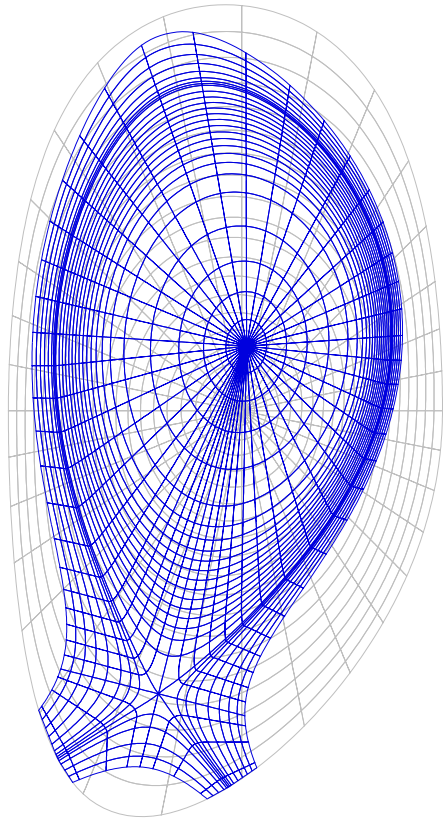


Figure: Aligned grid

Spatial discretization

- The equations in the poloidal plane are discretized using finite element method.
- For the toroidal direction: Fourier expansion.
- **Basis functions**: Cubic Bezier elements
  - Generalization of cubic Hermite elements.
  - The generalization allows the local refinement of each element essential for adaptive mesh refinement.
  - 4 degrees of freedom by node to describe a function.
  - The isoparametric formulation (discretization of (R, Z) using the Bezier elements) the finite elements can be accurately aligned with the equilibrium flux surfaces.
  - The Cubic Bezier elements assure a C^1 polynomial reconstruction.

Only moment with involutive constrains

- Potential writing of Magnetic field assure that $\mathrm{div}\,\mathbf{B} = 0$.

- *Bezier surfaces and finite elements for MHD simulations*, O. Czarny, G. Huysmans, JCP 2008.

Time scheme in Jorek code

- We recall the model $\partial_t A(\mathbf{U}) = B(\mathbf{U}, t)$
- For time stepping we use a Crank Nicholson or BDF2 scheme :

$$(1+\zeta)A(\mathbf{U}^{n+1}) - \zeta A(\mathbf{U}^{n}) + \zeta A(\mathbf{U}^{n-1}) = \theta \Delta t B(\mathbf{U}^{n+1}) + (1-\theta)\Delta t B(\mathbf{U}^{n})$$

- Defining $G(\mathbf{U}) = (1+\zeta)A(\mathbf{U}) - \theta \Delta t B(\mathbf{U})$ and

$$b(\mathbf{U}^{n}, \mathbf{U}^{n-1}) = (1+2\zeta)A(\mathbf{U}^{n}) - \zeta A(\mathbf{U}^{n-1}) + (1-\theta)\Delta t B(\mathbf{U}^{n})$$

  we obtain the nonlinear problem

$$G(\mathbf{U}^{n+1}) = -G(\mathbf{U}^{n}) + b(\mathbf{U}^{n}, \mathbf{U}^{n-1})$$

- First order linearization

$$\left(\frac{\partial G(\mathbf{U}^{n})}{\partial \mathbf{U}^{n}}\right) \delta \mathbf{U}^{n} = -G(\mathbf{U}^{n}) + b(\mathbf{U}^{n}, \mathbf{U}^{n-1})$$

  with $\delta \mathbf{U}^{n} = \mathbf{U}^{n+1} - \mathbf{U}^{n}$ and $\frac{\partial G(\mathbf{U}^{n})}{\partial \mathbf{U}^{n}}$ the jacobian of $G(\mathbf{U}^{n})$.

Time scheme in Jorek code

- Linear solver in Jorek:
  - Case 1: Direct solver using pastix (using when $n_{tor} = 1$)
  - Case 2: Iterative solver
- Iterative Solver step 1: Preconditioning
  - Extraction of submatrices associated to each toroidal mode.
  - We factorize and solve exactly (with Pastix) each subsystem.
  - We construct the initial vector of GMRES using the solutions of these systems.
- Iterative solver step 2: GMRES solver for the global matrix.
- The submatrices are used as right preconditioning for the GMRES method.
- This preconditioning is based on the assumption that the coupling between the toroidal mods is weak and linear.
- Finally we predict the solution neglecting the coupling between the mods.
- In practice for some test cases this coupling is strongly nonlinear.

Jorek code: convergence issues

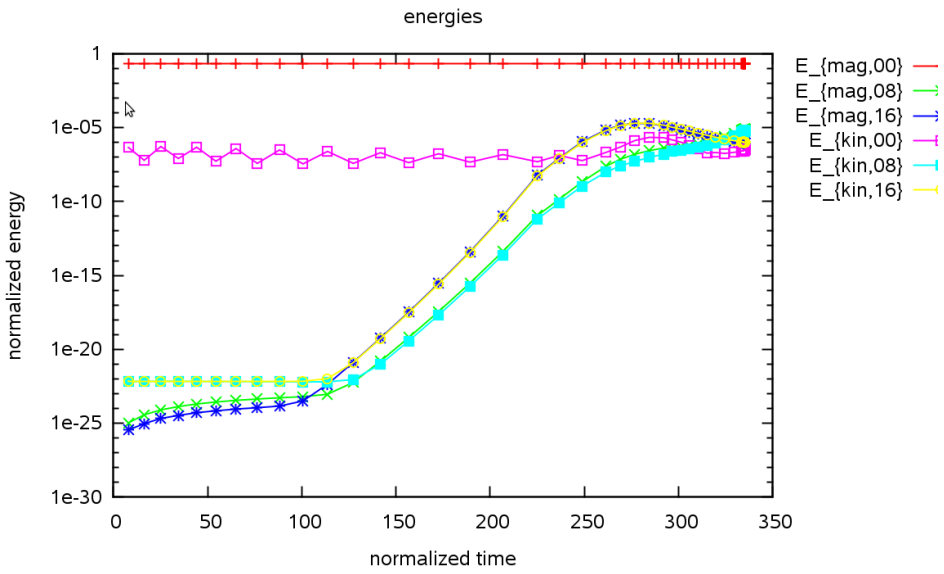
Problem :

- For some test cases the GMRES method does not converge in the non linear phase for large time step and small numerical viscosity terms.

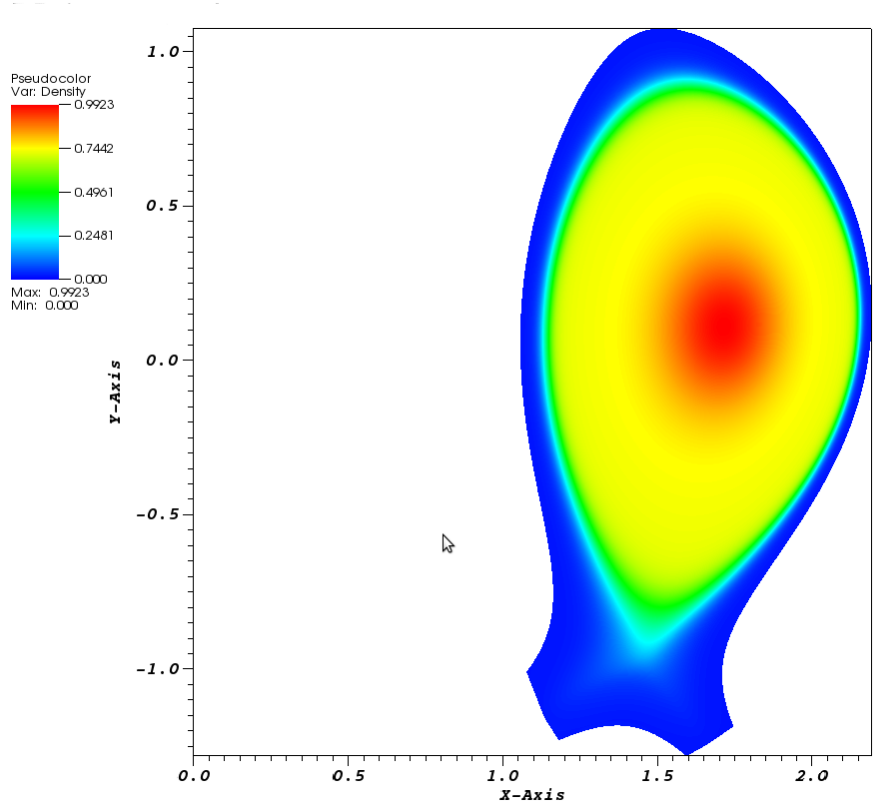
- Why ?
  - The preconditioning is not sufficient to obtain a robust GMRES method ?
  - The spatial poloidal and toroidal discretizations are not adapted ? Problem of positivity ?
  - The mesh is not adapted ? non C^1 polynomial reconstruction at the X-point (singularity of the mesh).
  - The models (linear and nonlinear hyperbolic parts) are not stables ?

Numerical example

- evolution of energy in time with adaptive time step.



- Density



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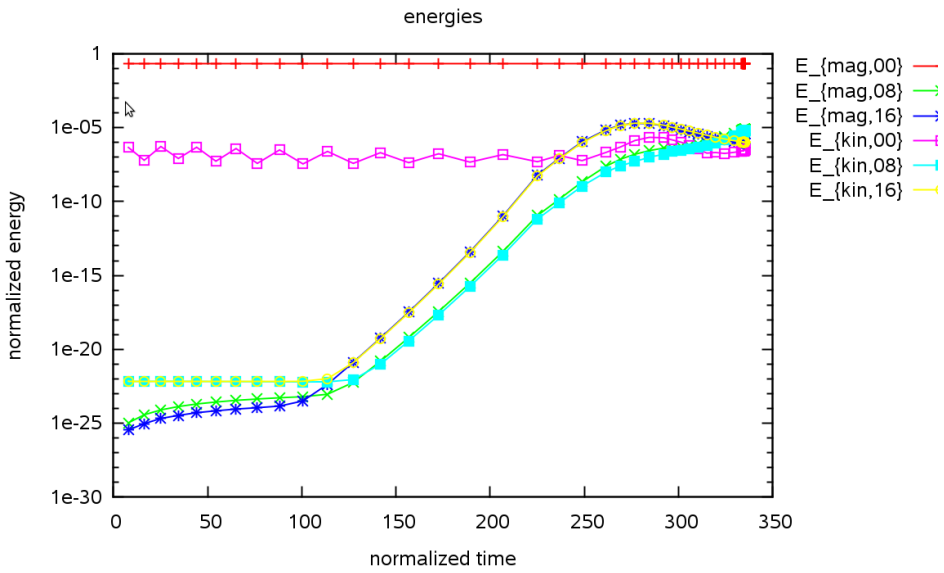

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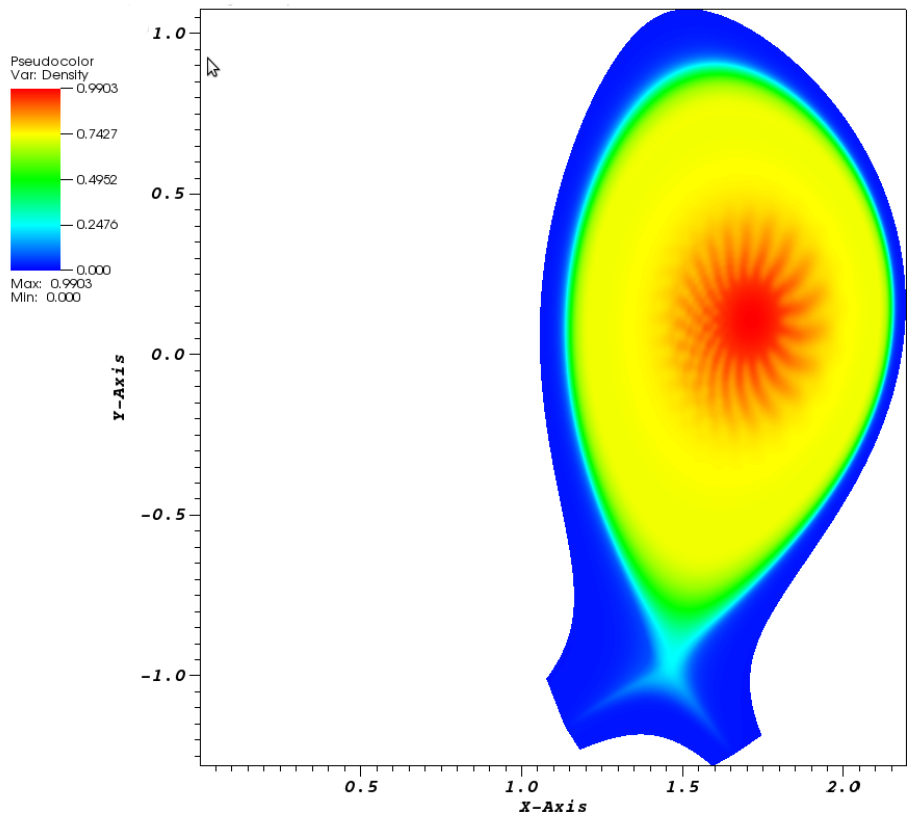

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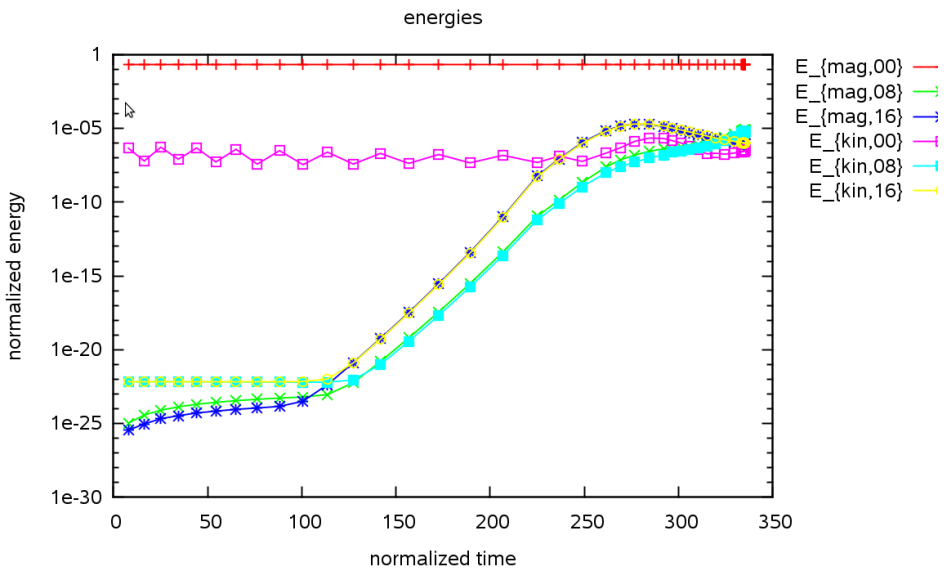

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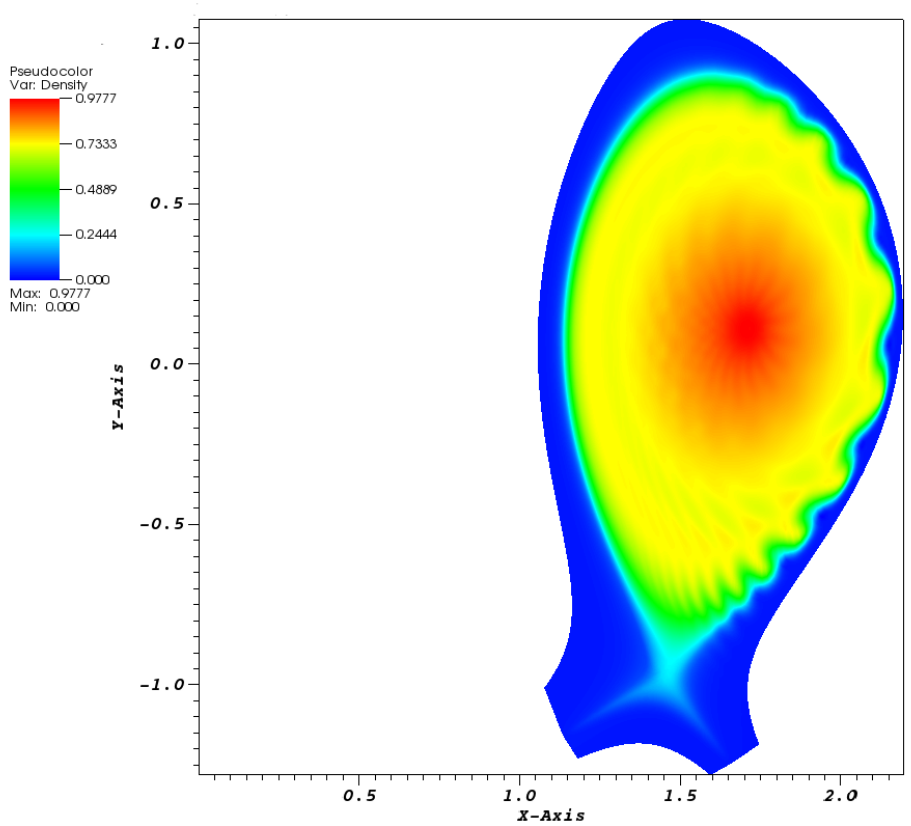

Newton and continuation methods, stabilization

Inexact Newton scheme

- At the time step n, we compute $b(\mathbf{U}^n, \mathbf{U}^{n-1})$, $G(\mathbf{U}^n)$
- We choose $\mathbf{U}_0 = \mathbf{U}^n$ and ε_0.
- Step k of the Newton procedure
  - We compute $G(\mathbf{U}_k)$ and $\left(\frac{\partial G}{\partial \mathbf{U}_k}\right)$
  - We solve the linear system with GMRES

$$\left(\frac{\partial G(\mathbf{U}_k)}{\partial \mathbf{U}_k}\right) \delta \mathbf{U}_k = \tilde{G}(\mathbf{U}_k) = b(\mathbf{U}^n, \mathbf{U}^{n-1}) - G(\mathbf{U}_k)$$

and the following convergence criterion

$$\frac{||\left(\frac{\partial G}{\partial \mathbf{U}_k}\right) \delta \mathbf{U}_k + \tilde{G}(\mathbf{U}_k)||}{||\tilde{G}(\mathbf{U}_k)||} \leq \varepsilon_k, \quad \varepsilon_k = \gamma \left(\frac{||\tilde{G}(\mathbf{U}_k)||}{||\tilde{G}(\mathbf{U}_{k-1})||}\right)^{\alpha}$$

  - We iterate with $\mathbf{U}_{k+1} = \mathbf{U}_k + \delta \mathbf{U}_k$.
  - We apply the convergence test (for example $||\tilde{G}(\mathbf{U}_k)|| < \varepsilon_a + \varepsilon_r ||\tilde{G}(\mathbf{U}^n)||$)
- If the newton procedure stop we define $\mathbf{U}^{n+1} = \mathbf{U}_{k+1}$.

Dissipation continuation: principle

- Nonlinear problem: $R(\mathbf{U}) = 0$ not easy to solve.
- **Idea**: replace the initial problem by the homotopy mapping $F(\mathbf{U}, \lambda) = 0$ easier to solve.
- **Algorithm**
- Set $d_0 = d_{max}$ and $R_0 = R(\mathbf{U}^0)$
- For $i = 1, 2, 3....$ do
  - Set $F_0 = F(\mathbf{U}^i, d_i)$
  - For $n = 1, 2, 3..n_{max}$ do
    - Compute residual $R(\mathbf{U})$
    - if $||R(\mathbf{U})|| \leq \varepsilon_1 ||R_0||$ then done
    - Compute homotopy map $F(\mathbf{U}, d_i)$
    - if $||F(\mathbf{U})|| \leq \varepsilon_2 ||F_0||$ then exit loop
    - Inexact Newton step : find $\delta\mathbf{U}$ such as $||A\delta\mathbf{U} + F|| \leq \eta F$.
    - $\mathbf{U} = \mathbf{U} + \delta\mathbf{U}$ and end for
  - set $d_{i+1} < d_i$
- Dissipation continuation: $F(\mathbf{U}, d) = 0 = R(\mathbf{U}) + dD(\mathbf{U})$ with D a diffusion operator.

Dissipation continuation: remarks

- Diffusion system used in jorek with additional visocisty

$$\begin{cases} \partial_t \dfrac{\Psi}{R^2} = \dfrac{\eta(T)}{R} z_j - \lambda_1 \nabla.(\nabla z_j) \\ R\nabla.(\hat{\rho}\nabla(\partial_t u)) = \nabla.(R\nu(T)\nabla w) + \lambda_2 \nabla.(\nabla w) \\ R\partial_t \rho = \nabla.(D\nabla\rho) + \lambda_5 \nabla.(\nabla\rho) \\ R\partial_t T = \nabla.(K\nabla T) + \lambda_6 \nabla.(\nabla T) \end{cases}$$

  with $\hat{\rho} = R^2\rho$, $z_j = R^2\nabla.(\frac{1}{R^2}\nabla\Phi)$ and $w = \nabla.(\nabla u)$.

- energy estimate and dissipative system

$$\frac{d}{dt}\frac{1}{2}\int\left(\frac{||\nabla\Psi||^2}{R^2} + R\hat{\rho}||\nabla u||^2 + p^2\right) = -\eta(T)\int\frac{||\triangle\Psi||^2}{R^2} - d_1\frac{z_j^2}{R^2} - \int(d_2 + R\nu(T))w^2$$

$$-\int(||D|| + I_d d_5)T^2||\nabla\rho||^2 - \int(||K|| + I_d d_6)\rho^2||\nabla T||^2$$

- Problem: non positive values for T imply $\nu(T)$ and $\eta(T)$ negative and the system can be non dissipative.
- Problem: In jorek we use absolute value which can generate non C^1 problem and the non convergence.

Remarks about continuation method

- Preconditioning for Newton procedure: algorithm which gives a good initial guess for Newton procedure.
- The continuation method can be view as an Preconditioning for Newton procedure.
- Efficiency and robustness depend on the formula to compute the new diffusion coefficient.
- Example : $d_i = \max\left(\min\left(\alpha d_{i-1}, (\frac{||G(\mathbf{U}^k)||}{||G(\mathbf{U}^0)||})^\beta\right), d_{min}\right)$ with $\alpha < 1$ and $\beta > 1$.
- The continuation method can be used to find way to explain the non convergence of solvers.

New global structure of the time step

- Computation of the residue $G(\mathbf{U}^n)$
- Inexact Newton procedure
  - Computation of the Residue $G(\mathbf{U}^k)$ and the Jacobian.
  - We factorize ans solve each submatrices to construct an initial guess.
  - GMRES solver with right preconditioning.
- If the Newton procedure converge quickly, we begin the following time step with a bigger step time.
- If the Newton procedure converge not quickly, we begin the following time step with the same or a smaller time step.
- However we restart the time step with the dissipation continuation procedure.
- If the continuation procedure converge, we begin the following time step.
- However we restart the time step with a smaller time step.

Stabilization of reduced MHD Models I

- B. Després and R. Sart have proposed a new method to derive the reduced MHD models using the moment method.
- They shows that to obtain energy estimate for a class of reduced models we must had a term on the poloidal magnetic flux Ψ equation.
- This term satisfies a elliptic equation.
- Results for this model: linear instabilities in the toroidal direction.

- Preliminary study shows that the class of model used in Jorek does not admit this problem.
- However the numerical and theoretical stabilities of the nonlinear reduced models are not clear.

- Stability of linear toroidal terms for model with $v_{||}$?

- *Derivation of hierarchies of reduced MHD models in Tokama geometry*, B. Després, Rémy Sart, 2013.

Stabilization of reduced MHD Models II

- To understand the difference between the different point of view: weak form.
- Weak form used in Jorek

$$\begin{cases} \int (\partial_t \rho + \nabla.(\rho \mathbf{v}))\rho^* = 0 \\ \int (\partial_t \rho \mathbf{v} + \rho \mathbf{v}.\nabla \mathbf{v} + \nabla(\rho T) - \mathbf{J} \times \mathbf{B}) R^2 \nabla u^* \times \nabla \phi = 0 \\ \int (\partial_t \rho T + \rho \mathbf{v}.\nabla T + (\gamma - 1)\rho T \nabla \mathbf{v}) T^* = 0 \\ \int (\partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \times \eta \mathbf{J}) \frac{\psi}{R} = 0 \end{cases} \tag{2}$$

- with $\mathbf{B} = \nabla\psi \times \nabla\phi$ and $\mathbf{v} = R^2 \nabla u \times \nabla\phi$.
- Interesting to study the impact of this small modification on the stability and conditioning.

Stabilization of reduced MHD Models II

- To understand the difference between the different point of view: weak form.
- Weak form used in the moment method

$$\begin{cases} \int (\partial_t \rho + \nabla.(\rho \mathbf{v}))\rho^* = 0 \\ \int (\partial_t \rho \mathbf{v} + \rho \mathbf{v}.\nabla \mathbf{v} + \nabla(\rho T) - \mathbf{J} \times \mathbf{B}) R^2 \nabla u^* \times \nabla \phi = 0 \\ \int (\partial_t \rho T + \rho \mathbf{v}.\nabla T + (\gamma - 1)\rho T \nabla \mathbf{v}) T^* = 0 \\ \int (\partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \times \eta \mathbf{J}) \nabla \psi^* \times \nabla \phi = 0 \end{cases} \quad (2)$$

- with $\mathbf{B} = \nabla\psi \times \nabla\phi$ and $\mathbf{v} = R^2 \nabla u \times \nabla \phi$.
- Interesting to study the impact of this small modification on the stability and conditioning.

Conclusion about continuation and Newton methods

- Test and validate the Dissipation continuation method for some reduced models.
- Global strategy:
  - Basic method : Newton procedure
  - If the Newton method does not converge, the time step is recomputed using the dissipation continuation method.
  - If the Continuation method does not converge, the time step is recomputed using a smaller time step.
  - If the Newton method converge quickly we increase the time step.
- Expected result: Use Continuation method when Newton does not converge avoid to use too small time step,
- Study numerical and theoretical stability of the models following the ideas B. Després and R. Sart

Preconditioning

Preconditioning idea I

- *An optimal, parallel fully implicit Newton-Krylov solver for 3D viscoresistive Magnetohydrodynamics*, L. Chacon, Phys. of plasma, 2008.
- *Scalable parallel implicit solvers for 3D magnetohydrodynamics*, L. Chacon, Journal of Phys. 2009.

- **Aim**: Construct an algorithm which give a good prediction of the solution and which is good conditioned
  - The algorithm must give a solution of $A\delta\mathbf{U}^n = -G(\mathbf{U}^n) + b(\mathbf{U}^n, \mathbf{U}^{n-1})$ with $A \approx \frac{\partial G(\mathbf{U}^n)}{\partial \mathbf{U}^n}$.
  - A must be good-conditioned. **Idea**: parabolization of the coupled hyperbolic equations.
- Example

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \partial_x u \end{cases} \longrightarrow \begin{cases} u^{n+1} = u^n + \Delta t \partial_x v^{n+1} \\ v^{n+1} = v^n + \Delta t \partial_x u^{n+1} \end{cases}$$

- We obtain $(1 - \Delta t^2 \partial_{xx}) u^{n+1} = u^n + \Delta t \partial_x v^n$.
- The matrix associated to $(1 - \Delta t^2 \partial_{xx})$ is diagonal dominant and good conditioned.

Preconditioning idea II

- To apply easily this method for more complicated equations, we use a other interpretation.
- We assume that the matrix associated to the previous linear system is

$$\begin{pmatrix} D_1 & U \\ L & D_2 \end{pmatrix}$$

- Using a Schur decomposition we obtain

$$\begin{pmatrix} D_1 & U \\ L & D_2 \end{pmatrix} = \begin{pmatrix} I & UD_2^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} P_{schur} & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ D_2^{-1}L & I \end{pmatrix}$$

$$\begin{pmatrix} I & -\Delta t\partial_x \\ -\Delta t\partial_x & I \end{pmatrix} = \begin{pmatrix} I & -\Delta t\partial_x \\ 0 & I \end{pmatrix} \begin{pmatrix} P_{schur} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Delta t\partial_x & I \end{pmatrix}$$

- The first and third matrices are triangular donc easily to invert.
- $P_{schur} = D_1 - UD_2^{-1}L = (1 - \Delta t^2 \partial_{xx})$ is good conditioned.

Preconditioning : preparation and Schur decomposition

- We apply the Schur decomposition to the model without $v_{||}$. The system solved is

$$\frac{\partial G(\mathbf{U}^n)}{\partial \mathbf{U}^n}\delta\mathbf{U}^n = \begin{pmatrix} D_\psi & 0 & D_{\psi,T} & D_{\psi,z_j} & 0 & U_{\psi,u} \\ 0 & D_\rho & 0 & 0 & 0 & U_{\rho,u} \\ 0 & 0 & D_T & 0 & 0 & U_{T,u} \\ D_{z_j,\psi} & 0 & 0 & D_{z_j} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_w & D_{w,u} \\ L_{u,\psi} & L_{u,\rho} & L_{u,T} & L_{u,z} & L_{u,w} & D_u \end{pmatrix} \delta\mathbf{U}^n = \tilde{G}(\mathbf{U}^n)$$

  with $\delta\mathbf{U}^n = (\delta\psi, \delta\rho, \delta T, \delta z_j, \delta w, \delta u)$ and $\tilde{G}(\mathbf{U}^n) = -G(\mathbf{U}^n) + b(\mathbf{U}^n, \mathbf{U}^{n-1})$.
- The terms D contains advection and diffusion operators.
- The terms L and U contains non linear coupling hyperbolic operators.
- We reduce the number on variable using the definition of w and z_j.

$$\frac{\partial G(\mathbf{U}^n)}{\partial \mathbf{U}^n}\delta\mathbf{U}^* = \begin{pmatrix} D^*_\psi & 0 & D^*_{\psi,T} & U_{\psi,u} \\ 0 & D_\rho & 0 & U_{\rho,u} \\ 0 & 0 & D_T & U_{T,u} \\ L^*_{u,\psi} & L^*_{u,\rho} & L^*_{u,T} & D^*_u \end{pmatrix} \delta\mathbf{U}^*$$

  with $\delta\mathbf{U} = (\delta\psi, \delta\rho, \delta T, \delta u)$

Preconditioning : Algorithm

- The final system with Schur decomposition is given by

$$\delta \mathbf{U}^n = \frac{\partial G(\mathbf{U}^n)}{\partial \mathbf{U}^n}^{-1} \tilde{G}(\mathbf{U}^n) = \begin{pmatrix} M & U \\ L & D_u \end{pmatrix}^{-1} \tilde{G}(\mathbf{U}^n)$$

$$= \begin{pmatrix} I & M^{-1}U \\ 0 & I \end{pmatrix} \begin{pmatrix} M^{-1} & 0 \\ 0 & P_{schur}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -LM^{-1} & I \end{pmatrix} \tilde{G}(\mathbf{U}^n)$$

  with $P_{schur} = D_u^* - LM^{-1}U$.
- M, D_u^* are associated to the advection and diffusion operators. L, U are associated to the hyperbolic coupling operators.

$$\begin{cases} \text{Predictor :} & M\delta \mathbf{v}_p^n = (-G_{\mathbf{v}}^n + B_{\mathbf{v}}^n) \\ \text{potential update :} & P_{schur}\delta u^n = \left(-L\delta \mathbf{v}_p^n - G_u^n + B_u^n\right) \\ \text{Corrector :} & M\delta \mathbf{v}^n = M\delta \mathbf{v}_p^n - U\delta u^n \\ \text{Current update :} & D_{z_j}\delta z_j^n = D_{z_j,\psi}\delta \psi^n \\ \text{Vorticity update :} & D_w \delta w^n = D_{w,u}\delta u^n \end{cases}$$

with $\delta v_p = (\delta\Psi, \delta\rho, \delta T)$, $G_{\mathbf{v}}$ and $B_{\mathbf{v}}$ the right hand side associated to the equations on Ψ, ρ and T.

Preconditioning : Approximation of the Schur complement

- For define the Schur complement $P_{schur} = D_u^* - LM^{-1}U$ we must know the matrix M^{-1}.
- Consequently we must approximate P_{schur}. Two approximations:
- Small flow approximation (L. Chacon)
  - In P_{schur} we assume that $M^{-1} \approx \Delta t$
  - **Mathematical problem**: estimate the operator LU.
- Arbitrary flow approximation (L. Chacon).
  - We introduce a operator M_* (in u-space) with $UM_* \approx MU$.
  - Consequently $P_{Schur} = (D_u M_* - LU)M_*^{-1}$ with LU given by the small flow approximation.
  - In this case the Potential udapte step in given by

$$\begin{cases} \text{potential update I :} & (D_u M_* - LU)\delta u^{*,n} = \left(-L\delta \mathbf{v}_p^n - G_u^n + B_u^n\right) \\ \text{potential update II :} & \delta u^n = M_* \delta u^{*,n} \end{cases}$$

  - **Mathematical problem**: estimate the operator M_*.

Extension for others reduced MHD and full MHD

- Extension of the PC-algorithm for the models with parallel velocity and full MHD.
- For the full-MHD we have

Helmholtz decomposition

$$\mathbf{v} = R\nabla u \times \mathbf{e}_\phi + Rw\mathbf{e}_\phi + \frac{1}{R^2}\nabla_\perp \chi$$

with u, w, χ scalar fluxes.

- u is associated mainly with the Alfven wave.
- w is associated mainly with the slow wave.
- χ is associated mainly with the fast wave.

- In the model 199, the choice of the velocity show the Alfven wave dominated.
- In the reduced MHD with parallel velocity and the full-MHD, the Schur complement is modified and the different type of wave are present.

AP scheme for anisotropic diffusion in Jorek

- Anisotropic diffusion

$$\partial_t \rho - \nabla.(D_{||} \nabla_{||} \rho + D_\perp \nabla_\perp \rho) = 0 \quad \text{with } D_\perp / D_{||} << 1$$

- It is known that the anisotropic diffusion operators are ill-conditioned.
- For instance the main problem of non convergence come from hyperbolic coupling. But it is possible that anisotropic diffusion impact the conditioning.
- In the future it will be interesting to study the impact of these terms and adapt the AP at the jorek code.

- *Asymptotic-Preserving schemes. Modeling, simulation and mathematical analysis of magnetically confined plasmas*, C. Negulescu.

Preconditioning: Remarks and schedule of work

- The PC preconditioning method use a prediction of the solution based on the approximation of the Schur complement.
- It is possible that this prediction of the solution is better than the prediction based on a jacobian when all the coupling terms between the mods are neglected.
- This preconditioning is compatible easily with free jacobian method.
- **Schedule of work**
  - Implement the Schur complement preconditioning in Pigasus code (Talk A. Ratnani) with multigrids methods for each subsystems.
  - Same study for some reduced models with density, temperature and parallel velocity.
  - In Pigasus study the efficient of AP schemes vs geometric multigrid methods (A. Ratnani) for anisotropic diffusion terms.
  - Implement the Schur complement preconditioning and multigrid methods in Jorek (objectives for semi-long time),

Thanks

Thanks for your attention